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Abstract

In this paper, we divide the players of a large game into countable different groups and assume that each player's payoff depends on her own action and the distribution of actions in each of the subgroups. Focusing on the interaction between Nash equilibria and the best response correspondence of the players, we characterize the pure-strategy equilibria in three settings of such large games, namely large games with countable actions, large games with countable homogeneous groups of players and large games with an atomless Loeb agent space. Furthermore, we also present a counterexample showing that a similar characterization result does not hold for large games under a more general setting.

1 Introduction

In this paper, we divide the players (agents) of a large game[1] into countable (finite or countably infinite) different groups and assume that each player's payoff depends on her own action and the distribution of actions in each of the subgroups. Such a game model is a generalization to the games considered in Khan and Sun

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[1]The large games discussed here are endowed with a separate agent space and called large non-anonymous games by some authors (see, eg, Khan and Sun (2002)).

(2002), where there is only a finite partition on the agent space. Beyond its own economic implications, this game model also has the motivation to connect with Bayesian games with countable players.[2] In such a large game, a pure-strategy action profile which assigns an action to each player is called a *pure-strategy Nash equilibrium* if no player has the incentive to change her assigned action. A distribution vector on the action space, whose components are the action distributions for respective groups, is called a *pure-strategy equilibrium distribution* if it is induced by a pure-strategy Nash equilibrium of the game.

In the past few decades, there have been a lot of famous existence or nonexistence results for pure-strategy Nash equilibria in different settings of large games (see, for example, the survey Chapter in Khan and Sun (2002)). However, very few studies focus on characterizing the pure-strategy Nash equilibria or equilibrium distributions. Clearly, good characterization results are also valuable since they can help us better understand the Nash equilibria and also provide alternative ideas for proving the existence of Nash equilibria. It is the aim of this paper to make some contributions in filling this gap. In particular, this paper presents three characterization results and a counterexample for the equilibrium distributions in large games.

The first result in this paper is for a large game with only *countable actions*. In this case, we show that a distribution vector on the action space is an equilibrium distribution iff for any subset or any finite subset of actions, the proportion of players in any group playing this subset of actions is no larger than the proportion of players in that group having a best response in this subset. A rather simple proof is given based on Khan and Sun (1995).

Our second result characterizes the Nash equilibria in a large game with *countable homogeneous groups of players*. The actions of the players is now allowed to be uncountable. Here, the homogeneousness of the groups means that the players in each group share a common payoff and play a common action set. Such a setting is reasonable and useful since in many situations, the partition of the agent space may well depend on some factors which influence or determine the payoff functions. Under this setting, we present a characterization result for the equilibrium distribution, which is essentially in the same form as our first result.

Both of the above two results have a countability assumption, either on the

[2]For example, our game model may find applications in the Bayesian games considered in Yannelis and Rustichini (1991) and Kim and Yannelis (1997).

action space or on the payoff space. It is surely desirable if we can obtain a similar characterization result for a large game without any restrictions of countability. However, it turns out that this wish cannot be granted easily. We show through a simple counterexample that if both the payoff and action spaces are uncountable and the agent space is a general probability space, say the Lebesgue measure space on the unit interval, then a simple generalization of the above characterization results *does not* hold in general.

However, following the insight in Khan and Sun (1996), we notice that the countability assumption can nevertheless be removed if we define the agent space to be some special probability space. In particular, a similar characterization result is shown for a large game with its agent space being *an atomless Loeb probability space*. Supported by the nice properties of a Loeb space, this result is not subject to any countability assumption and is shown by applying a result on the distribution of correspondences on Loeb spaces from Sun (1996).

It is also worth mention that in this paper, we do not confine the action space to a compact metric space. Our action space is generalized to be a *Polish space* and we assume that each player's action set is a compact subset of the whole action space. Such a generalization is useful in that it includes some commonly used spaces such as the natural numbers or the real numbers (see, eg, Yu and Zhang (2007) for more motivations for such a change). Furthermore, our paper also paves the way for proving the existence of the Nash equilibria in the above games by showing the existence of their characterizing counterparts. This work has recently been done by Fu (2008) where not only more general existence results are obtained but also much simpler proofs are presented. Such an application also highlights the value of our characterization results.

The paper is organized as follows. Section 2 establishes the general game model for this paper. Section 3, 4 and 6 present three characterization results for three settings of large games, namely games with countable actions, games with countable homogeneous groups of players and games with Loeb agent space, respectively. Section 5 gives a counterexample showing the nonexistence of such a characterization result for large games falling out of the above three settings. Section 7 is for discussion and all the proofs are given in Section 8.

2 Game model

Let $(T, \mathscr{T}, \lambda)$ be an atomless probability space of *agents* and I a countable (finite or countably infinite) index set. Let $(T_i)_{i\in I}$ be a measurable partition of T with positive λ-measures $(\alpha_i)_{i\in I}$. For each $i \in I$, let λ_i be the probability measure on T_i such that for any measurable set $B \subseteq T_i$, $\lambda_i(B) = \lambda(B)/\alpha_i$.

Let A be a Polish space[3] of *actions*, $\mathscr{B}(A)$ the Borel σ-algebra of A, $\mathscr{M}(A)$ the set of all Borel probability measures on A endowed with the topology of weak convergence of measures,[4] and $\mathscr{M}(A)^I$ the product space of $|I|$ copies of $\mathscr{M}(A)$ with the usual product topology. Suppose that each player $t \in T$ chooses her own action from an action set $K(t) \in A$, where $K : T \to A$ is a compact valued measurable correspondence. Since A is Polish, $\mathscr{M}(A)$ is Polish[5] and hence $A \times \mathscr{M}(A)^I$ is also Polish. For easy notation, we now let $\Omega := A \times \mathscr{M}(A)^I$.

Definition 1. A *large game* $\mathcal{G}$ is characterized by a Carathéodory function $U : T \times \Omega \to R$ such that for each $\omega \in \Omega$, the function $U^\omega = U(\cdot, \omega) : T \to R$ is measurable and for each $t \in T$, the function $U_t = U(t, \cdot) : \Omega \to R$ is continuous.[6] A measurable function $f : T \to A$ is called a *pure-strategy profile* if $f(t) \in K(t)$ for all $t \in T$. A pure-strategy profile f is called a *(pure-strategy) Nash equilibrium*[7] if

$$U[t, f(t), (\lambda_i f_i^{-1})_{i\in I}] \geq U[t, a, (\lambda_i f_i^{-1})_{i\in I}] \text{ for all } a \in K(t) \text{ and all } t \in T,$$

where f_i is the restriction of f to T_i and $\lambda_i f_i^{-1}$ the induced distribution on A. A distribution vector μ in $\mathscr{M}(A)^I$ is called a *(pure-strategy) equilibrium distribution*[8] if $\mu = (\lambda_i f_i^{-1})_{i\in I}$ for some Nash equilibrium f.

Recall that a correspondence F from T to A is said to be *measurable* if for each closed subset C of A, the set $F^{-1}(C) = \{t \in T : F(t) \cap C \neq \emptyset\}$ is measurable in $\mathscr{T}$. A function f from T to A is said to be a *measurable selection* of F if f is

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[3] A Polish space is a topological space homeomorphic to some complete separable metric space.

[4] In the following context, we reserve the notation $\mathscr{M}(X)$ to denote the space of all Borel probability measures on any topological space X.

[5] See, eg, Theorem 14.15 in Aliprantis and Border (1999).

[6] To be consistent with the existing literature, we can also define a large game to be a measurable function U from T to the space of all continuous real-valued functions on Ω, endowed with its compact-open topology, which is easily seen to be a Carathéodory function.

[7] Throughout this paper, we deal only with pure-strategy Nash equilibrium and pure-strategy equilibrium distribution. Thus we suppress the adjective 'pure-strategy' hereafter.

[8] More precisely, μ should be called an *equilibrium distribution vector*.

measurable and $f(t) \in F(t)$ for all $t \in T$. When F is measurable and closed valued, the classical Kuratowski-Ryll-Nardzewski Theorem (see, eg, Aliprantis and Border (1999, p.567)) says that F has a measurable selection.

Given $\mu \in \mathscr{M}(A)^I$, let $B^\mu(t) = \arg\max_{a \in K(t)} U(t, a, \mu)$ be the set of best responses for player t given action distribution μ. By the Measurable Maximum Theorem in Aliprantis and Border (1999, p.570), B^μ is a measurable correspondence from T to A, has nonempty compact values and admits a measurable selection. Let $B_i^\mu : T_i \twoheadrightarrow A$ be the restriction of B^μ to T_i. Thus, it is easy to check that μ is an equilibrium distribution iff $\mu = (\lambda_i f_i^{-1})_{i \in I}$, where f_i is one of the measurable selections of B_i^μ. Recall that for any $C \subseteq A$, the set $(B_i^\mu)^{-1}(C) = \{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\}$ denotes the set of the players who are in group i and have a best response in C.

The above notations will be used in the whole paper. Unless otherwise specified, the meaning of these notations remains the same.

3 Large games with countable actions

In this section, we consider large games with countable actions. Our main result is formulated as follows.

Theorem 1. *In a large game $\mathcal{G}$, if the action space A is a countable and complete metric space, then the following statements are equivalent:*

(i) $\mu = (\mu_i)_{i \in I} \in \mathscr{M}(A)^I$ *is an equilibrium distribution;*

(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^\mu)^{-1}(C)]$ for every subset C in A;*

(iii) *for each $i \in I$, $\mu_i(D) \leq \lambda_i[(B_i^\mu)^{-1}(D)]$ for every finite set D in A.*

Theorem 1 is essentially saying that a distribution vector on the action space is an equilibrium distribution iff for any subset or any finite subset of the actions, the proportion of players in any group playing this subset of actions is no larger than the proportion of players in that group having a best response in this subset.

Remark 1. The very special case that $|I| = 1$ and A is finite is the main result in Blonski (2005). Our proof for the above theorem uses a selection theorem in Khan and Sun (1995), which is not only much simpler than the proof for the special case considered in Blonski (2005) but also yields a much stronger result.

4 Large games with countable homogeneous groups of players

Very often, our partition of the agent space is not arbitrary but dependent on some important factors which may well influence or determine the payoff functions. In this section, we consider a simplified case where we assume that all the players in each group are *homogeneous*, that is, all the players in each subgroup share a common payoff function and play a common action set. Under such a setting, we can remove the restriction of the countability on the action space. We now let A be a general Polish space.

A large game $\mathcal{G}$ is said to have *countable homogeneous groups of players* if for each group $i \in I$, U_t and $K(t)$ do not change for all $t \in T_i$.

Theorem 2. *If a large game $\mathcal{G}$ has countable homogeneous groups of players, then the following statements are equivalent:*

(i) $\mu = (\mu_i)_{i \in I} \in \mathscr{M}(A)^I$ *is an equilibrium distribution;*

(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^{\mu})^{-1}(C)]$ for every Borel set C in A;*

(iii) *for each $i \in I$, $\mu_i(F) \leq \lambda_i[(B_i^{\mu})^{-1}(F)]$ for every closed set F in A;*

(iv) *for each $i \in I$, $\mu_i(O) \leq \lambda_i[(B_i^{\mu})^{-1}(O)]$ for every open set O in A.*

5 Large games with uncountable actions and payoffs - a counterexample

The above two sections present characterization results for large games restricted by either countable actions or countable payoffs. It is surely desirable to obtain a similar characterization result for a general game without the above restrictions. So one would like to ask the following question: can we still obtain a characterization result in the form of Theorem 2 without the assumption of homogeneousness of the groups? As we can see from the example below, it turns out the answer is no, that is, such a direct generalization does not hold in general. For simplicity, we only need to consider the case that $|I| = 1$, ie, there is no partition on the agent space.

Example 1. Consider a large game $\mathcal{G}$ in which the space of players is the Lebesgue unit interval $T = [0,1]$ with the Lebesgue measure denoted by λ, the action set A is the interval $[-1,1]$ and the payoffs are given by $U(t,a,\mu) = -|t - |a||$[9] where $t \in T$, $a \in A$ and $\mu \in \mathscr{M}(A)$, which, obviously, is a Carathéodory function.

Let the uniform distribution on $[-1,1]$ be denoted by η. Thus, given η, the best response set for player t is:

$$B^{\eta}(t) = \arg\max U(t,a,\eta) = \{t, -t\}.$$

Let C be any Borel set in A and define $C_1 = C \cap (0,1]$ and $C_2 = C \cap [-1,0]$. Then

$$\begin{aligned}
\lambda[(B^{\eta})^{-1}(C)] &= \lambda(\{t \in T : B^{\eta}(t) \cap C \neq \emptyset\}) \\
&= \lambda\{t \in T : t \in C_1 \text{ or } -t \in C_2\} \\
&\geq \max\{\lambda(C_1), \lambda(C_2)\} \\
&\geq \frac{\lambda(C_1) + \lambda(C_2)}{2}.
\end{aligned}$$

Since η is the uniform distribution on $[-1,1]$, $\eta(C) = \frac{\lambda(C)}{2} = \frac{\lambda(C_1 \bigcup C_2)}{2} = \frac{\lambda(C_1)+\lambda(C_2)}{2}$. Therefore, we have

$$\lambda[(B^{\eta})^{-1}(C)] \geq \eta(C).$$

Now we shall prove by contradiction that η can not be an equilibrium distribution.

Suppose η is an equilibrium distribution. Then, by definition, there exists a measurable selection f of B^{η} such that $\lambda f^{-1} = \eta$ and $f(t) \in B^{\eta}(t)$ for all $t \in T$. Let $D = f^{-1}((0,1])$. Then

$$f(t) = \begin{cases} t, & t \in D \\ -t, & t \notin D. \end{cases}$$

Note that $f^{-1}(D) = \{t : f(t) \in D\} = \{t : t \in D\} = D$. Hence, $\lambda(D) = \lambda f^{-1}(D) = \eta(D) = \frac{\lambda(D)}{2}$, which is a contradiction. Therefore, η cannot be an equilibrium distribution. ■

[9]This payoff function is similar to a payoff function used in Rath et al. (1995).

6 Large games with Loeb agent space

The usage of hyperfinite Loeb spaces in modeling large games is systematically studied in Khan and Sun (1996, 1999). By modeling the set of players as a Loeb space, Khan and Sun (1999) shows the existence of Nash equilibria in large games without any countability assumption on action or payoff space, which is false when the agent space is modeled by Lebesgue unit interval (see Khan et al., 1997). This major success, among others, led them to argue Loeb spaces as the 'right' tool for modeling games with a large number of players.

In this section, we provide a characterization result for large games with its agent space being an atomless Loeb probability space. This result is not subject to any countability assumption and is shown by applying a proposition from Sun (1996) on the distribution of correspondences on Loeb spaces.

Theorem 3. *In a large game $\mathcal{G}$, if the agent space $(T, \mathscr{T}, \lambda)$ is an atomless Loeb probability space , then the following statements are equivalent:*

(i) $\mu = (\mu_i)_{i \in I} \in \mathscr{M}(A)^I$ *is an equilibrium distribution;*

(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^\mu)^{-1}(C)]$ for every Borel set C in A;*

(iii) *for each $i \in I$, $\mu_i(F) \leq \lambda_i[(B_i^\mu)^{-1}(F)]$ for every closed set F in A;*

(iv) *for each $i \in I$, $\mu_i(O) \leq \lambda_i[(B_i^\mu)^{-1}(O)]$ for every open set O in A.*

Recall that Loeb probability spaces, even though constituted by nonstandard entities, are standard probability spaces. They satisfy the assumption of countable additivity, and hence any result proved for an abstract probability space applies to them. For more information about Loeb spaces, see Khan and Sun (1999) or Loeb and Wolff (2000).

7 Discussions

From Example 1, we see that although a generalized characterization result does not hold, there does exist a probability measure $\eta \in \mathscr{M}(A)$ such that $\eta(C) \leq \lambda(B^{\eta^{-1}}(C))$ for all $C \in \mathscr{B}(A)$. Thus one would like to guess that the existence of our characterizing probability measures hold in a more general sense. Furthermore, if this is the case, one would like to infer that the proof for the existence should not be difficult because of the generality. If this is also the case, then Theorems 1, 2 and 3, together with the existence of their characterizing counterparts, can also lead to the existence of the corresponding Nash equilibria. The above conjectures have mostly been confirmed by Fu (2008), which also highlights the value of our results. Finally, we hope that such an idea of characterization may also find applications in other works, for example in Bayesian games, and bring more benefits to economic research and study.

8 Proofs

8.1 Proof of Theorem 1

To prove this theorem, we need the following lemma from Khan and Sun (1995).

Lemma 1. *(Khan and Sun, 1995, Theorem 5) Let $(T, \mathscr{T}, \lambda)$ be an atomless probability space, S a countable metric space, F a measurable correspondence from T to S and $\mathscr{D}_F = \{\lambda f^{-1} : f$ is a measurable selection of $F\}$. Then a distribution $\nu \in \mathscr{M}(S)$ belongs to $\mathscr{D}_F$ iff for all finite $D \subseteq S$, $\nu(D) \leq \lambda(F^{-1}(D))$.*[10]

Proof of Theorem 1 For (i)⇒(ii), let μ be an equilibrium distribution. Then, there exists a Nash equilibrium $f : T \to A$ such that $\mu = ((\lambda_i f_i^{-1})_{i \in I})$ and $f_i(t) \in B_i^\mu(t)$ for all $t \in T_i$ and for all $i \in I$. Thus, for any $i \in I$ and for every $C \subseteq A$,

$$\begin{aligned}\mu_i(C) &= \lambda_i(f_i^{-1}(C)) = \lambda_i(\{t \in T_i : f_i(t) \in C\}) \\ &\leq \lambda_i(\{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\}) = \lambda_i[(B_i^\mu)^{-1}(C)].\end{aligned}$$

It is clear that (ii) ⇒ (iii).

To prove (iii) ⇒ (i), define $\mathscr{D}_{B_i^\mu} = \{\lambda_i f_i^{-1} : f_i$ is a measurable selection of $B_i^\mu\}$. Recall that B_i^μ is measurable. By definition, μ is an equilibrium distribution iff $\mu_i \in \mathscr{D}_{B_i^\mu}$ for all $i \in I$, but by Lemma 1, $\mu_i \in \mathscr{D}_{B_i^\mu}$ iff for all finite $D \in A$, $\mu_i(D) \leq \lambda_i[(B_i^\mu)^{-1}(D)]$. ■

[10] In the general setting of Khan and Sun (1995), S is assumed to be a countable compact metric space. But it is also straightforward to check that the compactness assumption of S, which is required in the other results of their work, can be removed in this result.

8.2 Proof of Theorem 2

To prove this theorem, we need to use the following lemma which is well know in this field and can be obtained by appropriately adjusting the proof of Theorem 3.11 in Skorokhod (1956).

Lemma 2. *(Skorokhod, 1956, Theorem 3.11)*[11] *Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space and S a Polish space. Then for any $\nu \in \mathscr{M}(S)$ there exists a measurable function $f : T \to S$ such that $\lambda f^{-1} = \nu$.*

Proof of Theorem 2. Let $\mu = (\mu_i)_{i \in I}$ be an element of $\mathscr{M}(A)^I$. Firstly, we want to make sure that for each $i \in I$ and every $C \in \mathscr{B}(A)$, $(B_i^\mu)^{-1}(C)$ is measurable. Now fix any $i \in I$. The fact that U_t and $K(t)$ do not change for all $t \in T_i$ implies that $B_i^\mu(t)$ also does not change for all $t \in T_i$. Thus we can let $C_i := B_i^\mu(t)$ for all $t \in T_i$. Then, for any $C \in \mathscr{B}(A)$, we have

$$(B_i^\mu)^{-1}(C) = \{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\} = \begin{cases} T_i & \text{if } C_i \cap C \neq \emptyset; \\ \emptyset & \text{otherwise,} \end{cases}$$

which is measurable.

To see (i)⇒(ii), let $\mu = (\mu_i)_{i \in I}$ be an equilibrium distribution. By assumption, there exists a Nash equilibrium $f : T \to A$ such that $\mu = (\lambda_i f_i^{-1})_{i \in I} \in \mathscr{M}(A)^I$ and $f(t) \in B^\mu(t)$ for all $t \in T$. Therefore, for any $C \in \mathscr{B}(A)$,

$$\begin{aligned} \mu_i(C) &= \lambda_i f_i^{-1}(C) = \lambda_i(\{t \in T_i : f_i(t) \in C\}) \\ &\leq \lambda_i(\{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\}) \\ &= \lambda_i[(B_i^\mu)^{-1}(C)]. \end{aligned}$$

It is clear that (ii) ⇒ (iii).

To see (iii) ⇒ (iv), let O be an open set in A. Then there is an increasing sequence $\{F_n\}_{n=1}^\infty$ of closed sets in A such that $O = \bigcup_{n=1}^\infty F_n$. For each n, we have $(B_i^\mu)^{-1}(F_n) \subseteq (B_i^\mu)^{-1}(O)$, which implies that $\mu_i(F_n) \leq \lambda_i[(B_i^\mu)^{-1}(F_n)] \leq \lambda_i[(B_i^\mu)^{-1}(O)]$. Thus, $\mu_i(O) \leq \lambda_i[(B_i^\mu)^{-1}(O)]$.

It remains to show (iv) ⇒ (i).

[11] This lemma is indirectly implied in the Theorem 3.11 in Skorokhod (1956). If any reader want to have a direct and separated proof for this lemma, please feel free to contact the corresponding author.

Recall that for all $i \in I$, the set $C_i := B_i^{\mu}(t)$ for any $t \in T_i$ is compact and hence also complete and separable. Fix any $i \in \mathbb{N}$. By the fact that the set $(A - C_i)$ is open, we have that

$$1 - \mu_i(C_i) = \mu_i(A - C_i) \leq \lambda_i[(B_i^{\mu})^{-1}(A - C_i)] = 0, \tag{1}$$

which gives $\mu_i(C_i) = 1$ for all i. Therefore, by Lemma 2, there exists a measurable function $f_i : T_i \to C_i$ such that $\mu_i = \lambda_i {f_i}^{-1}$. By definition, $f_i \in B_i^{\mu}$.

Define $f : T \to A$ by letting $f(t) = f_i(t)$ for all $t \in T_i$ and all $i \in I$. Thus f is a measurable selection of B^{μ} and $\mu = (\mu_i)_{i \in I} = (\lambda_i f_i^{-1})_{i \in I}$ is an equilibrium distribution. ■

8.3 Proof of Theorem 3

To prove this theorem, we need to use the following lemma in Sun (1996).

Lemma 3. *(Sun, 1996, Proposition 3.5) Let Γ be a closed valued measurable correspondence from an atomless Loeb probability space $(\Omega, \mathscr{F}, P)$ to a Polish space X. Let ν be a Borel probability measure on X. Then the following are equivalent:*

(i) *there is a measurable selection f of Γ such that $Pf^{-1} = \nu$;*

(ii) *for every Borel set C in X, $\nu(C) \leq P(\Gamma^{-1}(C))$;*

(iii) *for every closed set F in X, $\nu(F) \leq P(\Gamma^{-1}(F))$;*

(iv) *for every open set O in X, $\nu(O) \leq P(\Gamma^{-1}(O))$.*

Proof of Theorem 3. For any $i \in I$, notice that B_i^{μ} is a compact valued (and hence closed valued) measurable correspondence from an atomless Loeb probability space $(T_i, \mathscr{T}_i, \lambda_i)$ to the Polish space A. Thus, by applying Proposition 3.5 in Sun (1996) to B_i^{μ}, we see that $\mu_i = \lambda_i f_i^{-1}$ for some f_i being a measurable selection of B_i^{μ} iff for every Borel (closed, or open) set H in A, $\mu_i(H) \leq \lambda_i[(B_i^{\mu})^{-1}(H)]$.

Since the above result holds for all $i \in I$, thus $\mu = (\mu_i)_{i \in I}$ is an equilibrium distribution iff for each $i \in I$ and every Borel (closed, or open) set H in A, $\mu_i(H) \leq \lambda_i[(B_i^{\mu})^{-1}(H)]$. ■

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