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# Existence of Bifurcation in Macroeconomic Dynamics: Grandmont was Right<sup>1</sup>

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**Abstract:** Grandmont (1985) found that the parameter space of the most classical dynamic general-equilibrium macroeconomic models are stratified into an infinite number of subsets supporting an infinite number of different kinds of dynamics, from monotonic stability at one extreme to chaos at the other extreme, and with all forms of multiperiodic dynamics between.

But Grandmont provided his result with a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Hence he was not able to reach conclusions about the policy relevance of his dramatic discovery. As a result, Barnett and He (1999, 2001, 2002) investigated a Keynesian structural model, and found results supporting Grandmont's conclusions within the parameter space of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. That prototypical Keynesian model was produced from a system of second order differential equations. The model contains frictions through adjustment lags, displays reasonable dynamics fitting the UK economy's data, and is clearly policy relevant. In addition, initial results by Barnett and Duzhak (2006) indicate the possible existence of Hopf bifurcation within the parameter space of recent New Keynesian models.

Lucas-critique criticism of Keynesian structural models has motivated development of Euler equations models having policy-invariant deep parameters, which are invariant to policy rule changes. Hence, we continue the investigation of policy-relevant bifurcation by searching the parameter space of the best known of the Euler equations general-equilibrium macroeconometric models: the Leeper and Sims (1994) model. We find the existence of singularity bifurcation boundaries within the parameter space. Although never before found in an economic model, our explanation of the relevant theory reveals that singularity bifurcation may be a common property of Euler equations models. These results further confirm Grandmont's views.

Beginning with Grandmont's findings with a classical model, we continue to follow the path from the Bergstrom-Wymer policy-relevant Keynesian model, to New Keynesian models, and now to Euler equations macroeconomic models having deep parameters.

Grandmont was right.

*Keywords:*

Bifurcation, inference, dynamic general equilibrium, Pareto optimality, Hopf bifurcation, Euler equations, Leeper and Sims model, singularity bifurcation, stability.

*JEL Codes:*

C14, C22, E37, E32.

□

<sup>1</sup> We have benefited from the comments of Huabin He at Nanchang University.

## 1. Introduction

### 1.1. The History

Grandmont (1985) found that the parameter space of even the simplest, classical general-equilibrium macroeconomic models are stratified into bifurcation regions. This result changed the prior common view that different kinds of economic dynamics can only be produced by different kinds of structures. But he provided that result with a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Hence he was not able to reach conclusions about the policy relevance of his dramatic discovery. Years of controversy followed, as evidenced by papers appearing in Barnett, Deissenberg, and Feichtinger (2004) and Barnett, Geweke, and Shell (2005). The econometric implications of Grandmont's findings are particularly important, if bifurcation boundaries cross the confidence regions surrounding parameter estimates in policy-relevant models. Stratification of a confidence region into bifurcated subsets seriously damages robustness of dynamical inferences.

The dramatic transformation of views precipitated by Grandmont's paper was criticized for lack of policy relevance. As a result, Barnett and He (1999, 2001, 2002) investigated a continuous-time traditional Keynesian structural model, and found results supporting Grandmont's conclusions. Barnett and He found transcritical, codimension-two, and Hopf bifurcation boundaries within the parameter space of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. That highly regarded Keynesian model was produced from a system of second order differential equations. The model contains frictions through adjustment lags, displays reasonable dynamics fitting the UK economy's data, and is clearly policy relevant. See Bergstrom and Wymer (1976), Bergstrom (1996), Bergstrom, Nowman, and Wandasiewicz (1994), Bergstrom, Nowman, and Wymer (1992), and Bergstrom and Nowman (2006). Barnett and He found that bifurcation boundaries cross confidence regions of parameter estimates in that model, such that both stability and instability are possible within the confidence regions.

Barnett and Duzhak (2006) have begun to explore bifurcation within the more recent class of New Keynesian models. They studied forward-looking and current-looking models, as well as hybrid models having both forward and current-looking

features. They find the possibility of Hopf bifurcation, with the setting of the policy parameters influencing the existence and location of the bifurcation boundary. No other form of bifurcation is possible within the three-equations log-linearized New Keynesian models that they consider. One surprising result from their proofs is the finding that a common setting of a parameter in the future-looking New-Keynesian model can put the model directly onto a Hopf bifurcation boundary.

The Lucas critique has motivated development of Euler equations general-equilibrium macroeconomic models. Hence, we continue the investigation of policy relevant bifurcation by searching the parameter space of the best known of the policy relevant Euler-equations macroeconometric models: the Leeper and Sims (1994) model. The results further confirm Grandmont's views, but with the finding of an unexpected form of bifurcation: singularity bifurcation. Although known in engineering and mathematics, singularity bifurcation has not previously been encountered in economics. Barnett and He (2004, 2006) have made clear the mathematical nature of singularity bifurcation and why it is likely to be common in the class of modern Euler equation models rendered important by the Lucas critique.

Leeper and Sims' model consists of differential equations with a set of algebraic constraints. Our analysis reveals the existence of a singularity bifurcation boundary within a small neighborhood of the estimated parameter values. When the parameter values approach the singularity boundary, one eigenvalue of the linearized part of the model moves rapidly to infinity, while other eigenvalues remain bounded. This phenomenon implies nearly instantaneous response of some variables to changes in other variables. On the singularity boundary, the number of differential equations will decrease, while the number of algebraic constraints will increase. Such change in the order of dynamics has not previously been found with macroeconometric models. But we find from the relevant theory that singularity bifurcation may be a common property of Euler equations models.

Beginning with Grandmont's findings with a classical model, we continue to follow the path from the Bergstrom-Wymer policy-relevant Keynesian model, to New Keynesian macroeconometric models, and now to Euler equations models having deep parameters. At this stage of our research, we believe that Grandmont's conclusions appear to hold for all categories of dynamic macroeconomic models, from the oldest to the newest.

## **1.2 The Leeper and Sims Model**

There is growing interest in rigorous analysis of macroeconomics through the study of mathematical models. For this purpose, various dynamic macroeconomic general-equilibrium models have been established in the literature.<sup>2</sup> Of particular importance is the Leeper and Sims (1994) Euler equations stochastic-dynamic general-equilibrium model intended to address such issues as the Lucas critique (Lucas (1976)). Similar models are developed in Kim (2000).

The dimension of the state space in the Leeper and Sims model is substantially lower than in the Bergstrom, Norman, and Wymer UK model. However, the dimension is still too high for complete analysis by generally available analytical approaches. By numerical methods complementing theoretical analysis, we find that the dynamics of the Leeper and Sims model is complicated by its inclusion, not only of differential equations, but also of algebraic constraints; and by the model's lack of existence of closed form solutions.

In this paper, we are interested in how the dynamic behavior of the model is affected by its parameter settings. We find that the order of the dynamics of the Leeper and Sims model can change within a small neighborhood of the estimated parameter values. In particular, one eigenvalue of the linearized part of the model can move quickly from finite to infinite and back again to finite. A large stable eigenvalue characterizes the case in which some variables can respond rapidly to changes of other variables, while a large unstable eigenvalue corresponds to the case in which rapid diversion occurs of one variable from other variables. Infinity eigenvalue implies existence of pure algebraic relationships among the variables.

The resulting change in the order of the dynamic part of the system is a fundamental property of Leeper and Sims model and corresponds to a class of bifurcations known to engineers and mathematicians as "singularity" bifurcations. To our knowledge, this is the first discovery of singularity bifurcation in

□

<sup>2</sup> Among those models that have direct relevance to this research are the high dimension continuous time macroeconomic models of Bergstrom, Nowman and Wymer (1992), Bergstrom, Nowman, and Wandasiewicz (1994), Bergstrom and Wymer (1976), Grandmont (1998), Leeper and Sims (1994), Powell and Murphy (1997) and Kim (2000). Surveys of relevant macroeconomic models are available in Bergstrom (1996) and in several textbooks such as Gandolfo (1996) and Medio (1992). General theory of economic dynamics is provided, in Boldrin and Woodford (1990) and Gandolfo (1996). Various bifurcation phenomena are reported in Bala (1997), Benhabib (1979), Medio (1992), Gandolfo (1996), and Nishimura and Takahashi (1992). Focused studies of stability are conducted in Grandmont (1998), Scarf (1960), and Nieuwenhuis and Schoonbeek (1997). Barnett and Chen (1988) empirically found chaotic dynamics in economics. Bergstrom, Nowman, and Wandasiewicz (1994) investigate stabilization of macroeconomic models using policy control. Wymer (1997) describes several mathematical frameworks for the study of the structural properties of macroeconomic models.

macroeconometric models; but appears to be closely connected with the structure of Euler equations models.

## 2. The Model

The Leeper and Sims (1994) Euler-equations general-equilibrium model includes the dynamic behavior of consumers, firms, and government. With the parameters of consumer and firm behavior retained as the deep parameters of tastes and technology, these parameters are invariant to government policy rule changes.<sup>3</sup> These models contain dynamic subsystems consisting of ordinary differential equations and algebraic constraints. Such systems are called differential/algebraic systems in systems theory.

In the Leeper and Sims model, both consumers and firms maximize their respective objective functions. The government provides monetary and tax policies to satisfy an intertemporal government budget constraint and to the pursuit of countercyclical policy objectives. The detailed derivation of the models is available in Leeper and Sims (1994). The resulting model is summarized in this section.

The model contains the following 12 state variables.

$L$  = labor supply

$C^*$  = consumption net of transactions costs

$M$  = consumer demand for non-interest-bearing money

$D$  = consumer demand for interest-bearing money

$K$  = capital

$Y$  = factor income from capital and labor, excluding interest on government debt.

$C$  = gross consumption

$Z$  = investment

$X$  = consumer goods aggregate price

$Q$  = investment goods price

$V$  = income velocity of money

$P$  = general price level

□

<sup>3</sup> Several similar models have been developed in Kim (2000) and in Binder and Pesaran (1999).

The model assumes that the consumer maximizes<sup>4</sup>

$$E[\int_0^{\infty} \exp(-\int_0^t \beta(s) ds) \frac{(C^* \pi (1-L)^{1-\pi})^{1-\gamma}}{1-\gamma} dt]$$

subject to

$$XC + QZ + \tau + \frac{\dot{M} + \dot{D}}{P} = Y + \frac{iD}{P},$$

$$XC^* + \phi VY = XC,$$

$$\dot{K} = Z - \delta K,$$

$$Y = rK + wL + S,$$

$$V = \frac{PY}{M},$$

where  $\pi \in (0,1)$  and  $\gamma > 0$  are parameters;  $0 \leq \beta(s) \leq 1$  is the subjective rate of time preference at time  $s$ ,  $\tau$  is the level of lump-sum taxes paid by the representative consumer;  $i$  is the nominal rate of return earned on government bonds;  $S$  is the sum of dividends received by the representative consumer,  $w$  is the wage rate;  $\phi > 0$  is the transaction cost per unit of  $VY$ ;  $\delta \geq 0$  is the rate of depreciation of capital; and  $r =$  rental rate of return on capital.<sup>5</sup> As we shall see below, parameters in this stochastic dynamic general-equilibrium model are not necessarily assumed to be constant or deterministic.

The firms' optimization problem is

$$\max\{X(C + g) + QI^* + A(\alpha K^\sigma + L^\sigma)^{1/\sigma} - rK - wL - ((C + g)^\mu + \theta I^{*\mu})^{1/\mu}\},$$

where  $g$  is the level of government purchases. The following are parameters:  $A > 0$ ,  $\alpha > 0$ ,  $\theta > 0$ ,  $\mu \geq 0$ , and  $0 \leq \sigma \leq 1$ . Investment goods produced by the firm,  $I^*$ , include both those bought by the existing population,  $Z$ , and those purchased by the

□

<sup>4</sup> Leeper and Sims describe the model's consumer as a "representative consumer" maximizing utility subject to constraints in total consumption of goods and leisure. This convention is unusual, since in aggregation theory, Gorman's representative consumer makes decisions in per capita variables, not totals. But as used empirically by Leeper and Sims, the resulting Euler equations are equivalent to those that would have resulted from a per capita decision for the representative consumer.

<sup>5</sup> Transactions are assumed by Leeper and Sims to be proportional to  $V$  and  $Y$ , with  $\phi$  being the

government for distribution to the newborn. Thus, a market-clearing condition is  $I^* = Z + nK$ , where  $n$  = the fraction of existing capital purchased by the government for distribution to the newborn.<sup>6</sup>

In this model, the state variables satisfy the following differential equations:

$$\frac{1}{P}(\dot{M} + \dot{D}) = Y - XC - QZ + \frac{iD}{P} + \tau, \quad (1)$$

$$\dot{K} = Z - \delta K, \quad (2)$$

$$(1 - \pi(1 - \gamma)) \frac{\dot{C}^*}{C^*} + (1 - \gamma)(1 - \pi) \frac{\dot{L}}{1 - L} + \frac{\dot{X}}{X} + \frac{\dot{P}}{P} = i - \beta + \frac{\dot{\pi}}{\pi} + \dot{\pi}(1 - \gamma) \log\left(\frac{C^*}{1 - L}\right), \quad (3)$$

$$\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} = i + \delta - (1 - 2\phi V) \frac{r}{Q}, \quad (4)$$

where equation (1) represents the consumers' budget constraint, (2) is the law of motion for capital, and (3) and (4) are first-order conditions from the consumers' optimization decision.

In addition to satisfying the four dynamic equations, the state variables satisfy the following algebraic constraints:

$$X = \left( \frac{Y}{C + g} \right)^{1 - \mu}, \quad (5)$$

$$Q = \theta \left( \frac{Y}{Z + nK} \right)^{1 - \mu}, \quad (6)$$

$$r = A^\sigma \alpha \left( \frac{Y}{K} \right)^{1 - \sigma}, \quad (7)$$

$$w = A^\sigma \left( \frac{Y}{L} \right)^{1 - \sigma}, \quad (8)$$

$$XC^* + \phi V Y = XC, \quad (9)$$

---

proportionality constant. The overdot is used throughout to designate time derivative.

<sup>6</sup> We are using Leeper and Sims' definitions, which we ourselves are not advocating. Another view could equate  $n$  with population growth rate so that  $nK$  could be interpreted to include capital endowed to the young generation by the old.



$$Y = rK + wL + S, \quad (10)$$

$$V = \frac{PY}{M}, \quad (11)$$

$$X(C + g) + Q(Z + nK) = Y, \quad (12)$$

$$(1 - 2\phi V) \frac{w}{X} = \frac{1 - \pi}{\pi} \frac{C^*}{1 - L}, \quad (13)$$

$$i = \phi V^2. \quad (14)$$

The relations (5)-(8) are obtained from the first-order conditions by maximizing the firms' objective function. Equation (9) defines consumption net of transactions costs, with total output serving as a measure of the level of transactions at a given point in time. Equation (10) defines income. Equation (11) is the income velocity of money. Equation (12) is the social resources constraint. Equations (13)-(14) are obtained from the first-order conditions for the consumers' decision.

The control variables are the government policy variables, consisting of the nominal rate of return on government bonds,  $i$ , and the level of lump-sum taxes,  $\tau$ . Leeper and Sims (1994) introduced the following monetary and tax policies into the model. The monetary policy rule is

$$\frac{\dot{i}}{i} = a_p \log\left(\frac{P}{\bar{P}}\right) + a_{int} \frac{\dot{P}}{P} + a_i \log\left(\frac{i}{\bar{i}}\right) + a_L \log\left(\frac{L}{\bar{L}}\right) + \varepsilon_i, \quad (15)$$

and the tax policy is

$$\frac{d}{dt} \frac{\tau}{C} = b_\tau \left( \frac{\tau}{C} - \frac{\bar{\tau}}{\bar{C}} \right) + b_L \log\left(\frac{L}{\bar{L}}\right) + b_{inf} \frac{\dot{P}}{P} + b_x \left( \frac{D}{PY} - \frac{\bar{D}}{\bar{P}\bar{Y}} \right) + \varepsilon_\tau. \quad (16)$$

The overscored variables denote steady state values, so that  $\bar{D}/\bar{Y}$  is the steady state debt-to-income level, where income is measured by Leeper and Sims as GNP. The free parameters are  $\bar{D}/\bar{Y}$ , the steady state price level,  $\bar{P}$ , the  $a$ 's, and the  $b$ 's. The disturbance noises are  $\varepsilon_i$  and  $\varepsilon_\tau$ .

In this model, it is conventional to use  $\tau_c = \tau/C$ , rather than  $\tau$ , as a control. Therefore, the control variables are  $i$  and  $\tau_c$ . The parameters and exogenous variables,  $n$ ,  $g$ ,  $\pi$ ,  $\delta$ ,  $\theta$ ,  $\alpha$ ,  $A$ , and  $\phi$ , are specified by Leeper and Sims to follow logarithmic first-order autoregressive (AR) processes in continuous time, while  $\beta$  is

specified to be a logarithmic first-order AR in unlogged form. However, we analyze the structural properties of (1)-(14) without external disturbances. As a result, in equation (3), we set  $\dot{\pi} = 0$  and treat  $\pi$  as a fixed parameter, along with the model's other parameters, that are all treated as fixed. We treat the exogenous variables as realized at their measured values. The extension of our analysis to the case of stochastic bifurcation is a subject for future research.

The original form (1)-(14) has 12 state variables and 14 equations. For analytical investigation, it is best to have as few state variables as possible. For this purpose, we next reduce the dimension of the problem by temporarily eliminating some state variables. We contract to the following 7 state variables

$$\mathbf{x} = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \end{bmatrix}. \quad (17)$$

The remaining state variables can be written as unique functions of  $\mathbf{x}$ .

By eliminating  $M, C^*, V, Q, X$  from the independent state variables, we can determine directly from (1)-(14) that  $\mathbf{x}$  satisfies the following equations.

$$\begin{aligned} & \frac{1}{P}\dot{D} + \frac{Y\sqrt{\phi i}}{P}\dot{P} + (\sqrt{\phi i})\dot{Y} \\ & = Y + \frac{iD}{P} - \left(\frac{Y}{C+g}\right)^{1-\mu} C - \theta \left(\frac{Y}{Z+nK}\right)^{1-\mu} L - \tau_c C + \frac{Y\sqrt{i/\phi}}{2V^2\phi} \dot{i}, \end{aligned} \quad (18)$$

$$\begin{aligned} & (1-\pi(1-\gamma)) \left( \frac{1-\phi V Y^\mu (1-\mu)(C+g)^{-\mu}}{C-\phi V Y^\mu (C+g)^{1-\mu}} - \frac{1-\mu}{C+g} \right) \dot{C} \\ & - \left( \frac{(1-\pi(1-\gamma))\phi V \mu Y^{\mu-1} (C+g)^{1-\mu}}{C-\phi V Y^\mu (C+g)^{1-\mu}} + \frac{1-\mu}{Y} \right) \dot{Y} + \frac{\dot{P}}{P} + \frac{(1-\gamma)(1-\pi)}{1-L} \dot{L} \\ & = i - \beta + \frac{Y^\mu (C+g)^{1-\mu}}{C-\phi V Y^\mu (C+g)^{1-\mu}} \frac{1}{2\sqrt{i\phi}} \dot{i}, \end{aligned} \quad (19)$$

$$\frac{\dot{P}}{P} + (1-\mu) \left( \frac{\dot{Y}}{Y} - \frac{\dot{Z} + n\dot{K}}{Z+nK} \right) = -(1-2\phi V) \frac{a^\sigma \alpha}{\theta} Y^{\mu-\sigma} (Z+nK)^{1-\mu} K^{\sigma-1} + i + \delta, \quad (20)$$

$$\dot{K} = Z - \delta K, \quad (21)$$

$$0 = (C + g)^\mu + \theta(Z + nK)^\mu - Y^\mu, \quad (22)$$

$$0 = \alpha K^\sigma + L^\sigma - a^{-\sigma} Y^{-\sigma}, \quad (23)$$

$$0 = (1 - 2\phi V) \frac{a^\sigma Y^{\mu-\sigma} (C + g)^{1-\mu}}{L^{1-\sigma}} + \frac{1-\pi}{\pi} \frac{\phi V}{1-L} Y^\mu (C + g)^{1-\mu} - \frac{1-\pi}{\pi} \frac{C}{1-L}. \quad (24)$$

For the ease of notation, we denote equations (18)-(24) as

$$\mathbf{h}(\mathbf{x}, \mathbf{u}) \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad (25)$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}), \quad (26)$$

where  $\mathbf{x}$  is the state vector,  $\mathbf{u}$  is the vector of controls,  $\mathbf{h}(\mathbf{x}, \mathbf{u})$  is a matrix having dimension  $4 \times 7$ , and  $\mathbf{f}(\mathbf{x}, \mathbf{u})$  is a  $4 \times 1$  vector. The dimension of the matrix  $\mathbf{g}(\mathbf{x}, \mathbf{u})$  is  $3 \times 7$ . Equation (25) describes the nonlinear dynamical behavior of the model, and (26) represents the algebraic constraints, which are nonlinear. Many systems can be described in the form of (25) and (26). Models in that form are called nonlinear descriptor systems in the mathematical literature on nonlinear dynamics.<sup>7</sup>

We shall use  $m$ ,  $m_1$ ,  $m_2$ , and  $l$  (with  $m = m_1 + m_2$ ), to denote respectively the dimension of  $\mathbf{x}$ , the number of differential equations in (25), the number of algebraic constraints in (26), and the dimension of the vector of control variables,  $\mathbf{u}$ . With the Leeper and Sims model,  $m = 7$ ,  $m_1 = 4$ ,  $m_2 = 3$ , and  $l = 2$ .

The steady state of (25)-(26) can be solved from the following equations:

$$\mathbf{0} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad (27)$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}). \quad (28)$$

We denote the steady states of  $\mathbf{x}$  and  $\mathbf{u}$  by  $\bar{\mathbf{x}}$  and  $\bar{\mathbf{u}}$ , respectively. The vector,  $\bar{\mathbf{u}}$ , is the solution of (25) and (26) in the steady state, when external stochastic disturbances are zero. Hence,

□

<sup>7</sup> The model developed in Kim (2000) is also in that form.

$$\begin{aligned}
\bar{i} &= \beta, \\
\bar{i} &= 0, \\
\bar{\tau}_c &= \frac{\bar{\tau}}{\bar{C}}.
\end{aligned} \tag{29}$$

The values  $\bar{\mathbf{x}}$  and  $\bar{\mathbf{u}}$  are solutions to (27)-(28), and (29). The resulting steady state is the equilibrium of (25)-(26), when the control variables are set at their steady state.

The vector of parameters in the steady state system is

$$\mathbf{p} = [\pi \ \beta \ \theta \ \alpha \ a \ \phi \ \delta \ \mu \ \gamma \ \sigma]',$$

where the prime denotes transpose. Leeper and Sims (1994) estimate the parameters with quarterly data from 1959 to 1992. Although  $g$  is not a parameter of tastes or technology, it is taken as a fixed value by the private sector at its setting by the government.

The constraints on the parameter values and  $g$  are:

$$0 < \pi < 1, \ \gamma > 0, \ 0 \leq \sigma \leq 1, \ \mu \geq 1, \ \delta \geq 0, \ 0 \leq \beta \leq 1, \ \delta > 0, \ g \geq 0. \tag{30}$$

### 3. Singularity Bifurcation in the Leeper and Sims Model

We explore the structural properties of the Leeper and Sims model in a small neighborhood of the steady state,  $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ , by using local linearization around the steady state. The linearized version of the system, (25) - (26), is

$$\mathbf{E}_1 \dot{\mathbf{x}} = \mathbf{A}_1 \mathbf{x} + \mathbf{B}_1 \mathbf{u} \tag{31}$$

$$\mathbf{0} = \mathbf{A}_2 \mathbf{x} + \mathbf{B}_2 \mathbf{u} \tag{32}$$

where

$$\mathbf{E}_1 = \mathbf{h}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{m_1 \times m},$$

$$\mathbf{A}_1 = \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_1 \times m}, \ \mathbf{A}_2 = \frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_2 \times m},$$

$$\mathbf{B}_1 = \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_1 \times m}, \ \mathbf{B}_2 = \frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_2 \times m}.$$

The linearized system, (31)-(32), is solvable if it is regular. Using the relevant regularity condition from Gantmacher (1974), we have the following solvability

condition, which must hold for some values of the determinant's parameter,  $s$ :

$$\det\left(\begin{bmatrix} s\mathbf{E}_1 - \mathbf{A}_1 \\ -\mathbf{A}_2 \end{bmatrix}\right) \neq 0.$$

If that regularity condition is violated for all  $s$ , the linearized system either has multiple solutions or no solution. We randomly chose parameter values within the theoretically feasible region and observed that the Leeper and Sims model, as expected, is regular.

To study the structural properties of the Leeper and Sims model, we further transform the linearized system (31)-(32) into the following form.

**Definition 3.1** Two systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (33)$$

and

$$\tilde{\mathbf{E}}\dot{\mathbf{y}} = \tilde{\mathbf{A}}\mathbf{y} + \tilde{\mathbf{B}}\mathbf{u} \quad (34)$$

are said to be restricted system equivalent (r.s.e.) if there exist two nonsingular matrices,  $\mathbf{T}_1$  and  $\mathbf{T}_2$ , such that

$$\mathbf{T}_1\mathbf{E}\mathbf{T}_2 = \tilde{\mathbf{E}}, \mathbf{T}_1\mathbf{A}\mathbf{T}_2 = \tilde{\mathbf{A}}, \mathbf{T}_1\mathbf{B} = \tilde{\mathbf{B}}, \mathbf{y} = \mathbf{T}_2\mathbf{x}.$$

The form (34) can be obtained by using the coordinate transform,  $\mathbf{y} = \mathbf{T}_2\mathbf{x}$ , and then multiplying both sides of (33) by  $\mathbf{T}_1$  from the left. The relationship of r.s.e. permits transforming a system into a convenient form, while preserving important properties.

We next transform (31)-(32) into a suitable r.s.e. form. First, denote

$$r_E = \text{rank}(\mathbf{E}_1).$$

Then there exist nonsingular matrices  $\mathbf{T}_1$  and  $\mathbf{T}_2$  such that

$$\mathbf{T}_1\mathbf{E}_1\mathbf{T}_2 = \begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Consider the following coordinate transform

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{T}_2^{-1}\mathbf{x}, \mathbf{x}_1 \in \mathbb{R}^{r_E}, \mathbf{x}_2 \in \mathbb{R}^{m_1 - r_E}.$$

Then

$$\mathbf{x} = \mathbf{T}_2 \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.$$

Substituting that form of  $\mathbf{x}$  into (31)-(32) and then multiplying both sides of (31) by  $\mathbf{T}_1$ , we find that (31)-(32) is r.s.e. to

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 + \mathbf{B}_{11}\mathbf{u} \\ \mathbf{0} &= \mathbf{A}'_{21}\mathbf{x}_1 + \mathbf{A}'_{22}\mathbf{x}_2 + \mathbf{B}'_{12}\mathbf{u} \\ \mathbf{0} &= \mathbf{A}_{31}\mathbf{x}_1 + \mathbf{A}_{32}\mathbf{x}_2 + \mathbf{B}_2\mathbf{u} \end{aligned} \quad (35)$$

where

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}'_{21} & \mathbf{A}'_{22} \end{bmatrix} = \mathbf{T}_1 \mathbf{A}_1 \mathbf{T}_2, \quad \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}'_{12} \end{bmatrix} = \mathbf{T}_1 \mathbf{B}_1, \quad \begin{bmatrix} \mathbf{A}_{31} & \mathbf{A}_{32} \end{bmatrix} = \mathbf{A}_2 \mathbf{T}_2.$$

Combining the second and the third equations in (35), we have

$$\dot{\mathbf{x}}_1 = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 + \mathbf{B}_{11}\mathbf{u}, \quad (36)$$

$$\mathbf{0} = \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 + \mathbf{B}_{12}\mathbf{u}, \quad (37)$$

where

$$\mathbf{A}_{21} = \begin{bmatrix} \mathbf{A}'_{21} \\ \mathbf{A}_{31} \end{bmatrix}, \quad \mathbf{A}_{22} = \begin{bmatrix} \mathbf{A}'_{22} \\ \mathbf{A}_{32} \end{bmatrix}, \quad \mathbf{B}_{12} = \begin{bmatrix} \mathbf{B}'_{12} \\ \mathbf{B}_2 \end{bmatrix}.$$

If  $\mathbf{A}_{22}$  is nonsingular (or invertible), it is possible to solve for  $\mathbf{x}_2$  from the algebraic constraint equation (37). In fact, in this case, we have

$$\mathbf{x}_2 = -\mathbf{A}_{22}^{-1}(\mathbf{A}_{21}\mathbf{x}_1 + \mathbf{B}_{12}\mathbf{u}).$$

Substituting the form of  $\mathbf{x}_2$  into (36), we obtain

$$\dot{\mathbf{x}}_1 = (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})\mathbf{x}_1 + (\mathbf{B}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{B}_{12})\mathbf{u}.$$

Hence,  $\mathbf{x}_1$  can be described by a system of ordinary differential equations and an algebraic relationship between  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .

However, the previously described transformation would not be possible, if  $\mathbf{A}_{22}$  were singular and thereby not invertible. In fact, as explained below, the dynamics of (25)-(26) could be dramatically different from those of ordinary linear differential equations, if  $\mathbf{A}_{22}$  were singular.

To see what could happen when  $\mathbf{A}_{22}$  is singular, we now revisit the linearized system (36)-(37), which can be re-written as

$$\begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix} \mathbf{u}. \quad (38)$$

If the Leeper and Sims model is regular, so is the matrix pair

$$\left( \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \right),$$

which is in the form of a matrix pencil.

For a regular matrix pencil, there exist nonsingular matrices  $\tilde{\mathbf{T}}_1$  and  $\tilde{\mathbf{T}}_2$  such that<sup>8</sup>

$$\tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{I}_{\tilde{m}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \text{ and } \tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tilde{\mathbf{T}}_2 = \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\tilde{m}_2} \end{bmatrix},$$

where  $\tilde{m}_1 + \tilde{m}_2 = m$  and  $\mathbf{N}$  is a nilpotent matrix. By the definition of nilpotent matrix, there exists a positive integer  $d > 1$  such that

$$\mathbf{N}^d = \mathbf{0}.$$

The smallest such integer  $d$  is called the nilpotent index of  $\mathbf{N}$ .

The following is an example of a nilpotent matrix:

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \dots & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (39)$$

A matrix is nilpotent if and only if it is similar to the following block diagonal matrix

$$\text{diag}(\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_p),$$

in which each  $\mathbf{N}_i$  has the form of (39).<sup>9</sup>

Consider the coordinate transform

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \tilde{\mathbf{T}}_2 \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \text{ or equivalently } \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \tilde{\mathbf{T}}_2^{-1} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.$$

Multiplying both sides of (38) by  $\tilde{\mathbf{T}}_1$ , we have another r.s.e. form of (31)-(32),

$$\dot{\mathbf{y}}_1 = \tilde{\mathbf{A}}_1 \mathbf{y}_1 + \tilde{\mathbf{B}}_1 \mathbf{u}, \quad (40)$$

□

<sup>8</sup> See Gantmacher (1974).

<sup>9</sup> Two matrices,  $\mathbf{A}$  and  $\mathbf{B}$ , are said to be similar, if there exists an invertible matrix,  $\mathbf{T}$ , such that

$$\mathbf{N}\dot{\mathbf{y}}_2 = \mathbf{y}_2 + \tilde{\mathbf{B}}_2 \mathbf{u}, \quad (41)$$

where

$$\begin{bmatrix} \tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{B}}_2 \end{bmatrix} = \tilde{\mathbf{T}}_1^{-1} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix}.$$

The solutions to (40) and (41) are respectively,

$$\mathbf{y}_1 = e^{\tilde{\mathbf{A}}_1(t-t_0)} \mathbf{y}_1(0) + \int_{t_0}^t e^{\tilde{\mathbf{A}}_1(t-\xi)} \tilde{\mathbf{B}}_1 \mathbf{u}(\xi) d\xi$$

$$\mathbf{y}_2 = - \sum_{k=1}^{d-1} \delta^{(k-1)}(t) \mathbf{N}^k \mathbf{y}_2(0) - \sum_{k=0}^{d-1} \mathbf{N}^k \tilde{\mathbf{B}}_{12} \mathbf{u}^{(k)}(t),$$

where  $t_0 \geq 0$  is the initial time,  $\delta^{(k-1)}(t)$  is the derivative of order  $k-1$  of the Dirac delta function, and  $\mathbf{u}^{(k)}$  denotes the  $k$ -th order derivative of  $\mathbf{u}$ .<sup>10</sup>

Unless  $\mathbf{N} = \mathbf{0}$  or the initial state  $\mathbf{y}_2(0) = \mathbf{0}$ , there exist the impulse terms in the first summation in the solution for  $\mathbf{y}_2$ , as well as the smooth derivative terms of  $\mathbf{u}$  in the second summation. In fact when  $\mathbf{N} = \mathbf{0}$ , the above solution for  $\mathbf{y}_2$  does not apply, although the solution for  $\mathbf{y}_1$  above remains valid. This solution structure with nonzero  $\mathbf{N}$  is very different from that of ordinary differential equations, such as (40) for  $\mathbf{y}_1$ .

The first summation in the solution for  $\mathbf{y}_2$  could produce shock effects to the state response of  $\mathbf{y}_2$ . In fact, the Dirac delta, which is  $\delta^{(k-1)}$  when  $k=1$ , is often called the unit impulse function.<sup>11</sup> But if  $\mathbf{N} = \mathbf{0}$ , we have from (41) that

$$\mathbf{y}_2 = -\tilde{\mathbf{B}}_2 \mathbf{u},$$

which is a smooth algebraic relationship between  $\mathbf{y}_2$  and  $\mathbf{u}$ . This bifurcation

$\mathbf{A} = \mathbf{T}^{-1} \mathbf{B} \mathbf{T}$ .

<sup>10</sup> We use  $e$  raised to a matrix power to designate the matrix of  $e$  to the power of each element of the matrix power. Regarding the form of the solutions to (40) and (41), see Cobb (1982, 1983). The discrete analog of the delta function is the Kronecker delta.

<sup>11</sup> The Lebesgue integral of the Dirac delta function from minus infinity to plus infinity is 1.0. Formally the Dirac delta,  $\delta(t)$ , is not a function but the limit of a sequence of functions (the nascent delta functions). In that limit, the Dirac delta is a measure with unit mass at the origin and is often called the unit impulse function. The antiderivative of the Dirac delta is the Heaviside (unit) step function, so that the Dirac delta can be viewed as the derivative of the step function. Since the Dirac delta is a measure, its derivatives require careful definition. Those derivatives are higher order "singularity functions" called "doublets," "triplets," etc. It can be shown that the  $n$ 'th derivative of  $\delta(t)$  is  $\delta^{(n)} = (-1)^n n! \delta(t)/t^n$ . Note that if  $\delta(x)$  is the unit impulse at  $t=0$ , then  $\delta^{(n)}(t)$  is a rescaled impulse at  $t=0$ .



phenomenon at  $\mathbf{N} = \mathbf{0}$  is consistent with the following theorem.

**Theorem 3.1.** If both (40)-(41) and (36)-(37) are r.s.e. forms of the same linearized system (31)-(32), then

$$\mathbf{N} = \mathbf{0},$$

if and only if  $\mathbf{A}_{22}$  is nonsingular, i.e.,  $\det(\mathbf{A}_{22}) \neq 0$ .

**Proof.** If  $\mathbf{N} = \mathbf{0}$ , then (40)-(41) and (36)-(37) have the same form with  $\mathbf{A}_{22} = \mathbf{I}_{\tilde{m}_2}$ , which is nonsingular.

Converseley, assume  $\mathbf{A}_{22}$  is nonsingular. Then choose

$$\tilde{\mathbf{T}}_1 = \begin{bmatrix} \mathbf{I}_{\tilde{m}_1} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0} & \mathbf{I}_{\tilde{m}_2} \end{bmatrix}, \quad \tilde{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0} & \mathbf{I}_{\tilde{m}_2} \end{bmatrix}. \quad (42)$$

Direct verification confirms that

$$\tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{I}_{\tilde{m}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tilde{\mathbf{T}}_2 = \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_2} \end{bmatrix}$$

with

$$\tilde{\mathbf{A}}_1 = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}.$$

Therefore, we have  $\mathbf{N} = \mathbf{0}$ . This completes the proof.  $\square$

With the linearized model, (31)-(32), singularity of  $\mathbf{A}_{22}$  results in completely different dynamical solution behavior. As a result, we say a singularity bifurcation occurs, when

$$\det(\mathbf{A}_{22}) = 0. \quad (43)$$

The preceding condition has another form in terms of the original coefficient matrices, as shown in the following theorem.

**Theorem 3.2** Assume that  $\mathbf{E}_1$  has full row rank, so that

$$\text{rank}(\mathbf{E}_1) = m_1. \quad (44)$$

Then  $\mathbf{A}_{22}$  is nonsingular if and only if

$$\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}$$

is nonsingular, so that

$$\text{rank}\left(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}\right) = m. \quad (45)$$

**Proof.** Denote

$$\hat{\mathbf{T}}_1 = \begin{bmatrix} \tilde{\mathbf{T}}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}}_1 \end{bmatrix}, \quad \hat{\mathbf{T}}_2 = \begin{bmatrix} \tilde{\mathbf{T}}_2 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}}_2 \end{bmatrix},$$

where  $\tilde{\mathbf{T}}_1$  and  $\tilde{\mathbf{T}}_2$  are defined as in (42). Then both  $\tilde{\mathbf{T}}_1$  and  $\tilde{\mathbf{T}}_2$  are non-singular.

Consider the following matrix

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{E}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then we have

$$\hat{\mathbf{T}}_1 \mathbf{\Lambda} \hat{\mathbf{T}}_2 = \begin{bmatrix} \tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}_2 & \tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \tilde{\mathbf{T}}_2 \\ \mathbf{0} & \tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\tilde{m}_1} & \mathbf{0} & \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{21} & \mathbf{A}_{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{\tilde{m}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\text{rank}(\hat{\mathbf{T}}_1 \mathbf{\Lambda} \hat{\mathbf{T}}_2) = 2\tilde{m}_1 + \text{rank}(\mathbf{A}_{22}). \quad (46)$$

But if  $\mathbf{E}_1$  has full row rank,  $\tilde{m} - 1 = m_1$ , then

$$\text{rank}(\mathbf{E}_1) = \tilde{m}_1 = m_1,$$

and

$$\begin{aligned} \text{rank}(\mathbf{\Lambda}) &= \text{rank}\left(\begin{bmatrix} \mathbf{E}_1 & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{E}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right) = \text{rank}(\mathbf{E}_1) + \text{rank}\left(\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{E}_1 \end{bmatrix}\right) \\ &= m_1 + \text{rank}\left(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}\right). \end{aligned}$$

Combining the previous equation with (42), we obtain

$$\text{rank}\left(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}\right) = m_1 + \text{rank}(\mathbf{A}_{22}). \quad (47)$$

Note that  $\mathbf{A}_{22} \in \mathbb{R}^{\tilde{m}_2 \times \tilde{m}_2}$  and  $\tilde{m}_2 = m_2$ . Hence equation (47) says that  $\mathbf{A}_{22}$  is nonsingular if and only

$$\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}$$

is nonsingular. □

Therefore, the following condition for singularity bifurcation is provided by Theorem 3.2:

$$\det\left(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}\right) = 0. \quad (48)$$

Note that  $\mathbf{x}_2$  is solvable from (37) alone if  $\mathbf{A}_{22}$  is nonsingular. Therefore, singularity condition implies the case in which  $\mathbf{x}_{22}$  is not readily solvable from the algebraic (37) alone. We need to take into account of the dynamic constraint (36).

We next introduce another property to have a closer look at the singularity condition.

**Corollary 3.1.** Consider the following system

$$\begin{aligned} \mathbf{E}_1 \dot{\mathbf{x}} + \mathbf{E}_{12} \dot{\mathbf{y}} &= \mathbf{A}_1 \mathbf{x} + \mathbf{A}_{12} \mathbf{y} + \mathbf{B}_1 \mathbf{u} \\ \dot{\mathbf{y}} &= \mathbf{A}_y \mathbf{y} + \mathbf{B}_y \mathbf{u} \\ \mathbf{0} &= \mathbf{A}_2 \mathbf{x} + \mathbf{A}_{22} \mathbf{y} + \mathbf{B}_2 \mathbf{u} \end{aligned}, \quad (49)$$

where  $\mathbf{A}_y$  is an arbitrary matrix of dimension  $m \times m$ ,  $\mathbf{B}_y$  is an arbitrary matrix of dimension  $m \times l$ , and the other matrices are as defined above. Then the singularity condition for (49) is the same as that for (31)-(32).

**Proof.** According to Theorem 4.2, the singularity condition for (49) is

$$\det\left(\begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_{12} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{A}_2 & \mathbf{A}_{22} \end{bmatrix}\right) = 0.$$

By eliminating the second column, that determinant condition is equivalent to (48),

which is the singularity condition for (31)-(32).  $\square$

Corollary 3.1 says that adding (or deleting) state variables that can be modeled by ordinary differential equations does not change the singularity condition. This property is useful in reducing the dimension of the problem under consideration. For example, we could drop the Leeper and Sims' model's state variable  $K$  from its state vector, (17), in the system (31)-(32), without affecting the singularity condition.

By thereby dropping the state variable  $K$ , the singularity condition becomes

$$\det\left(\begin{bmatrix} \mathbf{E}'_1 \\ \mathbf{A}'_2 \end{bmatrix}\right) = 0, \quad (50)$$

in which

$$\mathbf{E}'_1 = \begin{bmatrix} \frac{1}{P} & \frac{Y}{PV} & 0 & 0 & 0 & \frac{1}{V} \\ 0 & \frac{1}{P} & e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & 0 & e_{26} \\ 0 & \frac{1}{P} & 0 & 0 & -\frac{1-\mu}{Z+nK} & \frac{1-\mu}{Y} \end{bmatrix}$$

and

$$\mathbf{A}'_2 = \begin{bmatrix} 0 & 0 & \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & \mu Y^{\mu-1} \\ 0 & 0 & a_{23} & a_{24} & 0 & a_{26} \\ 0 & 0 & 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma} \sigma Y^{\sigma-1} \end{bmatrix},$$

where

$$e_{23} = \frac{1-\pi(1-\gamma)}{C^*} [1-\phi V Y^{\mu} (\mu-1)(C+g)^{\mu-2}] - \frac{1-\mu}{C+g},$$

$$e_{26} = \frac{1-\pi(1-\gamma)}{C^*} [-\phi V Y^{\mu} \mu (C+g)^{\mu-1}] + \frac{1-\mu}{Y},$$

$$a_{23} = (1-2\phi V) A^{\sigma} Y^{\mu-\sigma} L^{\sigma-1} (1-\mu)(C+g)^{-\mu} - \frac{1-\pi}{\pi} \frac{1}{1-L},$$

$$a_{24} = (1-2\phi V) A^{\sigma} Y^{\mu-\sigma} (\sigma-1) L^{\sigma-2} (C+g)^{1-\mu} - \frac{1-\pi}{\pi} \frac{C}{(1-L)^2},$$

$$a_{26} = (1 - 2\phi V)A^\sigma (\mu - \sigma)Y^{\mu - \sigma - 1}L^{\sigma - 1}(C + g)^{1 - \mu}.$$

Direct calculation shows that (50) is equivalent to

$$\det \begin{pmatrix} e_{23} & \frac{(1 - \gamma)(1 - \pi)}{1 - L} & \frac{1 - \mu}{Z + nK} & e'_{26} \\ \mu(C + g)^{\mu - 1} & 0 & \theta\mu(Z + nK)^{\mu - 1} & -\mu Y^{\mu - 1} \\ a_{23} & a_{24} & 0 & a_{26} \\ 0 & \sigma L^{\sigma - 1} & 0 & A^{-\sigma} \sigma Y^{\sigma - 1} \end{pmatrix} \quad (51)$$

where

$$e'_{26} = \frac{1 - \pi(1 - \gamma)}{C^*} [-\phi V Y^\mu \mu (C + g)^{\mu - 1}].$$

As we shall demonstrate later, singularity does occur within the theoretically feasible parameter regions.

In systems theory, bifurcation is said to occur if change of structural dynamic solution properties occurs, when a parameter crosses a certain value. Such a critical value is called a bifurcation point. Many types of bifurcation are known, such as saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation. Bifurcation analysis is particularly useful in locating subsets of the parameter space supporting various dynamical behaviors of a system, such as existence of limit cycles, multiperiodic instability, monotonic stability, or damped stability.

We find that the Leeper and Sims model has structural changes in its dynamics, and the boundary determined by (51) is a singularity-induced bifurcation boundary. To the best of our knowledge, this is the first time that this type of bifurcation has been found in a macroeconomic model.

Leeper and Sims (1994) proposed government policy control using the monetary policy (15) and the tax policy (16). To investigate bifurcation of the closed-loop system under the control of government policies, let us expand the state variable to

$$\mathbf{x}_c = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \\ i \\ \tau_c \end{bmatrix}. \quad (52)$$

With this new state variable, the linearized system (31)-(32) becomes

$$\mathbf{E}_1^c \dot{\mathbf{x}}_c = \mathbf{A}_1^c \mathbf{x}_c, \quad (53)$$

$$\mathbf{0} = [\mathbf{A}_2 \quad \mathbf{0}] \mathbf{x}_c, \quad (54)$$

where  $\mathbf{E}_1^c \in \mathbb{R}^{m_1^c \times m^c}$ ,  $\mathbf{A}_1^c \in \mathbb{R}^{m_1^c \times m^c}$ ,  $m_1^c = m_1 + 2$ , and  $m^c = m + 2$ .

#### 4. Numerical Results

In this section, we numerically locate the singularity-induced bifurcation boundaries. We use the condition (51) applied to the closed-loop system (54).<sup>12</sup>

We first test all pairs of parameters to determine those pairs that reach bifurcation boundaries, when varied with all other parameters set at their point estimates.<sup>13</sup> Pairs of parameters permitted to vary about their point estimates are allowed to take values within the intersection of their theoretically feasible ranges and their 95% confidence intervals of their estimated values. In particular, the intersection,  $\mathcal{H}$ , of (30) and

$$p(i) \in [\bar{p}(i) - \bar{c}\sigma_i, \bar{p}(i) + \bar{c}\sigma_i]$$

where  $\bar{p}(i)$  is the estimated value of parameter  $p(i)$ ,  $\sigma_i$  is the standard error of the estimate, and  $\bar{c}$  is the critical value of the 95th-percentile confidence interval for  $N(0,1)$ .<sup>14</sup>

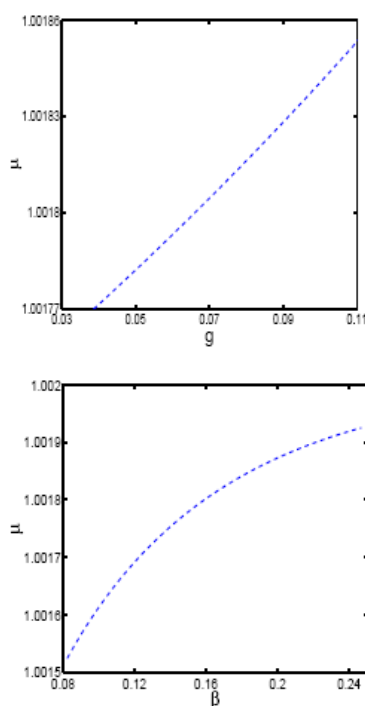
□

<sup>12</sup> Regarding numerical stability, we limited our computations to the theoretical procedure mentioned earlier. We did not use additional algorithms to check for numerical stability. But since we encountered no stability problems with MatLab software, we had no indication of the need for redundant checks of numerical stability.

<sup>13</sup> Hyperplanes along which only two parameters vary can fail to intersect bifurcation boundaries, even if they exist at other settings of some parameters.

<sup>14</sup> For some parameters, standard errors are not provided in Leeper and Sims (1994). In such unfortunate cases, we permitted parameter values to take values within 50% of the point estimates. Such a range of parameter values keep the parameters well within the theoretically feasible region.

Figures 4.1 and 4.1 show some of the sections of the singularity-induced bifurcation boundary that we located. Figure 4.1 displays 2-dimensional sections with the other parameters set at their point estimates, while figure 4.2 displays 3-dimensional sections with the other parameters set at their point estimates.<sup>15</sup> In the first section of figure 4.2, we display a section varying  $\mu$  and  $g$ , while in the second section, we display  $\mu$  versus  $\beta$ . The range of the plots' axes are within the  $\mathcal{H}$  intervals about each parameter's estimate. Table 1 provides the point estimates, standard errors, and  $\mathcal{H}$  intervals used in producing figures 4.1 and 4.2.<sup>16</sup>



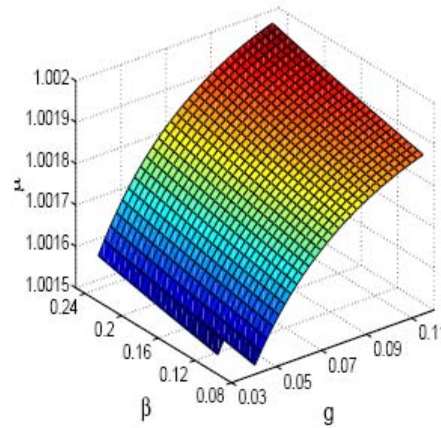
**Figure 4.1. Two-dimensional sections of a singularity-induced bifurcation boundary.**

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Another complication is produced by the fact that Leeper and Sims did not report covariances of parameter estimators. Hence, in our three dimensional searches we do not have 3-dimensional confidence regions, but rather use the Cartesian products of the pairwise confidence intervals.

<sup>15</sup> Since the parameter space is a high dimensional space, we investigated many sections of the space within  $\mathcal{H}$ . We display only the sections that we found to be particularly informative about the location and nature of the singularity boundary.

<sup>16</sup> While  $\beta$  and  $\mu$  are parameters,  $g$  is an exogenous variable. What we display as “estimate” and “standard error” for  $g$  is the sample mean and standard deviation of  $g$ .



**Figure 4.2. Three-dimensional sections of a singularity-induced bifurcation boundary.**

The estimation information for the parameters  $\mu$ ,  $g$ , and  $\beta$  used in figures 4.1 and 4.2 are in Table 1. All estimation information is taken directly from the Leeper and Sims paper. We make no changes in their models, in their reported point estimates, or their reported standard errors. Our experiments are conditional upon what Leeper and Sims have published, without modification.

**Table 1. Estimation of  $\mu$ ,  $g$ , and  $\beta$**

parameter	estimate	standard error	$\mathcal{H}$ interval
$\mu$	1.0248	0.324	[1, 1.6598]
$g$	0.0773*	0.292*	[0, 0.6496]
$\beta$	0.1645	0.288	[0, 0.7290]

\*Since  $g$  is an exogenous variable, rather than a parameter, the “estimate” is the sample mean and the “standard error” is the sample standard deviation.

To illustrate what happens when parameter values cross the singularity boundary, consider the parameter  $\beta$ . Table 2 displays the changes of finite eigenvalues,  $\lambda_1, \dots, \lambda_8$ , when  $\beta$  varies.



**Table 2. Eigenvalue changes**

$\beta$	0.080	0.120	0.160	0.165	0.170	0.200	0.240
$\lambda_1$	1.002	1.002	1.002	1.002	1.002	1.002	1.002
$\lambda_2$	0.080	0.120	0.160	0.165	0.170	0.200	0.240
$\lambda_3$	-0.303	-0.262	-0.220	-0.215	-0.210	-0.178	-0.135
$\lambda_4$	-3.558	-3.559	-3.561	-3.561	-3.561	-3.563	-3.566
$\lambda_5$	-0.098	-0.084	-0.077	-0.076	-0.075	-0.072	-0.069
$\lambda_6$	-0.002	-0.003	-0.003	-0.003	-0.003	-0.004	-0.004
$\lambda_7$	3.101	5.177	8.237	8.682	9.254	13.416	28.401
$\lambda_8$	-117.790	-204.703	-1811.413	$\infty$	1456.294	195.888	58.059
note			boundary				

The first row in Table 2 are settings of  $\beta$  that we explore. The second through the ninth rows are the corresponding finite eigenvalues of the linearized model at each setting of  $\beta$ . There are three more eigenvalues, which are not shown in the table. Those eigenvalues are infinite. The table shows that when the value of  $\beta$  increases and crosses the bifurcation boundary,  $\lambda_8$  decreases rapidly to  $-\infty$ , spikes suddenly from  $-\infty$  to  $+\infty$ , and then decreases from  $+\infty$ .

Table 2 clearly shows that the Leeper and Sims model has a structural change in dynamics, when  $\beta$  crosses the singularity-induced bifurcation boundary. The two regions separated by the boundary exhibits drastically different dynamical behaviors. Also note the very small range of values of  $\mu$  displayed along its axis in figures 4.1 and 4.2. That displayed range consists of a small subset of values within the interval  $\mathcal{H}$ . As a result, very small changes in  $\mu$  can cause bifurcation, independently of the settings of  $g$  or  $\beta$ .

As shown by Table 2, such singularity bifurcations can have dramatic effects. The number of dynamic equations and the number of algebraic equations change, when the singularity-induced bifurcation boundary is crossed.

## 5. Conclusions

The Leeper and Sims Euler equations macroeconomic model is representative of a larger class of systems, designed to address the Lucas critique. The most distinguishing characteristic of this class of system is the models' form,

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

in which the matrix  $\mathbf{E}$  could become singular at some settings of the parameters. In this paper, we have examined the basic properties of such model, the important Leeper and Sims model, proposed an approach for bifurcation analysis of such models, and most importantly discovered the existence of singularity-induced bifurcations for the first time in a macroeconometric model.

Within a small region of the estimated parameter values, we locate and characterized the nature of the singularity-induced bifurcation. The dynamic order of the system changes in a dramatic manner, when parameter values cross the bifurcation boundary. In a theoretical survey paper of types of bifurcation, Barnett and He (2006) have argued that singularity bifurcation may not be unusual in Euler equations models; and we have, in the current paper, illustrated that theoretical speculation in a well known model in that class.

With the policy-relevant Bergstrom and Wymer Keynesian second-order differential-equations macroeconometric model of the UK economy, Barnett and He (1998,1999, 2001, 2002) found three types of bifurcation boundaries within the parameters' confidence regions. Subsequently with New Keynesian models, Barnett and Duzhak (2006) are finding Hopf bifurcation boundaries. Now in the current paper, we have found a particularly dramatic type of bifurcation with an Euler equations, having deep parameters that are invariant to policy rule changes and thereby immune to the Lucas critique.

In all of these studies, the models used are highly policy relevant and were not modified from their influential previously-published forms. While Grandmont's model has been criticized for its lack of policy relevance, we believe from the accumulating evidence that Grandmont's conclusions are correct and are highly relevant to policy. In particular, these results cast into doubt the dynamical inferences acquired in the traditional manner, by simulating macroeconometric models at their parameter point estimates. To be able to achieve robustness of dynamical inferences, such simulations must be made at various settings throughout the confidence region about the parameters' point estimates.

## References

Bala V. A pitchfork bifurcation in the tatonement process. *Economic Theory* 1997; 10; 521-530.

Barnett WA., Chen P. The aggregation theoretic monetary aggregates are chaotic and have strange attractors: an econometric application of mathematical chaos. In: Barnett WA, Berndt ER, White H (Eds.), *Dynamic econometric modeling*. Cambridge University Press: Cambridge, UK; 1988. p.199-246.

Barnett WA and Duzhak EA. Non-robust dynamic inferences from macroeconomic models: bifurcation of confidence regions. University of Kansas Working Paper No. 200608; 2006.

Barnett WA., He Y. Analysis and control of bifurcations in continuous time macroeconomic systems. *Proceedings of the 37th IEEE Conference on Decision and Control*. December 16-18, 1998; Tampa, Florida; p. 2455-2460.

Barnett WA., He Y. Stability analysis of continuous time macroeconomic systems. *Studies in Nonlinear Dynamics and Econometrics* 1999; 3; 169-188.

Barnett WA, He Y. Nonlinearity, chaos, and bifurcation: a competition and an experiment. In: Negishi T, Ramachandran R, Mino K (Eds.), *Economic theory, dynamics and markets: essays in honor of Ryuzo Sato*. Kluwer Academic Publishers: Amsterdam; 2001. p. 167-187.

Barnett WA., He Y. Stabilization policy as bifurcation selection: would stabilization policy work if the economy really were unstable? *Macroeconomic Dynamics* 2002; 6; 713-747.

Barnett WA., He Y. Bifurcations in macroeconomic models. In: Dowrick S, Rohan P, Turnovsky S (Eds), *Economic growth and macroeconomic dynamics: recent developments in economic theory*. Cambridge University Press: Cambridge, UK; 2004. p. 95-112.

Barnett W., He Y. Singularity bifurcations. *Journal of Macroeconomics* 2006; 28; 5-22.

Barnett WA., Deissenberg C, Feichtinger G. Economic complexity: non-linear dynamics, multi-agents economies, and learning. North Holland: Amsterdam; 2004.

Barnett, WA., Geweke J, Shell K. Economic complexity: chaos, sunspots, bubbles, and nonlinearity. Cambridge University Press: Cambridge, UK; 2005.

Benhabib, J. The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth. *Journal of Economic Theory* 1979; 21; 421-444.

Bergstrom AR Survey of continuous time econometrics. In: Barnett WA, Gandolfo G, Hillinger C. (Eds.), *Dynamic disequilibrium modeling*. Cambridge University Press: Cambridge, UK; 1996. p. 3-26.

Bergstrom AR, Nowman KB. A continuous time econometric model of the United Kingdom with stochastic trends. Cambridge University Press: Cambridge, UK; 2006.

Bergstrom AR, Wymer CR. A model of disequilibrium neoclassic growth and its application to the United Kingdom. In: Bergstrom AR (Ed.), *Statistical inference in continuous time economic models*. North Holland: Amsterdam; 1976, p. 267-327.

Bergstrom AR, Nowman KB, Wandasiewicz S. Monetary and fiscal policy in a second-order continuous time macroeconomic model of the United Kingdom. *Journal of Economic Dynamics and Control* 1994; 18; 731-761.

Bergstrom AR., Nowman KB., Wymer CR. Gaussian estimation of a second order continuous time macroeconomic model of the United Kingdom. *Economic Modelling* 1992; 9; 313-352.

Binder M, Pesaran MH. Stochastic growth models and their econometric implications. *Journal of Economic Growth* 1999; 4; 139-183.

Boldrin M and Woodford M. Equilibrium models displaying endogenous fluctuations and chaos: a survey. *Journal of Monetary Economics* 1990; 25; 189-222.

Cobb JD. On the solutions of linear differential equations with singular coefficients. *Journal of Differential Equations* 1982; 46; 310-323.

Cobb JD. A further interpretation of inconsistent initial conditions in descriptor variable systems. *IEEE Transactions on Automatic Control* 1983; 28; 920-922.

Gandolfo G. *Economic Dynamics*. Springer: New York; 1996

Gantmacher FR. *The Theory of Matrices*. Chelsea: New York; 1974.

Grandmont JM. Expectations formation and stability of large socioeconomic systems. *Econometrica* 1998; 66; 741-782.

Kim J. Constructing and estimating a realistic optimizing model of monetary policy. *Journal of Monetary Economics* 2000; 45; 329-359.

Leeper E, Sims C. Toward a modern macro model usable for policy analysis. *NBER Macroeconomics Annual* 1994; 81-117.

Lucas RE. Econometric policy evaluation: A critique. In Brunner K, Meltzer AH (Eds.), *The Phillips curve and labor markets*. *Journal of Monetary Economics*, supplement; 1976. p. 19-46.

Medio A. *Chaotic dynamics: theory and applications to economics*. Cambridge University Press: Cambridge, UK; 1992.

Nieuwenhuis HJ, Schoonbeek L. Stability and the structure of continuous-time economic models. *Economic Modelling* 1997; 14; 311-340.

Nishimura K, Takahashi H. Factor intensity and Hopf bifurcations. In Feichtinger G (Eds.), *Dynamic Economic Models and Optimal Control*; 1992. p. 135-149.

Powell A, Murphy C. Inside a modern macroeconomic model: a guide to the Murphy model. Springer: Heidelberg; 1997

Scarf H. Some examples of global instability of competitive equilibrium. International Economic Review 1960; 1; 157-172.

Wymer CR. Structural nonlinear continuous time models in econometrics. Macroeconomic Dynamics 1997; 1; 518-548.