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Biased Contests for Symmetric Players*

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Abstract

In a *biased contest*, one of the players has an advantage in the winner determination process. We characterize a novel class of biased contest success functions pertaining to such contests and provide necessary and sufficient conditions for zero bias to be a critical point of arbitrary objectives satisfying certain symmetry restrictions. We, however, challenge the common wisdom that unbiased contests are always *optimal* when contestants are symmetric *ex ante* or even *ex post*. We show that contests with arbitrary favorites, i.e., biased contests of symmetric players, can be optimal in terms of various objectives such as expected aggregate effort, the probability to reveal the stronger player as the winner or expected effort of the winner.

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Keywords: biased contest, biased contest success function, aggregate effort, predictive power, winner's effort.

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1 Introduction

In contests, players expend effort or other resources to win a valuable prize. Examples range from rent seeking (Congleton, Hillman and Konrad, 2008) and sports (Szymanski, 2003) to competition for promotion and bonuses in firms (Lazear, 1995; Prendergast, 1999; Connelly et al., 2014). The key element of a simple contest game is the winner determination process that can be characterized, in a reduced form, by a *contest success function* (CSF) mapping a vector of players' efforts into the winning probability for each player. A contest is procedurally fair, or *unbiased*, if its CSF has the anonymity property (Skaperdas, 1996): If the efforts of any two players are swapped, so will be their probabilities of winning.

It is generally understood that unbiased contests are most effective, from the organizer's perspective, when players are homogeneous in their ability. Thus, the literature on *biased* contests, or contests with handicaps, studies how to bias a contest optimally when the players are heterogeneous (e.g., Dukerich, Weigelt and Schotter, 1990; Schotter and Weigelt, 1992; Fain, 2009; Epstein, Mealem and Nitzan, 2011; Franke, 2012; Franke et al., 2013; Lee, 2013).[1,2] In these and other papers on biased contests, specific tractable contest models have been used and biases have been introduced in a number of *ad hoc* ways.

In this paper, we systematically explore biased contests in a very general setting and provide general results in the case of symmetric players. We introduce a class of biased CSFs that includes as special cases the commonly used additive and multiplicative biases but also allows for other types of biases. Our first contribution is to show that zero bias is a critical point of a general objective function of the contest designer if and only if the CSF belongs to this class. The general objective function includes as special cases the aggregate effort, the winner's effort, the winner's ability and predictive power, i.e., the probability that the highest ability player wins. In other words, the first derivative of almost any objective function used in the literature with respect to the bias is zero at zero bias under very general conditions.

The second contribution of the paper is to study whether a biased or an unbiased

[1] The idea of using handicaps to restore efficiency in tournaments of heterogeneous agents goes back to Lazear and Rosen (1981) and O'Keeffe, Viscusi and Zeckhauser (1984); see also Tsoulouhas, Knoeber and Agrawal (2007).

[2] In this paper, we focus on imperfectly discriminating contests with smooth contest success functions. There is also a parallel literature using the all-pay auction model of contests, e.g., Lien (1990), Clark and Riis (2000), Konrad (2002), Fu (2006), Feess, Muehlheusser and Walzl (2008), Li and Yu (2012), Kirkegaard (2012, 2013).

contest is optimal when players are symmetric and to show that biased contests are often optimal.[3] As an example, consider a Lazear and Rosen (1981) type tournament model with two risk-neutral players $i \in \{1, 2\}$ in which player i's output (y_i) is her effort (e_i) distorted by a zero-mean additive shock (u_i): $y_i = e_i + u_i$. Player i's cost of effort is $\frac{2}{3}e_i^{\frac{3}{2}}$. The player with the highest output wins and receives the prize equal to 1, while the other player receives zero. Similar to Meyer (1991), Konrad (2009), Ederer (2010) and Brown and Minor (2014), assume that $u_1 - u_2$ is uniformly distributed on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Bias $\beta \geq 0$ favors player 1 by increasing her effort (at no cost) to $(1+\beta)e_1$ and simultaneously decreasing player 2's effort to $(1-\beta)e_2$;[4] the unbiased contest is obtained at $\beta = 0$. Assuming an interior equilibrium (e_1^*, e_2^*), the first-order conditions for expected payoff maximization for each player are $1+\beta = \sqrt{e_1}$ and $1-\beta = \sqrt{e_2}$. It is easy to see that in this model the aggregate equilibrium effort is $e_1^* + e_2^* = 2(\beta^2 + 1)$. While its derivative with respect to β is zero at zero bias, the aggregate effort increases with the bias. The intuition (confirmed formally in Section 4.1) is that the bias creates a mean-preserving variation in the marginal benefit of effort across the players. Such variation then increases (respectively, decreases) total effort if the marginal cost function is concave (respectively, convex). As discussed below, this intuition is similar to the one arising in the literature on dynamic contests.

Contests may also be used as selection mechanisms that are characterized by predictive power, i.e., the probability to reveal the best player as the winner (Hvide and Kristiansen, 2003; Ryvkin and Ortmann, 2008). Continuing with the example from the previous paragraph, suppose now that player i's cost of effort is $\frac{2}{3}t_i e_i^{\frac{3}{2}}$, where $t_i > 0$ is player i's type. Assume that the two players are symmetric *ex ante* but may be heterogeneous *ex post*, with t_i drawn independently for each player to be equal to t_L or $t_H > t_L$ with probabilities $\frac{1}{2}$. It is straightforward to show that in the interior equilibrium the predictive power of this contest, defined as the probability that a player with type t_L wins against a player with type t_H conditional on the players being heterogeneous *ex post*, is $\frac{1}{2} + (3\beta^2 + 1)\frac{t_H^2 - t_L^2}{t_H^2 t_L^2}$.[5] Again, while its derivative is zero at zero bias, the predictive power increases with the bias.

[3] A biased contest is automatically optimal for symmetric players whenever the CSF does *not* belong to the class mentioned above. However, biased contests are also optimal for many CSFs in the class. In this paper, we focus mainly on the CSFs in this class because it is for these CSFs that the zero bias is a critical point for many objectives, and hence the optimality of biased contests for symmetric players is the most counterintuitive. The class also happens to include the most popular CSFs used in the literature.

[4] Thus, the probability of player 1 winning is $p(e_1, e_2; \beta) = \frac{1}{2} + (1+\beta)e_1 - (1-\beta)e_2$ for e_1, e_2 and β such that this expression is between zero and one.

[5] As long as this expression is less than one.

The second contribution of this paper is thus to show that the above examples are by no means exceptional. We provide some general results and many examples showing that it might be optimal to bias a contest in favor of one of the two symmetric players. Results and examples include Tullock (1980) type contests and Lazear and Rosen (1981) type tournaments; contests with players who are symmetric *ex post* or only *ex ante*; contests in which players' types are public or private information for the players. While in the examples we focus on the two most popular objectives of the principal discussed in the literature, maximization of aggregate effort and predictive power, some of our results apply to other objectives such as maximization of the winner's effort or of the winner's ability. In the model used in the examples above, both the winner's ability and the winner's effort are also increasing with the bias. Our examples show that at least for some parameterizations optimal biases in contests of symmetric players can be large and lead to substantial improvements in the principal's objectives.

The results of our paper are relevant in situations when one would like, or is institutionally obligated, to use biased contests but is concerned about their costs. Suppose there is positive discrimination and hence, the contest designer has to favor some participants over others. Our results can help the designer to turn this obligation to his or her advantage and reach a better outcome in terms of essentially any possible objective. Another application, as discussed below in more detail, is that of dynamic contests in which it may seem fair, or is indeed customary, to favor those who had early success at later stages. Our results can guide the contest designer to create a contest in which there would be no trade-off between rewarding early success and generating subsequent performance. In both cases, the contest designer effectively uses the institutional constraints for introducing a bias that is hard to justify otherwise. Finally, our paper is important from a methodological perspective in showing the limits of the "leveling the playing field" and "competitive balance" ideas in the design of contests with asymmetric players.

The "common wisdom" prevailing in the literature that it is optimal not to bias the contest when players are symmetric (and thus it is optimal to "level the playing field" when players are different) has an obvious intuitive appeal. However, we believe that it is based on a coincidence that this is true in the two specifications of biased contests used most commonly in the literature: multiplicative bias in the Tullock contest (see Epstein, Mealem and Nitzan, 2011; Franke, 2012; Franke et al., 2013) and additive bias in the Lazear-Rosen tournament (see Dukerich, Weigelt and Schotter, 1990; Schotter and Weigelt, 1992; Fain, 2009; Lee, 2013).[6] We provide a general condition that gives these

[6] In the latter case it has been noted that zero bias is optimal only under the (most natural) assumption

two results as special cases for any effort cost functions. However, as soon as the bias is introduced differently, for example, additively in the Tullock contest or multiplicatively in the Lazear-Rosen tournament, the unbiased contest may no longer be optimal.

The two papers closest to ours are by Kawamura and Moreno de Barreda (2014) and Pérez-Castrillo and Wettstein (2015) who provide examples of optimal biased contests when players are symmetric *ex ante*, both in the all-pay auction setting. Specifically, Kawamura and Moreno de Barreda (2014) show that an additive bias may be optimal when there are two types, public information and the principal's objective is predictive power. Pérez-Castrillo and Wettstein (2015) also show that a bias in the form of player-dependent prizes may be optimal in a setting with private information, continuum of types and with the principal maximizing the sum of the winner's type and effort. Our results are much more general in that we allow for arbitrary (smooth) CSFs and ways the bias is introduced. We also show that biased contests may be optimal even when players are symmetric *ex post*, as in the example above. The rest of the literature on biased contests (and all-pay auctions) studies how to bias contests when players are not symmetric *ex post* and, when there are types, not symmetric *ex ante*.[7]

Our results are also related to models of dynamic contests (see Meyer, 1991, 1992; Lizzeri, Meyer and Persico, 1999, 2002; Höffler and Sliwka, 2003; Aoyagi, 2010; Ederer, 2010).[8] These models typically use a two-period tournament setting where the first-period contest is unbiased. One major question is whether the first-period winner should be favored in the second-period contest (see Meyer (1991, 1992) and, to some extent, Höffler and Sliwka (2003)). A crucial observation there is that a small bias in the second period leads to a second-order loss in the second period and to a first-order gain in the first period and hence, is optimal. Our result that zero bias is a critical point in a very general setting thus generalizes these papers to many CSFs, ways to introduce the bias and objective functions. Our results on the optimality of biased contests imply that in some cases there is no trade-off: Favoring the first-period winner in the second period generates higher efforts in both the first and the second periods.

Another major question in the literature on dynamic contests is whether information

that the distribution of the difference of the noise terms is unimodal (see Lizzeri, Meyer and Persico, 1999; Aoyagi, 2010).

[7] Moroni (2015) shows that identical agents might not be treated in the same way. However, her setting is very different from ours as she considers a dynamic contest with externalities and several "milestones." If one agent reaches a milestone, all other agents can work towards the next one. Then, at any moment, each agent might prefer to wait until some other agent reaches the current milestone. *Ex ante* asymmetric contracts reduce these free-riding incentives and might be optimal.

[8] We are grateful to Margaret Meyer for pointing to these connections.

about who won the first-period contest (and by how much) should be disclosed, as in Lizzeri, Meyer and Persico (1999, 2002), Aoyagi (2010) and Ederer (2010). These models use the Lazear-Rosen tournament in which the performance of each player is the sum of her efforts and noise terms over two periods. Therefore, if the players know their first-period performance, the contest in the second period has effectively an additive bias since one player has (generically) a higher performance in the first period. This creates a variability of the second-period marginal benefit of efforts but does not change its average by the law of iterated expectations. Thus, providing information increases total effort if and only if the marginal cost is concave which is exactly the same result and a very similar intuition as in our example above.

The rest of the paper is organized as follows. In Section 2, we introduce a general model of a biased two-player contest and discuss properties of biased CSFs. In Section 3, we show when zero bias is a critical point of an objective function of the contest designer. In Section 4 we provide general conditions for when a biased contest is optimal. In Section 5, we provide examples of models and parameterizations for which unbiased contests of symmetric players are not optimal. Section 6 provides an extension to the general case of $n \geq 2$ players. Section 7 concludes. All proofs are contained in Appendix A. Appendix B contains the most general form of second-order conditions for two objectives – aggregate effort and predictive power – and provides sufficient conditions for each to have a local maximum or minimum when the contest is unbiased.

2 Biased contests

2.1 Model setup

There are two risk-neutral players and a risk-neutral principal. The players indexed by $i = 1, 2$ compete in a contest by simultaneously exerting efforts $e_i \geq 0$. Player i's cost of effort is $C(e_i, t_i)$, where $t_i > 0$ is player i's type; $C(\cdot, \cdot)$ is a thrice continuously differentiable function with $C_1 \geq 0$, $C_{11} \geq 0$ and $C_2 \geq 0$. The types are drawn from a commonly known joint distribution $F(t_1, t_2)$, which is symmetric, with $F(t_1, t_2) = F(t_2, t_1)$ for all (t_1, t_2) in its support.

The probability of player 1 winning the contest is given by a smooth contest success function (CSF) $p(e_1, e_2; \beta)$ with $0 \leq p \leq 1$, $p_1 \geq 0$, $p_2 \leq 0$. Parameter β characterizes the *bias* in the contest.[9] The winner of the contest receives a fixed prize normalized to one,

[9]We introduce players' types through the effort cost function. Alternatively, types can be introduced

while the other player receives zero prize. Each player's payoff is her prize less her cost of effort.

Definition 1 *The contest is unbiased at* $\beta = \bar{\beta}$ *if for all* $e_1, e_2 \geq 0$

$$p(e_1, e_2; \bar{\beta}) = 1 - p\left(e_2, e_1; \bar{\beta}\right). \tag{1}$$

When the contest is unbiased, we obtain a standard symmetric CSF. Property (1) has been called "perfect symmetry" by Dixit (1987) and "anonymity" by Skaperdas (1996).

In order to ensure that β is indeed a bias parameter and not just some parameter the CSF depends on, we assume that there exists an open interval B such that $\bar{\beta} \in B$ is unique, and restrict attention to the values of β in this interval.[10] Further, we assume that for all admissible effort combinations (e_1, e_2) the derivative $p_\beta(e_1, e_2; \beta)$ does not change sign in B, i.e., an increase in the bias always benefits one of the players.[11] Without loss of generality, we can assume it benefits player 1, i.e., $p_\beta(e_1, e_2; \beta) \geq 0$. This inequality must be strict for at least some values of the arguments because otherwise $\bar{\beta}$ would not be unique.

In what follows, we consider two versions of the contest model that differ by the structure of information about the players' types (t_1, t_2). In the public information version, types t_1 and t_2 are observable by both players, while in the private information version each player i only observes her own type t_i. In both cases, we assume that the principal does not observe (t_1, t_2).

The principal's choice variable is the bias parameter β, and her goal is maximization of the objective

$$Q(\beta) = \int q(e_1, t_1; e_2, t_2; \beta) dF(t_1, t_2). \tag{2}$$

Objective (2) is the expectation, over types, of a function q that may depend on the equilibrium effort levels e_i, types t_i and bias parameter β. For example, $q = e_1 + e_2$ for a

directly into the CSF, which then becomes $p(e_1, t_1; e_2, t_2; \beta)$, keeping the effort cost function the same for all types (see, e.g., Meyer, 1991; Höffler and Sliwka, 2003; Ederer, 2010, in a Lazear-Rosen tournament framework). All of our results can be reproduced in such a setting as well.

[10] This assumption excludes from consideration CSFs such as $p(e_1, e_2; \beta) = \frac{e_1+\beta}{e_1+e_2+2\beta}$, which is unbiased at any β. In this example, β is not a bias parameter. At the same time, CSFs such as $p(e_1, e_2; \beta) = \frac{e_1+\beta-\beta^3}{e_1+e_2}$ are admissible even though it is unbiased for $\bar{\beta} = -1$, 0 and 1. Around each of these values of $\bar{\beta}$ there is an interval in which $\bar{\beta}$ is unique.

[11] This assumption is not critical for the theory developed below; it is reasonable, however, because it makes the interpretation of the bias more natural in applications. It excludes CSFs such as $p(e_1, e_2; \beta) = \frac{e_1+\beta^2}{e_1+e_2}$, which is unbiased at $\bar{\beta} = 0$ but nonmonotonic in β in any interval around $\bar{\beta}$.

principal maximizing aggregate effort; $q = p(e_1, e_2, \beta)I(t_1 < t_2)+[1-p(e_1, e_2, \beta)]I(t_1 > t_2)$ for a principal maximizing the probability of the best player winning. Note that the equilibrium effort levels e_i may themselves be functions of t_i and β, but we also allow for explicit dependence on t_i and β in q. For convenience, we assume that q is differentiable in e_i and β (see, however, the discussion after Example 3 in Section 3.3). Below, we impose additional symmetry restrictions on q that ensure that objective (2) has a critical point at $\beta = \bar{\beta}$.

In the analysis below we rely heavily on the systems of first-order conditions for equilibrium effort levels. Thus, we essentially restrict attention to interior pure strategy equilibria. In general, multiple such equilibria can exist in the contest game, and in that case the results apply to any such equilibrium. The results do not apply to mixed-strategy equilibria and to equilibria with effort levels at the boundary of the domain of CSF p where first-order conditions are not satisfied.

2.2 Properties of biased CSFs

Biased CSF $p(e_1, e_2; \beta)$ represents an extended class of CSFs. Bias can be introduced into a CSF in a variety of ways. Suppose $p^0(e_1, e_2)$ is an unbiased CSF satisfying the anonymity property $p^0(e_1, e_2) = 1 - p^0(e_2, e_1)$. A biased CSF can be defined, for example, with an additive bias as $p(e_1, e_2; \beta) = p^0(e_1 + \beta, e_2)$, with $\bar{\beta} = 0$; or with a multiplicative bias as $p(e_1, e_2; \beta) = p^0(e_1\beta, e_2)$, with $\beta \geq 0$ and $\bar{\beta} = 1$; or with a different form of additive bias as $p(e_1, e_2; \beta) = p^0(e_1 + \beta, e_2 - \beta)$, with $\bar{\beta} = 0$. In this section, we introduce a property of biased CSFs that we call *locally symmetric bias*. As we show below, this property leads to a certain permutational symmetry in the dependence of equilibrium efforts on the bias, which makes it equivalent, under additional symmetry restrictions on the principal's objective (2), to the point $\beta = \bar{\beta}$ being a critical point of $Q(\beta)$.

Definition 2 (Locally symmetric bias) *Contest success function $p(e_1, e_2; \beta)$ has a locally symmetric bias at $\beta = \bar{\beta}$ if for all $e_1, e_2 \geq 0$*

$$p_{1\beta}(e_1, e_2; \bar{\beta}) - p_{2\beta}(e_2, e_1; \bar{\beta}) = 0. \tag{3}$$

This condition can be interpreted as follows: $p_{1\beta}(e_1, e_2; \bar{\beta})$ is the marginal effect of the bias on the marginal benefit of player 1's effort. Similarly, $-p_{2\beta}(e_2, e_1; \bar{\beta})$ is the marginal effect of the bias on the marginal benefit of player 2's effort with the players' efforts swapped. Thus, the locally symmetric bias condition (3) states that the total "symmetrized" marginal effect of the bias on the marginal benefit of the two players is zero.[12]

It is straightforward to show via integration that the locally symmetric bias condition (3) is equivalent to the condition,

$$p_\beta(e_1, e_2; \bar{\beta}) = p_\beta(e_2, e_1; \bar{\beta}), \tag{4}$$

which is easier to check. However, for $n > 2$ players it is impossible to express (the generalization of) (3) in a form similar to (4), see Section 6.

The locally symmetric bias condition (3) is *ordinal*, in the sense that it is invariant to smooth monotonic transformations of the bias, as stated in the following straightforward lemma.

Lemma 1 *Suppose $p(e_1, e_2; \beta)$ is a biased CSF that is unbiased at $\beta = \bar{\beta} \in B$, $\tau : B \to \mathcal{R}$ is a continuously differentiable strictly monotonic function, and $\tilde{\beta} = \tau^{-1}(\bar{\beta})$. Then,*
(i) CSF $p(e_1, e_2; \tau(\beta))$ is unbiased at $\beta = \tilde{\beta}$;
(ii) $p(e_1, e_2; \beta)$ satisfies condition (3) at $\beta = \bar{\beta}$ if and only if $p(e_1, e_2; \tau(\beta))$ satisfies condition (3) at $\beta = \tilde{\beta}$.

We conclude this Section by providing several examples of biased CSFs that have the locally symmetric bias and those that do not. Checking condition (4) in each case is straightforward.

Example 1 *The following CSFs satisfy locally symmetric bias:*

(i) Multiplicative bias in the Tullock contest: $p(e_1, e_2; \beta) = \frac{\beta e_1^r}{\beta e_1^r + e_2^r}$, $r > 0$;

(ii) Additive bias in the Tullock contest: $p(e_1, e_2; \beta) = \frac{e_1^r + \beta}{e_1^r + e_2^r}$;

(iii) Additive bias in the Lazear-Rosen tournament: $p(e_1, e_2; \beta) = \Pr\{e_1 + u_1 + \beta \geq e_2 + u_2\}$, where u_1, u_2 are zero-mean i.i.d. shocks;

(iv) Multiplicative bias in the Lazear-Rosen tournament: $p(e_1, e_2; \beta) = \Pr\{(1 + \beta)\, e_1 + u_1 \geq (1 - \beta)\, e_2 + u_2\}$.

(v) The Tullock contest with a combination of biases: $p(e_1, e_2; \beta) = \frac{(1+\beta)e_1^r + \beta^3}{(1+\beta)e_1^r + e_2^r}$.

Example 2 *The following CSFs do not satisfy locally symmetric bias:*

(i) Another form of additive bias in the Tullock contest: $p(e_1, e_2; \beta) = \frac{e_1^r + \beta}{e_1^r + \beta + e_2^r}$;

[12] The swapping of efforts in the marginal effect for player 2 is a special case of cyclical permutation of efforts that is part of the corresponding condition in the general case of $n \geq 2$ players, see Section 6.

(ii) Another form of multiplicative bias in the Lazear-Rosen tournament: $p(e_1, e_2; \beta) = \Pr\{\beta e_1 + u_1 \geq e_2 + u_2\}$*;*

(iii) A contest in which with probability β *player* 1 *wins for sure and with probability* $1 - \beta$ *there is an unbiased contest:* $p(e_1, e_2; \beta) = \beta + (1 - \beta) p^0(e_1, e_2)$*, where* $p^0(e_1, e_2)$ *is symmetric, i.e.,* $p^0(e_1, e_2) + p^0(e_2, e_1) = 1$*.*

3 Properties of unbiased contests

3.1 Public information contests

We start the analysis with the public information case where the players observe each others' types. The expected payoffs of players 1 and 2 are

$$\pi_1 = p(e_1, e_2; \beta) - C(e_1, t_1), \quad \pi_2 = 1 - p(e_1, e_2; \beta) - C(e_2, t_2).$$

In what follows, we assume that for all (t_1, t_2) in the support of F and for all β in some open neighborhood of $\bar{\beta}$ the contest has an equilibrium in pure strategies, $e_i^*(t_1, t_2; \beta)$, $i = 1, 2$, that is characterized by the system of first-order conditions[13]

$$p_1(e_1, e_2; \beta) = C_1(e_1, t_1), \quad -p_2(e_1, e_2; \beta) = C_1(e_2, t_2). \tag{5}$$

When the contest is unbiased, swapping the players' identities correspondingly swaps the equilibrium effort levels, i.e., $e_1^*(t_1, t_2; \bar{\beta}) = e_2^*(t_2, t_1; \bar{\beta})$. The following lemma shows that the local symmetry property of the CSF is necessary and sufficient for a zero total change in the effort levels with respect to β.

Lemma 2 *(i) Suppose contest success function* $p(e_1, e_2; \beta)$ *has the locally symmetric bias. Then in any equilibrium characterized above we have*

$$e_{1\beta}^*(t_1, t_2; \bar{\beta}) = -e_{2\beta}^*(t_2, t_1; \bar{\beta}). \tag{6}$$

[13] Thus, we require that the CSF p be "sufficiently concave" in e_1. For example, for a Lazear and Rosen (1981) type tournament model, this would imply a sufficiently large variance of additive noise; for a Tullock (1980) type contest model, this would imply a sufficiently low discriminatory power of the contest. An additional, complementary, requirement is that the effort cost function C be "sufficiently convex" in effort.

(ii) Suppose in any equilibrium characterized above we have

$$e^*_{1\beta}(t_1,t_2;\bar\beta)+e^*_{1\beta}(t_2,t_1;\bar\beta)=-e^*_{2\beta}(t_1,t_2;\bar\beta)-e^*_{2\beta}(t_2,t_1;\bar\beta). \tag{7}$$

Then contest success function p has the locally symmetric bias.

Proof. See Appendix A. ■

Lemma 2 plays a key role in the proof of the equivalence of the locally symmetric bias condition (3) and $\beta=\bar\beta$ being the critical point of $Q(\beta)$ in the case of public information in Proposition 1 below. Indeed, it is seen immediately from (6) that, due to the symmetry of the distribution of types $F(t_1,t_2)$, $\beta=\bar\beta$ is a critical point of the expected aggregate effort $Q^{\mathrm{E}}(\beta)=\int[e^*_1(t_1,t_2;\beta)+e^*_2(t_1,t_2;\beta)]dF(t_1,t_2)$. For a general objective $Q(\beta)$, cf. (2), we have

$$Q'(\bar\beta)=\int[q_{e_1}e^*_{1\beta}(t_1,t_2,\bar\beta)+q_{e_2}e^*_{2\beta}(t_1,t_2,\bar\beta)+q_\beta]dF(t_1,t_2),$$

and the result then follows provided function $q(e_1,t_1;e_2,t_2;\beta)$ satisfies appropriate symmetry restrictions that ensure that its derivatives q_{e_i} are symmetric, and q_β is anti-symmetric, with respect to a permutation of players (see Definition 3 below).

3.2 Private information contests

In this environment, each player i only observes her own type t_i, and an equilibrium in pure strategies has the form of bidding functions $b_i(t;\beta)$, $i=1,2$. Such an equilibrium with non-increasing bidding functions exists under a wide range of conditions (Wasser, 2013; Ewerhart, 2014; Brookins and Ryvkin, 2015; He and Yannelis, 2015). As above, we will assume that the equilibrium is characterized by the first-order conditions that in this case take the form of a system of integral equations:

$$\begin{aligned}\int p_1(b_1(t;\beta),b_2(t';\beta);\beta)dF(t'|t)&=C_1(b_1(t;\beta),t),\\ -\int p_2(b_1(t';\beta),b_2(t;\beta);\beta)dF(t'|t)&=C_1(b_2(t;\beta),t).\end{aligned} \tag{8}$$

Here, $F(t'|t)$ is the conditional distribution of cost parameters. At $\beta=\bar\beta$ we have a symmetric equilibrium with $b_1(t;\bar\beta)=b_2(t;\bar\beta)$. The following lemma is the analog of Lemma 2 for the case of private information.

Lemma 3 *Contest success function $p(e_1, e_2; \beta)$ has the locally symmetric bias if and only if in any equilibrium in pure strategies characterized above*

$$b_{1\beta}(t; \bar{\beta}) = -b_{2\beta}(t; \bar{\beta}). \tag{9}$$

Proof. See Appendix A. ■

Lemma 3 plays a key role in the proof of the equivalence of the locally symmetric bias condition (3) and $\beta = \bar{\beta}$ being the critical point of $Q(\beta)$ in the case of private information in Proposition 1 below. Similar to the case of public information, it is seen immediately from (9) that $\beta = \bar{\beta}$ is a critical point of the expected aggregate effort $Q^{\mathrm{E}}(\beta) = \int [b_1(t; \beta) + b_2(t; \beta)] dF(t)$.[14] For a general objective $Q(\beta)$ the result then follows similarly to the case of public information as discussed after Lemma 2.

3.3 First-order conditions for maximization of principal's objectives

We now turn to analyzing the principal's objective function Q defined by (2). The following definition ensures that the objective is symmetric when the contest is unbiased and, if it depends explicitly on β, the bias in the objective is locally symmetric in a way similar to the local symmetry property of the CSF.

Definition 3 *Objective $Q(\beta) = \int q(e_1, t_1; e_2, t_2; \beta) dF$ is*

(i) symmetric if $q(e_1, t_1; e_2, t_2; \bar{\beta}) = q(e_2, t_2; e_1, t_1; \bar{\beta})$;

(ii) locally symmetrically biased if $q_\beta(e_1, t_1; e_2, t_2; \bar{\beta}) = -q_\beta(e_2, t_2; e_1, t_1; \bar{\beta})$,

for all effort pairs $e_1, e_2 \geq 0$ and types (t_1, t_2) in the support of F.

In what follows, for the sake of style and brevity, we will sometimes refer to $\beta = \bar{\beta}$ as "zero bias."

Proposition 1 *In both cases of public and private information, zero bias $\beta = \bar{\beta}$ is a critical point of any symmetric and locally symmetrically biased objective Q, i.e., $Q'(\bar{\beta}) = 0$, if and only if the CSF p has a locally symmetric bias.*

Proof. See Appendix A. ■

Proposition 1 is the central result of this section. It shows that for a large class of objectives the first-order condition with respect to bias is satisfied by the unbiased contest. The next obvious Corollary shows a simple way to find when a biased contest is optimal.

[14] Here, with a slight abuse of notation, $F(t)$ denotes the marginal of $F(t_1, t_2)$.

Corollary 1 *In both cases of public and private information, if the CSF p does not have a locally symmetric bias, a biased contest is optimal for any symmetric and locally symmetrically biased objective Q.*

Examples of principal's objectives that satisfy the conditions of Proposition 1 and Corollary 1 include the following.

Example 3 *The following objectives satisfy Definition 3 when the CSF has a locally symmetric bias:*

(i) Aggregate effort: $q = e_1 + e_2$;

(ii) Predictive power: $q = p(e_1, e_2; \beta)I(t_1 < t_2) + [1 - p(e_1, e_2; \beta)]I(t_1 > t_2)$;

(iii) Expected ability of the winner: $q = p(e_1, e_2; \beta)a(t_1) + [1 - p(e_1, e_2; \beta)]a(t_2)$, *where ability* $a(\cdot)$ *decreases with the type,* $a' < 0$.

(iv) Winner's expected effort: $q = p(e_1, e_2; \beta)e_1 + [1 - p(e_1, e_2; \beta)]e_2$.

Aggregate effort is one of the most commonly studied objectives in the literature on contests. Predictive power, or selection efficiency of a contest, is defined here as the probability of the best player winning (Hvide and Kristiansen, 2003; Ryvkin and Ortmann, 2008). It is relevant in environments such as recruitment and promotion tournaments in organizations or lobbying for public procurement. The expected ability of the winner is relevant in similar environments (e.g., Höffler and Sliwka, 2003; Ryvkin, 2010). The expected winner's effort can emerge as an objective in R&D competition where the value of the innovation that ends up being patented depends positively on the winner's R&D investment (Baye and Hoppe, 2003; Serena, 2015).

As mentioned in Section 2.1, we assume that q is differentiable in e_i and β at $\beta = \bar{\beta}$, which is the case for most objectives typically used, cf. Example 3. If q is not differentiable at $\beta = \bar{\beta}$, then the notion of "critical point" has to be extended to situations when the derivative $Q'(\beta)$ is not defined. For example, consider a Tullock contest with an additive bias, $p(e_1, e_2; \beta) = \frac{e_1+\beta}{e_1+e_2}$, as in Example 1(ii), and suppose the principal's objective is maximal effort, $q = \max\{e_1, e_2\}$ (Denter and Sisak, 2015). When both players have effort cost function $c(e) = \frac{1}{2}e^2$, the equilibrium efforts are $e_1^* = \frac{1-\beta}{2}$ and $e_2^* = \frac{1+\beta}{2}$, and $q = \frac{1+|\beta|}{2}$ is not differentiable at $\beta = 0$, although $Q(\beta)$ reaches the minimum there. In general, however, our analysis does not apply to such cases.

Note that condition (ii) of Definition 3 and the requirement that the bias in CSF p is locally symmetric are two independent conditions. In cases when objective Q does not include p explicitly, such as in Example 3(i), locally symmetric bias is still necessary for

Q to have a critical point at $\bar{\beta}$. At the same time, when Q includes p condition (ii) of Definition 3 is still necessary even if p has a locally symmetric bias. In Examples 3(ii)-(iv), it is satisfied automatically provided p is locally symmetric, but this does not have to be the case in general. An alternative, albeit less general approach, is to impose a structural restriction on Q. The following definitions and corollary cover Examples 3(i)-(iv) and provide a more intuitive alternative to Definition 3 and Proposition 1 in terms of the primitives of Q.

Definition 4 *Objective $Q(\beta) = \int q(e_1, t_1; e_2, t_2; \beta) dF$ has the expectation form if there are functions v and w such that*

$$q(e_1, t_1; e_2, t_2; \beta) = p(e_1, e_2; \beta) v(e_1, t_1; e_2, t_2) + [1 - p(e_1, e_2; \beta)] w(e_1, t_1; e_2, t_2).$$

Definition 5 *Objective $Q(\beta) = \int q(e_1, t_1; e_2, t_2; \beta) dF$ of the expectation form is symmetric if for all $e_1, e_2 \geq 0$ and for all (t_1, t_2) in the support of F*

$$v(e_1, t_1; e_2, t_2) = w(e_2, t_2; e_1, t_1).$$

Corollary 2 *In both cases of public and private information, zero bias $\beta = \bar{\beta}$ is a critical point of any symmetric objective Q of the expectation form, i.e., $Q'(\bar{\beta}) = 0$, if and only if the CSF p has a locally symmetric bias.*

4 Optimality of biased contests

Proposition 1 establishes that zero bias $\beta = \bar{\beta}$ is a critical point of essentially any reasonable objective function. However, checking the second-order conditions is crucial since, as this section shows, they are not satisfied in many cases. Hence, this section provides some general results on when biased or unbiased contests are optimal. In particular, Section 4.1 considers the case when players are identical *ex post* and the contest designer maximizes the aggregate effort. Section 4.2 considers maximization of the aggregate effort when players are identical *ex ante* and there are two possible types. Finally, Section 4.3 analyzes maximization of predictive power when CSF is linear in efforts.

In Appendix B we provide general conditions for when biased contests are optimal for aggregate effort and predictive power, both under private and public information (Propositions B1-B4). However, if no additional assumptions are made, they are very complicated and hard to verify and interpret.

4.1 Aggregate effort for *ex post* symmetric types

Here we consider the simplest case of *ex post* symmetric players, $t_1 = t_2$, and public information. Note that, in general, the players' types may still be random (i.e., not observable by the principal), but it is assumed here that they are perfectly positively correlated. Let $c(e_i)$ denote each player's effort cost function, in which the identical cost parameter argument is suppressed. The first-order conditions for equilibrium efforts (5) take the form

$$p_1(e_1, e_2; \beta) = c'(e_1), \quad -p_2(e_1, e_2; \beta) = c'(e_2). \tag{10}$$

The second-order conditions $p_{11} - c''(e_1) < 0$ and $-p_{22} - c''(e_2) < 0$ are assumed to be satisfied in equilibrium. Let $e^* = e_1^* = e_2^*$ denote the symmetric solution of (10) for $\beta = \bar{\beta}$. Checking the sign of $e^*_{1\beta\beta} + e^*_{2\beta\beta}$ at $(e_1, e_2; \beta) = (e^*, e^*, \bar{\beta})$ leads to the following result.

Proposition 2 *Consider the case of ex post symmetric players and public information. Suppose p has the locally symmetric bias property. Aggregate effort is maximized in a biased contest if*

$$p_{1\beta\beta} - p_{2\beta\beta} > 2(e^*_{1\beta})^2(c''' - p_{111} + 3p_{112}) + 4e^*_{1\beta}(p_{12\beta} - p_{11\beta}), \tag{11}$$

where $e^*_{1\beta} = \frac{p_{1\beta}}{c''-p_{11}}$ and all the functions are evaluated at $(e_1, e_2; \beta) = (e^*, e^*, \bar{\beta})$. If the sign in (11) is reversed, then aggregate effort reaches a local maximum in the unbiased contest.

Proof. See Appendix A. ■

While the exact interpretation of (11) is difficult, two points can be made. First, the left-hand side of (11) is the rate of change in aggregate marginal benefits of efforts (see (3) and its interpretation). When it is higher, it is more likely that (11) is satisfied and, hence, a biased contest is optimal. Second, the right-hand side of (11) contains the third derivative of the cost function. If it is positive, that is, the marginal costs are convex, a spread in the marginal benefits of efforts decreases the total effort, other things being equal. Then it is more likely that (11) is not satisfied and the unbiased contest is (locally) optimal. Overall, however, since the bias affects the effort of each player directly and through the change in the effort of the other player, all kinds of third derivatives of the CSF enter condition (11).

In some examples, condition (11) is easy to check. There are two types of such examples. First, when the CSF is linear or quadratic in efforts so that all or most of its third derivatives are zero. Consider the example from the Introduction where $p(e_1, e_2; \beta) = \frac{1}{2} + (1+\beta)e_1 - (1-\beta)e_2$. All third derivatives are zero and (11) reduces to $c''' < 0$. Thus, as we mentioned in the Introduction, the mean-preserving variation in the marginal benefit of effort across the players created by the bias increases (respectively, decreases) total effort if the marginal cost function is concave (respectively, convex).

A related (and even more striking) example is obtained if the bias is introduced as $p(e_1, e_2; \beta) = \frac{1}{2} + \beta e_1 - \frac{1}{\beta} e_2$. Since efforts enter linearly, all third derivatives of p on the right-hand side of (11) are zero. Condition (11) then reduces to $1 > (e^*_{1\beta})^2 c'''$. Concavity of marginal costs is now sufficient but not necessary for the optimality of the biased contest. Intuitively, the aggregate marginal benefit of effort, $\beta + \frac{1}{\beta}$, increases as the bias moves further away from $\bar{\beta} = 1$ and hence, even if marginal costs of effort are slightly convex, aggregate effort increases with the bias. Using explicit expressions for equilibrium efforts it is easy to check that condition (11) is satisfied for any convex cost function of the form $c(e) = e^z$, $z > 1$. In other words, increasing aggregate marginal benefit of effort dominates increasing marginal costs of effort and the optimal contest is always biased.[15]

The second type of examples in which condition (11) is easy to check is when $e^*_{1\beta} = p_{1\beta} = 0$ at the equilibrium of the unbiased contest. Then, condition (11) reduces to $p_{1\beta\beta} - p_{2\beta\beta} > 0$. Intuitively, $e^*_{1\beta} = 0$ means that the bias has only a second-order effect on equilibrium efforts and hence the equilibrium interdependence of efforts and a change in the costs are negligible. The effect of the bias is then determined only by its effect on the aggregate marginal benefits of efforts. In particular, the cost function has no influence on the optimality of the (un)biased contest.

As a first example of this type, take the Tullock contest with the multiplicative bias considered in Example 1(i).[16] Then, $p_{1\beta} = r e_1^{r-1} e_2^r \frac{e_2^r - \beta e_1^r}{\left(e_2^r + \beta e_1^r\right)^3}$ which is zero at $\bar{\beta} = 1$ and equal efforts. The aggregate marginal benefit of effort, $p_1 - p_2 = \frac{2r}{e^*} \frac{\beta}{(1+\beta)^2}$, is concave in the bias and $p_{1\beta\beta} - p_{2\beta\beta} = -\frac{1}{4} \frac{r}{e^*} < 0$ at $\bar{\beta} = 1$. The unbiased contest is (locally) optimal for any cost function.

Another example of this type is a Lazear-Rosen tournament with additive bias con-

[15]Note that when efforts enter additively into the CSF, the optimal effort of each player does not depend on the effort of the other player. Then, when there are types, optimal effort of each player depends only on his or her type but not on the type of the other player. The cases of *ex post* identical players (or equivalently, perfectly correlated types), *ex ante* identical players under public information, and *ex ante* identical players under private information are all equivalent. In the simplest case of a CSF linear in efforts, $p(e_1, e_2; \beta) = \gamma_1(\beta)e_1 + \gamma_2(\beta)e_2 + \gamma(\beta)$, it is easy to see that Propositions 2, B1 and B2 all lead to the condition $p_{1\beta\beta} - p_{2\beta\beta} > 2(e^*_{1\beta})^2 C_{111}$ for the optimality of a biased contest.

[16]See also Section 5.1 for the analysis of this example when there are two types.

sidered in Example 1(iii). Let G denote the cdf of the difference of the noise terms, $u_2 - u_1$, and suppose the corresponding pdf g is differentiable. The CSF of this contest is $p(e_1, e_2; \beta) = G(e_1 - e_2 + \beta)$, and the two first-order conditions (10) become $g(e_1 - e_2 + \beta) = c'(e_i)$, $i = 1, 2$. Since g is symmetric around zero, $g'(0) = 0$, which implies that at $\bar{\beta} = 0$ we have $e^*_{1\beta} = \frac{p_{1\beta}}{c'' - p_{11}} = 0$ and condition (11) reduces to $g''(0) > 0$. Thus, the optimal contest is biased for any cost function if g has an even number of peaks. If g has an odd number of peaks, the (locally) optimal contest is always unbiased. If g is unimodal, it is maximized at 0 and hence, the globally optimal contest is unbiased as has been noted by Lizzeri, Meyer and Persico (1999) and Aoyagi (2010).

Finally, in some cases condition (11) just happens to be very simple. Take the Tullock contest with the additive bias considered in Example 1(ii) with $r = 1$.[17] It is easy to check that $p_{1\beta\beta} = p_{2\beta\beta} = 0$, $p_{11\beta} = p_{12\beta}$ and, at $\beta = 0$ and equal efforts, $3p_{112} - p_{111} = 0$. Then, condition (11) reduces to $c''' < 0$. As in the example in the Introduction (see above), the optimality of an (un)biased contest is determined by the convexity or concavity of the marginal cost function.

Effect of the bias on individual efforts

It may seem intuitive that the bias "encourages" player 1 and hence, increases his or her effort, and "discourages" player 2 whose effort then decreases. However, this intuition may be misleading for two reasons. First, the marginal benefit of each player's effort depends, in general, on the other player's effort. A change in the bias changes both players' efforts, and the effect on their marginal benefits is ambiguous. The second reason is that, even holding the other player's effort fixed, the effect of the bias depends on how the bias is introduced. For example, a multiplicative bias does increase the marginal benefit of player 1's effort but an additive bias may decrease it.

To illustrate the ambiguous effect of the bias on individual efforts, we consider now the three CSFs that we use in Section 5 below. In Section 5.1 we consider a Tullock contest with a multiplicative bias, $p\left(e_1, e_2; \beta\right) = \frac{\beta e_1}{\beta e_1 + e_2}$, as in Example 1(i). When both players have cost function $c(e) = \frac{1}{2}e^2$, the equilibrium efforts are the same, $e^*_1 = e^*_2 = \frac{\sqrt{\beta}}{\beta+1}$, and decrease with the bias.

In Section 5.2 we consider a Tullock contest with an additive bias, $p\left(e_1, e_2; \beta\right) = \frac{e_1+\beta}{e_1+e_2}$, as in Example 1(ii). When both players have cost function $c(e) = \frac{1}{2}e^2$, the equilibrium efforts are $e^*_1 = \frac{1-\beta}{2}$ and $e^*_2 = \frac{1+\beta}{2}$. Thus, player 1 exerts a lower effort than player 2,

[17] See also Section 5.2 for the analysis of this example when there are two types.

and the difference increases with the bias. Note that the marginal benefits of efforts are $p_1 = \frac{e_2-\beta}{(e_1+e_2)^2}$ and $-p_2 = \frac{e_1+\beta}{(e_1+e_2)^2}$ for players 1 and 2, respectively. Keeping the effort of the other player constant, a higher bias decreases (increases) the marginal benefit of effort for player 1 (player 2).

Finally, in Section 5.3 we consider a Lazear-Rosen tournament with a multiplicative bias as in Example 1(iv). Denote by g the pdf of the difference in the noise terms, $u_1 - u_2$. The first-order conditions for the players' equilibrium efforts are

$$(1+\beta)g((1+\beta)e_1 - (1-\beta)e_2) = c'(e_1), \qquad (1-\beta)g((1+\beta)e_1 - (1-\beta)e_2) = c'(e_2),$$

which gives $\frac{c'(e_1^*)}{1+\beta} = \frac{c'(e_2^*)}{1-\beta}$, and hence for $\beta > 0$ we have $e_1^* > e_2^*$ in equilibrium.[18]

4.2 Aggregate effort for two correlated types

In this section we suppose that players are symmetric *ex ante* but may be asymmetric *ex post*. As we will see, there is more scope for a biased contest to be optimal. Indeed, with some probability players are different *ex post*. Then, with probability $\frac{1}{2}$ the bias will favor the stronger player and lead to a more lopsided competition than the unbiased contest, while with probability $\frac{1}{2}$ the bias will favor the weaker player and lead to a more leveled contest than the unbiased one. It might be that the expected gain in the principal's objective from the latter will exceed the expected loss from the former. Thus, even if the unbiased contest is optimal when the players are symmetric *ex post*, a biased contest may be optimal when the types of players are not too positively correlated.

Consider the case of public information with two player types, $t_L < t_H$, such that $\Pr(t_i = t_L) = 1 - \Pr(t_i = t_H) = \lambda$, $i = 1, 2$, and $\mathrm{Corr}(t_1, t_2) = \rho$. Let $q_{ij} = \Pr(t_1 = t_i, t_2 = t_j)$, $i, j \in \{L, H\}$. Then

$$\begin{aligned} q_{LL} &= \lambda\left(1-(1-\lambda)(1-\rho)\right), \quad q_{HH} = (1-\lambda)\left(1-\lambda(1-\rho)\right), \\ q_{LH} &= q_{HL} = \lambda(1-\lambda)(1-\rho). \end{aligned}$$

Note that the restriction $\rho \geq \max\{-\frac{\lambda}{1-\lambda}, -\frac{1-\lambda}{\lambda}\}$ has to be satisfied in order for q_{LL} and q_{HH} to be non-negative. Let $Q^{\mathrm{E}}(\beta; \lambda, \rho)$ denote the expected aggregate effort in the

[18] Note that if the bias is additive as in Example 1(iii), then the two players exert the same effort characterized by the first-order condition $g(\beta) = c'(e_i^*), i = 1, 2$.

equilibrium for given values of parameters λ and ρ:

$$\begin{aligned} Q^{\mathrm{E}}(\beta;\lambda,\rho) &= q_{LL}[e_1^*(t_L,t_L;\beta)+e_2^*(t_L,t_L;\beta)] + q_{HH}[e_1^*(t_H,t_H;\beta)+e_2^*(t_H,t_H;\beta)] \\ &+ q_{HL}[e_1^*(t_H,t_L;\beta)+e_2^*(t_H,t_L;\beta)+e_1^*(t_L,t_H;\beta)+e_2^*(t_L,t_H;\beta)]. \end{aligned} \tag{12}$$

Differentiating (12) twice with respect to β and using the expressions for q_{ij} above, we arrive at the following lemma.

Lemma 4 *In a contest with public information and two correlated types, the second derivative of aggregate effort at $\beta = \bar{\beta}$ can be written in the form*

$$Q^{\mathrm{E}}_{\beta\beta}(\bar{\beta};\lambda,\rho) = Q^{\mathrm{E}}_{\beta\beta}(\bar{\beta};\lambda,1) - 2\lambda(1-\lambda)(1-\rho)\left[Q^{\mathrm{E}}_{\beta\beta}\left(\bar{\beta};\frac{1}{2},1\right) - Q^{\mathrm{E}}_{\beta\beta}\left(\bar{\beta};\frac{1}{2},-1\right)\right]. \tag{13}$$

Proof. See Appendix A. ■

As seen from (13), $Q^{\mathrm{E}}_{\beta\beta}(\bar{\beta};\lambda,\rho)$ is linear in ρ, which is expected since all probabilities q_{ij} are linear in ρ. The interesting result of Lemma 4 is that the dependence of $Q^{\mathrm{E}}_{\beta\beta}(\bar{\beta};\lambda,\rho)$ on ρ is determined entirely by the two extreme cases – with perfectly positively and negatively correlated types. When $\lambda = \frac{1}{2}$, (13) simplifies to

$$Q^{\mathrm{E}}_{\beta\beta}\left(\bar{\beta};\frac{1}{2},\rho\right) = \frac{1+\rho}{2}Q^{\mathrm{E}}_{\beta\beta}\left(\bar{\beta};\frac{1}{2},1\right) + \frac{1-\rho}{2}Q^{\mathrm{E}}_{\beta\beta}\left(\bar{\beta};\frac{1}{2},-1\right). \tag{14}$$

When players have different types, the optimal contest is often biased and $Q^{\mathrm{E}}_{\beta\beta}\left(\bar{\beta};\frac{1}{2},-1\right) > 0$. Then, by continuity (14) implies that the optimal contest is biased for *ex ante* symmetric players when ρ is negative enough. In other words, there exists a critical $\hat{\rho}$ such that $Q^{\mathrm{E}}_{\beta\beta}(\bar{\beta};\frac{1}{2},\rho) > 0$ for $\rho \in [-1,\hat{\rho})$. By continuity, the same happens for values of λ that are different but close enough to $\frac{1}{2}$.

Thus, even if under positive correlation between types the unbiased contest is optimal, but under perfect negative correlation it is not then there is a range of negative correlations for which a biased contest will be optimal. This is exactly what happens in the Tullock contest with a multiplicative bias. As we know from Section 4.1, the unbiased contest is optimal when players have the same types. However, as we will see in Section 5.1, introducing a bias is optimal when the types are sufficiently negatively correlated.

4.3 Predictive power for a CSF linear in efforts

Consider the case of a CSF linear in efforts, $p(e_1, e_2; \beta) = \gamma_1(\beta)e_1 + \gamma_2(\beta)e_2 + \gamma(\beta)$. The unbiasedness condition (1) implies that $\gamma_1(\bar{\beta}) = -\gamma_2(\bar{\beta})$, and locally symmetric bias condition (3) implies that $\gamma_1'(\bar{\beta}) = \gamma_2'(\bar{\beta})$. We will explore the contest's predictive power, i.e., the probability that the winner is a player with a lower type, that we denote by $Q^S(\beta)$. Propositions B3 and B4 in Appendix B provide general expressions for the second derivative $Q^S_{\beta\beta}(\bar{\beta})$ for the cases of public and private information, respectively. For the CSF linear in effort they lead to the following Corollary.

Corollary 3 *Suppose $p(e_1, e_2; \beta) = \gamma_1(\beta)e_1 + \gamma_2(\beta)e_2 + \gamma(\beta)$ and p has the locally symmetric bias property. Predictive power is maximized in a biased contest under both public and private information if for all $t_1 < t_2$ in the support of F*

$$\begin{aligned}&[4\gamma_1'(\bar{\beta})^2 + \gamma_1(\bar{\beta})(\gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta}))]\left[\frac{1}{C_{11}(e_1^*, t_1)} - \frac{1}{C_{11}(e_2^*, t_2)}\right] \\ &+ 2\gamma_1(\bar{\beta})\gamma_1'(\bar{\beta})^2\left[\frac{C_{111}(e_2^*, t_2)}{C_{11}(e_2^*, t_2)^3} - \frac{C_{111}(e_1^*, t_1)}{C_{11}(e_1^*, t_1)^3}\right] + [\gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta})](e_1^* - e_2^*) > 0.\end{aligned} \tag{15}$$

If the sign in (15) is reversed, then predictive power reaches a local maximum in the unbiased contest.

Proof. See Appendix B. ■

This condition is quite involved despite the CSF being linear in efforts. Indeed, a higher bias increases the probability that player 1 wins for any configuration of types. Take a pair of types (t_1, t_2), $t_1 < t_2$. Predictive power is proportional to the difference between the probabilities of player 1 winning when the types are (t_1, t_2) and when they are (t_2, t_1). Since both probabilities increase, the overall effect is ambiguous. The linearity of the CSF in efforts implies that a higher bias increases player 1's effort and decreases player 2's effort for any configuration of types. While this is a significant simplification, the second-order condition still must involve the second derivatives of bias functions $\gamma_i(\beta)$ and equilibrium efforts leading to the second and third derivatives of the cost function evaluated at two different points.[19]

An immediate observation from inspecting (15) is that the additive bias $\gamma(\beta)$ does not enter the expression. Indeed, an additive bias increases the probability that the first

[19] Another simplification brought about by the linearity in efforts, as we mentioned in fn. 15, is that the effort of each player depends only on his or her type. Thus, whether the player knows the type of the other player is irrelevant and the cases of public and private information coincide.

player wins by the same amount whether he has a higher or a lower type and hence, does not affect predictive power.

The first two terms in (15) come from the effect of the bias through changes in efforts while the last term is the direct effect of the bias keeping the efforts fixed. Note that $e_1^* > e_2^*$ since $t_1 < t_2$ and $\gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta}) = p_{1\beta\beta} - p_{2\beta\beta}$. Thus, the direct effect of convex aggregate marginal benefit of efforts is to make it more likely that a biased contest is optimal. For the case of multiplicative types, $C(e,t) = tc(e)$, it can be easily shown that $\frac{1}{C_{11}(e_1^*,t_1)} - \frac{1}{C_{11}(e_2^*,t_2)} > 0$ if and only if $c(e)$ exhibits increasing absolute risk aversion (IARA). If this is the case, increasing the aggregate marginal benefit of efforts unambiguously helps a biased contest to be optimal.

For some cost functions, condition (15) simplifies significantly. The simplest case is when the cost function is exponential, $C(e,t) = t(\exp(e) - 1)$. Then, $C_1(e,t) = C_{11}(e,t) = C_{111}(e,t)$ and in the equilibrium $C_1(e_1^*, t_1) = C_1(e_2^*, t_2) = \gamma_1(\bar{\beta})$ at $\beta = \bar{\beta}$. Condition (15) reduces to $\gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta}) > 0$.

Another simple case is that of the power cost function $C(e,t) = \frac{1}{z} t e^z$ with $z > 1$. Condition (15) becomes

$$\frac{2}{z-1} \frac{\gamma_1'(\bar{\beta})^2}{\gamma_1\left(\bar{\beta}\right)} + \gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta}) > 0 \tag{16}$$

The first term in (16) is positive since $z > 1$ and $\gamma_1(\beta) > 0$, while the sign of $\gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta}))$ is ambiguous. For the example in the Introduction where $p(e_1, e_2; \beta) = \frac{1}{2} + (1+\beta)e_1 - (1-\beta)e_2$ and hence, $\gamma_1''(\beta) = \gamma_2''(\beta) = 0$ this implies that predictive power has a local minimum at zero bias for any $z > 1$. Take another example considered in Section 4.1 in which $p(e_1, e_2; \beta) = \frac{1}{2} + \beta e_1 - \frac{1}{\beta} e_2$. Then, $\gamma_1''(\beta) - \gamma_2''(\beta) = \frac{1}{\beta^2} > 0$ and hence, predictive power is maximized in a biased contest for any $z > 1$ for this CSF as well.

5 Examples

5.1 Example: Tullock contest, multiplicative bias

Consider the Tullock (1980) contest success function, $p^0(e_1, e_2) = \frac{e_1}{e_1+e_2}$, and introduce the multiplicative bias as in Example 1(i): $p\left(e_1, e_2; \beta\right) = \frac{\beta e_1}{\beta e_1 + e_2}$. This CSF is unbiased at $\bar{\beta} = 1$.

We already know from Section 4.1 that when the types are identical the optimal contest in this case is unbiased for any cost function. We will now find the conditions under which the optimal contest is biased when there are two correlated types distributed as in Section 4.2. Indeed, from previous work (see Franke et al. (2013) for the most general treatment) we know that the optimal contest is biased when players are different and have linear costs. Lemma 4 then implies that the optimal contest is biased when the correlation between the two types is negative and sufficiently strong.

Consider the case of public information. The first-order conditions (5) write as

$$\frac{\beta e_2}{(\beta e_1 + e_2)^2} = t_1, \qquad \frac{\beta e_1}{(\beta e_1 + e_2)^2} = t_2,$$

where t_1 and t_2 are the constant marginal costs of player 1 and 2, respectively. Thus, $\frac{e_2}{e_1} = \frac{t_1}{t_2}$ and the equilibrium effort levels are

$$e_1^* = \frac{\beta t_2}{(t_1 + \beta t_2)^2}, \qquad e_2^* = \frac{\beta t_1}{(t_1 + \beta t_2)^2}. \tag{17}$$

When the two players have the same marginal costs t, $e_1^* = e_2^* = \frac{\beta}{(1+\beta)^2 t}$.

The expected aggregate effort is

$$Q^{\mathrm{E}}(\beta) = \frac{2\beta}{(1+\beta)^2}\left(\frac{q_{LL}}{t_L} + \frac{q_{HH}}{t_H}\right) + \beta\left(t_H + t_L\right)\left(\frac{q_{HL}}{(t_H + \beta t_L)^2} + \frac{q_{LH}}{(t_L + \beta t_H)^2}\right)$$

Then,

$$Q^{\mathrm{E}}_{\beta}(\beta) = 2\frac{1-\beta}{(1+\beta)^3}\left(\frac{q_{LL}}{t_L} + \frac{q_{HH}}{t_H}\right) + \left(t_H + t_L\right)\left(q_{HL}\frac{t_H - \beta t_L}{(t_H + \beta t_L)^3} + q_{LH}\frac{t_L - \beta t_H}{(t_L + \beta t_H)^3}\right)$$

Note as an illustration of Proposition 1 that $Q^{\mathrm{E}}_{\beta}\left(\bar{\beta}\right) = 0$. Indeed, the first term is zero and, given $q_{LH} = q_{HL}$, the second term is also zero.

The second derivative of $Q^{\mathrm{E}}(\beta)$ is

$$Q^{\mathrm{E}}_{\beta\beta}(\beta) = -4\frac{2-\beta}{(1+\beta)^4}\left(\frac{q_{LL}}{t_L} + \frac{q_{HH}}{t_H}\right) + 2\left(t_H + t_L\right)\left(q_{HL}t_L\frac{\beta t_L - 2t_H}{(t_H + \beta t_L)^4} + q_{LH}t_H\frac{\beta t_H - 2t_L}{(t_L + \beta t_H)^4}\right)$$

and at $\beta = \bar{\beta}$, using $q_{LH} = q_{HL}$, it becomes

$$Q^{\mathrm{E}}_{\beta\beta}\left(\bar{\beta}\right) = -\frac{1}{4}\left(\frac{q_{LL}}{t_L} + \frac{q_{HH}}{t_H}\right) + 2q_{HL}\frac{t_H^2 + t_L^2 - 4t_H t_L}{\left(t_H + t_L\right)^3}$$
$$= \frac{1}{t_L}\left(-\frac{1}{4}\left(q_{LL} + \frac{q_{HH}}{h}\right) + 2q_{HL}\frac{h^2 - 4h + 1}{\left(h+1\right)^3}\right), \tag{18}$$

where $h = \frac{t_H}{t_L}$, a measure of heterogeneity between the two players. If (18) is positive, the optimal contest is biased. Thus, $h^2 - 4h + 1 > 0$, that is, $h > 2 + \sqrt{3}$ is necessary for the optimal contest to be biased. In Figure 1 we plot the region where (18) is positive and, therefore, the optimal contest is biased.[20]

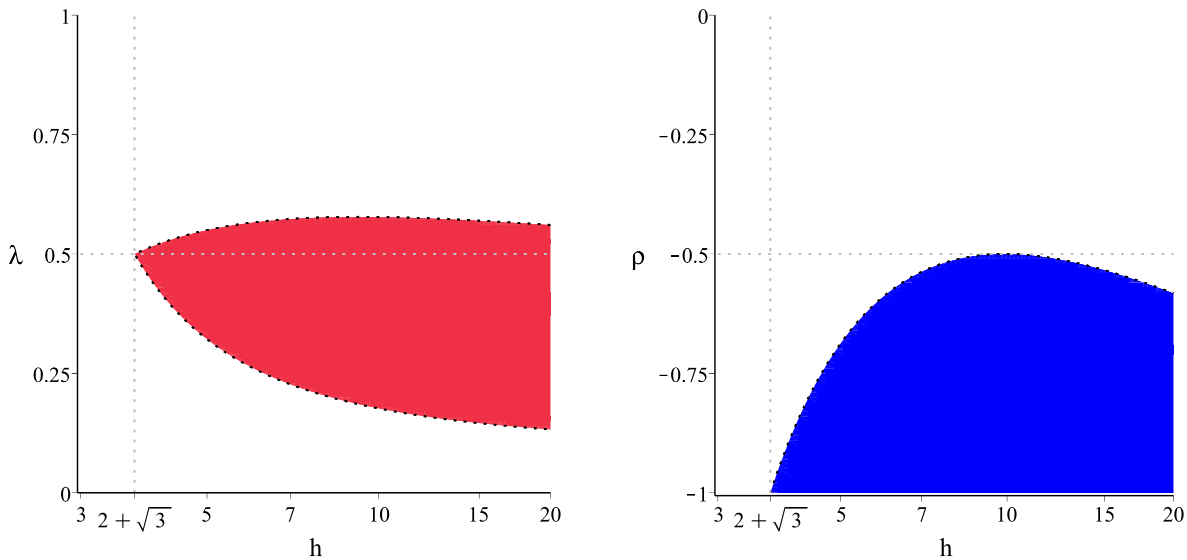

Figure 1: The region where the aggregate effort Q^{E} in a Tullock contest with a multiplicative bias is not maximized at no bias. The cost function is $C(e,t) = te$; in the left figure $\rho = \max\{-\frac{\lambda}{1-\lambda}, -\frac{1-\lambda}{\lambda}\}$ and in the right figure $\lambda = \frac{1}{2}$.

When the types are perfectly negatively correlated, there is a closed-form solution for the optimal bias, as described in the following Proposition.

Proposition 3 *In the Tullock contest with multiplicative bias and perfectly negatively correlated types, the optimal bias is* $\bar{\beta} = 1$ *(no bias) if* $h \leq 2 + \sqrt{3}$ *and otherwise it is*

[20] When the information is private, there is no closed-form solution for equilibrium efforts and, thus, the region where unbiased contest is not optimal cannot be determined analytically. However, numerical examples are easily found with the same cost parameters and negative correlation strong enough.

equal to

$$\widehat{\beta} = \frac{1}{2h\left(h^2+1\right)}\left(\left(h^2-1\right)\sqrt{\left(h^2+4h+1\right)\left(h^2-4h+1\right)} + h^4 - 6h^2 + 1\right).$$

Moreover, $\widehat{\beta}$ is strictly increasing in the heterogeneity between the players h.

In Figure 2 we plot the optimal bias and also the ratio of the total efforts in an optimally biased contest and in an unbiased contest.

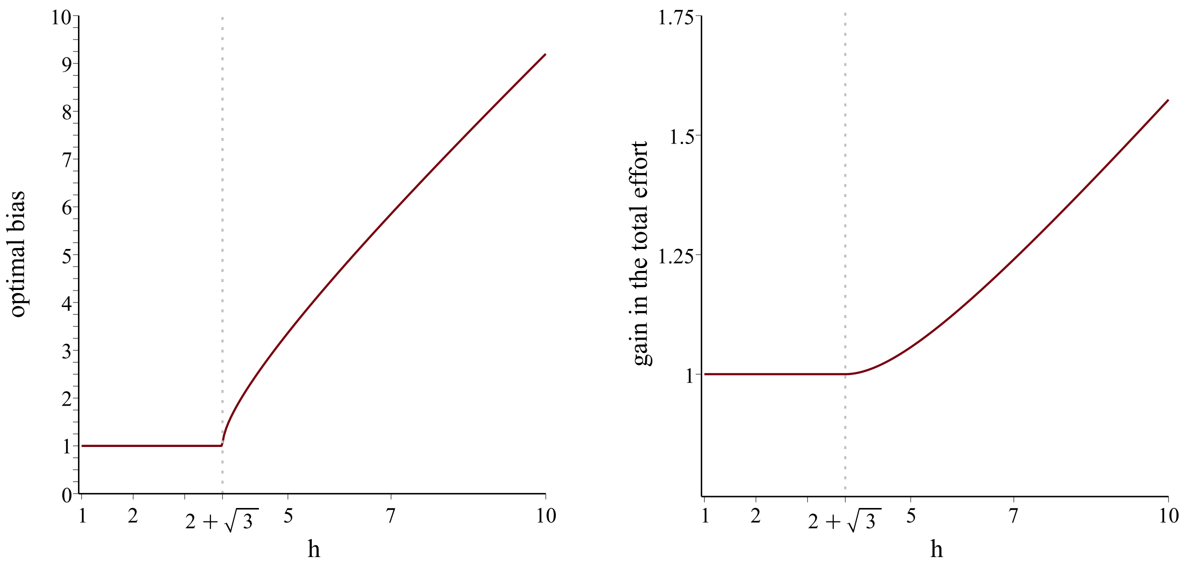

Figure 2: The optimal bias (left) and the ratio of the total efforts in an optimally biased contest and in an unbiased contest (right).

5.2 Example: Tullock contest, additive bias

Consider the Tullock (1980) contest success function, $p^0(e_1, e_2) = \frac{e_1}{e_1+e_2}$ and introduce the additive bias as in Example 1(ii): $p\left(e_1, e_2; \beta\right) = \frac{e_1+\beta}{e_1+e_2}$. This CSF is unbiased at $\bar{\beta} = 0$.

We already know from Section 4.1 that when the types are identical the optimal contest is biased (unbiased) if $C''' < (>)0$. Since, in general, there is no closed-form solution for the equilibrium efforts unless $C(e, t)$ is linear in e, we will now consider two numerical examples with two uncorrelated types. In Figure 3, we plot the aggregate effort, the winner's effort (see Example 3(iv)) and predictive power (see Example 3(ii)), which becomes, in the case of public information with two types,

$$Q^S(\beta) = \frac{1}{2}\left[p\left(e_1^*(t_L, t_H; \beta), e_2^*(t_L, t_H; \beta); \beta\right) + 1 - p\left(e_1^*(t_H, t_L; \beta), e_2^*(t_H, t_L; \beta)\right)\right]. \quad (19)$$

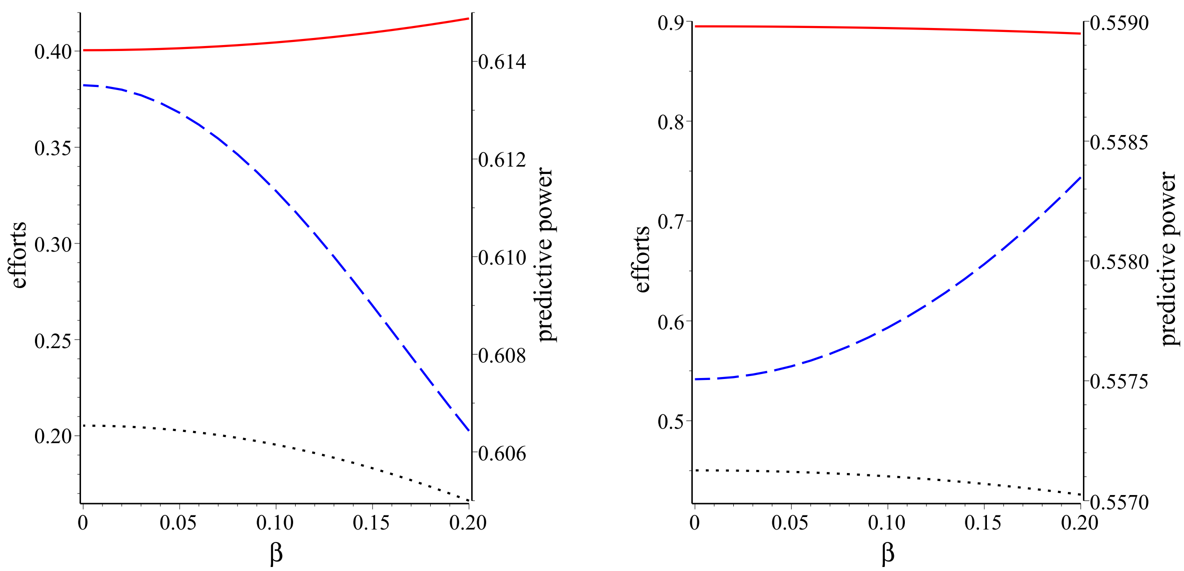

Figure 3: The expected aggregate effort (solid line) and the winner's effort (dotted line) (left scale) and the predictive power (dashed line, right scale) of a Tullock contest with an additive bias and public information as a function of bias β. The parameters are $t_L = 2$, $t_H = 4$, $\lambda = \frac{1}{2}$, $\rho = 0$ and $C(e, t) = \frac{2}{3}te^{\frac{3}{2}}$ (left figure) and $C(e, t) = \frac{1}{3}te^3$ (right figure).

Under private information, predictive power takes the form

$$Q^S(\beta) = \frac{1}{2}\left[p\left(b_1(t_L; \beta), b_2(t_H; \beta), \beta\right) + 1 - p\left(b_1(t_H; \beta), b_2(t_L; \beta)\right)\right], \quad (20)$$

and the graphs are qualitatively similar. If the types are the same, the graphs for aggregate and winner's efforts are also qualitatively similar.

5.3 Example: Lazear-Rosen tournament, multiplicative bias, normally distributed noise difference

Take Lazear-Rosen tournament with multiplicative bias as in the example in the Introduction and Example 1(iv) and suppose that $u_1 - u_2$ is distributed normally.[21] There is no closed-form solution for the equilibrium efforts. In Figure 4, we plot the aggregate effort, the winner's effort and predictive power (19) as functions of β for two different cost functions.

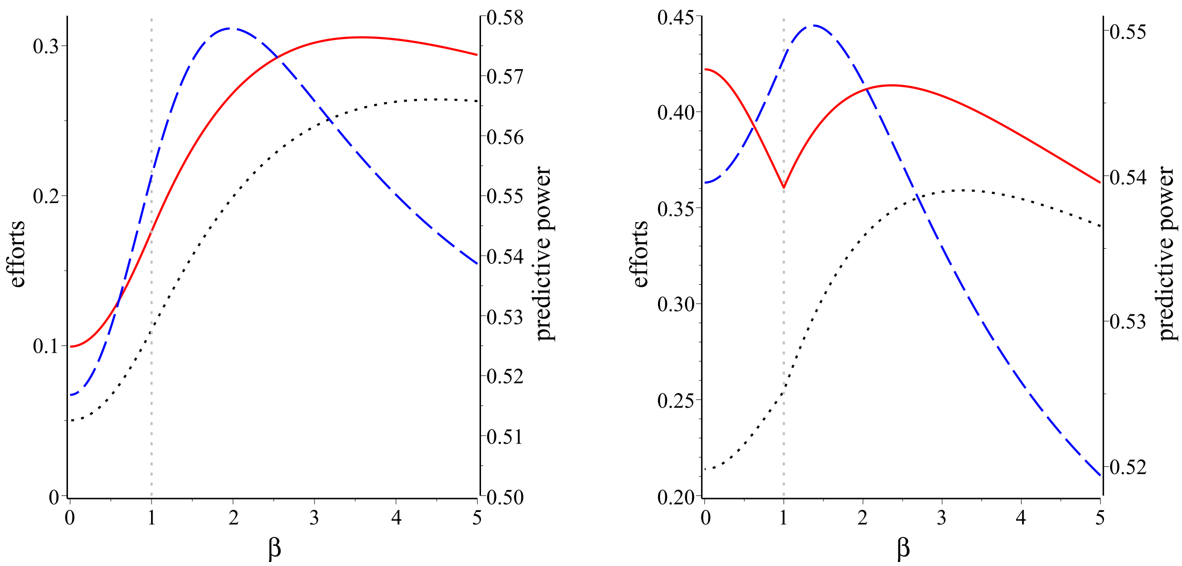

Figure 4: The aggregate effort (solid line) and the winner's effort (dotted line) (left scale) and the predictive power (dashed line, right scale) of a Lazear-Rosen tournament with multiplicative bias, $u_1 - u_2 \sim \text{Normal}(0, 2)$ and public information as a function of bias β. The parameters are $t_L = 1$, $t_H = 2$, $\lambda = \frac{1}{2}$, $\rho = 0$ and $C(e,t) = \frac{2}{3}te^{\frac{3}{2}}$ (left figure) and $C(e,t) = \frac{1}{2}te^2$ (right figure). Note that for $\beta \geq 1$ the second player does not exert any effort.

Under private information the graphs are qualitatively similar. If the types are the same, the graphs for aggregate and winner's efforts are also qualitatively similar.

[21] For the purposes of this example, we extend the definition of the CSF to allow for $\beta > 1$ (the case of $\beta < -1$ is symmetric and can be treated similarly):

$$p(e_1, e_2; \beta) = \begin{cases} \Pr\{(1+\beta)e_1 + u_1 \geq (1-\beta)e_2 + u_2\}, & \text{if } |\beta| \leq 1 \\ \Pr\{(1+\beta)e_1 + u_1 \geq u_2\}, & \text{if } \beta > 1 \end{cases}$$

Clearly, for $\beta \geq 1$ the equilibrium effort of player 2 is zero.

5.4 Expected ability of the winner

To conclude this section, note that we have not said anything about the expected ability of the winner, another possible goal of the contest organizer (see Example 3(iii)). The reason is that when there are two types, $t_L < t_H$, it is an affine transformation of predictive power. Indeed, let $a_L = a(t_L)$, $a_H = a(t_H)$, and rewrite the expected ability of the winner in the case of private information as (the case of public information is similar)

$$\begin{aligned}
&p\left(b_1\left(t_L;\beta\right), b_2\left(t_H;\beta\right);\beta\right) a_L + \left[1 - p\left(b_1\left(t_L;\beta\right), b_2\left(t_H;\beta\right);\beta\right)\right] a_H \\
&+ p\left(b_1\left(t_H;\beta\right), b_2\left(t_L;\beta\right);\beta\right) a_H + \left[1 - p\left(b_1\left(t_H;\beta\right), b_2\left(t_L;\beta\right);\beta\right)\right] a_L \\
&= a_L + a_H + \left(a_L - a_H\right)\left[p\left(b_1\left(t_L;\beta\right), b_2\left(t_H;\beta\right);\beta\right) - p\left(b_1\left(t_H;\beta\right), b_2\left(t_L;\beta\right);\beta\right)\right].
\end{aligned}$$

The last line is an affine transformation of predictive power, cf. (20). Thus, the effects of the bias on predictive power and the expected ability of the winner are qualitatively the same. In particular, as can be seen from Figures 3 and 4, in some cases bias increases predictive power and, hence, the expected ability of the winner.

6 The general case of $n \geq 2$ players

Most of our results readily extend to the case of an arbitrary number of players $n \geq 2$. Consider a contest defined by a family of biased CSFs $p^i(e_1, \ldots, e_n; \beta)$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n p^i = 1$. Here, p^i is the probability of player i winning the contest given the vector of effort levels $e = (e_1, \ldots, e_n)$ of all players, and β is the bias parameter. As before, player i's cost of effort is $C(e_i, t_i)$, where t_i is the player's type. The joint distribution of types $F(t_1, \ldots, t_n)$ is symmetric.

Let $\sigma_{ij} : \mathcal{R}^n \to \mathcal{R}^n$ denote the swap operator such that if $x' = \sigma_{ij}(x)$ then $x'_i = x_j$, $x'_j = x_i$ and $x'_k = x_k$ for $k \neq i, j$.

Definition 6 (Generalized unbiased contest) *The contest of n players is unbiased at $\beta = \bar{\beta}$ if*
(i) for any i, j, $p^i(e; \bar{\beta}) = p^j(\sigma_{ij}(e); \bar{\beta})$;
(ii) for any $k \neq i, j$, $p^k(e; \bar{\beta}) = p^k(\sigma_{ij}(e); \bar{\beta})$.

The following definition provides a general form of the locally symmetric bias condition (3).

Definition 7 (Generalized locally symmetric bias) *Contest success functions* $p^i(e;\beta)$ *have locally symmetric bias at* $\beta = \bar{\beta}$ *if for all admissible effort levels* $(e_1, \ldots, e_n)$

$$\begin{aligned} & p^1_{1\beta}(e_1, e_2, \ldots, e_{n-1}, e_n; \bar{\beta}) + p^2_{2\beta}(e_n, e_1, \ldots, e_{n-2}, e_{n-1}; \bar{\beta}) \\ & + p^3_{3\beta}(e_{n-1}, e_n, e_1, \ldots, e_{n-3}, e_{n-2}; \bar{\beta}) + \ldots + p^n_{n\beta}(e_2, e_3, \ldots, e_n, e_1; \bar{\beta}) = 0. \end{aligned} \tag{21}$$

Definition 7 states that the sum of the marginal effects of the bias on the marginal benefits of players with cyclically permuted efforts must be zero.

Note that here and in the previous sections restricting the analysis to one bias parameter is without loss of generality. All the results also apply to CSFs with multiple bias parameters $(\beta_1, \ldots; \beta_m)$. Definition 6 would then be formulated for a vector of bias parameters $(\bar{\beta}_1, \ldots, \bar{\beta}_m)$, and Definition 7 would be replaced by m equations for each bias parameter.

Continue with our example from the Introduction. Suppose there are n *ex post* symmetric players, each with the same cost function $\frac{2}{3}e_i^{\frac{3}{2}}$. The CSF is biased in favor of the first player. The bias increases the effort of the first player to $(1+\beta)e_1$ and reduces the efforts of all other players to $(1 - \frac{\beta}{n-1})e_j$, $j \geq 2$, that is,

$$\begin{aligned} p^1\left(e_1, \ldots, e_n; \beta\right) &= \frac{1}{n} + (1+\beta)\, e_1 - \frac{1}{n-1}\left(1 - \frac{\beta}{n-1}\right)\sum_{i=2}^{n} e_i, \\ p^j\left(e_1, \ldots, e_n; \beta\right) &= \frac{1}{n} + \left(1 - \frac{\beta}{n-1}\right)e_j - \frac{1+\beta}{n-1}e_1 - \frac{1}{n-1}\left(1 - \frac{\beta}{n-1}\right)\sum_{i=2, i\neq j}^{n} e_i, \quad j \geq 2, \end{aligned}$$

provided all these expressions are between zero and one. It is easy to see that this CSF has the generalized locally symmetric bias (21).

The equilibrium efforts are $e_1^* = (1+\beta)^2$ and $e_j^* = \left(1 - \frac{\beta}{n-1}\right)^2$, $j \geq 2$. Hence, the aggregate effort is

$$Q^{\mathrm{E}}\left(\beta\right) = (1+\beta)^2 + (n-1)\left(1 - \frac{\beta}{n-1}\right)^2 = n\left(1 + \frac{\beta^2}{n-1}\right),$$

which increases with β. At $\beta = \bar{\beta} = 0$ the first-order condition is satisfied and the aggregate effort reaches its minimum.

We will now show how the generalized locally symmetric bias condition (21) is related to the aggregate effort having a critical point at $\beta = \bar{\beta}$.[22] In this section, we restrict

[22]As shown in the proof of Proposition 1, it is then straightforward to extend the results to arbitrary

attention to the case of public information. It is instructive to start with the simple case of *ex post* symmetric types. As before, let e^* denote the symmetric equilibrium effort at $\beta = \bar{\beta}$ and let $c(e_i)$ denote the cost function of effort.

Lemma 5 *When players are ex post symmetric, the marginal effect of the bias on aggregate equilibrium effort takes the form*

$$\sum_{i=1}^{n} e_{i\beta}^* = \frac{\sum_{i=1}^{n} p_{i\beta}^i \left(e^*, ...e^*; \bar{\beta}\right)}{c''(e^*) - p_{11}^1(e^*, \ldots, e^*; \bar{\beta}) - (n-1)p_{12}^1(e^*, \ldots, e^*; \bar{\beta})}.$$

Proof. See Appendix A. ■

Lemma 5 shows that for *ex post* symmetric players the marginal effect of the bias on aggregate effort is proportional to the sum of the marginal effects of the bias on the marginal benefits of players, and hence the generalized local symmetry condition (21) is equivalent to $\beta = \bar{\beta}$ being a critical point of aggregate effort.

Consider now the general case of n *ex post* asymmetric players. Let $e_i^*(t;\beta)$ denote the equilibrium effort of player i in the contest given the vector of player types $t = (t_1, \ldots, t_n)$ and the bias parameter β. The expected aggregate effort in the contest is $Q^{\mathrm{E}}(\beta) = \int \sum_i e^*(t,\beta) dF(t)$.

Proposition 4 *In a biased contest of n players CSFs $p^i(e_1, \ldots; \beta)$ satisfy the generalized locally symmetric bias condition (21) if and only if $Q_\beta^{\mathrm{E}}(\bar{\beta}) = 0$.*

Proof. See Appendix A. ■

7 Conclusion

In this paper we have provided arguably the first systematic study of biased contests. The first contribution of the paper is to introduce and characterize a class of biased CSFs that includes as special cases the commonly used additive and multiplicative biases but also allows for other types of biases. We show that exactly how a bias is introduced into a CSF is crucial for zero bias to be a critical point of various principal's objectives, i.e., for whether or not a small bias will have a first-order effect on each of the objectives. Specifically, we identify necessary and sufficient conditions on the shape of a biased CSF for a general class of symmetric objectives (that includes, but is not limited to, aggregate

symmetric objectives; see also the discussion after Lemma 2.

effort, predictive power, expected effort of the winner or expected ability of the winner) to have a zero first derivative with respect to the bias at zero bias. The conditions are very general and are satisfied by most biased CSFs used in the literature.

The second contribution of the paper is to provide some general results and numerous examples when biased contests are optimal when players are symmetric. Examples include Tullock (1980) type contests and Lazear and Rosen (1981) type tournaments; contests with players identical *ex post* or only *ex ante*; contests with public information and private information; and the principal's objective functions mentioned in the paragraph above.

One important type of contest models not covered by our analysis is all-pay auctions, in which the CSF is not smooth. Such games have equilibria in mixed strategies under complete information which are not linked directly to the pure strategy equilibria we exploit in this paper. However, the pure strategy equilibria of all-pay auctions under incomplete information are smooth bidding functions that can be viewed as the zero-noise limit of equilibrium bidding functions from private information contests. Thus, our analysis informs on the effect of biases in all-pay auctions of incomplete information with arbitrarily small but nonzero noise. To what extent the results also apply to the limit of zero-noise all-pay auctions is still an open question.

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Appendix A. Proofs

Proof of Lemma 2. Differentiate (5) with respect to β to obtain (omit arguments of $e^*_{i\beta}\left(t_1, t_2; \beta\right)$ and $p_{ij}\left(e^*_1(t_1, t_2; \beta), e^*_2(t_1, t_2; \beta); \beta\right)$ for brevity)

$$\begin{aligned} p_{11}e^*_{1\beta} + p_{12}e^*_{2\beta} + p_{1\beta} &= C_{11}\left(e^*_1, t_1\right) e^*_{1\beta}, \\ -p_{12}e^*_{1\beta} - p_{22}e^*_{2\beta} - p_{2\beta} &= C_{11}\left(e^*_2, t_2\right) e^*_{2\beta}. \end{aligned} \tag{22}$$

Rewrite it as

$$\begin{aligned} \left[p_{11} - C_{11}\left(e^*_1, t_1\right)\right] e^*_{1\beta} + p_{12}e^*_{2\beta} &= -p_{1\beta}, \\ p_{12}e^*_{1\beta} + \left[p_{22} + C_{11}\left(e^*_2, t_2\right)\right] e^*_{2\beta} &= -p_{2\beta}. \end{aligned}$$

Then,

$$\begin{aligned} e^*_{1\beta}(t_1, t_2; \beta) &= -\frac{p_{1\beta}\left[p_{22} + C_{11}\left(e^*_2, t_2\right)\right] - p_{2\beta}p_{12}}{\left[p_{11} - C_{11}\left(e^*_1, t_1\right)\right]\left[p_{22} + C_{11}\left(e^*_2, t_2\right)\right] - p^2_{12}} \\ e^*_{2\beta}(t_1, t_2; \beta) &= -\frac{p_{2\beta}\left[p_{11} - C_{11}\left(e^*_1, t_1\right)\right] - p_{1\beta}p_{12}}{\left[p_{11} - C_{11}\left(e^*_1, t_1\right)\right]\left[p_{22} + C_{11}\left(e^*_2, t_2\right)\right] - p^2_{12}} \end{aligned} \tag{23}$$

From now on, set $\beta = \bar{\beta}$. Consider the symmetric cost pair, (t_2, t_1). At $\beta = \bar{\beta}$, using (1), rewrite (5) as

$$p_1\left(e_1, e_2; \bar{\beta}\right) = C_1\left(e_1, t_2\right), \quad p_1\left(e_2, e_1; \bar{\beta}\right) = C_1\left(e_2, t_1\right). \tag{24}$$

Note that the first of equations in (24) can be transformed into the second one by replacing t_1 with t_2 and e_1 with e_2. This implies that equilibrium effort levels e_1^* and e_2^* have the following symmetry:

$$e_1^*\left(t_1, t_2; \bar{\beta}\right) = e_2^*\left(t_2, t_1; \bar{\beta}\right). \tag{25}$$

We can derive $e_{i\beta}^*\left(t_2, t_1; \beta\right)$ analogous to (23). Then, using (25), replace $e_1^*\left(t_2, t_1; \bar{\beta}\right)$ and $e_2^*\left(t_2, t_1; \bar{\beta}\right)$ with $e_2^*\left(t_1, t_2; \bar{\beta}\right)$ and $e_1^*\left(t_1, t_2; \bar{\beta}\right)$, respectively, in the arguments of p_{ij}. The resulting expressions are as in (23) but with all p_{ij} evaluated at the point $\left(e_2^*(t_1, t_2; \bar{\beta}), e_1^*(t_1, t_2; \bar{\beta}); \bar{\beta}\right)$, that is, with the reversed order of equilibrium efforts:

$$\begin{aligned} e_{1\beta}^*\left(t_2, t_1, \bar{\beta}\right) &= -\frac{\tilde{p}_{1\beta}\left[\tilde{p}_{22} + C_{11}\left(e_1^*, t_1\right)\right] - \tilde{p}_{2\beta}\tilde{p}_{12}}{\left[\tilde{p}_{11} - C_{11}\left(e_2^*, t_2\right)\right]\left[\tilde{p}_{22} + C_{11}\left(e_1^*, t_1\right)\right] - \tilde{p}_{12}^2} \\ e_{2\beta}^*\left(t_2, t_1; \bar{\beta}\right) &= -\frac{\tilde{p}_{2\beta}\left[\tilde{p}_{11} - C_{11}\left(e_2^*, t_2\right)\right] - \tilde{p}_{1\beta}\tilde{p}_{12}}{\left[\tilde{p}_{11} - C_{11}\left(e_2^*, t_2\right)\right]\left[\tilde{p}_{22} + C_{11}\left(e_1^*, t_1\right)\right] - \tilde{p}_{12}^2} \end{aligned} \tag{26}$$

Here, $\tilde{p}_{ij}$ denotes $p_{ij}(e_2^*, e_1^*; \bar{\beta})$, and the arguments of $e_i^*(t_1, t_2; \bar{\beta})$ are suppressed for brevity.

Differentiating (1) with respect to e_1 twice and with respect to e_1 and e_2 obtain the relationships

$$p_{11}\left(e_1, e_2; \bar{\beta}\right) = -p_{22}\left(e_2, e_1; \bar{\beta}\right), \quad p_{12}\left(e_1, e_2; \bar{\beta}\right) = -p_{12}\left(e_2, e_1; \bar{\beta}\right), \tag{27}$$

which imply, in particular, that in equilibrium

$$p_{11} = -\tilde{p}_{22}, \quad p_{22} = -\tilde{p}_{11}, \quad p_{12} = -\tilde{p}_{12}. \tag{28}$$

(i) Comparing the expressions for $e_{1\beta}^*(t_1, t_2; \bar{\beta})$ and $e_{2\beta}^*(t_2, t_1; \bar{\beta})$ in (23) and (26), respectively, note that, due to conditions (27), the denominators are the same. Suppose that p has a locally symmetric bias, i.e., $p_\beta(e_1, e_2; \bar{\beta}) = p_\beta(e_2, e_1; \bar{\beta})$, cf. (4). Differentiating both sides with respect to e_1 and setting e_1 and e_2 to the equilibrium efforts, obtain $p_{1\beta} = \tilde{p}_{2\beta}$. Similarly, $p_{2\beta} = \tilde{p}_{1\beta}$. Comparing the numerators of $e_{1\beta}^*(t_1, t_2; \bar{\beta})$ and $e_{2\beta}^*(t_2, t_1; \bar{\beta})$, note that they only differ by sign, which proves (6).

(ii) Suppose now that (7) is true. As shown above, the denominators of all four terms are equal, therefore the numerators should sum up to zero. The sum of the numerators in (7) is (without the minus sign)

$$\begin{aligned} & p_{1\beta}\left[p_{22}+C_{11}\left(e_2^*, t_2\right)-p_{12}\right]+p_{2\beta}\left[p_{11}-C_{11}\left(e_1^*, t_1\right)-p_{12}\right] \\ & +\tilde{p}_{1\beta}\left[\tilde{p}_{22}+C_{11}\left(e_1^*, t_1\right)-\tilde{p}_{12}\right]+\tilde{p}_{2\beta}\left[\tilde{p}_{11}-C_{11}\left(e_2^*, t_2\right)-\tilde{p}_{12}\right]=0. \end{aligned} \tag{29}$$

Using (28) rewrite the last line of (29) as

$$\tilde{p}_{1\beta}\left[-p_{11}+C_{11}\left(e_1^*, t_1\right)+p_{12}\right]+\tilde{p}_{2\beta}\left[-p_{22}-C_{11}\left(e_2^*, t_2\right)+p_{12}\right].$$

Finally, rewrite (29) as

$$\left(p_{1\beta}-\tilde{p}_{2\beta}\right)\left[p_{22}+C_{11}\left(e_2^*, t_2\right)-p_{12}\right]+\left(p_{2\beta}-\tilde{p}_{1\beta}\right)\left[p_{11}-C_{11}\left(e_1^*, t_1\right)-p_{12}\right]=0. \tag{30}$$

Note that (30) must be equal to zero for any (t_1, t_2) and an arbitrary function C (provided that the equilibrium in pure strategies exists). Suppose C is quadratic in effort, then C_{11} is a constant, and the only way for (30) to be zero for any (t_1, t_2) is to have $p_{1\beta}=\tilde{p}_{2\beta}$ and $p_{2\beta}=\tilde{p}_{1\beta}$ for any (t_1, t_2). Thus, it must be that $p_{1\beta}(e_1, e_2; \bar{\beta})=p_{2\beta}(e_2, e_1; \bar{\beta})$ for any (e_1, e_2), i.e., (3) holds. Writing this expression as $p_{1\beta}(t, e_2; \bar{\beta})=p_{2\beta}(e_2, t; \bar{\beta})$ and integrating both parts over t from e_2 to e_1, obtain

$$p_\beta(e_1, e_2; \bar{\beta})-p_\beta(e_2, e_2; \bar{\beta})=p_\beta(e_2, e_1; \bar{\beta})-p_\beta(e_2, e_2; \bar{\beta}),$$

which leads to (4). ■

Proof of Lemma 3. Differentiate both sides of equations (8) over β:

$$\begin{aligned} & \int\left[p_{11}(b_1(t; \beta), b_2(t'; \beta); \beta) b_{1\beta}(t; \beta)+p_{12}(b_1(t; \beta), b_2(t'; \beta); \beta) b_{2\beta}(t'; \beta)\right. \\ & \left.+p_{1\beta}(b_1(t; \beta), b_2(t'; \beta); \beta)\right] dF(t'|t)=C_{11}(b_1(t; \beta), t) b_{1\beta}(t; \beta), \\ & -\int\left[p_{12}(b_1(t'; \beta), b_2(t; \beta); \beta) b_{1\beta}(t'; \beta)+p_{22}(b_1(t'; \beta), b_2(t; \beta); \beta) b_{2\beta}(t; \beta)\right. \\ & \left.+p_{2\beta}(b_1(t'; \beta), b_2(t; \beta); \beta)\right] dF(t'|t)=C_{11}(b_2(t; \beta), t) b_{2\beta}(t; \beta). \end{aligned} \tag{31}$$

From this point on, set $\beta=\bar{\beta}$. Recall that $b_1(t; \bar{\beta})=b_2(t; \bar{\beta})$, $p_{11}(e_1, e_2; \bar{\beta})=-p_{22}(e_2, e_1; \bar{\beta})$

and $p_{12}(e_1, e_2; \bar{\beta}) = -p_{12}(e_2, e_1; \bar{\beta})$. The system of equations (31) then gives

$$\begin{aligned}
&\int [p_{11}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta}) b_{1\beta}(t;\bar{\beta}) + p_{12}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta}) b_{2\beta}(t';\bar{\beta}) \\
&+ p_{1\beta}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta})] dF(t'|t) = C_{11}(b_1(t;\bar{\beta}), t) b_{1\beta}(t;\bar{\beta}), \qquad (32) \\
&\int [p_{12}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta}) b_{1\beta}(t';\bar{\beta}) + p_{11}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta}) b_{2\beta}(t;\bar{\beta}) \\
&- p_{2\beta}(b_1(t';\bar{\beta}), b_1(t;\bar{\beta}); \bar{\beta})] dF(t'|t) = C_{11}(b_1(t;\bar{\beta}), t) b_{2\beta}(t;\bar{\beta}).
\end{aligned}$$

Let $y(t) \equiv b_{1\beta}(t;\bar{\beta}) + b_{2\beta}(t;\bar{\beta})$. Adding the two equations (32), obtain the following Fredholm integral equation of the second kind for function $y(t)$:

$$\begin{aligned}
&\left[C_{11}(b_1(t;\bar{\beta}), t) - \int p_{11}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta}) dF(t'|t)\right] y(t) \qquad (33) \\
&= \int p_{12}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta}) y(t') dF(t'|t) \\
&+ \int [p_{1\beta}(b_1(t;\bar{\beta}), b_1(t';\bar{\beta}); \bar{\beta}) - p_{2\beta}(b_1(t';\bar{\beta}), b_1(t;\bar{\beta}); \bar{\beta})] dF(t'|t).
\end{aligned}$$

(i) Suppose that p has locally symmetric bias, i.e., $p_{1\beta}(e_1, e_2; \bar{\beta}) = p_{2\beta}(e_2, e_1; \bar{\beta})$. Then the last integral in (33) is zero, and $y(t) = 0$ is a solution of the resulting homogeneous Fredholm equation. Although it is possible for the equation to have other solutions, those would have to be eigenfunctions of the corresponding integral operator, which only exist for very special configurations of parameters. The trivial solution $y(t) = 0$ is the only "generic" solution that exists for arbitrary functions F and p. We conclude that if a pure strategy equilibrium in the contest with private information exists for a measurable set of parameterizations, it has to satisfy $y(t) = 0$.

(ii) Suppose now that $y(t) = 0$. This implies that the last integral in (33) has to be zero for all distributions F. This is only possible if the integrand is identically zero for all t and t', i.e., $p_{1\beta}(e_1, e_2; \bar{\beta}) = p_{2\beta}(e_2, e_1; \bar{\beta})$ for all admissible effort levels e_1 and e_2, i.e., p has a locally symmetric bias. ■

Proof of Proposition 1. Start with the case of public information. Differentiating $Q(\beta)$ over β and setting $\beta = \bar{\beta}$, obtain

$$\begin{aligned}
Q'(\bar{\beta}) = &\int [q_{e_1}(e_1^*, t_1; e_2^*, t_2; \bar{\beta}) e_{1\beta}^*(t_1, t_2; \bar{\beta}) + q_{e_2}(e_1^*, t_1; e_2^*, t_2; \bar{\beta}) e_{2\beta}^*(t_1, t_2; \bar{\beta}) \\
&+ q_\beta(e_1^*, t_1; e_2^*, t_2; \bar{\beta})] dF(t_1, t_2).
\end{aligned}$$

Function q_β is antisymmetric in $(e_1, t_1; e_2, t_2)$ at $\bar{\beta}$, and hence in equilibrium it is also antisymmetric in (t_1, t_2); therefore, the last term integrates to zero. Swapping the variables of integration in the second term and using the symmetry of q in $(e_1, t_1; e_2, t_2)$, obtain

$$Q'(\bar{\beta}) = \int q_{e_1}(e_1^*, t_1; e_2^*, t_2; \bar{\beta})[e_{1\beta}^*(t_1, t_2; \bar{\beta}) + e_{2\beta}^*(t_2, t_1; \bar{\beta})]dF(t_1, t_2).$$

Suppose p has a locally symmetric bias; then the expression in square brackets is equal to zero due to part (i) of Lemma 2. Conversely, consider $q = e_1 + e_2$, which gives $q_{e_1} = 1$ and

$$Q'(\bar{\beta}) = \int [e_{1\beta}^*(t_1, t_2; \bar{\beta}) + e_{2\beta}^*(t_2, t_1; \bar{\beta})]dF(t_1, t_2) = 0.$$

This implies that function $e_{1\beta}^*(t_1, t_2; \bar{\beta}) + e_{2\beta}^*(t_2, t_1; \bar{\beta})$ is antisymmetric in (t_1, t_2), i.e., condition (7) is satisfied, and part (ii) of Lemma 2 implies that p has a locally symmetric bias.

Consider now the case of private information. The objective function is written as

$$Q(\beta) = \int q(b_1(t_1; \beta), b_2(t_2; \beta); \beta)dF(t_1, t_2).$$

Differentiating with respect to β and setting $\beta = \bar{\beta}$, obtain

$$Q'(\bar{\beta}) = \int [q_{e_1}(b_1, t_1; b_2, t_2; \bar{\beta})b_{1\beta}(t_1; \bar{\beta}) + q_{e_2}(b_1, e_1; b_2, t_2; \bar{\beta})b_{2\beta}(t_2; \bar{\beta})$$
$$+ q_\beta(b_1, t_1; b_2, t_2; \bar{\beta})]dF(t_1, t_2).$$

Recall that $b_1(t; \bar{\beta}) = b_2(t; \bar{\beta})$. Due to property (ii) of Definition 3, function q_β at $\bar{\beta}$ is antisymmetric in $(b_1, t_1; b_2, t_2)$; therefore, in equilibrium it is also antisymmetric in (t_1, t_2), and the last term integrates to zero. Property (i) of Definition 3 implies $q_{e_1}(b_1, t_1; b_2, t_2; \bar{\beta}) = q_{e_2}(b_2, t_2; b_1, t_1; \bar{\beta})$. Swapping the variables of integration in the second term and using the symmetry of F obtain, similar to the case of public information,

$$Q'(\bar{\beta}) = \int q_{e_1}(b_1(t_1; \bar{\beta}), t_1; b_1(t_2; \bar{\beta}), t_2; \bar{\beta})[b_{1\beta}(t_1; \bar{\beta}) + b_{2\beta}(t_1; \bar{\beta})]dF(t_1, t_2).$$

Suppose p has a locally symmetric bias. Then the expression in square brackets is equal to zero, due to Lemma 3. Conversely, suppose $Q'(\bar{\beta}) = 0$ for any symmetric and locally symmetrically biased objective Q. Consider the objective with $q = e_1 + e_2$, i.e., with $q_{e_1} = 1$. In this case, the integrand in the expression for $Q'(\bar{\beta})$ depends on t_1 only, i.e., the whole integral is equal to an integral of $b_{1\beta}(t;\bar\beta)+b_{2\beta}(t;\bar\beta)$ over a positive measure, which implies that (9) is satisfied and, by Lemma 3, the CSF has a locally symmetric bias. ■

Proof of Proposition 2. Differentiating both sides of each of the equations in (10) with respect to β, obtain

$$p_{11}e^*_{1\beta}+p_{12}e^*_{2\beta}+p_{1\beta}=c''(e^*_1)e^*_{1\beta},\quad -p_{12}e^*_{1\beta}-p_{22}e^*_{2\beta}-p_{2\beta}=c''(e^*_2)e^*_{1\beta}. \tag{34}$$

At $\beta=\bar\beta$ we have $e^*_1=e^*_2$ and

$$p_1=-p_2,\quad p_{11}=-p_{22},\quad p_{12}=0,\quad p_{111}=-p_{222},\quad p_{112}=-p_{122}. \tag{35}$$

The first of the equations (34) then gives

$$e^*_{1\beta}=-e^*_{2\beta}=\frac{p_{1\beta}}{c''-p_{11}}. \tag{36}$$

Since we assume that $c''-p_{11}>0$, the sign of $e^*_{i\beta}$ is determined by the sign of $p_{1\beta}$.

Differentiating (34) with respect to β one more time, obtain

$$\begin{aligned}
&(p_{111}e^*_{1\beta}+p_{112}e^*_{2\beta}+p_{11\beta})e^*_{1\beta}+p_{11}e^*_{1\beta\beta}+(p_{112}e^*_{1\beta}+p_{122}e^*_{2\beta}+p_{12\beta})e^*_{2\beta}+p_{12}e^*_{2\beta\beta}\\
&+p_{11\beta}e^*_{1\beta}+p_{12\beta}e^*_{2\beta}+p_{1\beta\beta}=c'''(e^*_1)(e^*_{1\beta})^2+c''(e^*_1)e^*_{1\beta\beta},\\
&-(p_{112}e^*_{1\beta}+p_{122}e^*_{2\beta}+p_{12\beta})e^*_{1\beta}-p_{12}e^*_{1\beta\beta}-(p_{122}e^*_{1\beta}+p_{222}e^*_{2\beta}+p_{22\beta})e^*_{2\beta}-p_{22}e^*_{2\beta\beta}\\
&-p_{12\beta}e^*_{1\beta}-p_{22\beta}e^*_{2\beta}-p_{2\beta\beta}=c'''(e^*_2)(e^*_{2\beta})^2+c''(e^*_2)e^*_{2\beta\beta}.
\end{aligned}$$

Now let $\beta=\bar\beta$ (and, consequently, $e^*_1=e^*_2$) and use the relations (35):

$$\begin{aligned}
&(p_{111}e^*_{1\beta}-p_{112}e^*_{1\beta}+2p_{11\beta})e^*_{1\beta}+p_{11}e^*_{1\beta\beta}-(p_{112}e^*_{1\beta}+p_{112}e^*_{1\beta}+2p_{12\beta})e^*_{1\beta}+p_{1\beta\beta}\\
&=c'''(e^*_{1\beta})^2+c''e^*_{1\beta\beta},\\
&-(p_{112}e^*_{1\beta}+p_{112}e^*_{1\beta}+2p_{12\beta})e^*_{1\beta}+(-p_{112}e^*_{1\beta}+p_{111}e^*_{1\beta}+2p_{22\beta})e^*_{1\beta}+p_{11}e^*_{2\beta\beta}-p_{2\beta\beta}\\
&=c'''(e^*_{1\beta})^2+c''e^*_{2\beta\beta}.
\end{aligned}$$

Adding the two equations, obtain

$$2(e^*_{1\beta})^2(p_{111}-3p_{112}-c''')+4e^*_{1\beta}(p_{11\beta}-p_{12\beta})+p_{1\beta\beta}-p_{2\beta\beta}=(e^*_{1\beta\beta}+e^*_{2\beta\beta})(c''-p_{11}).$$

Here, we used the fact that $p_{11\beta}=p_{22\beta}$ due to the locally symmetric bias. Finally, using the expression (36) for $e^*_{1\beta}$, obtain the result. ■

Proof of Lemma 4. The second derivative of $Q^{\mathrm{E}}(\beta;\lambda,\rho)$ at $\beta=\bar\beta$ is

$$\begin{aligned}
&Q^{\mathrm{E}}_{\beta\beta}(\bar\beta;\lambda,\rho)=q_{LL}[e^*_{1\beta\beta}(t_L,t_L;\bar\beta)+e^*_{2\beta\beta}(t_L,t_L;\bar\beta)]\\
&+q_{HH}[e^*_{1\beta\beta}(t_H,t_H;\bar\beta)+e^*_{2\beta\beta}(t_H,t_H;\bar\beta)]\\
&+q_{HL}[e^*_{1\beta\beta}(t_H,t_L;\bar\beta)+e^*_{2\beta\beta}(t_H,t_L;\bar\beta)+e^*_{1\beta\beta}(t_L,t_H;\bar\beta)+e^*_{2\beta\beta}(t_L,t_H;\bar\beta)].
\end{aligned}$$

Using the expressions for q_{ij}, this can be written as

$$\begin{aligned}
&Q^{\mathrm{E}}_{\beta\beta}(\bar\beta;\lambda,\rho)=\lambda[e^*_{1\beta\beta}(t_L,t_L;\bar\beta)+e^*_{2\beta\beta}(t_L,t_L;\bar\beta)]\\
&+(1-\lambda)[e^*_{1\beta\beta}(t_H,t_H;\bar\beta)+e^*_{2\beta\beta}(t_H,t_H;\bar\beta)]\\
&-\lambda(1-\lambda)(1-\rho)[e^*_{1\beta\beta}(t_L,t_L;\bar\beta)+e^*_{2\beta\beta}(t_L,t_L;\bar\beta)+e^*_{1\beta\beta}(t_H,t_H;\bar\beta)\\
&+e^*_{2\beta\beta}(t_H,t_H;\bar\beta)-e^*_{1\beta\beta}(t_H,t_L;\bar\beta)-e^*_{2\beta\beta}(t_H,t_L;\bar\beta)-e^*_{1\beta\beta}(t_L,t_H;\bar\beta)\\
&-e^*_{2\beta\beta}(t_L,t_H;\bar\beta)].
\end{aligned}$$

Note that only the last term depends on ρ. The first two terms combined represent $Q^{\mathrm{E}}_{\beta\beta}(\bar\beta;\lambda,1)$, whereas the expression in the square brackets in the last term can be rewritten in the form $2[Q^{\mathrm{E}}_{\beta\beta}\left(\bar\beta;\frac{1}{2},1\right)-Q^{\mathrm{E}}_{\beta\beta}\left(\bar\beta;\frac{1}{2},-1\right)]$. ■

Proof of Lemma 5. The FOC for player i is

$$p^i_i\left(e_1,\ldots,e_n;\beta\right)=c'(e_i).$$

Let e^* denote the symmetric equilibrium effort at $\beta=\bar\beta$. Differentiating the FOC with respect to β and setting $e_1=\ldots=e_n=e^*$ and $\beta=\bar\beta$, obtain (suppressing the arguments for brevity)

$$\sum_{j=1}^n p^i_{ij}e^*_{j\beta}+p^i_{i\beta}=c''(e^*)e^*_{i\beta}.$$

Summing these up for all the players get

$$\sum_{i=1}^n\sum_{j=1}^n p^i_{ij}e^*_{j\beta}+\sum_{i=1}^n p^i_{i\beta}=c''(e^*)\sum_{i=1}^n e^*_{i\beta}.$$

Rewrite the first term as $\sum_{j=1}^n e^*_{j\beta}\sum_{i=1}^n p^i_{ij}$ and note that $\sum_{i=1}^n p^i_{ij}$ is the same for all j.

Indeed, in the symmetric equilibrium we have

$$\sum_{i=1}^{n} p^i_{ij} = p^i_{ii} + \sum_{i \neq j} p^i_{ij} = p^1_{11} + (n-1)p^1_{12}.$$

Then, expressing the sum $\sum_{i=1}^{n} e^*_{i\beta}$ leads to the result. ■

Proof of Proposition 4. For the ease of exposition, we first prove the sufficiency result for $n = 3$, then generalize it to an arbitrary n, and conclude with a proof of the necessity result. For $n = 3$, the equilibrium we consider solves the system of FOCs

$$p^i_i(e_1, e_2, e_3; \beta) = C_1(e_i, t_i), \quad i = 1, 2, 3.$$

Let $e^*_i(t; \beta)$ denote the solution of this system of equations. The expected aggregate effort is

$$Q^{\mathrm{E}}(\beta) = \int \sum_i e^*_i(t; \beta) dF(t).$$

Differentiating the FOCs with respect to β and setting $\beta = \bar{\beta}$ obtain the system of equations

$$p^i_{i1} e^*_{1\beta} + p^i_{i2} e^*_{2\beta} + p^i_{i3} e^*_{3\beta} + p^i_{i\beta} = C_{11}(e^*_i, t_i), \quad i = 1, 2, 3.$$

Let $C^i_{11} \equiv C_{11}(e^*_i, t_i)$. The system of equations above has the determinant

$$M = \begin{vmatrix} p^1_{11} - C^1_{11} & p^1_{12} & p^1_{13} \\ p^2_{12} & p^2_{22} - C^2_{11} & p^2_{23} \\ p^3_{13} & p^3_{23} & p^3_{33} - C^3_{11} \end{vmatrix}.$$

Let us first show that M is the same for all permutations of types $t = (t_1, t_2, t_3)$. It is sufficient to show that it is the same for t and $\sigma_{12}(t)$. Note that at $\beta = \bar{\beta}$ we have $e^*_i(t; \bar{\beta}) = e^*_j(\sigma_{ij}(t); \bar{\beta})$ and $e^*_k(t; \bar{\beta}) = e^*_k(\sigma_{ij}(t); \bar{\beta})$ for $k \neq i, j$, which implies $e^*_1(t_1, t_2, t_3; \bar{\beta}) = e^*_2(t_2, t_1, t_3; \bar{\beta})$ and $e^*_3(t_1, t_2, t_3; \bar{\beta}) = e^*_3(t_2, t_1, t_3; \bar{\beta})$. Abusing notation, let $e^* = (e^*_1, e^*_2, e^*_3)$ denote the vector of equilibrium efforts. Thus, a permutation of types is equivalent to the corresponding permutation of efforts e^* in the arguments of p^i. For the types (t_2, t_1, t_3) this gives the determinant

$$\tilde{M} = \begin{vmatrix} p^1_{11}(\sigma_{12}(e^*); \bar{\beta}) - C^2_{11} & p^1_{12}(\sigma_{12}(e^*); \bar{\beta}) & p^1_{13}(\sigma_{12}(e^*); \bar{\beta}) \\ p^2_{12}(\sigma_{12}(e^*); \bar{\beta}) & p^2_{22}(\sigma_{12}(e^*); \bar{\beta}) - C^1_{11} & p^2_{23}(\sigma_{12}(e^*); \bar{\beta}) \\ p^3_{13}(\sigma_{12}(e^*); \bar{\beta}) & p^3_{23}(\sigma_{12}(e^*); \bar{\beta}) & p^3_{33}(\sigma_{12}(e^*); \bar{\beta}) - C^3_{11} \end{vmatrix}.$$

From the unbiasedness condition, we have $p^1_{11}(e_2, e_1, e_3; \bar\beta) = p^1_{22}(e_1, e_2, e_3; \bar\beta)$, $p^1_{12}(e_2, e_1, e_3; \bar\beta) = p^2_{12}(e_1, e_2, e_3; \bar\beta)$, $p^1_{13}(e_2, e_1, e_3; \bar\beta) = p^2_{23}(e_1, e_2, e_3; \bar\beta)$, $p^3_{13}(e_2, e_1, e_3; \bar\beta) = p^3_{23}(e_1, e_2, e_3; \bar\beta)$ and $p^3_{33}(e_2, e_1, e_3; \bar\beta) = p^3_{33}(e_1, e_2, e_3; \bar\beta)$. This gives

$$\tilde M = \begin{vmatrix} p^2_{22}(e^*;\bar\beta) - C^2_{11} & p^2_{12}(e^*;\bar\beta) & p^2_{23}(e^*;\bar\beta) \\ p^1_{12}(e^*;\bar\beta) & p^1_{11}(e^*;\bar\beta) - C^1_{11} & p^1_{13}(e^*;\bar\beta) \\ p^3_{23}(e^*;\bar\beta) & p^3_{13}(e^*;\bar\beta) & p^3_{33}(e^*;\bar\beta) - C^3_{11} \end{vmatrix}.$$

By swapping the first two rows and then the first two columns, determinant $\tilde M$ is transformed into M, which implies that $\tilde M = M$. More generally, this implies that the determinant of the system of equations for $e^*_{i\beta}$ is invariant to a permutation of types.

For brevity, let $g^i \equiv p^i_{ii}(e^*;\bar\beta) - C^i_{11}$. Using Kramer's rule, we can write

$$e^*_{i\beta} = -\frac{M^i}{M}, \quad M^1 = \begin{vmatrix} p^1_{1\beta} & p^1_{12} & p^1_{13} \\ p^2_{2\beta} & g^2 & p^2_{23} \\ p^3_{3\beta} & p^3_{23} & g^3 \end{vmatrix},$$
$$M^2 = \begin{vmatrix} g^1 & p^1_{1\beta} & p^1_{13} \\ p^2_{12} & p^2_{2\beta} & p^2_{23} \\ p^3_{13} & p^3_{3\beta} & g^3 \end{vmatrix}, \quad M^3 = \begin{vmatrix} g^1 & p^1_{12} & p^1_{1\beta} \\ p^2_{12} & g^2 & p^2_{2\beta} \\ p^3_{13} & p^3_{23} & p^3_{3\beta} \end{vmatrix}.$$

Further, using the symmetry of distribution F we can write

$$Q^{\mathrm{E}}_\beta(\bar\beta) = -\frac{1}{n!}\int \frac{1}{M}\sum_s [M^1(s) + M^2(s) + M^3(s)]dF(s),$$

where the summation goes over all permutations $s = (s_1, s_2, s_3)$ of vector t and $M^i(s)$ denotes the determinant M^i evaluated for the corresponding permutation.

Consider the sum over permutations in the equation above:

$$\begin{aligned} P = {} & M^1(t_1,t_2,t_3) + M^2(t_1,t_2,t_3) + M^3(t_1,t_2,t_3) \\ & + M^1(t_1,t_3,t_2) + M^2(t_1,t_3,t_2) + M^3(t_1,t_3,t_2) \\ & + M^1(t_2,t_1,t_3) + M^2(t_2,t_1,t_3) + M^3(t_2,t_1,t_3) \\ & + M^1(t_2,t_3,t_1) + M^2(t_2,t_3,t_1) + M^3(t_2,t_3,t_1) \\ & + M^1(t_3,t_1,t_2) + M^2(t_3,t_1,t_2) + M^3(t_3,t_1,t_2) \\ & + M^1(t_3,t_2,t_1) + M^2(t_3,t_2,t_1) + M^3(t_3,t_2,t_1). \end{aligned}$$

Note that $M^1(t_1,t_2,t_3) = M^1(t_1,t_3,t_2)$, $M^2(t_1,t_2,t_3) = M^1(t_3,t_2,t_1)$ and $M^3(t_1,t_2,t_3) = M^3(t_2,t_1,t_3)$. This gives

$$\begin{aligned}
P = 2[&M^1(t_1,t_2,t_3) + M^1(t_2,t_3,t_1) + M^1(t_3,t_1,t_2) \\
&+ M^2(t_1,t_2,t_3) + M^2(t_2,t_3,t_1) + M^2(t_3,t_1,t_2) \\
&+ M^3(t_1,t_2,t_3) + M^3(t_2,t_3,t_1) + M^3(t_3,t_1,t_2)].
\end{aligned}$$

We will find conditions for this sum to be equal to zero for any configuration of types. This is equivalent to the requirement that the corresponding sum of determinants be equal to zero for any admissible vector of efforts (e_1,e_2,e_3). Consider the first three terms in the sum above:

$$\begin{aligned}
&\begin{vmatrix}
p^1_{1\beta}(e_1,e_2,e_3;\bar\beta) & p^1_{12}(e_1,e_2,e_3;\bar\beta) & p^1_{13}(e_1,e_2,e_3;\bar\beta) \\
p^2_{2\beta}(e_1,e_2,e_3;\bar\beta) & g^2(e_1,e_2,e_3;\bar\beta) & p^2_{23}(e_1,e_2,e_3;\bar\beta) \\
p^3_{3\beta}(e_1,e_2,e_3;\bar\beta) & p^3_{23}(e_1,e_2,e_3;\bar\beta) & g^3(e_1,e_2,e_3;\bar\beta)
\end{vmatrix} \\
&+ \begin{vmatrix}
p^1_{1\beta}(e_2,e_3,e_1;\bar\beta) & p^1_{12}(e_2,e_3,e_1;\bar\beta) & p^1_{13}(e_2,e_3,e_1;\bar\beta) \\
p^2_{2\beta}(e_2,e_3,e_1;\bar\beta) & g^2(e_2,e_3,e_1;\bar\beta) & p^2_{23}(e_2,e_3,e_1;\bar\beta) \\
p^3_{3\beta}(e_2,e_3,e_1;\bar\beta) & p^3_{23}(e_2,e_3,e_1;\bar\beta) & g^3(e_2,e_3,e_1;\bar\beta)
\end{vmatrix} \\
&+ \begin{vmatrix}
p^1_{1\beta}(e_3,e_1,e_2;\bar\beta) & p^1_{12}(e_3,e_1,e_2;\bar\beta) & p^1_{13}(e_3,e_1,e_2;\bar\beta) \\
p^2_{2\beta}(e_3,e_1,e_2;\bar\beta) & g^2(e_3,e_1,e_2;\bar\beta) & p^2_{23}(e_3,e_1,e_2;\bar\beta) \\
p^3_{3\beta}(e_3,e_1,e_2;\bar\beta) & p^3_{23}(e_3,e_1,e_2;\bar\beta) & g^3(e_3,e_1,e_2;\bar\beta)
\end{vmatrix}.
\end{aligned}$$

Using the unbiasedness condition, the second and third determinants can be written with g^i and p^i_{jk} as functions of (e_1,e_2,e_3). Leaving out the arguments, obtain

$$\begin{aligned}
&\begin{vmatrix}
p^1_{1\beta}(e_1,e_2,e_3;\bar\beta) & p^1_{12} & p^1_{13} \\
p^2_{2\beta}(e_1,e_2,e_3;\bar\beta) & g^2 & p^2_{23} \\
p^3_{3\beta}(e_1,e_2,e_3;\bar\beta) & p^3_{23} & g^3
\end{vmatrix}
+ \begin{vmatrix}
p^1_{1\beta}(e_2,e_3,e_1;\bar\beta) & p^2_{23} & p^2_{12} \\
p^2_{2\beta}(e_2,e_3,e_1;\bar\beta) & g^3 & p^3_{13} \\
p^3_{3\beta}(e_2,e_3,e_1;\bar\beta) & p^1_{13} & g^1
\end{vmatrix} \\
&+ \begin{vmatrix}
p^1_{1\beta}(e_3,e_1,e_2;\bar\beta) & p^3_{13} & p^3_{23} \\
p^2_{2\beta}(e_3,e_1,e_2;\bar\beta) & g^1 & p^1_{12} \\
p^3_{3\beta}(e_3,e_1,e_2;\bar\beta) & p^2_{12} & g^2
\end{vmatrix}.
\end{aligned}$$

Using symmetry, obtain for the whole sum:

$$\begin{vmatrix} p^1_{1\beta}(e_1,e_2,e_3;\bar\beta) & p^1_{12} & p^1_{13} \\ p^2_{2\beta}(e_1,e_2,e_3;\bar\beta) & g^2 & p^2_{23} \\ p^3_{3\beta}(e_1,e_2,e_3;\bar\beta) & p^3_{23} & g^3 \end{vmatrix} + \begin{vmatrix} p^1_{1\beta}(e_2,e_3,e_1;\bar\beta) & p^2_{23} & p^2_{12} \\ p^2_{2\beta}(e_2,e_3,e_1;\bar\beta) & g^3 & p^3_{13} \\ p^3_{3\beta}(e_2,e_3,e_1;\bar\beta) & p^1_{13} & g^1 \end{vmatrix}$$
$$+ \begin{vmatrix} p^1_{1\beta}(e_3,e_1,e_2;\bar\beta) & p^3_{13} & p^3_{23} \\ p^2_{2\beta}(e_3,e_1,e_2;\bar\beta) & g^1 & p^1_{12} \\ p^3_{3\beta}(e_3,e_1,e_2;\bar\beta) & p^2_{12} & g^2 \end{vmatrix}$$
$$+ \begin{vmatrix} g^1 & p^1_{1\beta}(e_1,e_2,e_3;\bar\beta) & p^1_{13} \\ p^2_{12} & p^2_{2\beta}(e_1,e_2,e_3;\bar\beta) & p^2_{23} \\ p^3_{13} & p^3_{3\beta}(e_1,e_2,e_3;\bar\beta) & g^3 \end{vmatrix} + \begin{vmatrix} g^2 & p^1_{1\beta}(e_2,e_3,e_1;\bar\beta) & p^2_{12} \\ p^3_{23} & p^2_{2\beta}(e_2,e_3,e_1;\bar\beta) & p^3_{13} \\ p^1_{12} & p^3_{3\beta}(e_2,e_3,e_1;\bar\beta) & g^1 \end{vmatrix}$$
$$+ \begin{vmatrix} g^3 & p^1_{1\beta}(e_3,e_1,e_2;\bar\beta) & p^3_{23} \\ p^1_{13} & p^2_{2\beta}(e_3,e_1,e_2;\bar\beta) & p^1_{12} \\ p^2_{23} & p^3_{3\beta}(e_3,e_1,e_2;\bar\beta) & g^2 \end{vmatrix}$$
$$+ \begin{vmatrix} g^1 & p^1_{12} & p^1_{1\beta}(e_1,e_2,e_3;\bar\beta) \\ p^2_{12} & g^2 & p^2_{2\beta}(e_1,e_2,e_3;\bar\beta) \\ p^3_{13} & p^3_{23} & p^3_{3\beta}(e_1,e_2,e_3;\bar\beta) \end{vmatrix} + \begin{vmatrix} g^2 & p^2_{23} & p^1_{1\beta}(e_2,e_3,e_1;\bar\beta) \\ p^3_{23} & g^3 & p^2_{2\beta}(e_2,e_3,e_1;\bar\beta) \\ p^1_{12} & p^1_{13} & p^3_{3\beta}(e_2,e_3,e_1;\bar\beta) \end{vmatrix}$$
$$+ \begin{vmatrix} g^3 & p^3_{13} & p^1_{1\beta}(e_3,e_1,e_2;\bar\beta) \\ p^1_{13} & g^1 & p^2_{2\beta}(e_3,e_1,e_2;\bar\beta) \\ p^1_{23} & p^2_{23} & p^3_{3\beta}(e_3,e_1,e_2;\bar\beta) \end{vmatrix}.$$

We will now group the terms in the following way: 1+6+8, 2+4+9, 3+5+7. Terms 1, 6 and 8 combined produce

$$\begin{vmatrix} p^1_{1\beta}(e_1,e_2,e_3;\bar\beta) & p^1_{12} & p^1_{13} \\ p^2_{2\beta}(e_1,e_2,e_3;\bar\beta) & g^2 & p^2_{23} \\ p^3_{3\beta}(e_1,e_2,e_3;\bar\beta) & p^3_{23} & g^3 \end{vmatrix} + \begin{vmatrix} g^3 & p^1_{1\beta}(e_3,e_1,e_2;\bar\beta) & p^3_{23} \\ p^1_{13} & p^2_{2\beta}(e_3,e_1,e_2;\bar\beta) & p^1_{12} \\ p^2_{23} & p^3_{3\beta}(e_3,e_1,e_2;\bar\beta) & g^2 \end{vmatrix}$$
$$+ \begin{vmatrix} g^2 & p^2_{23} & p^1_{1\beta}(e_2,e_3,e_1;\bar\beta) \\ p^3_{23} & g^3 & p^2_{2\beta}(e_2,e_3,e_1;\bar\beta) \\ p^1_{12} & p^1_{13} & p^3_{3\beta}(e_2,e_3,e_1;\bar\beta) \end{vmatrix}$$
$$\begin{aligned} &= (g^2g^3 - p^2_{23}p^3_{23})[p^1_{1\beta}(e_1,e_2,e_3;\bar\beta) + p^2_{2\beta}(e_3,e_1,e_2;\bar\beta) + p^3_{3\beta}(e_2,e_3,e_1;\bar\beta)] \qquad (37) \\ &- (p^1_{12}g^3 - p^1_{13}p^3_{23})[p^2_{2\beta}(e_1,e_2,e_3;\bar\beta) + p^3_{3\beta}(e_3,e_1,e_2;\bar\beta) + p^1_{1\beta}(e_2,e_3,e_1;\bar\beta)] \\ &+ (p^1_{12}p^2_{23} - g^2p^1_{13})[p^3_{3\beta}(e_1,e_2,e_3;\bar\beta) + p^1_{1\beta}(e_3,e_1,e_2;\bar\beta) + p^2_{2\beta}(e_2,e_3,e_1;\bar\beta)]. \end{aligned}$$

Each of the expressions in square brackets is zero due to the locally symmetric bias condition (21).

Consider now the case of arbitrary $n \geq 2$. Similar to the special case above, write $e^*_{i\beta} = -\frac{M^i}{M}$, where M is a determinant with elements $m_{ij} = p^i_{ij} - \delta_{ij} C^i_{11}$, which is invariant to permutations of types, and M^i is the determinant M with the i-th column replaced by vector $(p^1_{1\beta}, \ldots, p^n_{n\beta})^T$.

The derivative of expected aggregate effort can be written as a sum over all permutations of types s:

$$Q^{\mathrm{E}}_\beta(\bar{\beta}) = -\frac{1}{n!} \int \frac{1}{M} \sum_s \sum_i M^i(s) dF(s).$$

Notice that $M^i(t)$ does not change with permutations of t as long as the i-th component of t stays fixed. Thus, for each i there are $(n-1)!$ identical terms in the sum that have $s_i = t_1$, $(n-1)!$ identical terms that have $s_i = t_2$, etc. We can, therefore, use the cyclical permutations of types to write the sum in the form

$$(n-1)! \sum_i [M^i(t_1, t_2, \ldots, t_n) + M^i(t_n, t_1, \ldots, t_{n-1}) + \ldots + M^i(t_2, \ldots, t_n, t_1)].$$

The sum in the expression above contains n^2 terms and can be divided into n groups of n terms each, where the first group is

$$M^1(t_1, t_2, \ldots, t_n) + M^2(t_n, t_1, \ldots, t_{n-1}) + \ldots + M^n(t_2, \ldots, t_n, t_1),$$

the second group is

$$M^1(t_n, t_1, \ldots, t_{n-1}) + M^2(t_{n-1}, t_n, t_1, \ldots, t_{n-2}) + \ldots + M^n(t_1, \ldots, t_n),$$

and the remaining groups are obtained by shifting the cyclical permutation one step forward in each term of the previous group.

We will now show that each of these groups of terms is equal to zero under the locally symmetric bias condition. Of course, it is sufficient to only prove this for one of the groups; therefore, we will focus on the first group. Thus, we will show that condition (21) implies

$$M^1(t_1, t_2, \ldots, t_n) + M^2(t_n, t_1, \ldots, t_{n-1}) + \ldots + M^n(t_2, \ldots, t_n, t_1) = 0. \tag{38}$$

Notice that the minor of element $p^1_{1\beta}$ from $M^1(t_1, t_2, \ldots, t_n)$ is the same as the minor of element $p^2_{2\beta}$ from $M^2(t_n, t_1, \ldots, t_{n-1})$. Indeed, element $p^1_{1\beta}$ is in the 11 position in $M^1(t_1, t_2, \ldots, t_n)$, therefore its minor is the same as the 11 minor of M. Element $p^2_{2\beta}$ is in the 22 position in $M^2(t_n, t_1, \ldots, t_{n-1})$, therefore its minor is the same as the 22 minor of M evaluated at the permutation of types $(t_n, t_1, \ldots, t_{n-1})$. Recall that M is invariant under permutations of types, and therefore so are the minors of its diagonal elements as long as the permutation does not change the type corresponding to that element. Thus, the 22 minor of M evaluated at $(t_n, t_1, \ldots, t_{n-1})$ is the same as the 22 minor of M evaluated at $\sigma_{12}(t)$, which is the same as the 11 minor of M evaluated at t.

Thus, we have shown that if we expand each determinant M^i in (38) in the elements of its i-th column, the coefficients on $p^1_{1\beta}$ in M^1, $p^2_{2\beta}$ in M^2, ..., on $p^n_{n\beta}$ in M^n are the same. Similarly, it follows that the coefficients on $p^2_{2\beta}$ in M^1, $p^3_{3\beta}$ in M^2, ..., on $p^1_{1\beta}$ in M^n are also the same, and so on. This implies that if condition (21) is satisfied then (38) is true.

In order to prove necessity, assume that $Q^{\mathrm{E}}_{\beta}(\bar{\beta}) = 0$ for all symmetric distributions of types F. This implies that the symmetrized marginal effect of β on total effort is zero, i.e.,

$$\frac{1}{n!} \sum_s \sum_i e^*_{i\beta}(s; \bar{\beta}) = 0,$$

where the summation goes over all permutations of the vector of types t. Recall that $e^*_{i\beta}(s; \bar{\beta}) = -\frac{M^i(s)}{M}$, where M is the same for all permutations of type, therefore $\sum_s \sum_i M^i(s) = 0$. As shown above, this implies that

$$\begin{aligned}
&\sum_i [M^i(t_1, t_2, \ldots, t_n) + M^i(t_n, t_1, \ldots, t_{n-1}) + \ldots + M^i(t_2, \ldots, t_n, t_1)] \\
&= M^1(t_1, t_2, \ldots, t_n) + M^2(t_n, t_1, \ldots, t_{n-1}) + \ldots + M^n(t_2, \ldots, t_n, t_1) \\
&+ M^1(t_n, t_1, \ldots, t_{n-1}) + M^2(t_{n-1}, t_n, t_1, \ldots, t_{n-2}) + \ldots + M^n(t_1, \ldots, t_n) \\
&+ \ldots \\
&+ M^1(t_2, \ldots, t_n, t_1) + M^2(t_1, t_2, \ldots, t_n) + \ldots + M^n(t_3, \ldots, t_n, t_1, t_2) = 0.
\end{aligned}$$

Here, we split the sum into the same n groups following cyclical permutations of types as described above. Each of these groups is a sum of terms like $K_s \sum_i p^i_{i\beta}$ where each $p^i_{i\beta}$ is evaluated at one of the cyclical permutations of types, and coefficients K_s are determined by the second derivatives p^i_{ij} and C^i_{11} (through g^i), cf. Eq. (37). The whole sum must equal zero for all configurations of types t, which are equivalent to arbitrary configurations of efforts in the arguments of p^i_{ij} and C^i_{11}. Note that the terms with C^i_{11} are only present in the coefficients K_s. Thus, for the sum to be identically zero, a restriction would have to be imposed involving the cost function of effort. Without imposing such restrictions, each of the coefficients K_s cannot be identically equal zero. Thus, the only way for the sum to be equal to zero without imposing restrictions on the cost function is to require that $\sum_i p^i_{i\beta} = 0$ for all cyclical permutations of efforts. ■

Appendix B. Second-order conditions

In this section, we derive general second-order conditions for maximization of two most popular objectives, expected aggregate effort and predictive power, under both public and private information. In doing so, we do not impose any assumptions on the CSF (except for locally symmetric bias) and distribution of types (except for its symmetry).

B1. Aggregate effort under public information

In the case of public information, define expected aggregate effort in the contest as

$$Q^{\mathrm{E}}(\beta) = \int [e_1^*(t_1, t_2; \beta) + e_2^*(t_1, t_2; \beta)] dF(t_1, t_2),$$

where $e_i^*(t_1, t_2; \beta)$ are the equilibrium effort levels satisfying the system of equations (5).

The following proposition provides conditions under which expected aggregate effort $Q^{\mathrm{E}}(\beta)$ has a local maximum or a local minimum at $\beta = \bar{\beta}$.

Proposition B1 *Suppose p has locally symmetric bias and the following condition is satisfied for all types (t_1, t_2) in the support of F at $\beta = \bar{\beta}$:*

$$[C_{11}(e_2^*, t_2) + p_{22} - p_{12}](2A_1 + p_{1\beta\beta} - \tilde{p}_{2\beta\beta}) < 0. \tag{39}$$

Here, $\tilde{p}_{2\beta\beta} \equiv p_{2\beta\beta}(e_2^*, e_1^*; \bar{\beta})$ and all other functions are evaluated at $(e_1^*, e_2^*; \bar{\beta})$;

$$\begin{aligned}
&A_1 = [p_{111}e_{1\beta}^* + p_{112}e_{2\beta}^* + 2p_{11\beta}]e_{1\beta}^* + [p_{112}e_{1\beta}^* + p_{122}e_{2\beta}^* + 2p_{12\beta}]e_{2\beta}^* - C_{111}(e_1^*, t_1)(e_{1\beta}^*)^2,\\
&e_{1\beta}^* = D^{-1}[p_{1\beta}(C_{11}(e_2^*, t_2) + p_{22}) - p_{12}p_{2\beta}],\\
&e_{2\beta}^* = -D^{-1}[p_{2\beta}(C_{11}(e_1^*, t_1) - p_{11}) + p_{12}p_{1\beta}],\\
&D = [C_{11}(e_1^*, t_1) - p_{11}][C_{11}(e_2^*, t_2) + p_{22}] + p_{12}^2.
\end{aligned} \tag{40}$$

Then $Q^{\mathrm{E}}(\beta)$ has a local maximum at $\beta = \bar{\beta}$. If the inequality in (39) is reversed, $Q^{\mathrm{E}}(\beta)$ has a local minimum at $\beta = \bar{\beta}$.

Proof. The system of first-order conditions for equilibrium efforts is

$$p_1(e_1, e_2; \beta) = C_1(e_1, t_1), \quad -p_2(e_1, e_2; \beta) = C_1(e_2, t_2).$$

Differentiating with respect to β, obtain

$$p_{11}e^*_{1\beta} + p_{12}e^*_{2\beta} + p_{1\beta} = C_{11}(e^*_1, t_1)e^*_{1\beta}, \quad -p_{12}e^*_{1\beta} - p_{22}e^*_{2\beta} - p_{2\beta} = C_{11}(e^*_2, t_2)e'_2.$$

Rewriting the system of equations above as

$$[C_{11}(e^*_1, t_1) - p_{11}]e^*_{1\beta} - p_{12}e^*_{2\beta} = p_{1\beta}, \quad p_{12}e^*_{1\beta} + [C_{11}(e^*_2, t_2) + p_{22}]e^*_{2\beta} = -p_{2\beta},$$

and solving it for $e^*_{1\beta}$ and $e^*_{1\beta}$, obtain the expressions given in the Proposition.

Differentiating with respect to β once more, and setting $\beta = \bar{\beta}$, obtain

$$\begin{gathered}
[p_{111}e^*_{1\beta} + p_{112}e^*_{2\beta} + p_{11\beta}]e^*_{1\beta} + p_{11}e^*_{1\beta\beta} + [p_{112}e^*_{1\beta} + p_{122}e^*_{2\beta} + p_{12\beta}]e^*_{2\beta} + p_{12}e^*_{2\beta\beta} \\
+p_{11\beta}e^*_{1\beta} + p_{12\beta}e^*_{2\beta} + p_{1\beta\beta} = C_{111}(e^*_1, t_1)(e^*_{1\beta})^2 + C_{11}(e^*_1, t_1)e^*_{1\beta\beta}, \\
-[p_{112}e^*_{1\beta} + p_{122}e^*_{2\beta} + p_{12\beta}]e^*_{1\beta} - p_{12}e^*_{1\beta\beta} - [p_{122}e^*_{1\beta} + p_{222}e^*_{2\beta} + p_{22\beta}]e^*_{2\beta} - p_{22}e^*_{2\beta\beta} \\
-p_{12\beta}e^*_{1\beta} - p_{22\beta}e^*_{2\beta} - p_{2\beta\beta} = C_{111}(e^*_2, t_2)(e^*_{2\beta})^2 + C_{11}(e^*_2, t_2)e^*_{2\beta\beta},
\end{gathered}$$

which can be rewritten as

$$\begin{aligned}
[C_{11}(e^*_1, t_1) - p_{11}]e^*_{1\beta\beta} - p_{12}e^*_{2\beta\beta} &= A_1 + p_{1\beta\beta}, \\
p_{12}e^*_{1\beta\beta} + [C_{11}(e^*_2, t_2) + p_{22}]e^*_{2\beta\beta} &= A_2 - p_{2\beta\beta},
\end{aligned}$$

where A_1 is given in (40) and

$$A_2 = -[p_{112}e^*_{1\beta} + p_{122}e^*_{2\beta} + 2p_{12\beta}]e^*_{1\beta} - [p_{122}e^*_{1\beta} + p_{222}e^*_{2\beta} + 2p_{22\beta}]e^*_{2\beta} - C_{111}(e^*_2, t_2)(e^*_{2\beta})^2. \tag{41}$$

The determinant of the system of equations for $e^*_{1\beta\beta}$ and $e^*_{2\beta\beta}$ is D, the same as the determinant of the system for $e^*_{1\beta}$ and $e^*_{2\beta}$. Solving the system gives

$$\begin{aligned}
e^*_{1\beta\beta} &= D^{-1}[(C_{11}(e^*_2, t_2) + p_{22})(A_1 + p_{1\beta\beta}) + p_{12}(A_2 - p_{2\beta\beta})], \qquad (42) \\
e^*_{2\beta\beta} &= D^{-1}[(C_{11}(e^*_1, t_1) - p_{11})(A_2 - p_{2\beta\beta}) - p_{12}(A_1 + p_{1\beta\beta})],
\end{aligned}$$

and

$$D(e^*_{1\beta\beta} + e^*_{2\beta\beta}) = [C_{11}(e^*_2, t_2) + p_{22} - p_{12}](A_1 + p_{1\beta\beta}) + [C_{11}(e^*_1, t_1) - p_{11} + p_{12}](A_2 - p_{2\beta\beta}).$$

Thus,

$$Q^{\mathrm{E}}_{\beta\beta}(\bar{\beta}) = \int (e^*_{1\beta\beta} + e^*_{2\beta\beta}) dF(t_1, t_2) = \int D^{-1}[C_{11}(e^*_2, t_2) + p_{22} - p_{12}](A_1 + p_{1\beta\beta}) dF(t_1, t_2)$$
$$+ \int D^{-1}[C_{11}(e^*_1, t_1) - p_{11} + p_{12}](A_2 - p_{2\beta\beta}) dF(t_1, t_2).$$

Using the symmetry of distribution F, we now swap the variables of integration (t_1, t_2) in the second integral, which is equivalent to swapping equilibrium efforts e^*_1 and e^*_2 in the arguments of all functions. Note that D is invariant under such a swap, whereas the expression in the square brackets becomes the same as in the first integral and, due to the locally symmetric bias condition, A_2 becomes A_1. This gives

$$\begin{aligned} Q^{\mathrm{E}}_{\beta\beta}(\bar{\beta}) &= \int D^{-1}[C_{11}(e^*_2, t_2) + p_{22} - p_{12}](A_1 + p_{1\beta\beta}) dF(t_1, t_2) \\ &+ \int D^{-1}[C_{11}(e^*_2, t_2) + p_{22} - p_{12}](A_1 - \tilde{p}_{2\beta\beta}) dF(t_1, t_2) \\ &= \int D^{-1}[C_{11}(e^*_2, t_2) + p_{22} - p_{12}](2A_1 + p_{1\beta\beta} - \tilde{p}_{2\beta\beta}) dF(t_1, t_2). \end{aligned}$$

Because $D > 0$ due to the second-order conditions for the equilibrium, the result follows. ■

B2. Aggregate effort under private information

In the case of private information, expected aggregate effort is defined as

$$Q^{\mathrm{E}}(\beta) = \int [b_1(t; \beta) + b_2(t; \beta)] dF(t),$$

where $b_i(t; \beta)$ are the equilibrium bidding functions satisfying the system of integral equations (8).

The following proposition provides conditions under which expected aggregate effort $Q^{\mathrm{E}}(\beta)$ has a local maximum or a local minimum at $\beta = \bar{\beta}$.

Proposition B2 *Suppose p has locally symmetric bias and $z(t)$ is a solution of the fol-*

lowing integral equation:

$$\left[C_{11}(b_1(t;\bar\beta),t)-\int p_{11}(b_1(t;\bar\beta),b_1(t';\bar\beta);\bar\beta)dF(t'|t)\right]z(t) \tag{43}$$
$$=\int p_{12}(b_1(t;\bar\beta),b_1(t';\bar\beta);\bar\beta)z(t')dF(t'|t)+B(t),$$

where

$$B(t)=-2C_{111}(b_1(t;\bar\beta),t)b_{1\beta}^2(t;\bar\beta)+\int[2(p_{111}b_{1\beta}(t;\bar\beta)+p_{112}b_{2\beta}(t';\bar\beta)+2p_{11\beta})b_{1\beta}(t;\bar\beta)$$
$$+2(p_{112}b_{1\beta}(t;\bar\beta)+p_{122}b_{2\beta}(t';\bar\beta)+2p_{12\beta})b_{2\beta}(t';\bar\beta)+p_{1\beta\beta}-\tilde p_{2\beta\beta}]dF(t'|t),$$

with $\tilde p_{2\beta\beta}\equiv p_{2\beta\beta}(b_1(t';\bar\beta),b_1(t;\bar\beta);\bar\beta)$ *and all other derivatives of* p *evaluated at* $(b_1(t;\bar\beta),b_1(t';\bar\beta);\bar\beta)$.

Then $Q^{\mathrm{E}}(\beta)$ *has a local maximum (respectively, local minimum) at* $\beta=\bar\beta$ *if* $z(t)$ *is negative (respectively, positive) for all* t.

Proof. Differentiating (8) over β twice and setting $\beta=\bar\beta$ obtain

$$\int[(p_{111}b_{1\beta}+p_{112}b'_{2\beta}+p_{11\beta})b_{1\beta}+p_{11}b_{1\beta\beta}+(p_{112}b_{1\beta}+p_{122}b'_{2\beta}+p_{12\beta})b'_{2\beta}+p_{12}b'_{2\beta\beta}$$
$$+p_{11\beta}b_{1\beta}+p_{12\beta}b'_{2\beta}+p_{1\beta\beta}]dF(t'|t)=C_{111}(b_1,t)b_{1\beta}^2+C_{11}(b_1,t)b_{1\beta\beta},$$
$$-\int[(p'_{112}b'_{1\beta}+p'_{122}b_{2\beta}+p'_{12\beta})b'_{1\beta}+p'_{12}b'_{1\beta\beta}+(p'_{122}b'_{1\beta}+p'_{222}b_{2\beta}+p'_{22\beta})b_{2\beta}+p'_{22}b_{2\beta\beta}$$
$$+p'_{12\beta}b'_{1\beta}+p'_{22\beta}b_{2\beta}+p'_{2\beta\beta}]dF(t'|t)=C_{111}(b_2,t)b_{2\beta}^2+C_{11}(b_2,t)b_{2\beta\beta}.$$

Here, p_{ij} and p_{ijk} are evaluated at $(b_1(t;\bar\beta),b_2(t';\bar\beta);\bar\beta)$ whereas p'_{ij} and p'_{ijk} are evaluated at $(b_1(t';\bar\beta),b_2(t;\bar\beta);\bar\beta)$. Similarly, $b_{i\beta}$ and $b_{i\beta\beta}$ are evaluated at $(t;\bar\beta)$ while $b'_{i\beta}$ and $b'_{i\beta\beta}$ are evaluated at $(t';\bar\beta)$.

Introduce $z(t)=b_{1\beta\beta}(t;\bar\beta)+b_{2\beta\beta}(t;\bar\beta)$. Recall that when p has a locally symmetric bias we have $b_1(t;\bar\beta)=b_2(t;\bar\beta)$, $b_{1\beta}(t;\bar\beta)=-b_{2\beta}(t;\bar\beta)$. This implies $p'_{12}=-p_{12}$, $p'_{22}=-p_{11}$, $p'_{112}=-p_{122}$, etc. Transforming the second equation, and summing up the two equations, obtain the integral equation (43). ■

B3. Predictive power under public information

The predictive power, i.e., the probability that the best player wins the contest (cf. Example 3(ii)), is equal to

$$Q^S(\beta) = \int_{t_1<t_2} p(e_1^*, e_2^*; \beta) dF(t_1, t_2) + \int_{t_1>t_2} [1 - p(e_1^*, e_2^*; \beta)] dF(t_1, t_2). \tag{44}$$

Here, integration in the first (respectively, second) term is over the set of types (t_1, t_2) such that $t_1 < t_2$ (respectively, $t_1 > t_2$) and $e_i^* \equiv e_i^*(t_1, t_2; \beta)$ are the equilibrium effort levels.

The following proposition provides conditions for $\bar{\beta}$ to be a local maximum or minimum of predictive power $Q^S(\beta)$.

Proposition B3 *Suppose p has locally symmetric bias and the following condition holds for all $t_1 < t_2$ in the support of F:*

$$\begin{aligned} &2(p_{11}e_{1\beta}^* + p_{12}e_{2\beta}^* + 2p_{1\beta})e_{1\beta}^* + 2(p_{12}e_{1\beta}^* + p_{22}e_{2\beta}^* + 2p_{2\beta})e_{2\beta}^* \\ &+ p_1(e_{1\beta\beta}^* + \tilde{e}_{2\beta\beta}^*) + p_2(\tilde{e}_{1\beta\beta}^* + e_{2\beta\beta}^*) + p_{\beta\beta} - \tilde{p}_{\beta\beta} < 0, \end{aligned}$$

where $e_{1\beta}^*$ and $e_{2\beta}^*$ are as in Proposition B1, $e_{1\beta\beta}^*$ and $e_{2\beta\beta}^*$ are given by (42), $\tilde{e}_{i\beta\beta}^*$ and $\tilde{p}_{\beta\beta}$ are evaluated at $(e_2^*, e_1^*; \bar{\beta})$, and all other functions are evaluated at $(e_1^*, e_2^*; \bar{\beta})$.

Then $Q^S(\beta)$ has a local maximum at $\beta = \bar{\beta}$. If the inequality is reversed, $Q^S(\beta)$ has a local minimum at $\beta = \bar{\beta}$.

Proof. Differentiating (44) over β and omitting the arguments of all functions for brevity, obtain

$$Q_\beta^S(\beta) = \int_{t_1<t_2} (p_1 e_{1\beta}^* + p_2 e_{2\beta}^* + p_\beta) dF - \int_{t_1>t_2} (p_1 e_{1\beta}^* + p_2 e_{2\beta}^* + p_\beta) dF.$$

Differentiating over β once more and setting $\beta = \bar{\beta}$, obtain

$$Q^S\beta\beta(\bar{\beta}) = \int_{t_1<t_2} [(p_{11}e^*_{1\beta} + p_{12}e^*_{2\beta} + p_{1\beta})e^*_{1\beta} + p_1 e^*_{1\beta\beta} + (p_{12}e^*_{1\beta} + p_{22}e^*_{2\beta} + p_{2\beta})e^*_{2\beta}$$
$$+ p_2 e^*_{2\beta\beta} + p_{1\beta}e^*_{1\beta} + p_{2\beta}e^*_{2\beta} + p_{\beta\beta}]dF - \int_{t_1>t_2} [\text{same expression}]dF$$
$$= \int_{t_1<t_2} [(p_{11}e^*_{1\beta} + p_{12}e^*_{2\beta} + 2p_{1\beta})e^*_{1\beta} + (p_{12}e^*_{1\beta} + p_{22}e^*_{2\beta} + 2p_{2\beta})e^*_{2\beta}$$
$$+ p_1 e^*_{1\beta\beta} + p_2 e^*_{1\beta\beta} + p_{\beta\beta}]dF - \int_{t_1>t_2} [\text{same expression}]dF.$$

Now, using the symmetry of F, swap the variables of integration in the second integral. Recall that $p_{11} = -\tilde{p}_{22}$, $p_1 = -\tilde{p}_2$, $p_{12} = -\tilde{p}_{12}$, and locally symmetric bias implies $e^*_{1\beta} = -\tilde{e}^*_{2\beta}$ and $p_{1\beta} = \tilde{p}_{2\beta}$. This gives

$$Q^S\beta\beta(\bar{\beta}) = \int_{t_1<t_2} [2(p_{11}e^*_{1\beta} + p_{12}e^*_{2\beta} + 2p_{1\beta})e^*_{1\beta} + 2(p_{12}e^*_{1\beta} + p_{22}e^*_{2\beta} + 2p_{2\beta})e^*_{2\beta}$$
$$+ p_1(e^*_{1\beta\beta} + \tilde{e}^*_{2\beta\beta}) + p_2(\tilde{e}^*_{1\beta\beta} + e^*_{2\beta\beta}) + p_{\beta\beta} - \tilde{p}_{\beta\beta}]dF,$$

and the result follows. ■

The conditions of Proposition B3 simplify substantially when p is linear in effort, i.e., it takes the form $p(e_1, e_2; \beta) = \gamma_1(\beta)e_1 + \gamma_2(\beta)e_2 + \gamma(\beta)$. The results are summarized in Corollary 3.

Proof of Corollary 3. Here we prove the case of public information. For the case of private information see Corollary 4.

When p is linear in e_1 and e_2, all second- and third-order partial derivatives of p with respect to effort are zero. The unbiasedness condition (1) implies that $\gamma_1(\bar{\beta}) = -\gamma_2(\bar{\beta})$, and locally symmetric bias condition (3) implies that $\gamma_1'(\bar{\beta}) = \gamma_2'(\bar{\beta})$. The expressions for A_1, A_2, D, $e^*_{i\beta}$ and $e^*_{i\beta\beta}$ from Propositions B1 and B3 are now simplified as follows:

$$D = C_{11}(e^*_1, t_1)C_{11}(e^*_2, t_2), \quad e^*_{1\beta} = \frac{\gamma_1'(\bar{\beta})}{C_{11}(e^*_1, t_1)}, \quad e^*_{2\beta} = -\frac{\gamma_1'(\bar{\beta})}{C_{11}(e^*_2, t_2)},$$

$$A_1 = -\frac{C_{111}(e^*_1, t_1)\gamma_1'(\bar{\beta})^2}{C_{11}(e^*_1, t_1)^2}, \quad A_2 = -\frac{C_{111}(e^*_2, t_2)\gamma_1'(\bar{\beta})^2}{C_{11}(e^*_2, t_2)^2},$$

$$e^*_{1\beta\beta} = \frac{A_1 + p_{1\beta\beta}}{C_{11}(e^*_1, t_1)} = \frac{1}{C_{11}(e^*_1, t_1)}\left[\gamma''_1(\bar\beta) - \frac{C_{111}(e^*_1, t_1)\gamma'_1(\bar\beta)^2}{C_{11}(e^*_1, t_1)^2}\right],$$
$$e^*_{2\beta\beta} = \frac{A_2 - p_{2\beta\beta}}{C_{11}(e^*_2, t_2)} = -\frac{1}{C_{11}(e^*_2, t_2)}\left[\gamma''_2(\bar\beta) + \frac{C_{111}(e^*_2, t_2)\gamma'_1(\bar\beta)^2}{C_{11}(e^*_2, t_2)^2}\right].$$

The criterion from Proposition B3 then becomes

$$\begin{aligned}
&4p_{1\beta}e^*_{1\beta} + 4p_{2\beta}e^*_{2\beta} + p_1(e^*_{1\beta\beta} + \tilde e^*_{2\beta\beta}) + p_2(\tilde e^*_{1\beta\beta} + e^*_{2\beta\beta}) + p_{\beta\beta} - \tilde p_{\beta\beta}\\
&= 4\gamma'_1(\bar\beta)^2\left[\frac{1}{C_{11}(e^*_1, t_1)} - \frac{1}{C_{11}(e^*_2, t_2)}\right]\\
&+ \gamma_1(\bar\beta)\left[\frac{\gamma''_1(\bar\beta)}{C_{11}(e^*_1, t_1)} - \frac{C_{111}(e^*_1, t_1)\gamma'_1(\bar\beta)^2}{C_{11}(e^*_1, t_1)^3} - \frac{\gamma''_1(\bar\beta)}{C_{11}(e^*_2, t_2)} + \frac{C_{111}(e^*_2, t_2)\gamma'_1(\bar\beta)^2}{C_{11}(e^*_2, t_2)^3}\right.\\
&\left.-\frac{\gamma''_2(\bar\beta)}{C_{11}(e^*_1, t_1)} - \frac{C_{111}(e^*_1, t_1)\gamma'_1(\bar\beta)^2}{C_{11}(e^*_1, t_1)^3} + \frac{\gamma''_2(\bar\beta)}{C_{11}(e^*_2, t_2)} + \frac{C_{111}(e^*_2, t_2)\gamma'_1(\bar\beta)^2}{C_{11}(e^*_2, t_2)^3}\right]\\
&+ \gamma''_1(\bar\beta)e^*_1 + \gamma''_2(\bar\beta)e^*_2 - \gamma''_1(\bar\beta)e^*_2 - \gamma''_2(\bar\beta)e^*_1,
\end{aligned}$$

and the result follows. ■

B4. Predictive power under private information

In the case of private information, define the predictive power of the contest as

$$\begin{aligned}
Q^S(\beta) = &\int_{t_1<t_2} p(b_1(t_1;\beta), b_2(t_2;\beta);\beta)dF(t_1, t_2)\\
&+ \int_{t_1>t_2}[1 - p(b_1(t_1;\beta), b_2(t_2;\beta);\beta)]dF(t_1, t_2).
\end{aligned} \tag{45}$$

Here, $b_i(t;\beta)$ are the equilibrium bidding functions satisfying the system of integral equations (8).

The following proposition provides conditions under which predictive power $Q^{\mathrm{E}}(\beta)$ has a local maximum or a local minimum at $\beta = \bar\beta$.

Proposition B4 *Suppose p has locally symmetric bias and the following condition holds for all $t_1 < t_2$ in the support of F:*

$$\begin{aligned}
&2\left(p_{11}b_1^2(t_1;\bar\beta) + 2p_{12}b_1(t_1;\bar\beta)b_1(t_2;\bar\beta) + p_{22}b_1^2(t_2;\bar\beta)\right.\\
&\left.+2p_{1\beta}b_{1\beta}(t_1;\bar\beta) + 2p_{2\beta}b_{2\beta}(t_2;\bar\beta)\right) + p_1z(t_1) + p_2z(t_2) + p_{\beta\beta} - \tilde p_{\beta\beta} < 0.
\end{aligned} \tag{46}$$

Here, all the derivatives of p are evaluated at $(b_1(t_1;\bar{\beta}), b_1(t_2;\bar{\beta});\bar{\beta})$; $z(t)$ is the solution of integral equation (43) in Proposition B2; and $\tilde{p}_{\beta\beta} \equiv p_{\beta\beta}(b_1(t_2;\bar{\beta}), b_1(t_1;\bar{\beta});\bar{\beta})$.

Then $Q^S(\beta)$ has a local maximum at $\beta = \bar{\beta}$. If the inequality is reversed, $S(\beta)$ has a local minimum at $\beta = \bar{\beta}$.

Proof. Differentiating the expression for $Q^S(\beta)$ with respect to β twice and setting $\beta = \bar{\beta}$, obtain

$$\begin{aligned} Q^S\beta\beta(\bar{\beta}) = \int_{t_1<t_2} &[(p_{11}b_{1\beta}(t_1;\bar{\beta}) + p_{12}b_{2\beta}(t_2;\bar{\beta}) + 2p_{1\beta})b_{1\beta}(t_1;\bar{\beta}) + p_1 b_{1\beta\beta}(t_1;\bar{\beta}) \\ &+ (p_{12}b_{1\beta}(t_1;\bar{\beta}) + p_{22}b_{2\beta}(t_2;\bar{\beta}) + 2p_{2\beta})b_{2\beta}(t_2;\bar{\beta}) + p_2 b_{2\beta\beta}(t_2;\bar{\beta}) + p_{\beta\beta}]dF(t_1,t_2) \\ &- \int_{t_1>t_2} [\text{same expression}]dF(t_1,t_2). \end{aligned}$$

Recall that in equilibrium at $\beta = \bar{\beta}$, with a locally symmetrically biased CSF, we have $b_1(t;\bar{\beta}) = b_2(t;\bar{\beta})$, $b_{1\beta}(t;\bar{\beta}) = -b_{2\beta}(t;\bar{\beta})$, $p_{11} = -\tilde{p}_{22}$, $p_{12} = -\tilde{p}_{12}$, and $p_{1\beta} = \tilde{p}_{2\beta}$. Swapping the variables of integration in the second integral and using the definition $z(t) = b_{1\beta\beta}(t;\bar{\beta}) + b_{2\beta\beta}(t;\bar{\beta})$, obtain the result. ■

For a CSF that is linear in effort, (46) takes a simpler form, and the result is similar to the case of public information.

Corollary 4 *Corollary 3 holds in the case of private information, with $e_1^* = b_1(t_1;\bar{\beta})$ and $e_2^* = b_1(t_2;\bar{\beta})$.*

Proof. For a linear CSF $p(e_1,e_2;\beta) = \gamma_1(\beta)e_1 + \gamma_2(\beta)e_2 + \gamma(\beta)$, Eqs. (32) become

$$\gamma_1'(\beta)\int dF(t'|t) = C_{11}(b_1(t;\bar{\beta}),t)b_{1\beta}(t;\bar{\beta}), \quad -\gamma_2'(\beta)\int dF(t'|t) = C_{11}(b_1(t;\bar{\beta}),t)b_{2\beta}(t;\bar{\beta}),$$

which gives, assuming $\gamma_1'(\bar{\beta}) = \gamma_2'(\bar{\beta})$ due to the locally symmetric bias condition,

$$b_{1\beta}(t;\bar{\beta}) = -b_{2\beta}(t;\bar{\beta}) = \frac{\gamma_1'(\bar{\beta})}{C_{11}(b_1(t;\bar{\beta}),t)}.$$

Furthermore, (43) gives

$$C_{11}(b_1(t;\bar{\beta}),t)z(t) = -2C_{111}(b_1(t;\bar{\beta}),t)b_{1\beta}^2(t;\bar{\beta}) + \gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta}),$$

$$z(t) = -\frac{2C_{111}(b_1(t;\bar{\beta}),t)\gamma_1'(\bar{\beta})^2}{C_{11}(b_1(t;\bar{\beta}),t)^3} + \frac{\gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta})}{C_{11}(b_1(t;\bar{\beta}),t)}.$$

The criterion from Proposition 46 is, in this case,

$$4\gamma_1'(\bar{\beta})[b_{1\beta}(t_1;\bar{\beta}) + b_{2\beta}(t_2;\bar{\beta})] + \gamma_1'(\bar{\beta})[z(t_1) + z(t_2)] + [\gamma_1''(\bar{\beta}) - \gamma_2''(\bar{\beta})][b_1(t_1;\bar{\beta}) - b_1(t_2;\bar{\beta})].$$

Plugging in the expressions for $b_{i\beta}$ and $z(\cdot)$, obtain the same result as in Corollary 3. ■

The fact that Corollary 3 holds for both public and private information, with equilibrium efforts appropriately redefined, is not unexpected. When the CSF is linear in effort, a player's equilibrium effort depends only on her own type, and hence information about the other player's type is irrelevant.