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Abstract

The cost of operational risk refers to the capital needed to afford the loss generated by ordinary activities of a firm. In this work we demonstrate how allocation principles can be used to the subdivision of the aggregate capital so that the firm can distribute this cost across its various constituents that generate operational risk. Several capital allocation principles are revised. Proportional allocation allows to calculate a relative risk premium to be charged to each unit. An example of fraud risk in the banking sector is presented and some correlation scenarios between business lines are compared.

Keywords: solvency, quantile, value at risk, copulas

1. Introduction and Motivation

Risk management in business is about anticipating the potential losses that can occur in a firm and to design methods that can either mitigate them or compensate them. It is a field of intense research given that security and protection is an essential part of quality control.

In ordinary business operations, there are risks of malfunctioning that are almost inevitable and that create a constant burden to the expected profits by substantially reducing those. These risks are called operational because they arise naturally in everyday business activities. They include, software failures, electricity cuts, human mistakes, internal and external fraud, among others. Expected

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operational losses can be accounted for as a fixed cost component of production, but holding a capital to be able to pay for the unexpected operational losses is necessary to respond to exceptional operational risk events that exceed the routine. We will address the cost of operational risk and to what proportion a every single produced unit should contribute to the total capital held for operational risk purposes. A constant allocation would mean that the total capital is divided by the number of product units in spite of the contribution of that unit to the aggregate operational risk. A proportional allocation would increase the contribution of those units whose production creates more risk than the others compared to the average.

We will examine an example in the context of fraud in banking. Our illustration is inspired in a typical simplified situation where a bank has only two lines of business, for instance credit cards and savings accounts. Losses due to fraud arise in these two business services and are an area of research for improving business performance [26, 2, 4, 5]. Managers can predict the annual average loss due to fraud in credit cards and savings accounts independently and include this expected loss as part of the general managing expenses of credit cards and savings accounts, respectively. Similar applications have been disussed in the context of automobile insurance before [33]. However, some additional capital must be held as a results of risk exposure due to fraud in any of the two lines and there are several ways to decide how much capital should be provided by the credit card business and how much from the savings account business. Moreover, assuming independence between lines of business is unrealistic. It is well known that fraud propensity fluctuates with exogenous factors that create spurious correlation between business units [32]. Factors such as economic recession, social networking where people share information about the *modus operandi* of successful fraud attempts and periods during the year when consumers are more prone to defraud affect all business lines at the same time (see, for instance Caudill et al. [15]). We will address how to cope with dependence between fraud risk in this two dimensional setting, here we consider fraud in credit cards and in savings accounts.

In general, companies wish to allocate capital to their business units for solvency reasons. Moreover, banks and insurance companies are legally required to set aside some amount of capital in order to remain solvent and they wish to associate the capital, and therefore the loss of returns, to every single unit as a price loading, also called a *risk premium*.

The mere existence of operational risk recommends that firms keep some capital, unless they prefer to purchase an insurance policy to cover operations failures, in which case instead of capital they need to pay for an insurance premium, which in our terms is an equivalent problem (see, Guillen et al. [21]).

Capital allocation of operational risk cost can be a useful tool and an indicator for performance measurement [7, 9]. Designing incentives schemes as managers'

performance can be assessed by the amount of capital allocated to their business units, which is an indicator of operational risk. Profit-and-loss analysis under loan pricing context and under general investment purposes are another reasons that motivate companies to carry out capital allocations.

Note that capital allocation, namely the contribution to risk of every unit, is the purpose of this work and we do not attempt going into details on how to determine the sum of economic capital to be allocated. We assume this capital is known and given, we are describing a way to determine the optimal proportions of this given capital for allocating them among different risk sources of the enterprise. The main problem to be solved is the so-called *allocation problem*. Based on the general framework proposed by [16] we provide explicit formulations for the proportion of capitals the manager should allocate on different risk sources based on a wide variety of risk measures.

We provide an exact functional forms of each allocation principle and also paying carefully attention to the numerical part, we analyze the *"correlation effect"* on the allocation principles. *Correlation effect* is considered to be the effect of changes in the allocated capital suggested by each principle when changing the correlation between the losses. We argue that correlations exist in practice [18, 29, 11, 12]. Our findings suggest that *correlation effect* exists.

The remainder of this article is arranged as follows. Section 2 discusses formally what the allocation problem is. Allocation principles are presented in Section 3 while the general framework for capital allocation, based on [16], is discussed in Section 4. An application is on fraud reported in Section 5. Some concluding remarks are in Section 6.

2. The General Capital Allocation Problem

Capital Allocation is a term referring to the subdivision of the aggregate capital held by the firm across its various constituents, for example, business lines, type of exposure, territories, or even individual products in a portfolio of insurance policies. This capital is often referred to as Economic Capital (EC) and is defined as the p-quantile of the loss distribution minus the expected value of the of loss distribution [28]. Formally, economic capital is a risk measure,

$$EC(p) = F_S^{-1}(p) - E(S) \quad \text{with,}$$
$$F_S^{-1}(p) = \inf\{s \in \mathbb{R} \mid F_S(s) \geq p\}, \qquad p \in (0, 1).$$

Since this definition of $EC(p)$ does not account for *"bad times"* episodes, then it is viewed as an *"all or nothing"* rule for capital definition. An alternative definition, according to [28], tries to incorporate such *"bad times"* in its formulation

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and treat it as a more *"optimistic"* event, this definition states that *EC* must be:

$$EC_K = E(S|S > K),$$

where this definition considers Economic Capital in average also enough to cushion losses even in bad times. Note that capital allocations in Section 4.2.2 are based on this capital definition.

Once the capital is defined, we have to define its counterpart, i.e. the loss. Consider a portfolio of n individual losses (random variables) $X_1, X_2, \ldots, X_n$ materializing at a fixed future date T. Assume that $(X_1, X_2, \ldots, X_n)$ is a random vector on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that any loss X_i has a finite mean. The distribution function $\mathbb{P}(X_i \leq x)$ of X_i will be denoted by $F_{X_i}(x)$.

The aggregate loss is defined by the sum of the individual losses:

$$S = \sum_{i=1}^{n} X_i, \tag{1}$$

where this aggregate loss can be interpreted as:

1. the total loss of a corporation, for example, an insurance company, with the individual losses corresponding to the losses of the respective business unit,
2. the loss from an insurance portfolio, with the individual losses being those arising from the different policies; or
3. the loss by a financial conglomerate, white the different individual losses correspond to the losses suffered by its subsidiaries.
4. the loss of a bank due to fraud in credit card and savings accounts, respectively[3].

Following [16] it is the first of these interpretations we will use throughout this article. Hence, S is the aggregate loss faced by a company and X_i is the loss of business unit i.

In order to clarify what the allocation problem is, one can view the problem from another perspective, namely, consider an investor who can invest in a fixed set of n different investment possibilities with losses represented by the random variables $X_1, X_2, \ldots, X_n$. We have the following economic interpretations depending on the area of application [27]:

1. **Performance measurement.** Here the investor is a financial institution and the X_i represent the Profit-and-Loss distribution of n different lines of business.

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[3]this example is similar to the illustration provided in the application section

2. **Loan pricing.** In this situation the investor is a loan book manager responsible for a portfolio of n loans.

3. **General investment.** Here we consider either an individual or institutional investor and the standard interpretation of X_i are profit-and-loss corresponding to a set of investments in various assets.

S is random, so usually we assume that the company has already determined the aggregate level of capital safely to face those losses and denote this total risk capital by K. The company now wishes to allocate this exogenously given total risk capital K across its various business units, that is, to determine non-negative real numbers $K_1, \ldots, K_n$ satisfying the full allocation requirement:

$$\sum_{i=1}^{n} K_i = K. \tag{2}$$

This allocation is in some sense a notional exercise; it does not mean that capital is physically shifted across the various units, as the company's assets and liabilities continue to be pooled. The allocation exercise could be made in order to rank the business units according to levels of profitability. This task can be performed, for example, by determining the returns on the allocated capital for the respective business units.

The general approach of capital allocation raises the question of what the appropriate risk capital for an individual investment opportunity might be. Thus the question of performance of the investment is intimately connected with the risk measurement chosen. A two-step procedure is used in practice [27].

1. Compute the overall risk capital $\rho(S)$, where S is defined in (1) and ρ is a particular risk measure, such as value at risk (VaR), expected shortfall (ES), or an economic capital (EC(p)) (see Dhaene et al. [17], **?**], Guillen et al. [22] and Abbasi and Guillen [1] for detailed explanations and applications and Alemany et al. [3] for estimation methods). Coherent measures will be more appropriate than non-coherent ones as they guarantee sub-additivity[4] Some new measures have been proposed in this area and they could generalize the interpretation [6]

2. Compute K as $\rho(S)$ and allocate the capital K to the individual units according to some mathematical *capital allocation principle* such that, if (K_i) denotes the capital allocated to i with potential loss X_i. The sum of K_i fulfills the requirement in (2).

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[4]See Dhaene et al. [17] for a definition of what a coherent measure implies.

We are interested in the second step of the procedure above; roughly speaking we require a mapping that takes as input the individual losses $X_1, X_2, \ldots, X_n$ and the risk measure ρ and yields as output the vector $(K_1, K_2, \ldots, K_n)$ such that:

$$\rho(S) = \sum_{i=1}^{n} K_i = K. \tag{3}$$

Such a mapping is called a *capital allocation principle*. The relation (3) is sometimes called the *full allocation property* [27] since all of the overall risk capital $\rho(S)$ (not more, not less) is allocated to the investment possibilities; [27] consider this property to be an integral part of the definition of an allocation principle.

Given that a capital allocation can be carried out in a countless number of ways, additional criteria must be set up in order to determine the most suitable form of determining the mapping. A reasonable start is to require the allocated capital amounts K_i to be "close" to their corresponding losses X_i in some appropriately defined sense. Prior to introducing the idea of "closeness" between individual loss and allocated capital, we revisit some well-known capital allocation methods.

3. Allocation Principles in Risk Management

A capital allocation principle in risk management is a general rule that assigns a capital K that is aimed to cover an aggregated loss S, to units that contribute to S and not necessarily independently. The reasons why firms want their total capital needs to cover risk to be allocated are [16]:

1. There is a need to redistribute the total (frictional or opportunity) cost associated with holding capital across various business lines so that this cost is equitably transferred back to the depositors or policyholders in the form of charges.
2. The allocation of expenses across lines of business is a necessary activity for financial reporting purposes.
3. Capital allocation provides for a useful device of assessing and comparing the performance of the different lines of business by determining the return on allocated capital for each line. Comparing these returns allows one to distinguish the most profitable business lines and hence may assist in remunerating the business line managers or in making decisions concerning business expansions, reductions or even eliminations.

Allocation principles are methods aimed to solve the allocation problem by providing capital to each business unit for them to face their losses. This means that allocation principles gives those K_i shown in (2) as solution to our main problem. Standard risk measures give rise to different allocation principles.

3.1. Haircut allocation principle

This a is straightforward allocation method consisting of allocating the capital $K_i = \gamma F_{X_i}^{-1}(p)$, $i = 1, \ldots, n$ to business unit i, where factor γ is chosen such that the full allocation requirement (2) is satisfied. This gives rise to the *haircut allocation principle*:

$$K_i = \frac{K}{\sum\limits_{i=1}^{n} F_{X_i}^{-1}(p)} F_{X_i}^{-1}(p), \quad i = 1, \ldots, n. \tag{4}$$

Haircut principle is based on the idea of measuring stand-alone losses using a VaR for a given (fixed) probability level p that is why it is a very common technique among banks and insurance companies. It boils down to a principle of single proportionality.

It should be noted that K is exogenously determined, it is considered as a given value. The capital allocated by this principle does not rely on the structure dependence of the losses X_i of the different business units. [16] consider haircut as a method which is independent of the portfolio context within which the individual losses are embedded, clearly this fact highlights the non-subadditivity property of the VaR.

The two more immediately consequences derived from non-subadditivity in the haircut principle context are: i) The portfolios do not benefit from a pooling effect (this is true even beyond haircut scope) and ii) It may happen that the allocated capitals K_i exceed the respective stand-alone capitals $F_{X_i}^{-1}(p)$.

3.2. CTE allocation principle (Overbeck type II allocation principle)

CTE principle is based on conditional tail expectation, we call this kind of allocation *Overbeck type II allocation principle*[5]. For a given probability level $p \in (0, 1)$, the CTE of the aggregate loss is defined as:

$$CTE_p[S] = E\left[S | S > F_{X_S}^{-1}(p)\right]. \tag{5}$$

expression (5) for a fixed level p, gives the average of the top $(1 - p)$ percent losses.

The *CTE allocation principle* for some fixed probability level $p \in (0, 1)$ has the form:

$$K_i = \frac{K}{CTE_p[S]} E\left[X_i | S > F_{X_S}^{-1}(p)\right], \quad i = 1, \ldots, n. \tag{6}$$

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[5]See Section 4.2.2 to find out why we call this principle this way.

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Unlike the *haircut allocation principle*, the *CTE principle* takes into account the dependence structure of the random losses $(X_1, X_2, \ldots, X_n)$. Interpreting the event $S > F_{X_S}^{-1}(p)$ as the "the aggregate loss S is large", we see from (6) that business units with larger conditional expected loss, given that the aggregate loss S is large, will be penalized with larger amount of capital required than those with lesser conditional expected loss.

3.3. Covariance allocation principle

The *Covariance allocation principle* takes the following form:

$$K_i = \frac{K}{Var[S]} Cov(X_i, S), \quad i = 1, \ldots, n, \tag{7}$$

where $Cov(X_i, S)$ is the covariance between the individual loss X_i and the aggregate loss S and $Var(S)$ is the variance of the aggregate loss. Because clearly the sum of the individual covariances is equal to the variance of the aggregate loss, the full allocation requirement in (2) is automatically satisfied in this case.

The *Covariance allocation principle* as well as the *CTE allocation principle* takes into account the dependence structure of the random losses. A nice interpretation that arises from the covariance principle is that "business units with a loss that is more correlated with the aggretate loss S are penalized by requiring them to hold a larger amount of capital than those that are less correlated" [16] .

3.4. Proportional allocations

[27] summarizes all the allocation methods explained in the previous sections into what they call *Proportional Allocations* which is a more general class encompassing the allocation principles described above. Depending on which risk measure ρ is chosen for attributing capital K_i is the key for obtaining one of them. This idea is formalized as:

$$K_i = \omega\rho(X_i), \quad i = 1, \ldots, n, \tag{8}$$

where K_i is the capital to be allocated to each business unit i, $\rho(\cdot)$ is risk measure and factor ω is chosen such that the full allocation requirement in (2) is satisfied, this factor takes the following form:

$$\omega = \frac{K}{\sum\limits_{i=1}^{n} \rho(X_i)}, \quad i = 1, \ldots, n. \tag{9}$$

Factor (9) can be seen as a weighting scheme for capital allocation, substituting (9) into (8) we have an explicit and general formulation encompassing all the

allocation principles discussed above:

$$K_i = \frac{K}{\sum\limits_{i=1}^{n} \rho(X_i)} \rho(X_i), \quad i = 1, \ldots, n. \tag{10}$$

4. Optimal Capital Allocations

As we have pointed out above, K is considered to be exogenous; because there are several allocation principles to aggregate capital K to n parts $K_1, \ldots, K_n$ corresponding to the different business units. [16] claim that "there seems to be a lack of a clear motivation for preferring to choose one over another, although it appears obvious that different capital allocations must in some sense correspond to different questions that can be asked within the context of risk management" and this is the main focus of the [16] becomes a key reference for systematizing capital allocation methods by viewing them as solutions to a particular decision problem. In order to achieve this goal they formulate a decision criterion, such as:

Capital should be allocated such that for each business unit the allocated capital and the loss are sufficiently close to each other [16].

In order to cast this statement in a more formal setting, consider the aggregate portfolio loss $S = X_1 + \ldots + X_n$ with aggregate capital K. Once the aggregate capital is allocated, the difference between the aggregate loss and the aggregate capital can be expressed as:

$$S - K = \sum_{i=1}^{n} (X_i - K_i), \tag{11}$$

where the quantity $(X_i - K_i)$ expresses the loss minus the allocated capital for unit i. It is important to notice that in this setting, the units are cross-subsidizing each other, in the sense that the occurrence of the event "$X_k > K_k$" does not necessarily lead to "ruin"; such unfavorable performance of subportfolio k may be compensated by a favorable outcome for one or more values $(X_l - K_l)$ of the other units. [16] propose to determine the appropriate allocation by the following optimization problem:

Definition 1. *Optimal Capital Allocation Problem* *Given the aggregate capital $K > 0$, determine the allocated capitals K_i, $i = 1, \ldots, n$, from the following optimization problem:*

$$\min_{K_1,\ldots,K_n} \sum_{i=1}^{n} \upsilon_i E\left[\zeta_i D\left(\frac{X_i - K_i}{\upsilon_i}\right) \right], \quad such\ that, \quad \sum_{i=1}^{n} K_i = K, \tag{12}$$

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where the v_i are non-negative real numbers such that $\sum_{i=1}^{n} v_i = 1$, the ζ_i are non-negative random variables such that $E(\zeta_i) = 1$, and D is a non-negative function.

Each of the component in the general optimal capital allocation problem in (12) are defined as follows:

v_i: The non-negative real number v_i is a measure of exposure or business volume of the ith unit, such as revenue, insurance premium, etc. These scalar quantities are chosen such that they sum to 1. Their inclusion in the expression $D\left(\dfrac{X_i - K_i}{v_i}\right)$ normalizes the deviations of loss from allocated capital across business units to make them relatively more comparable. At the same time, the v_is are used as weights attached to the different values of $E\left[\zeta_i D\left(\dfrac{X_i - K_i}{v_i}\right)\right]$ in the minimization problem in (12), in order to reflect the relative importance of the different business units.

$D\left(\dfrac{X_i - K_i}{v_i}\right)$: For simplicity, it is first assumed that $v_i = 1$ and also that $\zeta_i \equiv 1$. The terms $D(X_i - K_i)$ quantify the deviations of the outcomes of the losses X_i from their allocated capital K_i. Minimizing the sum of the expectations of these quantities essentially reflects the requirement that the allocated capitals should be "as close as possible" to the losses they are allocated to. Examples of distance measures are "squared or quadratic deviations" and "absolute deviations".

ζ_i The deviations of the losses X_i from their respective allocated capital levels K_i are measured by the terms $E\left[\zeta_i D(X_i - K_i)\right]$. These expectations involve non-negative random variables ζ_i with $E(\zeta_i) = 1$ that are used as weight factors to the different possible outcomes of $D(X_i - K_i)$. One possible choice for the ζ_i could be $\zeta_i = h(X_i)$ for some non-negative and non-decreasing function h. In this case, the heaviest weights are attached to deviations that correspond to states of the world leading to the largest outcomes of X_i. We will call allocations based on such a choice for the ζ_i *business unit driven allocations*.

Another choice is to let $\zeta_i = h(S)$ for some non-negative and non-decreasing function h, such that the outcomes of the deviations are weighted with respect to the aggregate portfolio performance. In this case, heavier weights are attached to deviations that correspond to states of the world leading to larger outcomes of S. Allocations based on such a choice for the random variables ζ_i will be called *aggregate portfolio driven allocations*.

A yet different approach is to let $\zeta_i = \zeta_M$ for all i, where ζ_M can be interpreted as the loss on a reference (or market) portfolio. In this case, the

weighting is market driven and the corresponding allocation is said to be a *market-driven allocation*.

The **Quadratic Optimization Criterion** is proposed by [16] as the *General Solution of the Quadratic Allocation Problem* by letting

$$D(x) = x^2. \tag{13}$$

This leads to (12) to

$$\min_{K_1,\dots,K_n} \sum_{i=1}^{n} E\left[\zeta_i \frac{(X_i - K_i)^2}{\upsilon_i}\right], \quad such\ that, \quad \sum_{i=1}^{n} K_i = K. \tag{14}$$

The solution to this minimization problem is given in the following theorem.

Theorem 1. *The optimal allocation problem in (14) has the following unique solution:*

$$K_i = E(\zeta_i X_i) + \upsilon_i \left(K - \sum_{i=1}^{n} E(\zeta_i X_j)\right), \quad i = 1,\dots,n. \tag{15}$$

A detailed proof of the solution for this minimization problem can be found in [16].

4.1. Business unit driven allocations

Following [16], in this subsection, we consider the case where the weighting random variables ζ_i in the quadratic allocation problem in (14) are given by

$$\zeta_i = h_i(X_i), \tag{16}$$

with h_i being a non-negative and non-decreasing function such that $E[h_i(X_i)] = 1$, for $i = 1,\dots,n$. Hence, for each business unit i, the states of the world to which we want to assign the heaviest weights are those under which the business unit performs the worst. As earlier pointed out, we call allocations based on (16) business unit driven allocations. In this case, the allocation rule in (15) can be rewritten as

$$K_i = E[X_i h_i(X_i)] + \upsilon_i \left(K - \sum_{i=1}^{n} E[X_i h_i(X_i)]\right), \quad i = 1,\dots,n. \tag{17}$$

For an exogenously given value of K, the allocations K_i are not influenced by the mutual dependence structure between the losses X_i of the different business units. In this sense, one can say that the allocation principle (17) is independent of the portfolio context within which the X_is are embedded and, hence, is

indeed business unit driven. Such allocations might be a useful instrument for determining the performance bonuses of the business unit managers, in case one assumes that each manager should be rewarded for the performance of his own business unit but not extra rewarded (or penalized) for the interrelationship that exists between the performance of his business unit and that of the other units of the company. One should however note that disregarding in this way diversification between business units, the allocation may give incentives to managers that are at odds with overall portfolio optimization criteria.

The law invariant risk measure $E[X_i h_i(X_i)]$ assigns to any loss X_i the expected value of the weighted outcomes of this loss, where higher weights correspond to larger outcomes of the loss, that is, to more adverse scenarios. Risk measures and premium principles of this general type are proposed and investigated in [25], [30], and [19].

Defining the volumes v_i by

$$v_i = \frac{E[X_i h_i(X_i)]}{\sum_{i=1}^{n} E[X_i h_i(X_i)]}.$$

(18)

The allocation principle could be found by substituting (18) in (17) and simplifying the expression as in:

$$K_i = E[X_i h_i(X_i)] + \frac{E[X_i h_i(X_i)]}{\sum_{i=1}^{n} E[X_i h_i(X_i)]}\left(K - \sum_{i=1}^{n} E[X_i h_i(X_i)]\right)$$

Now it can be easily seen from this last expression the allocation principle based on the *business unit driven* idea is given by:

$$K_i = \frac{K}{\sum_{i=1}^{n} E[X_i h_i(X_i)]} E[X_i h_i(X_i)].$$

(19)

Once we got to know the general form of the business unit driven allocation principle we are now able to choose different forms for $h_i(X_i)$ in order to achieve several capital allocation principles based upon the *business unit driven allocation* framework, this is exactly the main purpose of the subsequent sections.

4.1.1. (Pure) Conditional Tail Expectation principle

Once we know the allocation principle for allocating K_i using *business unit driven principle* we can set specific forms for $h_i(X_i)$, we can obtain several explicitly functional forms for K_i, for instance by choosing $h_i(X_i) = \frac{\mathbb{I}(X_i > F_{X_i}^{-1}(p))}{1 - F_{X_i}(F_{X_i}^{-1}(p))}$, then K_i will result in the *(Pure) Conditional Tail Expectation* principle.

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We call this principle *(Pure) Conditional Tail Expectation* because both the aggregate loss and each individual business unit losses are taken conditional expectation based on the average of the top $(1 - p)$ loss. Since $CTE(\cdot)$ is applied to S and X_i then we call it *(Pure) Conditional Tail Expectation* so that we can distinguish it from the *Conditional Tail Expectation* principle based on [28] which we call *Overbeck type II allocation principle* which is a special case of the *Aggregate Portfolio Driven Allocations*, see Section 4.2.

Lemma 1. *For an integrable loss X with continuous distribution function, F_X and any $p \in (0, 1)$ we have [16],*

$$ES_p = \frac{1}{1 - p} E(X : X \geq q_p(X)) = E(X|X \geq VaR_p).$$

By choosing $h_i(X_i) = \frac{\mathbb{I}(X_i > F_{X_i}^{-1}(p))}{1 - F_{X_i}(F_{X_i}^{-1}(p))}$ multiplying by X_i and taking expectations will lead us to:

$$E[X_i h_i(X_i)] = E\left[X_i \frac{\mathbb{I}(X_i > F_{X_i}^{-1}(p))}{1 - p}\right] = \frac{1}{1 - p} E[X_i|X_i > F_{X_i}^{-1}(p)].$$

From Lemma 1 the previous expressions reduces to the *Conditional Tail Expectation*:

$$E[X_i h_i(X_i)] = CTE_p[X_i].$$

Now replacing $E[X_i h_i(X_i)]$ by $CTE_p[X_i]$ in (19) we have:

$$K_i = \frac{K}{\sum_{i=1}^n CTE_p(X_i)} CTE_p(X_i) = \frac{K}{CTE_p(\sum_{i=1}^n X_i)} CTE_p(X_i).$$

$\sum_{i=1}^n CTE_p(X_i) = CTE_p(\sum_{i=1}^n X_i)$ follows from the additivity proporty of CTE.

Hence K_i takes the following form:

$$K_i = \frac{K}{CTE_p(S)} CTE_p(X_i). \tag{20}$$

4.1.2. Standard deviation principle

The *standard deviation principle* [14] can be easily obtained by choosing $h_i(X_i) = 1 + a\frac{X_i - E(X_i)}{\sigma_{X_i}}, \quad a \geq 0$, so that replacing it into $E[X_i h_i(X_i)]$ and then plug

it into (19) will have the so-called *standard deviation principle*.

In order to get an expression for K_i based upon the *standard deviation principle* we proceed as follows:

$$E[X_i h_i(X_i)] = E\left[X_i + a\frac{X_i^2 - X_i E(X_i)}{\sigma_{X_i}}\right]$$
$$= E(X_i) + a\sigma_{X_i}.$$

For $\sum_{i=1}^{n} E[X_i h_i(X_i)]$ to be explicitly found we proceed as follows:

$$\sum_{i=1}^{n} E[X_i h_i(X_i)] = \sum_{i=1}^{n} \{E(X_i) + a\sigma_{X_i}\}$$
$$= \sum_{i=1}^{n} E(X_i) + a\sum_{i=1}^{n} \sigma_{X_i}. \tag{21}$$

Expression (21) can be simplified to (22) if and only if $Cov(X_i, X_j) = 0 \quad \forall i \neq j$

$$E(S) + a\sigma_S, \tag{22}$$

this follows from the following operations:

$$\sum_{i=1}^{n} E(X_i) + a\sum_{i=1}^{n} \sigma_{X_i} = E\left(\sum_{i=1}^{n} X_i\right) + a\sqrt{Var\left(\sum_{i=1}^{n} X_i\right)} \Leftrightarrow Cov(X_i, X_j) = 0 \quad \forall i \neq j$$
$$= E(S) + a\sigma_S.$$

Consequently the form taken by K_i based upon the *standard deviation principle* is:

$$K_i = \frac{K}{E(S) + a\sigma_S}\left(E(X_i) + a\sigma_{X_i}\right). \tag{23}$$

A very interesting relationship between *Overbeck type I allocation principle* which we will be studied in (4.2.2) and the *Standard deviation allocation principle*, (32) and (23), respectively, is given by:

$$K_i = \frac{K}{E(S) + a\phi}\left(E(X_i) + \frac{a}{\sigma}\gamma\right). \tag{24}$$

Overbeck type I is retrieved by (24) when choosing $\phi = \sigma_S^2$ and $\gamma = Cov(X_i, S)$.

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Whereas the *standard deviation principle* is recovered when setting $\phi = \sigma_S$ and $\gamma = Cov(X_i, X_i) = Var(X_i) = \sigma_{X_i}^2$.

4.1.3. Esscher principle

If we let $h_i(X_i)$ be $\frac{e^{aX_i}}{E[e^{aX_i}]}$ with $a > 0$ then K will be allocated accordingly by the *Esscher Principle* [20], as we shall see below:

$$E[X_i h_i(X_i)] = E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right].$$

$$\sum_{i=1}^{n} E[X_i h_i(X_i)] = \sum_{i=1}^{n} E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right].$$

Thus, the optimal K_i will look like as (25):

$$K_i = \frac{K}{\sum_{i=1}^{n} E\left[\dfrac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]} E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]. \tag{25}$$

Table 1: Business Unit Driven Capital Allocation

Reference	$h_i(X_i)$	K_i
(Pure) Conditional Tail Expectation [28]	$\frac{\mathbb{I}(X_i > F_{X_i}^{-1}(p))}{1 - F_{X_i}(F_{X_i}^{-1}(p))} \quad p \in (0,1)$	$\frac{K}{CTE_p(S)} CTE_p(X_i)$
Standard deviation principle [14][6]	$1 + a\frac{X_i - E(X_i)}{\sigma_{X_i}}, \quad a \geq 0$	$\frac{K}{E(S) + a\sigma_S}\left(E(X_i) + a\sigma_{X_i}\right)$
Esscher principle [20]	$\frac{e^{aX_i}}{E(e^{aX_i})}, \quad a > 0$	$K_i = \frac{K}{\sum_{i=1}^{n} E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]} E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]$

4.2. Aggregate portfolio driven allocations

Unlike from the *Business Unit Driven Allocation rule*, this time [16] consider the case where

$$\zeta_i = h(S), \quad i = 1, \ldots, n, \tag{26}$$

with h being a non-negative and non-decreasing function such that $E[h(S)] = 1$. In this case, the states of the world to which we assign the heaviest weights are those under which the aggregate portfolio performs worst. Therefore, we call such allocations *aggregate portfolio driven allocations*. The allocation rule (15) can now be rewritten as:

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$$K_i = E[X_i h(S)] + v_i(K - E[S h(S)]), \quad i = 1, \ldots, n. \tag{27}$$

Hence, the capital K_i allocated to unit i is determined using a weighted expectation of the loss X_i, with higher weights attached to states of the world that involve a large aggregate loss S. Notice that the allocation principle (27) can be reformulated as[7]

$$K_i = E(X_i) + Cov[X_i, h(S)] + v_i(K - E[S h(S)]), \quad i = 1, \ldots, n. \tag{28}$$

This means that the capital allocated to the ith business unit is given by the sum of the expected loss $E[X_i]$, a loading that depends on the covariance between the individual and aggregate losses X_i and $h(S)$, plus a term proportional to the volume of the business unit. A strong positive correlation between X_i and $h(S)$, which reflects that X_i could be a substantial driver of the aggregate loss S, produces a higher allocated capital K_i.

Using aggregate portfolio driven allocations might be appropriate when one wants to investigate each individual portfolio's contribution to the aggregate loss of the entire company. In other words, the company wishes to evaluate the sub-portfolio performances, for example, the returns on the allocated capitals, in the presence of the other subportfolios. This can provide relevant information to the company within which it can further be used to evaluate either business expansions or reductions.

Defining the volumes v_i by

$$v_i = \frac{E[X_i h(S)]}{E[S h(S)]}, \quad i = 1, \ldots, n. \tag{29}$$

Plugging (29) into (27) and simplifying the resulting expression we end up having a proportional allocation rule:

$$K_i = \frac{K}{E[S h(S)]} E[X_i h(S)]. \tag{30}$$

Using the proportional allocation principle shown in (30) and choosing some structure for $h(S)$, the researcher/practitioner can be allowed to construct several ways for allocating K. For instance let us consider a particular choice for $h(S)$ to be $h(S) = S - E(S)$ this yields to the covariance allocation principle introduced in section 3 by means of determining the expression for both $E[X_i h(S)]$ and $E[S h(S)]$ and then plug them into (30) as it is shown below.

---

[7]This follows from the fact that $Cov(X_i, h(S)) = E(X_i h(S)) - E(X_i)E(h(S))$ solving for $E(X_i h(S))$ we end up with $E(X_i) + Cov[X_i, h(S)]$ since $E(h(S)) = 1$

4.2.1. Covariance allocation principle

This subsection is intended to derive the *Covariance allocation principle* from the general setting presented in the previous section by setting $h(S) = S - E(S)$ and using the philosophy of the *plug-in principle*.

Setting $h(S) = S - E(S)$ the aim is to determine $E[X_i h(S)]$ and $E[S h(S)]$.

For $E[X_i h(S)]$ we have:

$$
\begin{aligned}
E[X_i h(S)] &= E[X_i S - E(S)] \\
&= Cov(X_i, S).
\end{aligned}
$$

For $E[S h(S)]$ to be explicitly found we proceed as follows:

$$
\begin{aligned}
E[S h(S)] &= E[S(S - E(S))] \\
&= E(S^2) - [E(S)]^2 \\
&= Var(S).
\end{aligned}
$$

Once we have the expressions for $E[X_i h(S)]$ and $E[S h(S)]$ we can now plug them into (30) in order to have the expression for allocating capital K among the different business units (X_i with $i = 1, \ldots, n$) based on the *Aggregate Portfolio Driven idea*. So the allocation principle has the form:

$$
K_i = \frac{K}{Var[S]} Cov(X_i, S), \quad i = 1, \ldots, n. \tag{31}
$$

Precisely this is exactly the expression shown in (7) from this fact one can notice that *Covariance Principle* is a special case of the *Aggregate Portfolio Driven Allocation* when choosing $h(S) = S - E(S)$.

4.2.2. Overbeck allocation principles

Within this subsection we provide an explicit expression for the Aggregate Portfolio Driven Allocation principle based on [28]. We call *Overbeck Type I allocation principle* to the principle obtained by setting $h(S) = 1 + a\frac{S - E(S)}{\sigma_S}, a \geq 0$. And we will call *Overbeck Type II allocation principle* to that when using $h(S) = \frac{1}{1-p}\mathbb{I}(S > F_S^{-1}(p))$, with $p \in (0, 1)$.

As in the previous sections we now proceed to find an explicit expression for K_i by setting $h(S) = 1 + a\frac{S - E(S)}{\sigma_S}, a \geq 0$.

For $E[X_i h(S)]$ we have:

$$E[X_i h(S)] = E\left[X_i\left(1 + a\frac{S - E(S)}{\sigma_S}\right)\right]$$
$$= E(X_i) + \frac{a}{\sigma_S}Cov(X_i, S).$$

Working on $E[S h(S)]$ we find:

$$E[S h(S)] = E\left[S + \frac{aS(S - E(S))}{\sigma_S}\right]$$
$$= E(S) + a\sigma_S.$$

Applying the *plug-in principle* and substituting the respective expressions of $E[X_i h(S)]$ and $E[S h(S)]$ into the general framework presented in (30) we get the allocation principle we've just called *Overbeck Type I allocation principle* whose form is:

$$K_i = \frac{K}{E(S) + a\sigma_S}\left[E(X_i) + \frac{a}{\sigma_S}Cov(X_i, S)\right]. \tag{32}$$

Overbeck Type II allocation principle is determined by letting $h(S)$ be $\frac{1}{1-p}\mathbb{I}(S > F_S^{-1}(p))$ with $p \in (0, 1)$:

$$E[X_i h(S)] = \frac{1}{1 - p}E[X_i|\mathbb{I}(S > F_S^{-1}(p))]$$
$$E[S h(S)] = \frac{1}{1 - p}E[S|\mathbb{I}(S > F_S^{-1}(p))] = CTE_p(S).$$

Therefore, K_i could be written as:

$$K_i = \frac{K}{CTE_p(S)}E[X_i|\mathbb{I}(S > F_S^{-1}(p))]. \tag{33}$$

Note this principle is exactly the same one presented in (6) in Section 3.2

4.2.3. Wang allocation principle

Let us consider $h(S) = \frac{e^{aS}}{E[e^{aS}]}$ with $a > 0$, we can construct an allocation principle based on [34] and give an expression for K_i. In order to achieve our goal

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the procedure is similar to the ones used in previous sections.

Once we consider $h(S) = \frac{e^{aS}}{E[e^{aS}]}$, the expression for $E[X_i h(S)]$ is found in the following way:

$$
\begin{aligned}
E[X_i h(S)] &= E\left[S\, h(S) = \frac{e^{aS}}{E[e^{aS}]}\right] \\
&= \frac{1}{E(e^{aS})} E(X_i e^{aS}) \\
&= \frac{E(X_i e^{aS})}{E(e^{aS})}.
\end{aligned}
$$

Then $E[S\, h(S)]$ is:

$$
\begin{aligned}
E[S\, h(S)] &= E\left[X_i h(S) = \frac{e^{aS}}{E[e^{aS}]}\right] \\
&= \frac{E(S\, e^{aS})}{E(e^{aS})}.
\end{aligned}
$$

Therefore, the allocation of the exogenously given aggregate capital K to n parts $K_1, \ldots, K_n$ corresponding to the different business units can be carried out using:

$$
K_i = \frac{K}{E(S\, e^{aS})} E(X_i e^{aS}). \tag{34}
$$

4.2.4. Tsanaka allocation principle

If we let $\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})}$ be $h(S)$ with $a > 0$, then this leads us to the [31] principle. Expressions for constructing the K_i are as follow:

$$
E[X_i h(S)] = E\left[X_i \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right],
$$

$$
E[S\, h(S)] = E\left[S \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})}\right],
$$

where the K_i to be allocated takes the following form:

$$
K_i = \frac{K}{E\left[S \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right]} E\left[X_i \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right]. \tag{35}
$$

Letting Ψ be $\int_0^1 \frac{e^{\gamma a S}}{E(e^{\gamma a S})} d\gamma$, then K_i could be rewritten as:

$$K_i = \frac{K}{E(S\Psi)} E(X_i \Psi). \tag{36}$$

Table 2 summarizes the *Aggregate Portfolio Driven Allocations* by providing expressions for K_i.

Table 2: Aggregate Portfolio Driven Allocations

Reference	$h(S)$	K_i
Covariance principle	$S - E(S)$	$\frac{K}{Var[S]} Cov(X_i, S)$
Overbeck Type I [28]	$1 + a\frac{S - E(S)}{\sigma_S}, \quad a \geq 0$	$\frac{K}{E(S) + a\sigma_S}\left[E(X_i) + \frac{a}{\sigma_S} Cov(X_i, S)\right]$
Overbeck Type II [28]	$\frac{1}{1-p}\mathbb{I}(S > F_S^{-1}(p)), \quad p \in (0,1)$	$\frac{K}{CTE_p(S)} E[X_i \| S > F_S^{-1}(p)]$
[34]	$\frac{e^{aS}}{E(e^{aS})}, \quad a > 0$	$\frac{K}{E(S e^{aS})} E(X_i e^{aS})$
[31]	$\int_0^1 \frac{e^{\gamma a S}}{E(e^{\gamma a S})} d\gamma, \quad a > 0$	$\frac{K}{E\left[S \int_0^1 \frac{e^{\gamma a S}}{E(e^{\gamma a S})} d\gamma\right]} E\left[X_i \int_0^1 \frac{e^{\gamma a S}}{E(e^{\gamma a S})} d\gamma\right]$

5. An application to fraud analysis

We give the practical examples of capital allocation approaches and their impact on amounts of allocated capital when considering a typocal case of operational risk. For this purpose, we use *Public data risk no. 1* and *Public data risk no. 2* from [8], these data consist of 1000 and 400 observed loss amounts for categories 1 and 2, respectively.

Let us consider these data as operational losses in a banking environment. For *Public data risk no. 1* to have some sense in this context we consider it as bank transfer mistakes which means that a bank teller transfers more money than the required to a client's savings bank account and *Public data risk no. 2* is to be considered as fraudulent transactions, for instance, a client loses her credit card and another person uses it, if the bank's client reports this situation to bank then the non-authorized use of the credit card will charge some losses to bank.

In this section, we quantify capital requirements based on risk measures for those types of losses. Given an exogenous amount of total capital, K calculated as the empirical Value at Risk at 99% of the aggregate loss ($VaR_{99}(S)$), the goal is allocating to each loss source an optimal portion of this capital and comparing three well-known allocation principles: Haircut, Covariance and Overbeck type II allocation principles, all of them belonging to the proportional allocations, note that both the Covariance and Overbeck type II allocation principles belong to the Aggregate portfolio allocation principle described in Section 4.2.

Table 3: Descriptive statistics for numerical example data

	Public data risk no. 1	*Public data risk no. 2*
nobs	1000.00	400.00
NAs	0.00	0.00
Minimum	0.00	0.00
Maximum	5122.14	1027.53
1. Quartile	2.24	2.67
3. Quartile	8.46	8.62
Mean	42.06	20.89
Median	3.47	4.29
Sum	42059.41	8357.32
SE Mean	9.23	4.80
Variance	85242.64	9199.45
Stdev	291.96	95.91
Skewness	13.61	9.10
Kurtosis	210.87	89.20

The reason why we decide to use aggregate portfolio driven allocations is that we want to consider the dependence structure compared to the Haircut allocation principle, which is based on a stand-alone risk measure which does not consider the dependence structure. Dependence structures cannot be ignored in risk management [23, 24].

Some descriptive insights are provided in Table 3 where one eye-catching fact is the difference in the number of observations in each vector of losses, *Public data risk no. 1* has 1000 observations and *Public data risk no. 2* has 400 which represents a drawback for the configuration of the allocation principles where all of them implicitly assume identical length for vector of losses, we overcome this inconvenient by using two different re-sampling techniques: bootstrapping and an uniformly pairwise random extraction. Another important characteristic of these data is the strong non-normality suggested by the skewness and the kurtosis coefficients. Data are characterized by a strong right asymmetry since the mean is larger than the median for both vectors. This behaviour is typical in loss data analysis and has been mentioned by many authors [13, 10].

The numerical exercises presented below consists of two cases: the first one where the dependence structure between the two lines is removed by the simulation procedure and in the second one a strong dependence structure between losses is artificially created. The aim of these two scenarios is checking the performance of the allocations principles when two extreme situations might happen.

Table 4: Case I. Capital allocation based on different principles

	HAP	CAP	Overbeck II
X^* (dat1.boot)	62953.00	72497.53	66070.96
Y^* (dat2.boot)	12620.96	3076.43	9503.00
Total	75573.96	75573.96	75573.96

5.1. Case I: Lack of dependence structure

In this subsection, we assess the performance of allocation principles we are interested in when losses exhibit a low degree of linear dependence, this means that the correlation coefficient between the losses is close enough to zero.

Let X and Y be vectors consisting of 1000 and 400 observations on individual losses, moreover *Public data risk no. 1* is now denoted by X and *Public data risk no. 2* is denoted by Y. We will estimate risk using a Montecarlo simulation method as follows

1. Draw 1000 observations from X and 400 from Y using re-sampling with replacement and obtain X_1 and Y_1.
2. Generate $x_1^* = \sum_{i=1}^{1000} X_{1,i}$ and $y_1^* = \sum_{i=1}^{400} Y_{1,i}$.
3. Repeat steps 1) and 2) 10000 times to obtain two vectors of equal lengths: X^* and Y^* with $X^* = \{x_i^*\}_{i=1}^{10000}$ and $Y^* = \{y_i^*\}_{i=1}^{10000}$.

Once we have X^* and Y^*, we knowthe distribution of losses in each unit, i.e risk no. 1 and risk no. 2, respectively. We can now compute the allocations based on the principles previously discussed.

Summarizing we generate for both vectors of losses 10000 replications of size 1000 and 400 for *Public data risk no. 1* and for *Public data risk no. 2*, respectively in order to obtain two vectors of length 10000 over which we can apply the allocation principle we are interested in.

An aggregate capital amount of 50 416.73[8] monetary units would be enough for facing the total loss for this particular sample comprised by X and Y (*Public data risk no. 1* and *Public data risk no. 2*, respectively). Nevertheless, in order to guarantee a coverage even when large deviations might occur we use the empirical $\text{VaR}_{99}(S) = 75\,573.96$ that ensures 99% coverage of potential losses and this is why we set the exogenous capital to be this value. Aggregate capital to be allocated is 75 573.96 monetary units.

---

[8]This aggregate capital comes from summing 42059.41 and 8357.32, see row labelled *Sum* in Table 3

Table 4 shows the allocated capital to each vector of losses based on different capital allocation principles, these results show the amount of capital to be set aside for each risk source. Note that Haircut allocation principle (HAP) boils down to a simple proportion when there is not any dependence structure (in a linear sense) between the losses, this happens when the correlation coefficient between X^* and Y^* is close to zero and in this particular case such correlation is ≈ 0.00014, therefore results obtained from HAP will be identical to those obtained using:

$$K_i = \frac{K}{\sum_{i=1}^{n} X_i} X_i, \tag{37}$$

recalling the fact that $\sum_{i=1}^{n} X_i = S$, this "simple proportional" allocation principle (SPA) reduces to $(K/S)X_i$. When $K = 1$ and multiplying the result by 100 gives us the percentage of X_i as a portion of the aggregate loss S as it is shown in Table 5.

According to Table 3 the losses seem to be non-normal, therefore both Haircut and Overbeck type II allocations are computed using the normal and the t-student distribution, for the t-student we used several degrees of freedom and results do not differ from those ones reported when using a normal distribution, so in Table 4 only normal results are reported.

In order to assess how well the allocations fit, we now calculate the proportions of capital to be set aside instead of the amount of capital, we reach this goal by choosing $K = 1$ and the new results are reported in Table 5.

As it was expected, the Haircut allocation principle is a good choice since it does not take into account the dependence structure and since the correlation between X^* and Y^* is almost zero the best choice for this case is using (37) as the allocation principle, because its results are the a good enough approximation for HAP and its calculation is enormously simplified, furthermore it does not rely on any distributional assumption. Table 5 shows how the approximation to HAP using (37) performs compared to HAP results.

Table 5: Case I. Proportions of capital allocation based on different principles

	SPA	HAP	CAP	Overbeck II
dat1.boot	0.8344	0.833	0.9593	0.8743
dat2.boot	0.1656	0.167	0.0407	0.1257

In Table 5 there is an additional column: SPA which is the approximation to the HAP when correlation tends to zero, we present this information in order to compare the proportions based on each principle. We can see that HAP is identical to SPA, nevertheless the Covariance allocation principle overestimates the

contribution of the first vector and underestimates the second one in a stronger way than Overbeck II does. In a rough sense we can see that in absence of correlation between losses, the estimates of the Covariance allocation principle are more biased than those of Overbeck II.

Clearly in this part of the exercise we conclude that Covariance allocation principle performs the worst compared to the other two principles.

In the next section we introduce a strong dependence structure in order to assess the performance of the allocations which account for correlation among losses.

5.2. Case II: Strong dependence structure

This section can be seen as the counterpart of the previous one as now we go to the other extreme case where a strong dependence framework is involved.

In order to create two vectors of losses strongly correlated we base the sampling scheme on quantiles-based extractions, this means for each probability p_i with $i = 1, \ldots, 10000$, which is common for both vectors X and Y, recall that X is the label for *Public data risk no. 1* and Y is the label for *Public data risk no. 2*, we take the value located at quantile given by $F_X^{-1}(p_i)$ and $F_Y^{-1}(p_i)$, each p_i was randomly drawn from a $U(0, 1)$, to make this point clear, we go through the following steps:

1. Draw randomly 10000 values from a $U(0, 1)$ for probabilities such that p_1 is one realization of $U(0, 1)$, p_2 is another, and so on until p_{10000}.
2. Generate W and Z such that both are vectors of dimension 10000×1 holding $F_X^{-1}(p_i)$ and $F_Y^{-1}(p_i)$.
3. Constructing W and Z this way guarantees that when we have a small value for W we also have a small value for Z and when we have a large for one W we also have a large one for Z. We store W and Z into a matrix M of dimension 10000×2 so that W and Z are now matched (pairwise).
4. Resample row-wise with replacement from M and draw 10000 pairs of observations, sum them colwise and get m_1 which is a 1×2 vector, repeat this step 10000 times in order to get m_i with $i = 1, \ldots, 10000$.
5. The data set we are going to work with is the matrix M^* consisting of the colwise concatenation of m_i with $i = 1, \ldots, 10000$. M^* should look like:

$$M^* = \begin{pmatrix} m_{1,1} & m_{1,2} \\ \vdots & \vdots \\ m_{10000,1} & m_{10000,2} \end{pmatrix}$$

6. We call the first column of M^* as X' and the second one is called Y' where X' is the resampled observations of the transfer mistakes (*Public data risk no.*

1) and Y' is the resampled associated to the fraudulent transactions (*Public data risk no. 2*). Here the apostrophe does not mean transpose, it is just a way to name *X* and *Y* in order to distinguish them from the originals *X* and *Y*.

Given that we suffer from different lengths for vectors of losses, we base this part of the exercise on a resampling technique using a uniform distribution as described above, this consists of generating 10000 random numbers from a uniform distribution, $U(0, 1)$, then we use this numbers to extract the empirical quantiles from each vectors, this way we obtain two vector of length 10000 with a strong dependence structure since each time we draw a "small" value from the first vector we also get a "small" value from the second one, the same happens with "big" values, this is because we are using the 10000 uniform number as index for the inverse distribution function to retrieve those numbers.

The correlation coefficient enrolled in this case is ≈ 0.8875, this is the correlation between X' and Y', which is the "strong" dependence structure giving name to this section.

Following the same idea from the previous section, we consider the total capital to be allocated as exogenously determined and taken as given, so we consider this capital to be the empirical Value at Risk at 99% which is 628 724.6 monetary units.

Table 6: Case II. Capital allocation based on different principles

	HAP	CAP	Overbeck II
X'	412897.2	464021.7	414842.6
Y'	215827.3	164702.9	213882.0
Total	628724.6	628724.6	628724.6

Table 6 presents the total capital and the amounts to be allocated to each business units. Note that the first business unit, called X' is again the riskiest one, so more capital is allocated to it. One important point, when linear dependence between these two business unit becomes higher, is that all allocation principles are very close to each other, we were aware of this fact for both CAP and Overbeck II since they takes into account the dependence structure. Looking at the Haircut allocation principle (HAP) we can see that when correlation between losses is close to one then its results quietly differ from those obtained with (37), it is clearly seen since now risks are dependent each other and this is the key reason why allocations based on (the approximation to HAP) surely leads us to misleading allocations. Note that approximation provided by (37), when correlation is high, becomes biased.

Table 7: Case II. Proportions of capital allocation based on different principles

	SPA	HAP	CAP	OverbeckII
X'	0.6503	0.6567	0.7380	0.6598
Y'	0.3497	0.3433	0.2620	0.3402

In terms of proportions, Table 7 gives a picture of how the principles distribute the total capital between the business units. The first column represents the results using (37), this would be the allocation if correlation between risk sources were zero, in this case the optimal distribution of the total capital should be 65.03% allocated to the first business line (bank transfer mistakes) and 34.97% to the loss caused by fraudulent transactions. Since correlation between risk sources is 0.8910, then allocation based on (37) is biased, so principles that includes the linear dependence in its calculations are needed.

In spite of the fact that HAP is based on the idea of measuring stand-alone losses using a VaR (normal VaR in this case) it performs well enough even if the correlation is high, but one has to have in mind that VaR is not a coherent risk measure so in this case it is better off using a coherent risk measure for capital allocation, from this point we can choose either Covariance allocation principle or Overbeck type II allocation principle, but in practice HAP and CAP results are not so different.

6. Conclusions and Future Research

In this study we present the allocation problem and based on [16] we provide explicit formulation for K_i when using different specifications for business unit driven principles as well as aggregate portfolio driven allocations.

The numerical exercise carried out shows that the configuration of the allocations depends on the degree of linear dependence. Haircut allocation principle, even being a principle based on a non-coherent risk measure, experiences a good performance and it is less affected by the "*correlation effect*" (changes in the correlations). Haircut allocations are very similar to those suggested by Overbeck type II principle when correlation is high, this confirms the good performance of Haircut allocation principle.

We conclude that failure to account for correlation may lead to risk managing practices that are unfair to the units contributing to risk. An example using data from the banking sector, shows that operational risk evaluation and allocation of costs due to this sort of events depends significantly on the choice of the allocation principle.

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