A stylised macroeconomic model incorporating green capital in the framework of Viability Theory

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A STYLISED MACROECONOMIC MODEL INCORPORATING GREEN CAPITAL IN THE FRAMEWORK OF VIABILITY THEORY

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Abstract. New Zealand's Treasury, as illustrated by its Living Standards Framework, desires policy that not only promotes economic growth, but also sustainability and equity. This paper studies how taxation and abatement policy can work to keep an economy viable in regards to capital stocks, consumption, debt, environment and the relative factor share (a proxy for income inequality), as well as the trade-offs it faces in different levels of pollutant industry. This is done in the context of Viability Theory, a branch of mathematics suited for policy analysis. The results show that reducing an economies environmental impact is key for achieving the multi faceted growth laid out in the Living Standards Framework.

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CONTENTS

1. Introduction

It is increasingly the desire of national governments to ensure that economic growth is a multifaceted process. Ever more, growth alludes to not only GDP, but the welfare of the government's current and future constituents. Evidence of this can be seen in the report commissioned by Nicolas Sarkozy for France (Stiglitz, Sen, and Fitoussi (2009)) to publications by Australia and New Zealand's Treasurys: Treasury (2004) and Gleisner, Llewellyn-Fowler, and McAlister (2011). The measurement of welfare however is far from a hard science. Given this, it is exceedingly difficult to construct a model in which to test policy against the targets of growth and welfare.

In Arrow, Dasgupta, Goulder, Mumford, and Oleson (2012) it was suggested that output should be measured in the context of 'collective well-being'. This was a measure of household's consumption and utilisation of assets within an economy. The ability to consume or utilise these assets spoke to the 'collective consumption' enjoyed by each household or citizen. In a modelling sense, maximising collective consumption meant maximising not only the output of the economy but households access to and use of said resources and output.

This idea has been ingrained in New Zealand Treasury's Living Standards Model, which suggests that while economic growth is a key focus for policy, it should also work to manage risk, increase equity, create social infrastructure and ensure sustainability for future generations. Karacaoglu (2015) models this through a discrete time optimization model based on the work by Acemoglu, Aghion, Bursztyn, and Hemous (2012). Formal work is currently being undertaken to extend this model for proper policy analysis under the direction of Anita King, see King (2016). Karacaoglu (2015) does express however that adapting and applying the concepts of Arrow et al (2012) to Viability Theory, a recently developed area of mathematics, could be a direction taken in future research.

It is this direction that is examined in this paper. In Krawczyk and Townsend (2015c) and O'Keefe and Krawczyk (2016), a model was developed that looked at how taxation policy could be used to control and increase equity.[1] In this paper we look to extend the scope of that model by incorporating 'sustainability for the future', which in the Living Standards Models is defined as maintaining well-being for current and future generations. In a modelling sense this is proied by maintaining physical and human capital and preservation

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[1]It was found that an economy concerned with reducing income inequality needn't face a trade-off with efficiency, especially when utilising separate taxation rates

of the environment.[2] Together, these concepts look to increasing the 'collective consumption' of the economy, and contribute to the use of Viability Theory in adapting Arrow et al (2012).

Using Viability Theory and modifications to the stylised models used in Krawczyk and Townsend (2015c) and O'Keefe and Krawczyk (2016), this paper studies how taxation and abatement policy can be used to achieve economic growth, environmental sustainability and reduced income inequality. The trade-offs discovered are then contextualised with the work done by Nordhaus (1994), Stern (2009) and Acemoglu et al (2012).

It is the finding of this paper that environmental sustainability is explicitly linked to reductions in the environmental impact of industry and that achieving this leads to increased ability to reduce and sustain lower income inequality. The model used also suggests that movement toward a lower carbon footprint can also lead to higher economic growth (in opposition to Nordhaus (1994)), though this conclusion is tempered by limitations in the model used, namely the inability to track the cost of transitioning between different levels of environmental impact. Once this point is reached however, fiscal policy is a sufficient tool for maintaining such a state, with no drastic action in the style of Stern (2009) required.

In Section 2 we explain the inspiration for this work, the Living Standards Framework, and how it approaches 'collective consumption' as well as how the concepts of the Living Standards Framework were interpreted in the context of modelling within Viability Theory. Section 3 proceeds to give an explanation of the use and advantages of Viability Theory, justifying its choice as a framework for future analysis. Section 4.1 covers how the model was developed while section 4.2 lays out the model itself. Sections 5 and 5.1 detail the calibration of the model, while sections 6 and 7 give analysis of results under first a combined taxation regime followed by a separated taxation regime. Section 8 concludes. As stated above, we recognise the model used in this paper is limited in analysis of transitioning between environmental impact levels, so section 9 lays out a model intended for future research.

2. Modelling the Living Standards Framework

2.1. **Living Standards Framework.** Karacaoglu (2015) states that "the fundamental role of public policy is one of stewardship... the wellbeing-generating capacity of capital assets is sustained or enhanced, and shared". This has lead to the identification of five goals for proposed policy, that it must:

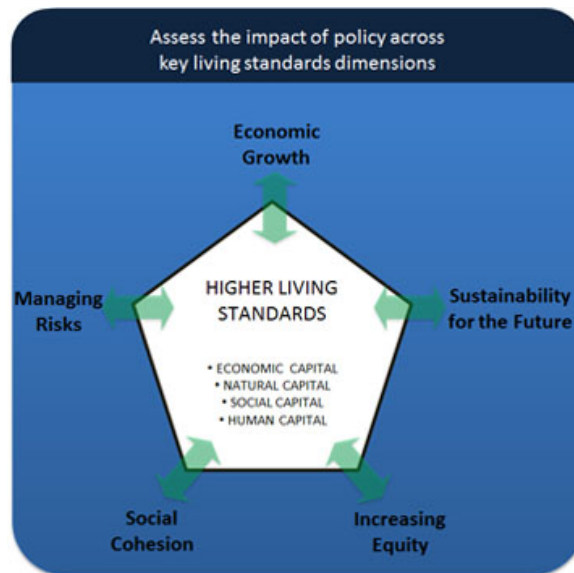[2]Which are of key relevance to consumer's utility functions

FIGURE 1. The Living Standards Framework detailing the five targeted facets of economic growth

- Promote Economic Growth
- Ensure and Promote Sustainability for the Future
- Increase the Equity within Society
- Increase Social Cohesion
- Manage Economic Risk

These targets are visualised in Figure 1 from Karacaoglu (2015)

2.2. **The Living Standards Model and Legacy.** New Zealand's Treasury has published a model that incorporates these facets as part of its Living Standards literature (Karacaoglu (2015)). The model draws inspiration from Acemoglu et al (2012) and Arrow et al (2012) as well as Chichilnisky, Heal, and Beltratti (1995) and Chichilnisky (1999). The model is in discrete time model with the option of innovation given to capitalists. By choosing to invest in clean technology, the economy can be lead to a more sustainable state, one with greater comprehensive consumption as per Arrow et al (2012). As stated earlier, this model is currently being extended and formalised for policy analysis in King (2016).

This paper takes a different direction, incorporating the ideas and applications into the area of Viability Theory, a branch of mathematics particularly useful for policy research due to its capture of 'satisficing' solutions. It is the opinion of this paper that this property

encapsulates Romer's idea of a government that makes policy with multiple objectives,[3] not one "that computes optimal solutions to social welfare maximands".

The first step of this was taken in Krawczyk and Townsend (2015a), Krawczyk and Townsend (2015b), Krawczyk and Townsend (2015c) and O'Keefe and Krawczyk (2016) where a dynamic model based on Krawczyk and Judd (2014) was augmented with a dynamic relative factor share that proxied income inequality.[4]

This paper looks to add additional dimensions to those models, adding dynamics of environmental quality and an additional control variable in the form of abatement spending. This model is laid out in section 4.2.

2.3. **Modelling Equity.** Before illustration of the model, the way in which the broad targets laid out in the Living Standards model have been incorporated into Viability Theory requires explanation.

The Living Standards Framework suggests increasing equity refers not only to ensuring people earn what they deserve and receive assistance when they are vulnerable, but also directing resources to create the maximum good. Understanding the degree to which these concepts overlap is difficult, but from a purely social perspective inequity can be interpreted as an inefficient use of resources. For the purpose of this paper we choose to model equity as the level of income inequality in our economy. Income inequality has been brought to the forefront of public consciousness by Piketty (2014), but it has been an issue for economists as far back as Ricardo.

Income inequality in our model is proxied by the relative factor share, the use of which is based upon Piketty's hypothesized $r > g$.[5] Piketty suggested that despite industrialization creating higher capital stocks, the marginal product of capital and thus the interest rate on it has failed to fall below the growth rate of the economy as a whole. This has lead to an ever

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[3]Viability Theory deems a set of starting conditions 'viable' if the evolution of all variables can be kept within their respective bound for the entirety of the systems evolution. Each variable can be thought of as a representation of an economies objective, for example stable capital stocks or low debt

[4] Justifiaction of this can be seen in Krawczyk and Townsend (2015a) where the factor share of New Zealand was regressed against income inequality data from The World Wealth and Income Database

[5]Where r is the interest rate on capital and g is output growth

increasing share of capital stocks going to those who already owned significant proportions of it.[6]

The relationship of the relative factor share to income inequality is based on the fact that capital wage is positively correlated with capital income. Piketty (2014) shows in Figures 1.2, 9.5 that the evolution of the factor share ratio has closely mirrored the evolution of income inequality in Europe. This relationship was shown to hold in New Zealand in Krawczyk and Townsend (2015b), especially at the top 1% and .1% of income shares.

We put forward that income inequality fundamentally affects the equity of access to capital through the natural growth of the factor ratio and also through the response of governments to decrease the flow of credit to low income households.[7] Thus by controlling it in our dynamic model, we effectively target equity while examining the response in economic growth.

2.4. **Modelling Sustainability.** As New Zealand Treasury suggests, the concept of sustainability is straight forward. Taking the line of reasoning supplied by Arrow et al (2012), if current and future generations want to enjoy 'comprehensive consumption', capital stocks must be maintained. New Zealand's Treasury specifies sustainability to relate to four capital stocks; environmental, economic, human and social. The stability of economic and environmental capital is explicitly targeted in the model through the imposition of a lower bound. For the two remaining, we argue that environmental sustainability fundamentally underpins human and social sustainability. This line of reasoning seems reasonable, and it appears to be New Zealand Treasuries sentiment that the three are related. (As shown by Figure 2) Thus by also imposing a lower bound on environmental stock, sustainability is endogenous to the model.

It should be noted that whether output and sustainability are mutually exclusive is debatable, many contesting that the environmental Kuznets curve represents the ability of consumers in well developed nations to distance themselves from the environmental degradation by

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[6]The factor share ratio representing inequality also requires that wage be more equally distributed than capital rent, which Piketty shows: Piketty (2014), tables 7.1, 7.2.

[7]This argument was put forward by former governor of the Reserve Bank of India Raghuram Rajan in his 2011 book Rajan (2011).
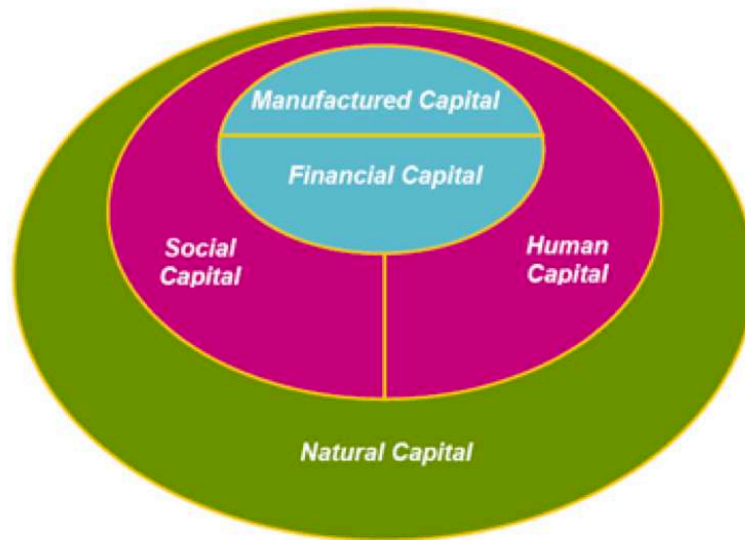
Figure 2. The Living Standards Framework Detailing how Enviroment Encapsulates Sustainability

'exporting it to underdeveloped areas'.[8] As our model is one of a closed economy however, any damage caused by production is undertaken by the economy itself.

The policies related to environmental sustainability cover a large spectrum; abatement, performance standards, pigouvian taxes, design standards or combinations of such. Additionally, there is friction between the optimal economic policy and what is perceived as plausible by policy makers.[9]

In the model used for analysis, we control sustainability through abatement costs, the economy sacrificing output to replenish the damage caused to the environment. The choice of incorporating the environmental level as a partial determinant of productivity, as opposed to an input, meant that taxing the use of dirty output (or subsidising clean output) was not possible. The model planned for future work does include this capability however.

3. Viability Theory

Viability Theory is the study of constrained dynamic systems, the target being to find a set of initial conditions that are deemed viable by the system. An evolution is deemed viable if for the life of the evolution the state variables are kept within a predetermined set called the

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[8]see Pearce, Warford et al (1993)

[9]For example, a performance standard is often perceived as difficult to measure, leading to the less efficient but easy to monitor solution of design standards

'constraint set', labeled K, given a chosen control set U. The set of points from which viable evolutions can be found are labeled the *Viability Kernel*. Using the program VIKAASA,[10] it is possible to determine whether a viability kernel exists and what its approximate boundaries are.

Viability Theory formalises the satisficing property laid out in Simon (1955), that is to say that it searches for an acceptable approach in the absence of one that is truly optimal. This was argued by Simon (1955) to better represent the decision making process of policy makers. An example of this being that if one were to attempt to accurately model firm behaviour, one would see firms don't target profit but perceive them more as a constraint. This thought process is ingrained in Viability Theory and is what makes it useful for policy analysis.

Contextualising this in terms of the Living Standards Model; using Viability Theory means modelling a policy-maker who recognises that economic growth involves balancing a variety of economic measures (capital, consumption and debt) with the target of environmental sustainability and reduced income inequality.[11]

We do not include a formal definition of Viability Theory in the mathematical sense as it has been covered in depth in a series of papers; Aubin and Cellina (2012), Krawczyk and Pharo (2013), Krawczyk and Townsend (2015c) among others. An indepth explanation of the program used to solve and graph the models can be found in Krawczyk and Pharo (2014).

4. Model Overview

4.1. **Modeling.** In Krawczyk and Townsend (2015c), viability theory was used to show how income inequality could be controlled though the use of taxation rates. This model aims to extend on this by incorporating environment as a dynamic variable.

Modeling in Viability Theory suffers from prolonged calculation times, so discretion was required when adapting the Living Standards Model from Karacaoglu (2015) to Viability Theory. The derived model incorporate a series of macroeconomic factors relating to policy, but takes total factor productivity to be constant. An evolving TFP would mean adding an additional state variable, seriously increasing calculation time. In addition, in the long run it would mean progressively higher levels of capital and consumption, so an evolving

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[10]An explanation of this program can be found in Krawczyk and Pharo (2014).

[11]By maintaining state variables within their bound it could be argued that Viability Theory endogenises risk aversion, at least to some degree.

upper bound would need to be integrated for those variables, a function not available in VIKAASA.

In order to understand how environment affects output, insipration was drawn from Lee (2012). Lee (2012) states: 'Environmental capital refers to natural objects, such as forest, soil air, and mineral resources. Climate and geographical conditions of a country are also important constituents of environmental capital." It is thus envisioned that a higher standard of environment lends itself to higher productivity for the portion of capital stock that is linked with it. (One can imagine this in terms of a blacksmith who has a set of 'capital' or tools. Some of the tools require water to run, and thus their productivity is directly related to the quality of the water). Replenishment of the environment is based on the abatement efforts of the economy, with abatement spending derived from some share of output.

The model is thus a stylised representation of pollution as a negative externality and can be used to set a benchmark for how different tax structures affect the level of production and resultant environmental damage. It also allows examination of how income inequality and environmental sustainability relate in terms of fiscal policy.

4.2. **A Growth Model with Environmental Productivity.** For the representative firm, capital is incorporated (pre-tax) at a rent cost r while labour can be hired at a wage w. Households receive r and w subject to the current taxation levels. They enjoy utility from consuming the product of the firms, but get dis-utility from supplying their labour.

The government taxes the output of the firms and spends it reducing its debt level B, exogenous government spending G and replenishing the environment v.

Output is captured through a representative firm whose output y is given by

$$(1) \qquad\qquad y = A\ell^{1-\alpha}[\beta k + (1-\beta)kR]^\alpha$$

with A as the productivity coefficient, ℓ as the labour input, k as the capital input and R as the environmental quality affecting the share of capital requiring it, ie $(1-\beta)$. We thus see that the output of the firm is increasing in productivity, capital and labour.

We use a Cobb-Douglas production function to define this relationship, meaning an elasticity of substitution of one. This is at odds with Pikkety's argument that income inequality is increasing in higher capital stocks due to an elasticity greater than one. He argues that

while an increasing capital stock will lead to lower interest rates from reduced marginal productivity, the fact that elasticity is greater than one means that the capitalists share of income will not be kept constant but rather increase.

As income inequality is examined through the relative factor share, the use of an elasticity of one needs justification. We first point out that empirical research, including that done in New Zealand, suggests an elasticity of less that one.[12]. Furthermore, elasticities in the short run tend to be lower. [13] Thus using a Cobb-Douglas production function is a conservative estimate for examining input dynamics.

From this production function we derive the marginal productivity of capital and labour.

$$
(2) \qquad \frac{dy}{dk} = \frac{A\ell^{1-\alpha}[\beta k + (1-\beta)R]}{[\beta k + (1-\beta)Rk]^{1-\alpha}}
$$

$$
(3) \qquad \frac{dy}{d\ell} = \frac{A(1-\alpha)(\beta k + (1-\beta)Rk)^{\alpha}}{\ell^{\alpha}}
$$

In the model, consumers are infinitely lived and receive positive utility from consumption but a dis-utility from labour;

$$
(4) \qquad u(c) = \frac{c^{1-\gamma}}{(1-\gamma)}
$$

but receive a dis-utility from working.

$$
(5) \qquad v(\ell) = \frac{V\ell^{1+\eta}}{(1+\eta)}
$$

As per standard, the after tax marginal product of capital is the marginal product of capital r less depreciation, reduced by taxation on capital τ_K;

$$
(6) \qquad \bar{r} = (1-\tau_K)\left(\frac{A\ell^{1-\alpha}[\beta k + (1-\beta)R]}{[\beta k + (1-\beta)Rk]^{1-\alpha}} - \delta\right)
$$

The after tax wage w received by employees is given by the marginal productivity of labour, reduced by the tax on labour;

$$
(7) \qquad \bar{w} = (1-\tau_L)A\frac{[\beta k + (1-\beta)kR]^{\alpha}}{\ell^{\alpha}}
$$

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[12]See Tipper (2012)

[13]Again, see Tipper (2012)

The government collects tax on labour and capital while allowing for depreciation of capital to be tax free. This gives the expression of total tax income T being

$$(8) \qquad T = \tau_K k \frac{d}{dk}(y - \delta k) + \tau_L \ell \frac{dy}{d\ell}$$

In order to characterize the economy in terms of its dynamics, we must derive time indexed equations for our measured variables: capital, consumption, debt and environment.

The growth of capital is determined by the Law of Motion, that is net output reduced by depreciation of capital, consumption c, environmental replenishment v as a fraction of output and exogenous government expenditure G. The fraction of output spent on abatement vy drains the growth of capital but goes toward increasing the state of R in (15). As it is a form of spending, it also increases the state of Debt (B) in (14)

$$(9) \qquad \begin{aligned} \frac{dk}{dt} &= (1-v)y - \delta k - c - G \\ &= (1-v)A\ell^{1-\alpha}[\beta k + (1-\beta)kR]^{\alpha} - \delta k - c - G \end{aligned}$$

We track abatement efforts as a portion of total output. In Filar, Krawczyk, and Agrawal (2014), it was put forward that as a country develops, an increasing percentage of its total output will have to be devoted to abatement. By pegging abatement to output we are able to explore whether this is the case or if it can instead be dealt with through an evolving taxation structure as well as allowing for easier interpretation.

Consumption growth can be derived by first calculating the private marginal rate of capital, and then inverting it through its relation to the marginal utility of consumption.

We take $\lambda > 0$ as the private marginal rate value of capital and assume that the coefficients V, γ and η from (4) and (5) are greater than 0. We can thus derive the evolution of λ in the time domain through maximization of the utility functions (4) and (5) on an infinite horizon.

$$(10) \qquad \frac{d\lambda}{dt} = \lambda(\rho - \bar{r})$$

Using the relationship

$$(11) \quad \frac{du}{dc} = \frac{1}{c^\gamma}$$

$$\lambda = \frac{du}{dc}$$

it is simple to see that

$$(12) \quad c = \frac{1}{\lambda^{1/\gamma}}$$

This equation for consumption can then be differentiated[14] with respect to time to derive the dynamic equation for consumption.

$$(13) \quad \frac{dc}{dt} = \frac{-1}{\gamma} \frac{1}{\lambda^{1+1/\gamma}} \frac{d\lambda}{dt}$$

$$= \frac{-c}{\gamma}(\rho - (1 - \tau_K)(\frac{dy}{dk} - \delta))$$

Thus if the marginal productivity of capital after depreciation is greater than the time discount ρ the economy will have increasing consumption, the rate of which being partially determined by the tax rate on capital. We also see that consumers internalise the structure of the economy in terms of the proportion of capital being 'damaging' through the marginal product of capital.

The evolution of government debt is governed by the interest payments on current debt, reduced by taxation and increasing in government spending, whether exogenous in the form of G or endogenous in replenishing environmental capital through vy.

$$(14) \quad \frac{dB}{dt} = \bar{r}B - T + G + vy$$

Finally, the evolution of environmental capital is governed by a natural rate of replenishment, damage inflicted though production and spending on replenishment by the government.

$$(15) \quad \frac{dR}{dt} = \mu R - \theta((1 - \beta)\alpha y) + \sigma vy$$

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[14]Assuming differentiability

where μ is the natural replenishment rate of environmental capital, θ is the coefficient dictating damage done by output and σ is the effectiveness of replenishment. The damage caused by production is defined as the share of output from our capital source that exploits environmental capital.

From this equation we see that abatement allows for direct replenishment of the environment, but to a smaller extent so does taxation as it allows us to control the level of capital used in output. Also of significant importance is the coefficient θ which dictates the level of damage from output and σ, the effectiveness of replenishment. If an economy is either better at limiting the damage caused by production or more effective at abating the caused pollution, output and environment quality would be expected to be more constant even in the face of rising capital stocks.

The final step required is to close equations (9), (13), (14) and (15) in terms of labour supply ℓ.

In equilibrium it must be the case for consumers that the marginal utility of consumption is equal to the dis-utility of labour, where consumption is weighted by the received wage $\bar{w}$.

$$(16) \qquad \bar{w}\frac{d}{dc}(u(c)) = \frac{d}{d\ell}(v(\ell))$$

Substituting in our expression for wage from 7 we can solve for ℓ, giving;

$$(17) \qquad \ell = \left[\frac{(1-\tau_L)(1-\alpha)A(\beta k + (1-\beta)kR)^\alpha)}{c^\gamma V}\right]^{1/(\alpha+\eta)}$$

Given this expression, we can derive closed dynamic equations for c, k, B and R by combining 9, 13, 14 and 15 with 17. The evolutions of these dynamic equations are dictated by the initial values of our inputs k,c,R,B,τ_K, τ_L, of which the government can control τ_K and τ_L to lead the economy to a stable state.

Finally, the factor share is determined by

$$(18) \qquad \chi = \frac{1-\tau_K}{1-\tau_L}\frac{kr}{\ell w}$$

which after substitution of substitution of variables and simplification can be expressed as

$$(19) \qquad \chi = \frac{1-\tau_K}{1-\tau_L}\left[\frac{\alpha}{1-\alpha} - \frac{\delta k}{(1-\alpha)y}\right]$$

14

This shows that in an environment where capital tax and labour tax are equal, taxation policy has no direct effect on inequality. From the end of the equation it is also clear that a high capital stock will serve to decrease income inequality. This is an expected result, as capital grows its marginal product decreases, lowering the rate at which capitalists could out-earn the labour force.

5. The Viability Kernel

Given the derived dynamic equations, it is possible to define the construct of the viability kernel. Let $x(t)$ be the state vector composed of capital k, consumption c, debt B, environment R, taxation rates τ_K, τ_L and abatement level υ. We ask whether the system dynamics $F(x(t))$, defined through the derived equations: ((9), (13), (14), (15)) are compatible with the viability constraints K:

$$
(20) \qquad K \equiv \left\{ (k, c, B, R, \tau) : \left|
\begin{array}{c}
\underline{k} \leq k(t) \leq \overline{k} \\[2ex]
\underline{c} \leq c(t) \leq \overline{c} \\[2ex]
\underline{B} \leq B(t) \leq \overline{B} \\[2ex]
\underline{R} \leq R(t) \leq \overline{R} \\[2ex]
\tau_K(t) \in [\tau_{K\min}, \tau_{K\max}] \\[2ex]
\tau_L(t) \in [\tau_{L\min}, \tau_{L\max}] \\[2ex]
\upsilon(t) \in [\upsilon_{\min}, \upsilon_{\max}] \\[2ex]
0 \leq \chi \leq \overline{\chi}
\end{array}
\right. \right\}.
$$

The values of the constraints on $k, c, B, R, \tau_K, \tau_L, \upsilon$ and $\chi - \underline{k}, \overline{k}, \underline{c}, \overline{c}$ etc. – will be explained in section 5.1

If the system's dynamics are compatible with K, there will exist a set of economic states from which there exist viable evolutions that respect the entire set of constraints. This is

the viability kernel discussed earlier, here given as

(21)

$$\mathcal{V}_F(K) \equiv \left\{ (k(0), c(0), B(0), R(0)\tau_K(0), \tau_L(0), v(0)) : \left| \begin{array}{c} \exists\ (k(\cdot), c(\cdot), B(\cdot), \tau_K(\cdot), \tau_L(\cdot), v(\cdot)), \\ \text{starting from } (k(0), c(0), B(0), R(0), \tau_K(0), \tau_L(0), v(0)) \\ \text{satisfying dynamics } F(x(t)), \\ u, s \in U \text{ and constraints (20)} \\ \forall\, t\, \in\, \Theta \end{array} \right. \right\}.$$

where U contains allowable taxation and abatement *adjustments* u and s for tax and abatement respectively (e.g., $\pm\, 20\%$ per year).

A regulator of the economy described by the dynamics $F(x(t))$ and the constraint set K will be seeking strategies $(u(\cdot), s(\cdot))$ that generate $k(\cdot)$, $c(\cdot)$, $B(\cdot)$, $R(\cdot)$, $\tau(\cdot)$, $v(\cdot)$ consistent with the above definition of $\mathcal{V}_F(K)$.

5.1. **Calibration.** Following the work laid out in Krawczyk and Judd (2015) and extended in Krawczyk and Townsend (2015c) and O'Keefe and Krawczyk (2016), analysis is of kernels produced for a "reasonably industrialized economy composed of rational agents interested in the near future, drawing a fair satisfaction from consumption and feeling, quite strongly, the burden of labour".

As such we assume $\rho = 0.04$, $\alpha = 0.33$, $\eta = 1$ and $\gamma = 0.5$.

As in Krawczyk and Judd (2015) we use a stylised natural state $\underline{k} = \underline{\ell} = \underline{R} = 1$ with no taxes or abatement and no government expenditure to calibrate A and V thus obtaining $A = 0.2727$, $V = 0.3872$. Following that we assume government expenditure G to be constant and set it to 10% of no-tax steady-state output; $G = 0.1 \cdot A = 0.02727$. As we take total factor productivity to be constant, even a small reduction in output can be interpreted as a positive result.

The share of capital requiring environmental resources is calibrated through the percentage of New Zealand's land use for agriculture to total land.[15] Thus β is the ratio of land used for agriculture to service and is calibrated as 0.60. Environmental recovery μ is difficult to

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[15]for the Environment and Statistics New Zealand (2015)

measure, so in the interest of examining the trade-offs between different facets of growth is stylised to 0.01.[16]

The damage coefficient θ reflecting dirty output's effect on the environment and the effectiveness of abatement σ are derived from Filar et al (2014), which contains a stylised model analysing the tipping point of environmental recovery. θ is calibrated as 0.35 for a pollutant economy and as 0.233 to represent an economy which through investment in cleaner capital practices has reduced its environmental impact. σ is calibrated as 0.07. This implies that if abatement was set to 0, output were at its natural level and environment was pristine, the environmental state would be decreasing at a rate of 0.45% in the pollutant economy and 0.03% per period in the reduced impact economy.

The constraints on the dynamic variable come from both model and normative sources due to the requirement to close K.

(1) **Capital** should be within 10% and 200% of no-tax steady state capital stock, $k \in [0.1, 2]$;

(2) **Consumption** should not deviate too far from a long-run equilibrium (see Krawczyk and Judd (2015)), $c \in [0.0330, 0.330]$;

(3) **Debt** may grow to 350% of the maximum steady-state capital stock and also drop somewhere below zero, $B \in [-1, 3.5]$;

(4) **Environment** must stay between 10% and 100% of the pristine state, $R \in [0.1, 1]$;

(5) **Tax and abatement** cannot be less than zero, with tax as most equal to 80% and abatement 100%, $\tau \in [0, 0.8]$, $\upsilon \in [0, 1]$;

(6) **Tax-rate adjustment speed** – the amount the regulator increases or decreases the tax rate within a year – will be less than 20 percentage points, $u \in [-0.2, 0.2]$.

(7) **Abatement adjustment speed** – the amount the regulator increases or decreases the abatement rate within a year – is unconstrained $s \in [-1, 1]$.

Results are calculated with no upper limit on χ, followed by ≤ 0.4, and then ≤ 0.25. All results will also require that χ be greater than 0, a choice that follows identical reasoning to that of Krawczyk and Townsend (2015c). A negative χ may be a viable point in terms of the mathematics of viability theory, but in real terms it requires negative interest rates as the marginal product of labour, or wages, will be positive in a Cobb-Douglas production

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[16]With more research a more precise environmental recovery rate function would assist in understanding the environmental trade-offs.

function. Negative interest rates would require that $\frac{dy}{dk} < \delta$. That isn't a viable situation, as investment will not occur with negative returns. Thus by requiring $\chi > 0$ we ensure we only get results that are the result of a long run steady state.

From this we have the constraint set K for tax and abatement, from which we will find the viability kernel;

$$(22) \qquad K = [0.1,\ 2] \ \times \ [0.033,\ 0.330] \ \times \ [-1,\ 3.5] \ \times \ [0.1,\ 1] \ \times \ [0,\ 0.8], \times \ [0,\ 1] \times [0,\ \overline{\chi}]\,,$$

where $\overline{\chi}$ is either undefined, 0.4 or 0.25.

6. Viability Kernel Results and Analysis in a Combined Tax System

This section provides analysis of the viability of economies with different relative factor share constraints and different damage coefficients imposed on the state of R by output. In order to simplify calculations and analysis, an added requirement of $\tau_K = \tau_L$ is initially imposed.

6.1. **The Effect of Environment and Inequality on Output.** The most pertinent question asked by this paper is how Environment and Income Inequality affect the level of output attained. As per 5.1, θ was calibrated at two levels; one to represent an economy whose output strongly affects environmental levels and one which via investment in a cleaner capital stocks has reduced its environmental impact. In Figure 3, 3D slices from the 7D kernel are displayed showing the relationship between the three dynamics, the first slice displaying the high impact economy, the second the lower impact economy. Both slices face no further constraints on debt, taxation or abatement level. Instead of plotting output, the kernels display NetY,[17] which is defined as output post abatement expenditure and better captures the success of the economy.

For both kernels,[18] output is maximised when inequality is around 0.35[19]. The level of environment has less of an effect on the current output level, a result likely due to the constant rate of substitution between capital and labour of one.[20] Most key to notice is that

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[17]$(1-v)y$

[18]First kernel has $\theta = 0.35$, latter has $\theta = 0.233$

[19]Which from Krawczyk and Townsend (2015b) represents the top 1% taking 5% of income

[20]A rate of substitution of one means high output can be attained through a large labour supply and low capital stock. As the kernel's bounds are displayed by the maximum values in regard to the variable, the impact of environment on capital based economies may be hidden
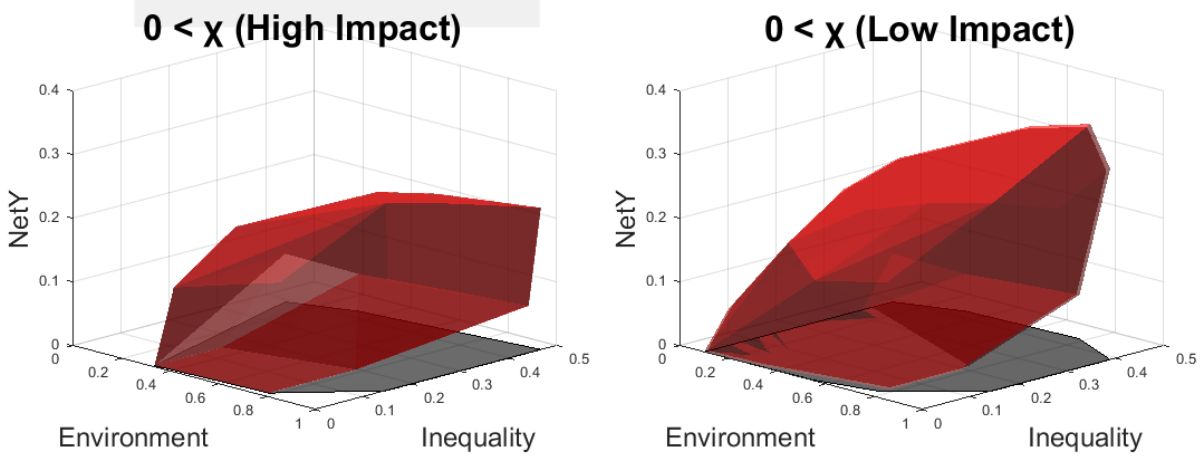
FIGURE 3. Viability Kernels Displaying Environment and Inequality's Effect on Output

the kernel representing the economy with lower environmental impact has a higher threshold on output. This was a logical conclusion of the model setup as lower abatement requirements allow higher capital stocks.

The Living Standards Framework is not simply about current output and environmental levels however, but about their sustainability for the future. It is thus useful to explore how viability is affected at different environmental levels and the evolution of the environment from those points. There are five possible starting conditions for R; [1, 0.7750, 0.550, 0.3250, 0.1]. Figure 4 displays the evolution of the viable points in terms of inequality and the various environmental level R. The top figure displays evolutions in the economy with higher environmental impact, the lower with the reduced environmental impact.

Figure 4 suggests that in the face of higher environmental impact, the environmental state will reach a point where fiscal policy is unable to prevent an evolution from escaping K significantly earlier than in the lower impact model.[21] Stern (2009) argues that to prevent environmental disaster, widespread policy change is required at a high cost of output. The model suggests that this is true, to prevent extreme environmental degradation the government must enact policies beyond simple fiscal measures. However if prior to reaching low levels of R the economy invests in a lower carbon footprint, fiscal policy can maintain viability at lower levels of R, as well as facilitating more opportunity for recovery.

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[21]This is seen in the top set of evolutions displaying no viable points for R < 0.325. In addition, the evolutions from the lower impact economy are able to stabilise at a level of R much closer to the original starting point, in some cases even increasing R
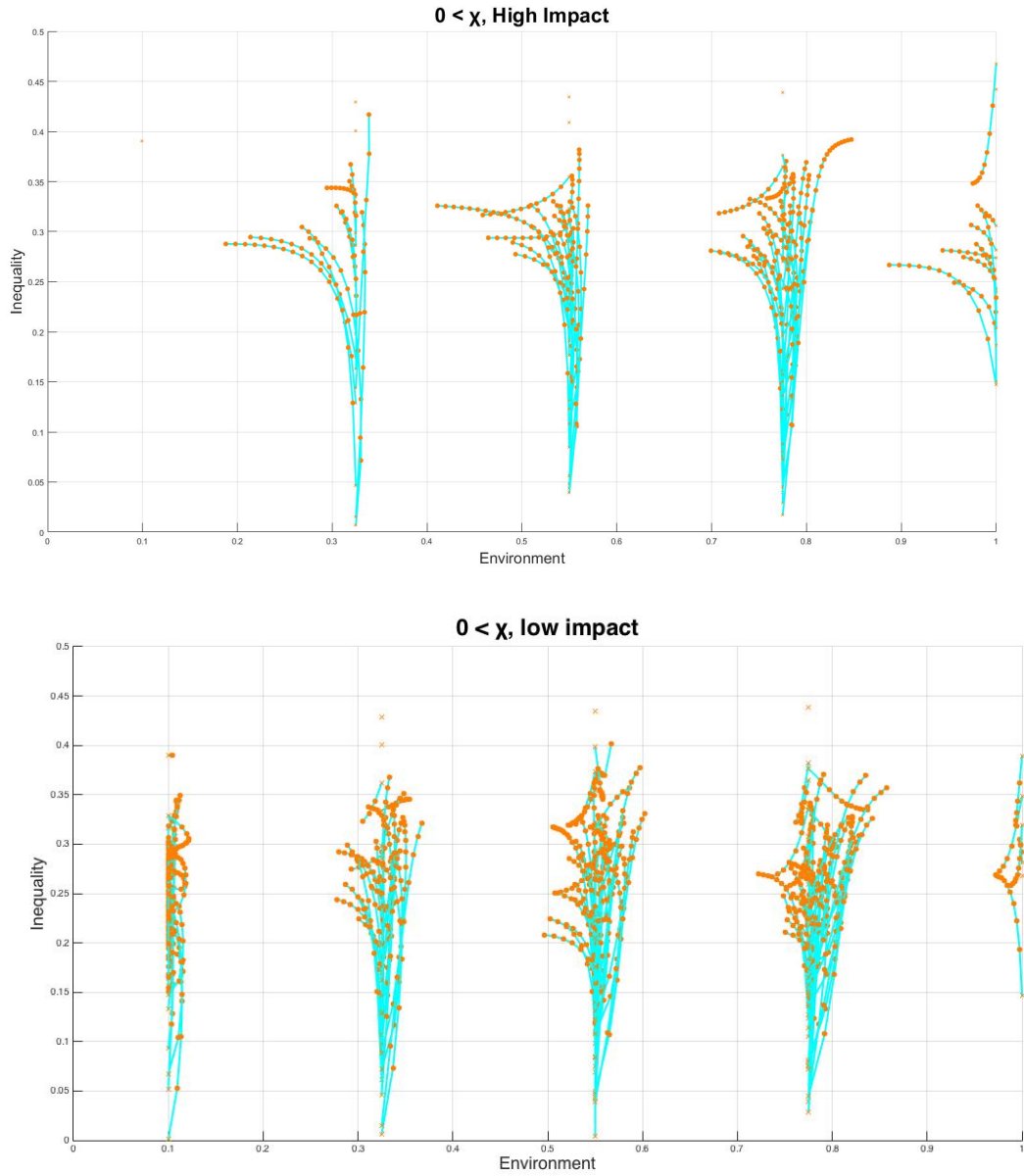
FIGURE 4. Viability slices of χ and environment evolutions

Analysis of Figures 3 and 4 leads to the conclusion that investing in reduced environmental impact maximises output. This result is expected, and must be tempered by the fact that attaining this state involves investment not accounted for in the model.

6.2. **Capital and Consumption in Reduced Income Inequality.** Section 6.1 broadly illustrates the benefit of a lower carbon footprint in terms of environmental sustainability. We now look at how this affects the attainment and sustainability of low income inequality, as well as the effect of the accompanying policy on capital and consumption.

Figure 5 holds kernels produced under different factor share constraints and high environmental impact.[22] The first slice shows the viable initial starting conditions for capital, consumption and χ (proxy for inequality) with no additional constraints on debt, taxation or abatement level. The second slice displays the effect of constraining χ to below 0.4. Constraining χ to below 0.25 produced no viable points so is thus excluded.
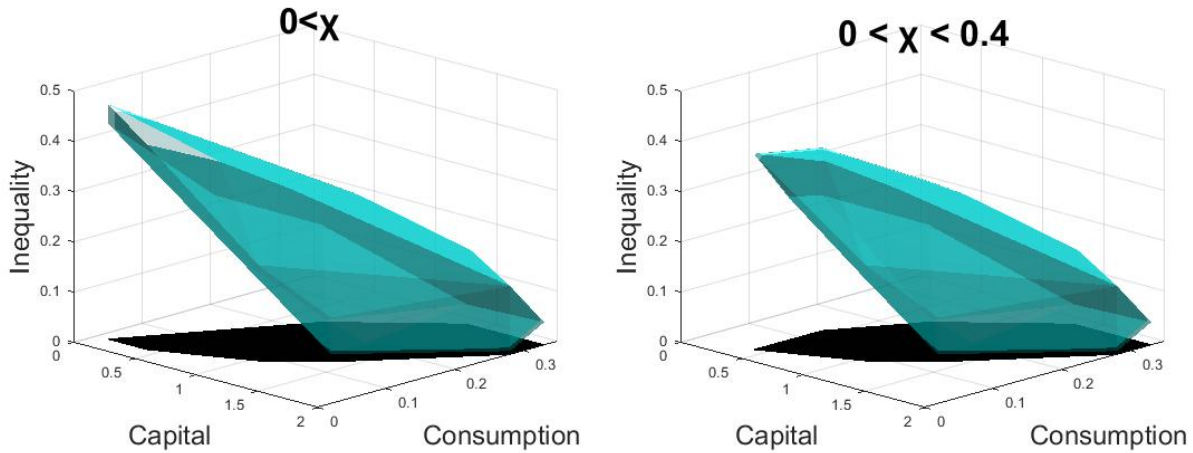

FIGURE 5. Viability Kernels for Different Relative Factor Share Constraints

Despite no limit on inequality in the first kernel, χ is naturally limited to below 0.45, thus there is little reduction in kernel size on imposition of the upper limit of 0.4.

The slices display a positive relationship between capital and consumption, with few viable points outside of that linear relationship. The appearance of this linear relationship is unsurprising, as the system is viable if all dynamics are able to be maintained within their bounds. If the system starts from a point of high (low) capital and low (high) consumption, capital will quickly ascend (fall) through its upper (lower) bound.

As can be seen, inequality is highest in low capital states, as was hypothesized in (19). Also of note is that both viability kernels cover points where χ is within the 0.25 realm, but when a requirement of the entire evolution being below 0.25 is set, VIKAASA returns no results. This would indicate that while an economy may start with low inequality, it will never maintain it.

This can be explored by looking at the evolution of points with low initial inequality. Figure 6 shows an example of an evolution with low initial inequality evolving beyond 0.25. The

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[22]$\theta = 0.35$

point starts from an initial capital stock of 200% of the natural state, maximum consumption of 0.33, negative debt, a moderate environmental state and a taxation policy of 45%.

In addition to the evolution of the state variables, the current output level and labour supply are displayed. NetY represents the remaining output after abatement while velocity represents the combined rate of change of all state dynamics.[23]
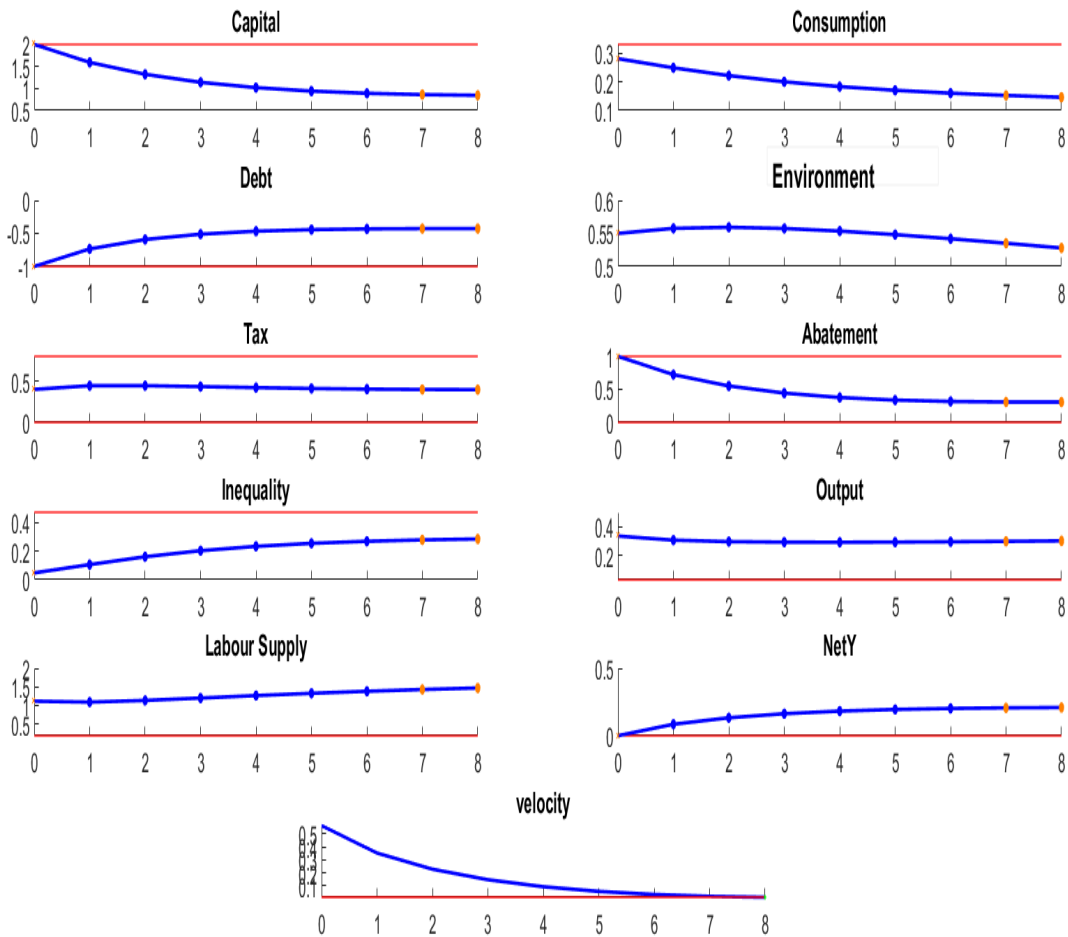

FIGURE 6. Time Profile of Initial Low Factor Share

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[23]A velocity of 0 means that the system has evolved to be maintained within the kernel and thus the starting conditions are viable.

The system is initially at a point of low inequality and high capital. The high capital stock is reduced over the evolution by abatement (in response to the pollution caused by the oversupply of capital) and the high consumption. In Krawczyk and Townsend (2015c) a similar dynamic of decreasing capital and increasing χ was found, but the reduction in capital was less due to the absence of abatement. As χ is decreasing in k, (as per the right hand side of (19)) the smaller reduction in k meant that the system reached viability before χ went above 0.25.

This suggests that if low income inequality is only achievable through high capital levels (which is the case when a social planner cant sperate taxation rates), then acknowledgement of pollution and the resulting abatement will make it unsustainable. This conclusion is only based on assumptions made in this model however, that labour tax and capital tax must be the same and that the dirty to clean capital stock is constant. In O'Keefe and Krawczyk (2016), dual tax regimes yielded a much broader base of viable points, especially in low inequality requirements. Thus it is expected that a dual pronged taxation policy would be able to sustain capital levels and reduce income inequality simultaneously. Also, increasing β[24] will mean a lower abatement requirement, which may mean sustaining high capital and low inequality.

6.3. **Capital with Lower Environmental Impact.** We now compare the findings from section 6.2 to the economy with reduced environmental impact, representing an economy that has successfully invested in cleaner production methods, limiting the leak of industry into the surrounding environment. Again, Figure 7 holds the 3-D slices from the kernels produced under different factor share constraints.

As before, the factor ratio is naturally limited to below 0.45, so little difference is evident between the first two kernels. The key difference is the existence of viable economic states that can maintain χ below 0.25. A lower damage coefficient would imply a lower abatement requirement, meaning capital could be maintained at a higher level, giving viability at lower levels of χ. To examine this, figure 8 displays the evolution of one of these points.

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[24]ie transitioning to an economy more reliant on service and clean capital than one based on production from dirty capital
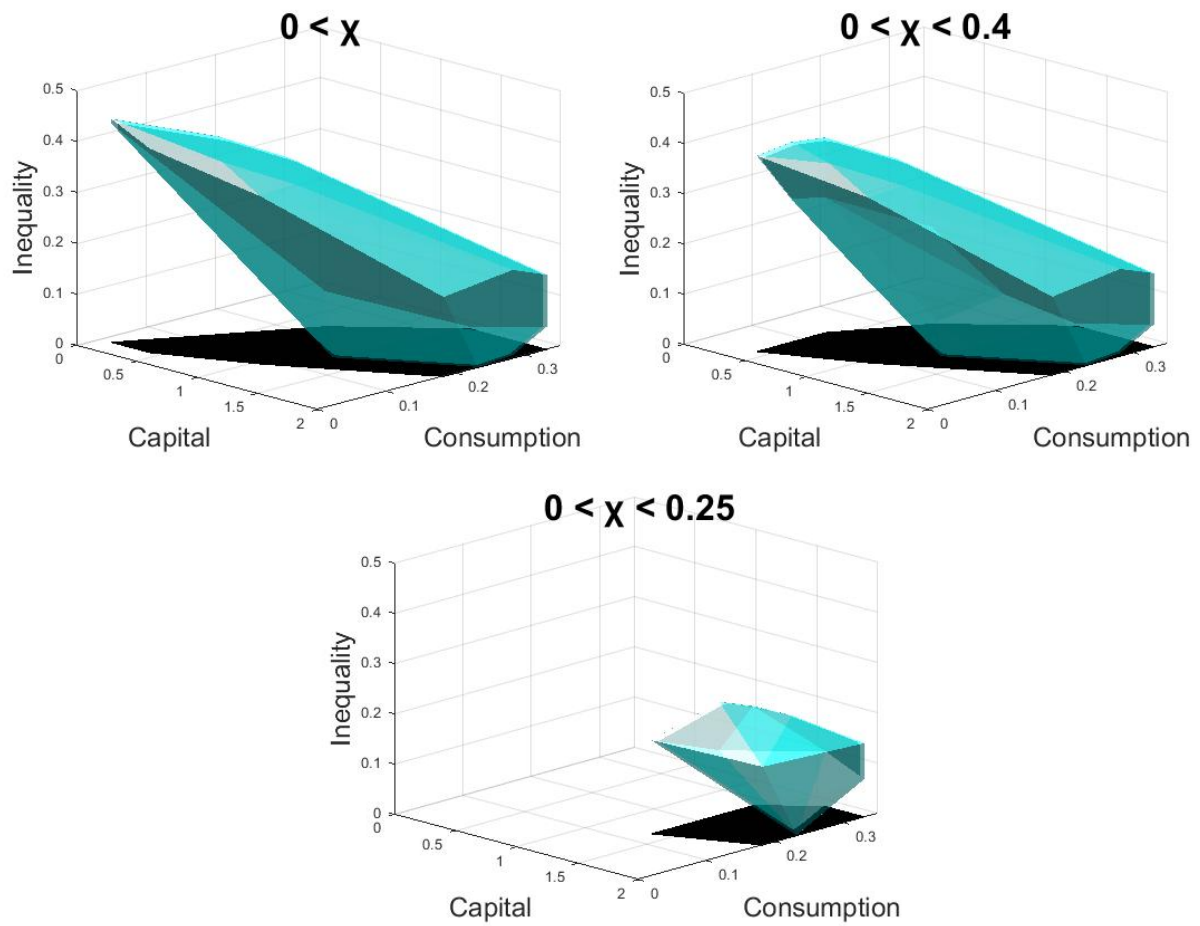
FIGURE 7. Viability Kernels for Different Relative Factor Share Constraints with Lower Environmental Impact
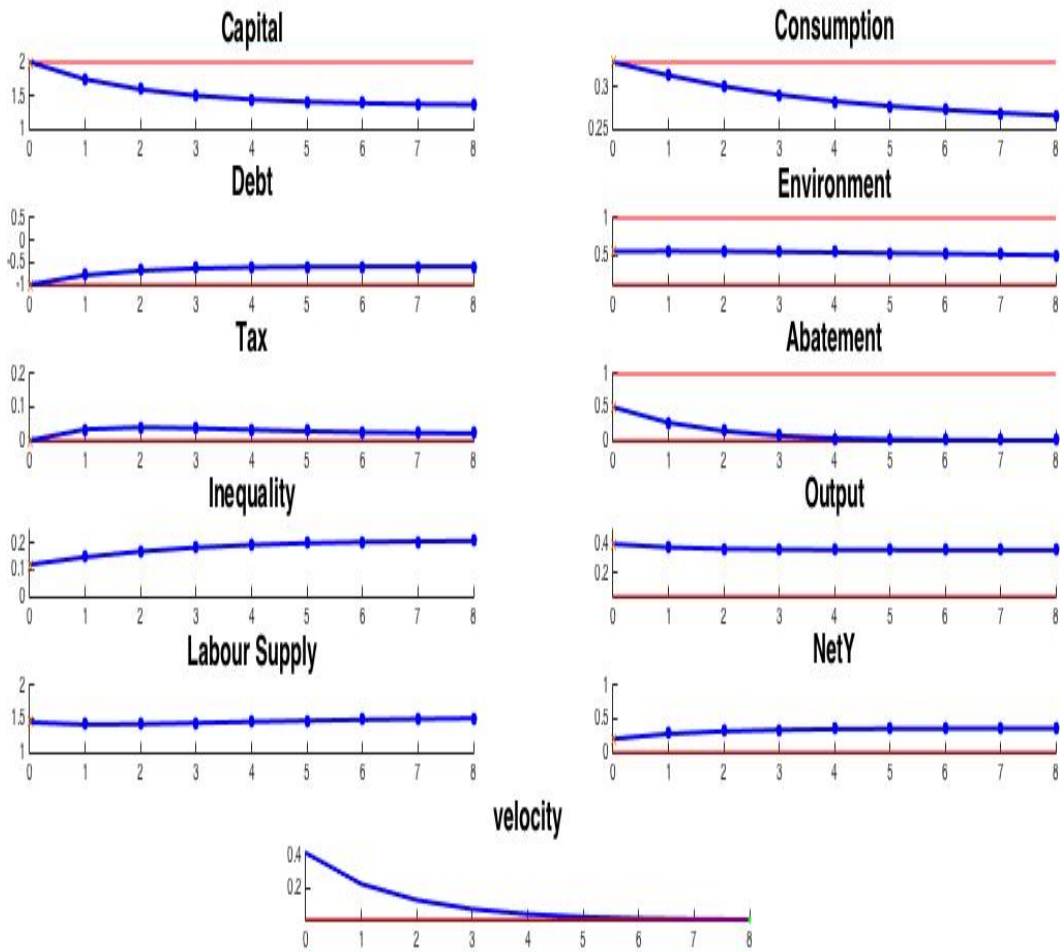
FIGURE 8. Time Profile of Low Impact Production Facilitating Low Income Inequality

The simulation displays what was theorised, by reducing the environmental impact of production there is a reduction in abatement requirements and capital reduction, which facilitates lower income inequality. Key to notice is that despite the attainment of lower inequality, output is near equal in either scenario. This correlates with the findings in O'Keefe and Krawczyk (2016) that policy maintaining lower income inequality does not suffer from Okun's trade-off. Net output however is significantly lower in Figure 6. This suggests that the ability to combat both environmental degradation and reduce income inequality is decreased when the economy is faced with highly pollutant industry.

How this is interpreted in regard to the Nordhaus (1994) theory on environmental sustainability, that gradual policy changes will lead to moderately lower growth, depends on whether total output or output post abatement spending is examined. Increased abatement did not lead to a reduction in output capability of the economy, but did lead to a reduction in effective output. There may however be a cost in either transitioning to or maintaining a lower carbon footprint. The model laid out in Section 9 should facilitate further analysis of this question in future work.

From these kernels and simulations, several conclusions can be made;

- Targeting and sustaining low income inequality is more difficult in an economy where the impact of capital on environment is more severe.
- Attaining a lower impact coefficient[25] facilitates not only a more stable environmental state but also increased capability to maintain higher capital stocks and lower income inequality.
- If environmental quality becomes too low, fiscal policy cannot achieve viability. The severity of this is increasing in the economy's environmental footprint. Again, attaining a lower carbon footprint does a great deal for increasing viability and facilitating environmental recovery.

Overall this gives the conclusion that investing in a reduced environmental impact will not only help sustain and replenish the environment, but also aid in the reduction of income inequality. While transition to a lower carbon footprint may present costs not recognised in this model, the attainment of such a state will mean that the multi faceted growth desired by the Living Standards Framework can be better attained.

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[25]*i.e.,* reducing the carbon footprint of industry

7. Viability Kernel Results and Analysis in a Separated Tax System

In this section we explore the benefits of separate taxation rates on capital and labour. Having additional controls should mean increased overall viability. More specifically, it allows more direct control over the relative factor share[26] and thus attainment of lower income inequality. As in Section 6.1, we first examine how net output levels fare in relation to environmental levels and income inequality. The kernels in Figure 12 are calculated under the same conditions as section 6.2
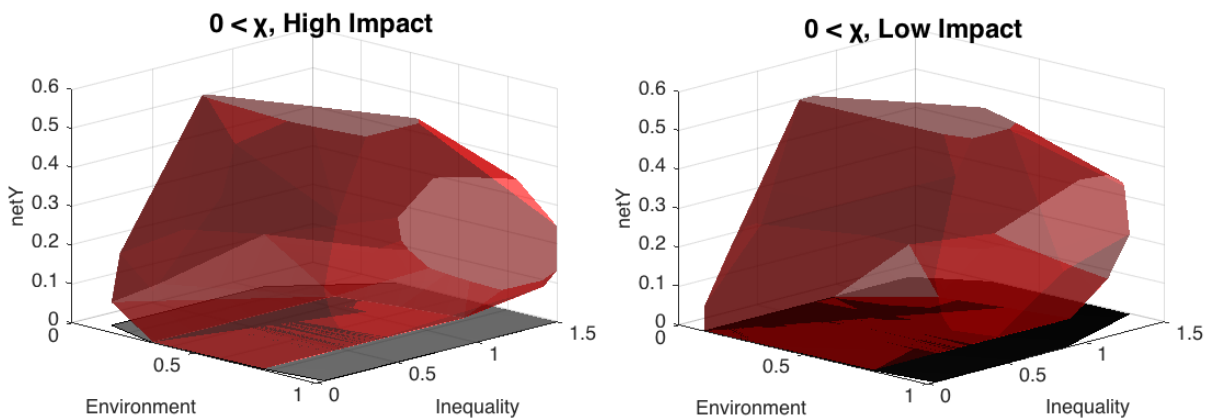

Figure 9. Viability Kernels Displaying Environment and Inequality's Effect on Net Output in a Separated Tax System

The figure illustrates similar results to those in section 6.1, output is maximised at inequality levels of 0.35. As opposed to the one tax model, a higher carbon footprint does not result in lower net output. This result is surprising, as in Section 6.1 it was concluded that investing in a lower carbon footprint leads to the socially preferred outcome of high equity and sustainability but with a possible hidden investment cost. In a two tax system, the benefit of investment seems negligible.

Important to note is that the Viability Kernels display the shell of the initial conditions from which viability can be obtained. In order to explore the nature of net output in the different environmental impact states, initial and final net output levels must be compared. Figure 10 displays the initial and final viable net output levels in our high impact model, while figure 11 displays the initial and final values in the low impact model.
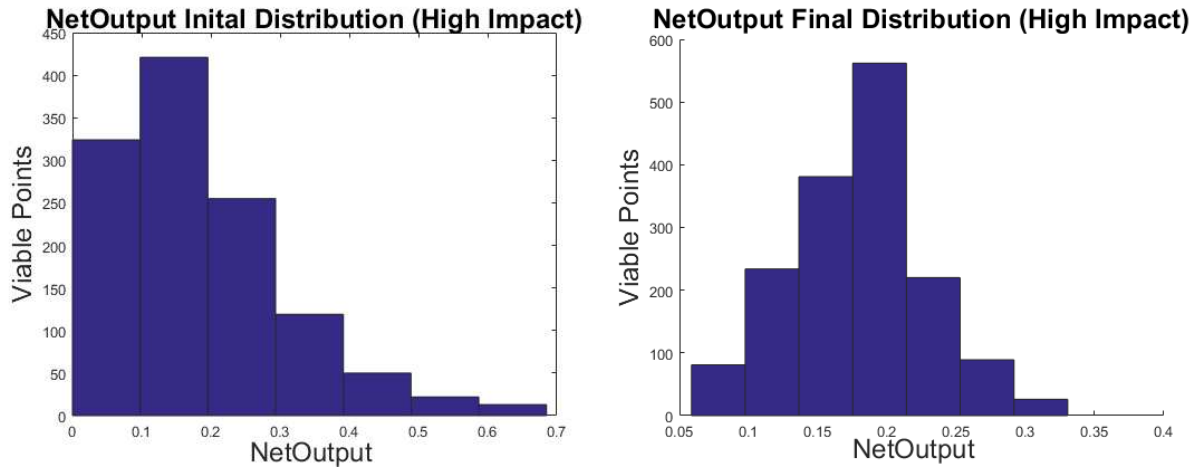
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[26]As can be seen in (19)

FIGURE 10. Histograms Displaying the Evolution of Net Output in the High Impact Model
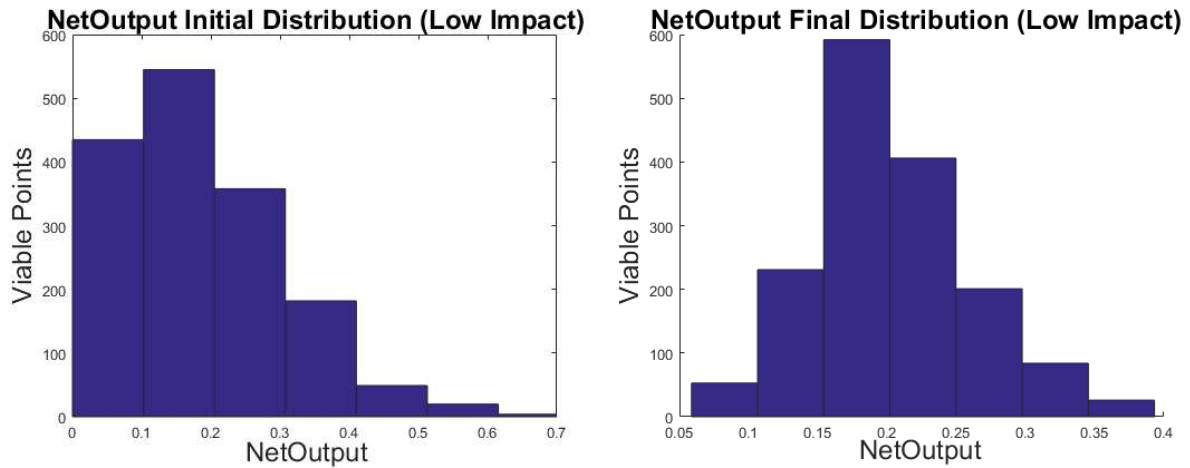

FIGURE 11. Histograms Displaying the Evolution of Net Output in the Low Impact Model

From figures 10, 11 it is clear that while having a two tax system facilitates higher initial output in both scenarios, in the long run the lower carbon footprint leads to higher equilibrium net output levels. A high environmental impact requires a reduction in capital levels, just as in section 6. The use of a two tax system allows this to be managed so as to maintain viability,[27] resulting in the appearance of high output in figure 9, but these output levels can not be maintained.

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[27]This is explained in section 7.1

This result then ratifies the conclusion of section 6, that attaining lower environmental impact is key for the multi faceted growth in the Living Standards Model, especially in regards to maintaining 'comprehensive consumption' for future generations.

7.1. **Capital and Consumption in a Separated Tax System.** We now examine how having a separated tax system affects the trade-offs between capital, consumption and attainment of low income inequality and environmental sustainability. The kernels in Figure 12 are calculated under the same conditions as Section 6.2[28]
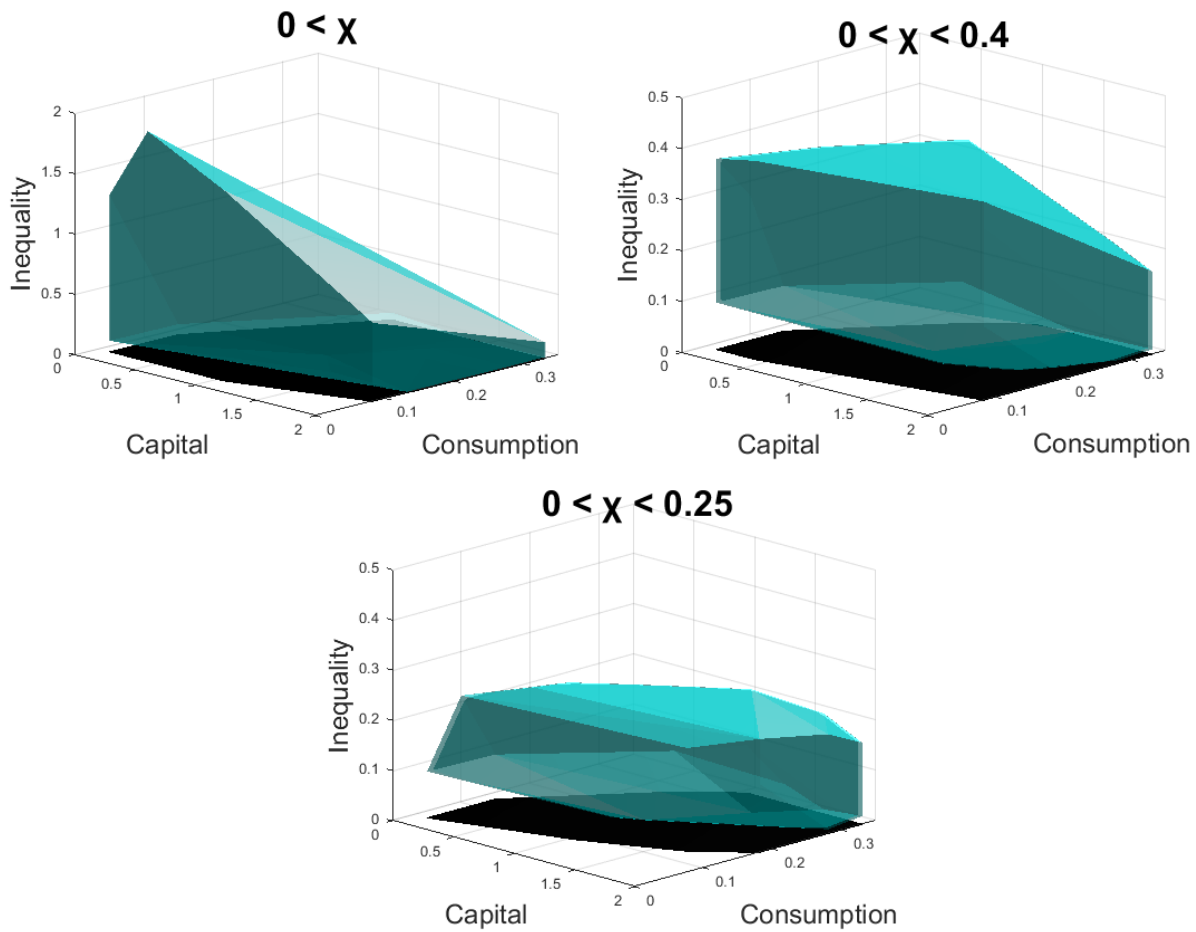

FIGURE 12. Viability Kernels for Different Relative Factor Share Constraints with High Environmental Impact and Separated Taxation Rates

The slices display a much broader base of consumption and capital combinations as a product of separate tax rates, but this comes at a cost of significantly more income inequality for some initial conditions. There is also little reduction in the slice size[29] as the restriction on income

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[28]The first slice has the inequality axis from 0 to 2 while latter slices have 0 0.5. This could be misleading but aids visualisation of the shape of the slices

[29]In terms of capital and consumption

inequality is increased. This lack in reduction in size is a positive result, as it suggests that in this model there is at most a marginal trade-off between efficiency and income inequality when using a separated tax system.

As stated earlier, from (19) it is clear that a high capital taxation rate will achieve low income inequality.[30] A high capital taxation rate is often undesirable however, as it can dissuade investment. It is thus pertinent to examine whether it is possible to attain low income inequality without excessive capital taxation, as well as stabilising environment.

One example of this is given in figure 13 below. The simulation starts from the same initial conditions as 6 and 8, i.e. with high capital and consumption and a surplus of savings.
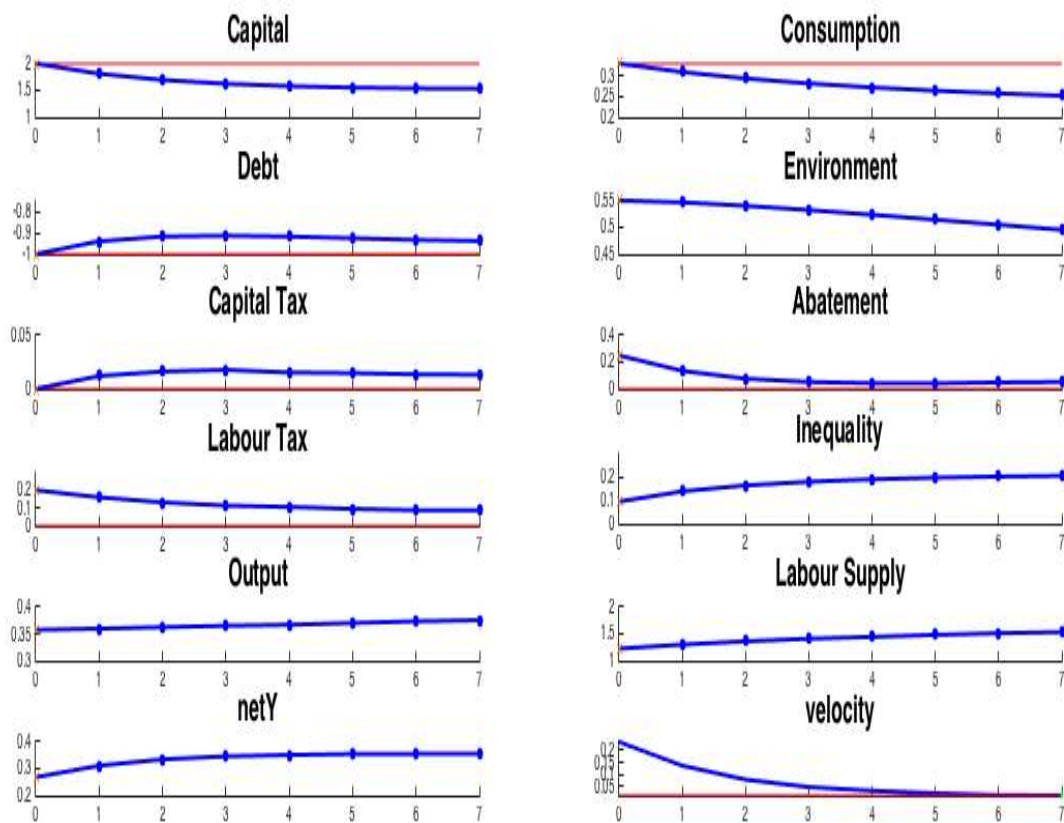

FIGURE 13. Time Profile of Low Capital Tax Achieving Low Inequality

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[30]This finding was confirmed in O'Keefe and Krawczyk (2016)

By imposing a low tax rate on capital and taxing labour at less than 20%, the economy is able to reach a point with low income inequality, a high capital stock and increasing output. The environment however is only viable in a reduced state.

The imposition of separated taxation rates gives the social planner increased ability to reduce income inequality while maintaining output. To maintain this however, low abatement is required, otherwise capital stocks would fall further and increase the returns on capital. Thus we see that even in a separated taxation regime, targeting low income inequality and environmental sustainability is impossible given an industries high environmental impact.

Analysis of the results in Section 7 thus leads to similar conclusions as Section 6.

- Utilising separated taxation rates, an economy is able to achieve high output (and net output) levels in the short run while keeping environment and income inequality within viable bounds.
- In the medium run however, the economy must accept increasing income inequality or lower environmental levels.
- In the long run, a higher environmental impact will lead to reduced net output.
- Thus while separate tax rates may appear to allow attainment of all facets of the Living Standards Framework, high environmental impact will make this impossible in the medium to long run.

8. Conclusion

This paper has taken the New Zealand Living Standards Framework as inspiration, attempting to integrate different facets of growth into one model. This was approached by augmenting an economic growth model originally laid out in Krawczyk and Judd (2014), integrating environmental level as a derivative of productivity as well as a dynamic variable and controlling for income inequality through the factor share. From this, kernels were derived for a combined tax system as well as a separated tax system.

In the combined tax system the benefit of investment in a reduced carbon footprint was significant. Attaining a lower carbon footprint allowed for targeting of low income inequality as well as environmental sustainability, all while maintaining high capital stocks and high consumption.

To further understanding of this, the model was complicated with separate taxation rates on capital and labour. It was found that this increased viability, allowing attainment of lower income inequality even in a highly pollutant economy. In the long run however, this lead to lower net growth and the inability to maintain both environment and low income inequality. Thus this model again displayed that investment in lower environmental impact holds significant benefit for realising the success of the Living Standards Framework.

Both models suggested that attainment of lower environmental impact would facilitate higher potential growth for the economy. We hesitate to fully stand by this due to the limitation of the model used, which does not speak to a cost of transitioning to a lower carbon footprint or the possible lower productivity it may bring.

It is hoped however that this paper has laid out how Viability Theory can be used to as a basis for the Living Standards Framework. While this model was heavily stylised, further refinement in calibration and in the model's structure may allow analysis of policy suggestions in relation to the Living Standards Framework, exploring how elasticity between different industries affects the cost and benefit of directed technical change.

As an indication of this, Section 9 lays out a new model structure that would allow improve analysis on the topics covered in this paper.

9. Future Research: A model of Capital Transition

The model laid out in section 4.2 provides ample area for analysis, but the inability to track costs associated with reduced environmental impact limits its analysis for the cost and optimal speed of transition. Thus it is difficult to compare its results to those laid out by Acemoglu et al (2012). To further understanding of this area, we now lay out the intended extensions to the work provided. This involves augmenting the model from section 4.2 with two capital stocks, each with different environmental impacts. The social planner continues to set taxation and abatement policy, but is now also able to subsidize low emission capital thus creating "directed techincal change".[31] In addition, a dictate can be set on how investment should be split between growing the stocks of different capitals. This will allow analysis of the impact on viability and different policies required in relation to different commitment levels to environmental sustainability.

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[31]Acemoglu et al (2012)

The output equation for the second model is thus;

$$(23) \qquad y = A((Bk)^\phi + X^\phi)^{\frac{\alpha}{\phi}} \ell^{1-\alpha}$$

The notation is the same in the previous model, but now with a second productivity coefficient $B > 1$, a second clean capital source X and an elasticity of substitution (ϕ) between k and X. The choice of an elasticity between capital types is based on the research done by Acemoglu et al (2012). It suggests that there is an elasticity between clean and dirty technologies greater than one given "we would expect successful clean technologies to substitute for the functions of dirty technologies"

Consumer utility functions are maintained; they receive positive utility from consumption and dis-utility from labour.

Investment in total capital K is given by the law of motion;

$$(24) \qquad \frac{dK}{dt} = \frac{dk}{dt} + \frac{dX}{dt} = Y - c - G - \delta_1 k - \delta_2 X$$

where δ_1 and δ_2 are the depreciation rates for k and X respectively.

We define the relationship between investment in these two capital stocks per below. ω represents the requirement set by the government that a higher proportion of investment go to growing X over k with the goal of a 'cleaner' economy.[32]

$$(25) \qquad \frac{dX}{dt} = \sigma \frac{dk}{dt}$$

where σ is greater than one.

Combining (24) and (25) we derive the equation for the growth of clean capital

$$(26) \qquad \begin{aligned} \frac{dk}{dt} + \frac{dX}{dt} &= y - c - G - \delta_1 k - \delta_2 X \\ \frac{dX}{dt}(1 + \frac{1}{\sigma}) &= y - c - G - \delta_1 k - \delta_2 X \\ \frac{dX}{dt} &= \frac{\sigma}{1 + \sigma}(y - c - G - \delta_1 k - \delta_2 X) \end{aligned}$$

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[32]This allows analysis of the different policies required in relation to different commitment levels to the 2015 UN Climate Change Conference in Paris

and the growth of dirty capital

$$(27) \qquad \frac{dk}{dt} = \frac{y - c - G - \delta_1 k - \delta_2}{1 + \sigma}$$

Government collects tax revenue the same way as before, but now with the additional option of taxing clean machinery and dirty capital separately. The subsidy on clean capital X is represented by τ_X and can be interpreted as a negative tax rate.

$$(28) \qquad T = \tau_K k(r - \delta_1) + \tau_L w\ell + \tau_X X(e - \delta_2)^{33}$$

Debt evolves in the same way as in equation (14).

$$(29) \qquad \frac{dB}{dt} = \bar{r}B - T + G$$

The interest rate on current debt is given by the marginal product of dirty capital k. The relationship between the marginal products of the two capital stocks is based on the difference in productivity. As dirty capital has higher productivity, it gives a higher return.[34] The interest on the debt is thus set to the higher interest rate, facilitating arbitrage in the economy.

We derive the consumption equation using Optimal Control theory, giving a micro foundation to our dynamic. The Hamiltonian is defined as;

$$(30) \qquad H = u(c) - v(\ell) + \lambda_k \frac{y(1 - v) - c - G - \delta_1 k - \delta_2 X}{1 + \sigma}$$
$$+ \lambda_X \frac{\sigma(y - c - G - \delta_1 k - \delta_2 X)}{1 + \sigma}$$

where λ_k, λ_x are the private marginal value of dirty and clean capital in the perfect-foresight household utility maximization of $u(c) - v(\ell)$ [35]. Households decide on how much labour to supply and how much to consume with the knowledge on how capital growth will affect output.

We expand output and government expenditure in terms of clean/dirty capital stock and labour supply with r, e and w as the relative marginal productivity

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[33]where e is the marginal productivity of clean capital X

[34]In the steady state the government subsidises the clean capital to the point at which return is equal

[35]$u(c)$ and $v(\ell)$ given by (4) and (5)

$$(31) \qquad\qquad y = rk + w\ell + eX$$

$$(32) \qquad\qquad G = \tau_K k(r - \delta_1) + \tau_L \ell w + \tau_X X(e - \delta_2)^{36}$$

and take first order conditions to determine the optimal choice of labour and consumption.

$$(33) \qquad \frac{dH}{dc} = u'(c) + \lambda_k \frac{-1}{1+\sigma} + \lambda_X \frac{-\sigma}{1+\sigma} = 0$$
$$\Rightarrow u'(c) = \lambda_k \frac{1}{1+\sigma} + \lambda_X \frac{\sigma}{1+\sigma}$$

$$(34) \qquad \frac{dH}{d\ell} = V\ell^\eta + \lambda_k \frac{(w - \tau_L w)}{1+\sigma} + \lambda_X \frac{\sigma(w - (\tau_L w)}{1+\sigma}$$

Using the time index identity we see that;

$$(35) \qquad \frac{d\lambda_k}{dt} = \rho\lambda_k - \frac{dH}{dk}$$

and

$$(36) \qquad \frac{d\lambda_X}{dt} = \rho\lambda_X - \frac{dH}{dX}$$

Differentiating the Hamiltonian with respect to the two capital stocks gives;

$$(37) \qquad \frac{dH}{dk} = \frac{\lambda_k}{1+\sigma}(r - \tau_K(r - \delta_1) - \delta_1) + \frac{\sigma\lambda_X}{1+\sigma}(r - \tau_k(r - \delta_1) - \delta_1)$$

and

$$(38) \qquad \frac{dH}{dX} = \frac{\lambda_k}{1+\sigma}(e - \tau_X(e - \delta_2) - \delta_2) + \frac{\sigma\lambda_X}{1+\sigma}(e - \tau_X(e - \delta_2) - \delta_2)$$

From (33) we can rearrange for c and take the derivative with respect to time. This leads to the expression

$$(39) \qquad \frac{dc}{dt} = \frac{-c^{1+\gamma}}{\gamma}\left[\frac{1}{1+\sigma}(\rho\lambda_k - \frac{dH}{dk}) + \frac{\sigma}{1+\sigma}(\rho\lambda_X - \frac{dH}{dx})\right]$$

Substituting (38) and (37) into (39) we are able to simplify and get a dynamic for consumption.

$$(40) \qquad \frac{dc}{dt} = \frac{-c}{\gamma}\left[\rho - \frac{(r - \delta_1)(1 - \tau_k)}{1+\sigma} - \frac{\sigma(e - \delta_2)(1 - \tau_x)}{1+\sigma}\right]$$

---

[36]So G is balanced by taxing production inputs and allowing for depreciation to be tax exempt

Similar to the previous model, we see that consumption is increasing when the future discount is less than the after tax marginal products of the two capital sources (allowing for depreciation) and weighted by the governments decision to push investment in clean capital. The taxation rates further dictate the effect the different capital stocks affect consumers consumption decisions.[37]

The environmental level is dictated by a natural replenishment function (μ), damage caused by industry, which is lower in X than k[38] and some abatement expenditure (ψ)

$$(41) \qquad \frac{dR}{dt} = \mu^{39} - \theta_1 \frac{dy}{dk} - \theta_2 \frac{dy}{dX} + \psi$$

As in the previous model we require closing the dynamic equations through ℓ. This is done again by equating the marginal utility of consumption weighted by after tax wage to the dis utility of labour.

$$(42) \qquad (1 - \tau_L) \frac{dy}{d\ell} \frac{du(c)}{dc} = \frac{d(v(\ell))}{d\ell}$$

$$\ell = \frac{(1 - \tau_L)(1 - \alpha)A((Bk)^\phi + X^\phi)^{\frac{\alpha}{\phi}}}{c^\gamma V}$$

This model should allow study of how an economy reacts to targeting of environmental sustainability and any cost associated with directed technical change

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[37]As the tax on clean capital is actually a government subsidy, it should be noted that consumption is increasing in τ_X

[38]ie $\theta_1 > \theta_2$

[39]Where μ is likely a logistic function

References

Acemoglu D, Aghion P, Bursztyn L, Hemous D (2012) The environment and directed technical change. The American Economic Review 102(1):131–166

Arrow KJ, Dasgupta P, Goulder LH, Mumford KJ, Oleson K (2012) Sustainability and the measurement of wealth. Environment and development economics 17(03):317–353

Aubin JP, Cellina A (2012) Differential inclusions: set-valued maps and viability theory, vol 264. Springer Science & Business Media

Chichilnisky G (1999) What is sustainable development? In: Man-Made Climate Change, Springer, pp 42–82

Chichilnisky G, Heal G, Beltratti A (1995) The green golden rule. Economics Letters 49(2):175–179

for the Environment M, Statistics New Zealand (2015) New zealand's environmental reporting series: Environment aotearoa 2015. http://www.mfe.govt.nz/publications/environmental-reporting/environment-aotearoa-2015-land/state-our-land

Filar JA, Krawczyk JB, Agrawal MR (2014) Sustainability screw: role of relative production and abatement time scales. Journal of the Operational Research Society URL http://www.palgrave-journals.com/jors/journal/vaop/ncurrent/full/jors201439a.html, published online 22 October 2014; doi:10.1057/jors.2014.39

Gleisner B, Llewellyn-Fowler M, McAlister F (2011) Working towards higher living standards for New Zealanders. URL http://purl.oclc.org/nzt/p-1387

Karacaoglu G (2015) The new zealand treasury's living standards framework-exploring a stylised model. MONTH 15:12

King A (2016) Living standards model documentation, toy08, new Zealand Treasury, unpublished

Krawczyk JB, Judd KL (2014) Which economic states today are sustainable under an slightly constrained tax-rate adjustment policy. Munich Personal RePEc Archive, URL http://mpra.ub.uni-muenchen.de/59027/

Krawczyk JB, Judd KL (2015) A note on determining viable economic states in a dynamic model of taxation. Macroeconomic Dynamics forthcoming

Krawczyk JB, Pharo A (2013) Viability theory: an applied mathematics tool for achieving dynamic systems' sustainability. Mathematica Applicanda 41:97–126, URL http://wydawnictwa.ptm.org.pl/index.php/matematyka-stosowana/article/view/409

Krawczyk JB, Pharo A (2014) Manual of VIKAASA 2.0: An application for computing and graphing viability kernels for simple viability problems. SEF Working Paper 08/2014, Victoria University of Wellington, URL http://researcharchive.vuw.ac.nz/handle/10063/3432

Krawczyk JB, Townsend W (2015a) Basic understanding of social inequality dynamics. Working Paper Series xxxx, Victoria University of Wellington, School of Economics and Finance

Krawczyk JB, Townsend W (2015b) New zealand inequality and the struggle between capital and labour. Working Paper Series xxxx, Victoria University of Wellington, School of Economics and Finance

Krawczyk JB, Townsend W (2015c) Viability of an economy with constrained inequality. Working Paper Series xxxx, Victoria University of Wellington, School of Economics and Finance

Lee J (2012) Examining a green growth model for policy implications. Seoul Journal of Economics 25(1):57–87

Nordhaus WD (1994) Managing the global commons: the economics of climate change, vol 31. MIT press Cambridge, MA

O'Keefe MD, Krawczyk JB (2016) Viability of an economy with constrained inequality in a two tax system

Pearce DW, Warford JJ, et al (1993) World without end: economics, environment, and sustainable development. Oxford University Press

Piketty T (2014) Capital in the Twenty-First Century. Belknap Press

Rajan RG (2011) Fault lines: How hidden fractures still threaten the world economy. Princeton University Press

Simon HA (1955) A behavioral model of rational choice. Quarterly Journal of Economics 69:99–118, DOI 10.2307/1884852

Stern N (2009) A blueprint for a safer planet: How to manage climate change and create a new era of progress and prosperity. Random House

Stiglitz J, Sen A, Fitoussi JP (2009) Report by the commission on the measurement of economic performance and social progress. URL http://www.stiglitz-sen-fitoussi.fr/documents/rapport_anglais.pdf

Tipper A (2012) Capital-labour substitution elasticities in new zealand: One for all industries? (12-01), URL http://www.stats.govt.nz/methods/research-papers/

working-papers-original/capital-labour-substitution-elasticities-nz-12-01.aspx

Treasury A (2004) Policy advice and treasuryís wellbeing frameworkí. Economic Roundup pp 1–20