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Abstract

We analyze the optimal auction of multiple non-identical objects when buyers are risk averse. We show that the auction formats that yield the maximum revenue in the risk neutral case are no longer optimal. In particular, selling the goods independently does not maximize the seller's revenue. We observe that seller's incentive for bundling arises solely due to the risk aversion of the buyers. The optimal auction which remains weakly efficient has the following properties: The seller perfectly insures all buyers against the risk of losing the object(s) for which they have high valuation. While the buyers who have high valuation for both objects are compensated if they do not win either object, the buyers who have low valuation for both objects incur a positive payment to the seller in the same event.

Key words: Multi-object Auctions, Optimal Auctions, Multi-dimensional Screening, Risk Averse Buyers, Bundling

JEL classification: D44, D81

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1 Introduction

Optimal selling mechanisms for multiple objects have been analyzed extensively due to their theoretical and practical importance (*e.g.*, the spectrum auctions, second hand car auctions).[1] One of the main assumptions in these studies is that the buyers are risk neutral. However, in many situations this assumption is violated and further analysis is needed.[2]

The optimal design problem in the presence of risk averse buyers can be described as follows: When the number of objects is limited, the buyers face the risk of not getting the object(s) they want. And in order to reduce this risk, the risk averse buyers, as compared to those that are risk neutral, bid more aggressively.[3] Therefore, in the presence of risk averse buyers, the seller will be tempted to increase the magnitude of the risk so as to induce aggressive bidding and, in turn, to increase the revenue from the sale. Yet, this comes with a trade-off, as the high type buyers (namely, the ones who value the good highly), when confronted with too much risk, may find it more profitable to follow the bidding strategy of the low type buyers or may even be discouraged to participate.[4] Therefore, a revenue maximizing selling scheme should impose "the right amount risk" on "the right type of buyers".

For the case of single object, Maskin and Riley [19], Matthews [21], and Eső [9] describe how the above mentioned trade-off should be balanced and observe that relaxing the risk neutrality assumption delivers quite different results. In his seminal work, Myerson [24] shows that if the buyers are risk neutral and their private valuations are independently distributed, then it is optimal to give the object to the buyer who has the highest *virtual valuation* (not the actual valuation) that exceeds the seller's outside option.[5] Thus, the standard auctions, including the "high bid"

[1] See for example, Harris and Raviv [12], Maskin and Riley [20], Levin [16], Figueroa and Skreta [10].

[2] In many auctions, the buyers are firms and they generally are risk neutral. Yet, firms whose ownership are non-diversified (e.g. most family owned companies), those that are bound by liquidity constraints or under a financial distress, and those that are subject to a nonlinear tax system should be assumed to be risk averse. (Asplund [4]) Even a firm which is owned by risk-neutral shareholders may behave in a risk-averse manner if the control of the firm is delegated to a risk-averse manager and his payment is linked to the firm's performance.(i.e. through stock options.)

Cox, et al. [7] show that the overbidding relative to Nash predictions (for the risk neutral environment) which has been observed in the first-price auction cannot be attributed to noisy-decision making, supporting the hypothesis that it must be due to the risk aversion of the buyers.

[3] See, for example, Maskin and Riley [19] and Matthews [21].

[4] As we know from the optimal auction literature, it may be desirable to exclude the low-type (and in some environments the high-type) buyers from the auction. (Exclusion Principle) Yet, if the seller imposes too much risk on all types then she will herself face the 'greatest' risk of no sale, hence ending up with zero profit.

[5] Virtual valuations are the adjusted valuations that take into account buyers' informational rents and, more precisely, are defined as $\psi_i(v_i) = v_i - [1 - F_i(v_i)]/f_i(v_i)$, if buyer i's valuation v_i is distributed according to cumulative

and "English" auctions, with appropriately chosen reserve price are all optimal. He further shows that any two auctions with the same allocation rule are revenue equivalent if the expected utility of each buyer in some benchmark case is the same, the celebrated *revenue equivalence theorem.* To the contrary, if the buyers are risk averse, the standard auctions with appropriate reserve price neither generate the same expected revenue nor are they optimal. (Maskin and Riley [19], Matthews [21]).[6] Another contrast is observed when the buyers' valuations are correlated: If the buyers are risk neutral, then the seller can fully extract the informational rents using an efficient auction (Crémer and McLean [8]), but she cannot do so if the buyers are risk averse, unless the correlation is sufficiently strong. (Eső [9]).[7]

In the light of these works, the current paper studies the optimal design problem for the case of multiple objects and seeks answers to the following two naturally-arising questions:

1. How does the optimal multi-object auction with *risk-averse* buyers compare to that with *risk-neutral* buyers?

2. Which features of the optimal *single-object* auction with risk averse buyers carry over to the optimal *multi-object* auction?

To answer the first question, we compare our results with those of Armstrong [1] who, in a binary model, characterizes the optimal multi-object auction for risk-neutral buyers.[8] This comparison provides a threefold answer.

One, in both problems, the optimal auction is weakly efficient.[9,10]

distribution function $F_i(.)$ with associated density function $f_i(.)$.

[6] In a second price auction, the buyers bid truthfully regardless of their risk preference. But in the first price auction, a risk-averse buyer shades his bid less than a risk-neutral buyer. As a result, the first price auction yields more revenue than the second price auction. Nevertheless, the first price auction is not optimal because it imposes too much risk on the high type buyers.

[7] Optimal auction should remove the risk from high type buyers, which requires providing insurance (and hence leaving some surplus) to them.

[8] Armstrong [1] inherited his setting from Armstrong and Rochet [2], who study a principal-agent problem. Both of these papers and the current paper assume that buyers/agents have multidimensional private information and, in this regard, differ from the references mentioned in footnote 1.

Manelli and Vincent [17] and Manelli and Vincent [18] also assume multidimensional private information, but different from the current paper, they assume a single buyer.

[9] Weak efficiency requires each object to be given to the buyer with the highest valuation whenever it is sold. Some of the objects can be kept by the seller eventhough there is a buyer who has valuation that exceeds that of the seller. For strong efficiency, on the other hand, the objects valued more highly by a buyer than the seller must always be sold. In this sense, the optimal auctions in Myerson [24] are weakly efficient.

[10] It must be noted, though, that the optimal multi-object auction is no longer weakly efficient when the model assumes a continuous type space.

Two, *none of the auction formats that are shown to be optimal in Armstrong* [1] *maximize the seller's revenue when the buyers are risk averse. In particular, it is not optimal to sell the two objects independently.* This is our main result. The sharp contrast is due to the way in which the objects are allocated in the state 'all buyers have low valuation for both objects'. (That is, 'all buyers are of type LL'.) The optimal auctions, in the risk neutral case, can take the form of independent auction, bundling auction, or mixed auction, depending on how buyers' valuations are correlated *across objects*.[11,12] These three formats allocate each object independently and randomly if all buyers are of type LL. However, doing so does not impose high enough risk on type LL. Contrarily, we show that, with risk averse buyers the optimal auction must give both objects to the same (LL type) buyer in this state.[13,14] Since the seller does not observe the buyers' types *ex ante*, she cannot sell the goods independently.

Three, in the risk neutral case, only the expected payments conditional on buyers' type matter in the design of the optimal scheme. That is, any two selling schemes with the same allocation rule but different payment rule, yield the same revenue to the seller if the buyers' *ex ante* expected payment conditional on their own type is the same. On the other hand, we show that, when the buyers are risk averse, the seller can improve the revenue by making the expected payments conditional *also* on the *type* and the *number* of the objects that the buyer wins. Moreover, it is not optimal to make these expected payments random.[15]

For the second question, we do a robustness check in order to see to what extent our results,

[11] In all three forms, the buyers have the same *expected* probability of winning the object(s) for which they have high valuation. These forms differ only in the *expected* probability of winning the objects for which buyers have low valuation. In a mixed auction, a buyer who has low valuation, say, for object A but high valuation for object B, is assigned object A more often than a buyer who has low valuation for both objects. While independent auctions don't distinguish between these two types for object A, bundling auction perfectly discriminates against the type that has low valuations for both objects. It should be noted that the bundling auction allows the goods to end up in the hands of different buyers.

[12] Avery and Hendershott [5] also consider risk-neutral buyers. While Armstrong [1] assumes that all buyers have demand for both objects, in Avery and Hendershott [5], only one buyer demands multiple objects and the remaining buyers demand only one or the other. Not surprisingly, the optimal auction in the latter paper may not be weakly efficient due to the good deal of asymmetry among buyers. Yet, even in that case, the optimal auction bundles the objects *probabilistically* for the multi-demand buyer.

[13] It is riskier to lose both objects than to lose a single object.

[14] In Armstrong [1], bundling is optimal only when buyers' valuations are negatively correlated across objects, or in other words, when a buyer's high value for one object, say A, is likely to be accompanied by a relatively low value for the other object, say B. The goods are bundled only for the types HL or LH. In this case, their incentive conditions in all directions are binding.

In the current paper, we show that the seller utilizes bundling not only to make the desired incentive conditions binding but also to increase the risk as much as possible for type LL.

[15] This also implies that it is not optimal to make the payments dependent on other buyers' reports.

which we obtain in a binary model, are comparable to those of the current literature which assumes continuous distribution of types. (Namely, Maskin and Riley [19] and Matthews [21])[16] We observe that the optimal single-object auction in the binary model replicates the behavior of that of the continuous model at the two extremes of the type space. This analogy helps us interpret our results regarding the features of the optimal multi-object auction: The seller perfectly insures all buyers against the risk of losing the object(s) for which they have the high(est) valuation. The buyers who are (most) eager to win both objects (namely, type HH) are compensated if they lose both objects. On the other hand, those (most) reluctant to win both objects (namely, type LL) must incur a positive payment to the seller if they lose both objects.[17]

The intuition for our results is as follows: While, on one hand, the seller would like to screen the buyers, on the other hand, she would like to confront them with risk. Screening the buyers requires leaving informational rents to (and, in turn, decreasing the risk for) the buyers who have high valuation for either or both objects. As a result, the buyers' marginal utility of income must remain the same regardless of whether they win or lose the objects for which they have high valuation. This also implies providing perfect insurance to type HH. On the other hand, the buyers who have low value for both objects must confront the highest risk from which the seller benefits in two ways: One, she makes imitating LL unattractive to the other types and two, she fully extracts the informational rents from type LL. Confronting these types with the highest risk involves not only bundling the objects whenever all buyers are LL but also collecting payments from them even when they lose both objects.

The current paper contributes substantially to the literature on bundling.[18] From the existing literature, some of the reasons as to why the bundling decision of a monopoly arises are: to take advantage of economies of scale and/or economies of scope, to reduce the transaction and information costs, to facilitate entering a new market, to signal the quality of the unknown product, to reduce the divergence in incentives, to acquire and maintain monopoly power and to exclude

[16] Matthews [21] studies the same problem as Maskin and Riley [19]. While the former assumes a particular form of utility function, namely CARA, and obtains necessary and sufficient conditions for an auction to be optimal, the latter considers different forms of risk aversion and characterize the properties of the optimal auction for all of these forms.

[17] A natural question to ask is how the punishment for type LL can be implemented in real life. When there is a single object, the optimal auction reduces to a modified first price auction for some parameter values. (Maskin and Riley [19]) The seller charges an entry fee, but she does not return it to the buyers with low valuation if they don't win the object.

[18] See Kobayshi [15] for a recent review of the literature.

possible entrants. Bundling may also arise when the goods are complementary, when valuations across objects are negatively correlated or when the positive correlation is weak. In the absence of the above-mentioned influences, we show that *the incentive for bundling results solely due to the risk aversion of buyers.*

We comment on the solution methods used in this paper: In section 2, we describe the optimal single object auction in *reduced form,* meaning we construct the buyers' *expected* probability of obtaining the object (contingent only on his own type), rather than his *actual* probability of winning as a function of all buyers' types. This technique was also utilized by Matthews [21] and Maskin and Riley [19] in order to avoid the computational complexity that risk aversion involves.[19] Yet, when solving the seller's optimal design problem in reduced form, in addition to the incentive constraints and the participation constraints, one must also impose the so-called *implementability* constraints that guarantee the existence of the *actual* probabilities.[20]

The number of implementability constraints increases exponentially with the number of goods (or more precisely with the number of elements in the type space), nevertheless Armstrong [1] was still able to solve the problem in reduced form. However in our problem with risk averse buyers, the correlation between the events of winning object A and object B also matters for the buyers (and in turn for the seller), making it very difficult, if not impossible, to characterize the implementability conditions that one needs to impose.[21] Therefore, in section 3, we describe the optimal auction in *non-reduced* form and construct the actual probabilities of the events that a buyer can possibly face as functions of the entire type profile (as reported by all participating buyers).[22] Since the buyers don't observe their opponents' types, only the *expected* probabilities of observing each event (conditional only on one's type) matter in the incentive conditions. Therefore, we also make use of these *expected* probabilities throughout our analysis.[23]

[19] The technique was introduced to the literature by Myerson [24].

[20] When there is a single object or when the buyers are risk neutral, these conditions take a very simple form, which, can be interpreted as the probability that a buyer whose type belongs to a given subset of the type space obtains a particular object cannot be higher than the probability that there is a buyer whose type is in that subset.

The implementability conditions need to be imposed because the seller has only a limited number of each type of good. A multi-product monopolist who has unlimited number of each type of good does not face this constraint. (See Manelli and Vincent [17] and Manelli and Vincent [18])

[21] Using the main result of Border[6] (Also footnote 28), Armstrong [1] was able to describe the implementability conditions. In his environment, the main difficulty is to identify the conditions that are binding at the optimum. In the current paper, on the other hand, Border[6]'s theorem is not applicable.

[22] These events are winning only object A, only object B, winning both objects and winning nothing.

[23] In regard to the solution method, this paper is also related to Menicucci [22] which extends Armstrong [1] by allowing for a synergy if the two goods end up in the hands of the same buyer. He shows that in this case the optimal

The remainder of the paper is organized as follows: In section 2, we construct the optimal single-object auction for risk averse buyers in a binary framework and analyze its properties. In Section 3, we assume two objects are for sale and we characterize the properties of the optimal auction when buyer valuations are strongly and positively correlated across objects. Finally, in section 4, we discuss the main results and their implications.

2 Optimal Single-Object Auctions

2.1 Description of the Problem

A single indivisible object is to be sold to one of $n \geq 2$ potential buyers, whose private valuations are discretely distributed according to a random variable v_i, which takes values v_H with probability $\alpha_H > 0$ and v_L with probability $\alpha_L > 0$ such that $\alpha_H + \alpha_L = 1$. Without loss of generality, we assume $v_H > v_L > 0$, so that v_H and v_L denote valuations of high-type (eager) and low-type (reluctant) buyers, respectively. Buyer valuations are distributed independently and identically. Buyers are risk-averse and have a constant measure of absolute risk aversion (CARA). In particular, their preferences are represented by a utility function $u(\omega) = -\frac{e^{-r\omega}}{r}$, where $r(> 0)$ measures the level of risk aversion. Note that, $u'(.) > 0$ and $u''(.) < 0$. Specifically, if a buyer with valuation v wins the object and incurs a *net* payment of τ then his utility is $u(v - \tau) = -\frac{e^{-r(v-\tau)}}{r}$. The seller is risk-neutral and her valuation for the object is zero. Both the seller and the buyers are expected utility maximizers.

The seller's problem is to design a selling scheme that maximizes her expected revenue.[24] Such a scheme generally consists of a message set, $M = M_1 \times \cdots \times M_n$, and an outcome function, $\psi : M \rightarrow \tilde{A}$, that maps the list of messages, $m \in M$, into a possibly random allocation $\tilde{a} \in \tilde{A} = \tilde{A}_1 \times \cdots \times \tilde{A}_n$.[25] Buyers' behavior is described by a Bayesian Nash equilibrium, $s = (s_1, ..., s_n)$, where $s_b : \Theta_b \rightarrow M_b$ is the equilibrium strategy of buyer b; $s_b(\theta_b)$ representing the message that maximizes buyer b's expected utility given that his type is θ_b and all buyers other than him follow the equilibrium

auction is likely to allocate the goods inefficiently.

[24] Milgrom [23] defines an auction to be a mechanism (scheme) to allocate resources among a group of bidders. Therefore, we use these three terms interchangeably.

[25] An allocation consists of a *decision* about who is going to get which object(s) and possibly negative monetary *transfers* from buyers to the seller.

strategy.[26] So, any selling scheme, in a given equilibrium, will result in an outcome represented by $\psi(s_1(\theta_1), ..., s_n(\theta_n))$, if the buyers' type profile is $(\theta_1, ..., \theta_n)$.

Alternatively, when looking for the optimal selling scheme, attention can be restricted to the *revelation schemes* in which the message space is the type space, Θ. This is because any allocation, $\psi(s_1(\theta_1), ..., s_n(\theta_n))$, resulting from an equilibrium of an arbitrary selling scheme can also be obtained in a revelation scheme in which the outcome is determined via the composite function $\psi \circ s : \Theta \to \tilde{A}$ and truth-telling is an equilibrium (Revelation Principle).[27] Thus, the seller's problem can be reduced to finding the optimal revelation scheme in which the buyers are willing to participate (*individual rationality)* and have incentive to truthfully report their type (*incentive compatibility)*.

Given a profile of reports, a selling scheme must, most generally, assign each buyer a probability of winning, a payment in case he wins and another payment in case he loses. That is, the outcome is determined by functions of the form $\psi_b(m) = (p_b(m), \tilde{t}_b^w(m), \tilde{t}_b^l(m))$ for $b = 1, ..., n$, where tildes represent the possibility that the payment functions are random. Since there is only one object for sale, a feasible scheme must satisfy $\sum_{b=1}^{n} p_b(m_1, ..., m_n) \leq 1$ for all $(m_1, ..., m_n)$.

Given an equilibrium, we can calculate buyer b's *expected* probability of winning and his expected random payments in case of winning and losing, respectively, as

$$\rho_b(m_b) = E_{-b}[p_b(m) \mid m_b] \tag{1}$$

$$\tilde{\tau}_b^w(m_b) = E_{-b}[\tilde{t}_b^w(m) \mid m_b] \tag{2}$$

$$\tilde{\tau}_b^l(m_b) = E_{-b}[t_b^l(m) \mid m_b]. \tag{3}$$

Since buyers are *ex ante* identical, only the schemes that treat them symmetrically need to be considered. This is because, for any asymmetric scheme, we can construct a symmetric scheme that generates the same revenue as the proposed asymmetric scheme. Symmetric schemes satisfy

[26] In this section, each type of a buyer corresponds to a possible valuation , namely $\Theta_j = \{v_H, v_L\}$ for all $j = 1, ..., n$, whereas, in the next section, there are four different types of buyers. That is, $\Theta_j = \{HH, HL, LH, LL\}$ for all $j = 1, ..., n$, where the first (second) letter in each type represents buyer j's value for object A (B).

[27] See Myerson [24].

the following condition:

$$\text{For any } b, b' \in \{1, ..., n\} \text{ and any } m, m' \in M,$$
$$\psi_b(m) = \psi_{b'}(m')$$
$$\text{if } m_b = m'_{b'}, m_{b'} = m'_b, \text{ and for all } b'' \neq b, b' \; m_{b''} = m'_{b''}.$$

Therefore, in a symmetric scheme, the *expected* probability and the *expected* payments of two different buyers submitting the same message are equal. Hence, we can drop the subscript on each of the functions in 1-3. Describing a selling scheme from the perspective of an arbitrary buyer, using $\rho(.), \tilde{\tau}^w(.)$, $\tilde{\tau}^l(.)$, is called *reduced form representation.*

Three points need to be emphasized about our approach to solving the seller's problem. First, using the Revelation Principle, we consider only the revelation schemes that satisfy two sets of conditions: individual rationality and incentive compatibility.

Second, we construct the optimal auction in reduced form. We justify this by imposing another set of conditions called *implementability conditions.*[28] These conditions make sure that the reduced form probability, $\rho(.)$, is *implementable*, that is, they make sure that there exists a symmetric auction with actual allocation probabilities, $p(.)$, which satisfies

$$\rho(m_b) = E[p(m) \mid m_b]. \tag{4}$$

The final point is that we initially consider only the schemes in which the *expected* payments contingent on winning and losing are nonrandom. In other words, we first construct the optimal

[28] Border [6] states the necessary and sufficient conditions, for the reduced form probabilities to be implementable. We include the proposition for easy reference:

Let (S, Υ) be a measurable space of possible types of bidders and $\lambda(.)$ be a probability measure on S. Define an auction to be a measurable function $p : S^n \to [0,1]^n$ satisfying $\sum_{i=1}^n p^i(s) \leq 1$ for all $s \in S^n$. Define an auction to be symmetric if $p^i(s)$ is independent of i. Given an auction, define

$$\rho^i(s_i) = \int_{S^{n-1}} p(s_1, ..., s_n) d\lambda(s_1, ., s_{i-1}, s_{i+1}, .., s_n)$$

to be the probability that a buyer i wins when he reports his type as s_i.

Then ρ is implementable by a symmetric auction if and only if for each measurable set of types $A \in \Upsilon$, the following inequality is satisfied:

$$\int_A \rho(s) d\lambda(s) \leq \frac{1 - \lambda(A^c)^n}{n}$$

Furthermore, if S is a topological space and λ is a regular Borel probability on S, then Υ may be replaced by either the open subsets or the closed subsets of S.

scheme within the class of schemes for which $\tilde{\tau}^w(.)$ and $\tilde{\tau}^l(.)$ are deterministic. (So, we drop the tildes.) Later, in proposition 6, we establish that this scheme is also optimal among *all* selling schemes, including those that assign random payments.

To summarize, the seller's problem is to construct the optimal revelation scheme, the reduced form of which can be represented by six variables, $\{\rho_i, \tau_i^w, \tau_i^l\}_{i=H,L}$, where $\rho_i \in [0,1]$ denotes the probability that a buyer wins the object when he reports a valuation of v_i, and $\tau_i^w, \tau_i^l \in \mathbb{R}$ denote the *net deterministic* payments that the same type of buyer incurs when he wins and loses the object, respectively. As mentioned above three sets of conditions are imposed:

If a buyer with valuation v_i reports v_j then his utility is equal to $\rho_j u(v_i - \tau_j^w) + (1-\rho_j)u(-\tau_j^l)$. Thus, buyers truthfully reveal their valuations if the auction satisfies the following two *incentive compatibility conditions*:

$$\begin{aligned} \rho_H u(v_H - \tau_H^w) + (1-\rho_H)u(-\tau_H^l) &\geq \rho_L u(v_H - \tau_L^w) + (1-\rho_L)u(-\tau_L^l) \\ \rho_L u(v_L - \tau_L^w) + (1-\rho_L)u(-\tau_L^l) &\geq \rho_H u(v_L - \tau_H^w) + (1-\rho_H)u(-\tau_H^l). \end{aligned}$$

Buyers are free to participate in the auction. Thus, participating buyers satisfy the *individual rationality conditions* of the form

$$\begin{aligned} \rho_H u(v_H - \tau_H^w) + (1-\rho_H)u(-\tau_H^l) &\geq u(0) \\ \rho_L u(v_L - \tau_L^w) + (1-\rho_L)u(-\tau_L^l) &\geq u(0). \end{aligned}$$

Finally, the *implementability conditions* take the following form in our binary model:

$$\begin{aligned} n(\alpha_L \rho_L + \alpha_H \rho_H) &\leq 1 && (IM_{\{H,L\}}) \\ n\alpha_H \rho_H &\leq 1 - \alpha_L^n && (IM_{\{H\}}) \\ n\alpha_L \rho_L &\leq 1 - \alpha_H^n. && (IM_{\{L\}}) \end{aligned}$$

One can interpret these conditions as follows: the probability the object is won by a buyer who belongs to a particular subset of the type space should be no greater than the probability that there *is* a buyer who belongs to that subset.[29]

The seller's revenue is the sum of the expected payments made by each buyer. Since buyers are *ex ante* identical the seller's revenue can be written in terms of the expected payments made by an arbitrary buyer (namely, the term in the bracket):

$$\pi = n[\alpha_H(\rho_H \tau_H^w + (1-\rho_H)\tau_H^l) + \alpha_L(\rho_L \tau_L^w + (1-\rho_L)\tau_L^l)].$$

To sum up, the seller's problem is to choose a reduced form scheme, $\{\rho_i, \tau_i^w, \tau_i^l\}_{i=H,L}$, that maximizes π subject to the two *incentive compatibility conditions*, the two *individual rationality conditions*, and the three *implementability conditions*.

For convenience, we define $c_i = e^{-rv_i}$ and $y_i^k = e^{r\tau_i^k}$. Note that, $0 < c_H < c_L < 1$ and $y_i^k > 0$ for all i and k. So, we can rewrite the seller's problem as

$$\max_{\{\rho_i, y_i^w, y_i^l\}_{i=H,L}} \pi = \frac{n}{r}[\alpha_H(\rho_H \ln y_H^w + (1-\rho_H)\ln y_H^l) + \alpha_L(\rho_L \ln y_L^w + (1-\rho_L)\ln y_L^l)] \tag{5}$$

subject to

$$\begin{aligned}
\rho_H c_H y_H^w + (1-\rho_H) y_H^l &\leq \rho_L c_H y_L^w + (1-\rho_L) y_L^l && (IC_H)\\
\rho_L c_L y_L^w + (1-\rho_L) y_L^l &\leq \rho_H c_L y_H^w + (1-\rho_H) y_H^l && (IC_L)\\
\rho_H c_H y_H^w + (1-\rho_H) y_H^l &\leq 1 && (IR_H)\\
\rho_L c_L y_L^w + (1-\rho_L) y_L^l &\leq 1 && (IR_L)\\
n(\alpha_L\rho_L + \alpha_H\rho_H) &\leq 1 && (IM_{\{H,L\}})\\
n\alpha_H\rho_H &\leq 1-\alpha_L^n && (IM_{\{H\}})\\
n\alpha_L\rho_L &\leq 1-\alpha_H^n && (IM_{\{L\}})
\end{aligned}$$

[29] Armstrong [1] alternatively calls these conditions *resource constraints*.

and the non-negativity conditions $\rho_H, \rho_L \geq 0$.

For convenience, we refer to the left-hand side of the inequalities in IR_H and IR_L as D_H and D_L, respectively. Similarly, right hand side of IC_H and IC_L are referred to as D_H^L and D_L^H, respectively. The subscripts denote a buyer's actual type, whereas superscripts denote the type he is imitating.

2.2 Solution to the Problem

Since $c_L > c_H$, IC_H and IR_L together imply IR_H.[30] Hence, this condition is redundant. For now, we also ignore IC_L when we solve the seller's problem. That is, we suppose that the low-type buyers do not have the incentive to misrepresent their types. Below, in proposition 5, we prove that this is indeed the case.

Definition 1 *The relaxed problem is defined to be a design problem that ignores the upward incentive constraints.*

The following lemma shows that when only the downward incentive conditions are considered, high-type's incentive condition and low-type's individual rationality condition must be binding.

Lemma 1 *In the relaxed problem, where IC_L is ignored, the constraints IC_H and IR_L must be binding.*

The seller may want to increase her revenue by excluding the low-type buyers from the auction if, for a given distribution of types, their valuation is small enough compared to that of the high-type buyers.[31] This results in an inefficiency, because with positive probability the seller keeps the object even if all buyers value the object more highly than her.

Inefficiency may also be due to a misallocation of the objects. To be consistent with Armstrong [1], we focus only on the latter kind of inefficiency, by assuming that the goods are always sold, i.e. $\rho_L > 0$.[32] In this case, it is optimal for the seller to leave informational rents to the high-type buyers.

[30] $D_H \leq D_H^L \leq D_L \leq 1$, where the second inequality is due to $c_H < c_L$.

[31] The same behavior is also observed when a monopolist implements second-degree price discrimination.

[32] Clearly, high-type buyers should not be excluded from participating in the auction if revenue is maximized. That is, ρ_H must be strictly positive. If not, then the incentive conditions would imply $\rho_L c_L \leq \rho_L c_H$, and since $c_L > c_H$ this in turn would imply $\rho_L = 0$, meaning the good is not sold, at all. Yet, the seller can always guarantee a positive profit by posting a fixed price of $v_L > 0$.

Lemma 2 *At the optimum, if the low-type buyers are not excluded from the auction, then IR_H must be slack.*

The following proposition states that it is not optimal to impose any risk on the high-type buyers. The risk is fully eliminated for them.

Proposition 1 *High-type buyers are fully insured against the risk of losing the object.*

Through insurance, a high-type's marginal utility of income in cases of winning and losing is made the same. Eliminating the risk rewards the high-type buyer for revealing his true type.

If the seller does not pay informational rents to the high type buyer ($\tau_H^w = v_H$), perfect (full) insurance requires that the seller sets the high type buyer's payment contingent on losing equal to zero ($\tau_H^l = 0$) in order to keep him at the same level of utility. However, when there is an information gap between the seller and the buyers, high-type buyers should receive information rent to be active. In this case (i.e. $\tau_H^w < v_H$), perfect insurance requires that the seller compensates the high type buyer ($\tau_H^l > 0$).

Proposition 2 *High-type buyers are compensated if they lose the object.*

Using proposition 1, we can write the seller's profit as

$$\pi = \frac{n}{r}[\alpha_H(\rho_H \ln \frac{1}{c_H} + \ln y_H^l) + \alpha_L(\rho_L \ln \frac{y_L^w}{y_L^l} + \ln y_L^l)] \tag{6}$$

Note that, since $0 < c_H < 1$, the seller's profit is strictly increasing with respect to ρ_H. Thus, given the values of other variables, ρ_H must be set as high as possible at the optimum. This implies that either $IM_{\{H\}}$ or $IM_{\{H,L\}}$, or both are binding.

The Kuhn-Tucker conditions with respect to y_L^w and y_L^l can be written as

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y_L^w} &= \alpha_L \rho_L \frac{1}{y_L^w} - \lambda_L \rho_L c_L + \mu_H \rho_L c_H = 0 \\
\frac{\partial \mathcal{L}}{\partial y_L^l} &= \alpha_L (1-\rho_L) \frac{1}{y_L^l} - \lambda_L (1-\rho_L) + \mu_H (1-\rho_L) = 0.
\end{aligned}$$

Since $\alpha_L \rho_L \frac{1}{y_L^w} > 0$, these two equations together yield

$$\frac{y_L^w}{y_L^l} = \frac{\lambda_L - \mu_H}{\lambda_L c_L - \mu_H c_H}. \tag{7}$$

Note that the right-hand side of equation 7 is smaller than $\frac{1}{c_H}$. So, we have

$$\frac{y_L^w}{y_L^l} < \frac{1}{c_H}. \tag{8}$$

This condition has a very nice implication: At the optimum, iso-revenue curve must be flatter than the line corresponding to the implementability condition $IM_{\{H,L\}}$.[33]

Thus, $IM_{\{H\}}$ and $IM_{\{H,L\}}$ are both binding and the optimal allocation probabilities can be calculated as

$$\rho_H = \tfrac{1-\alpha_L^n}{n\alpha_H}; \quad \rho_L = \tfrac{\alpha_L^{n-1}}{n} \tag{9}$$

which is the point where the iso-revenue curve (6) is tangent to the feasible set that is bound by the implementability conditions (Figure 1)

It is not surprising to see that the allocation probabilities that we have obtained in 9 are the same as those in the risk-neutral environment. The optimal allocation is monotonic with respect to buyer types in either case.

Note that, $n\alpha_L\rho_L = \alpha_L^n$, meaning the probability that the object is won by a low-type buyer is equal to the probability that all buyers are low-type. In other words, the object is won by a high-type buyer whenever there is one. Hence, the proposition follows.

Proposition 3 *The optimal auction is weakly efficient.*

Contrary to the insurance provided to the high-type buyers, the seller confronts the low-type buyers with risk by making their marginal utilities vary in cases of winning and losing. In this circumstance, a high-type buyer who considers imitating the low-type buyers would face a greater risk, and will eventually reveal his own true valuation. Hence, it is optimal for the seller to relax the high-type buyer's incentive constraint and not to offer insurance to the low-type buyers. The following proposition states that at the optimum low-type buyers' marginal utility of income is greater when he wins the object than when he loses it.

Proposition 4 *Low-type buyers are better off winning than losing:* $c_L y_L^w < y_L^l$. *Moreover, in*

[33]This condition is equivalent to $\frac{\alpha_L \ln(y_L^w/y_L^l)}{\alpha_H \ln(1/c_H)} < \frac{\alpha_L}{\alpha_H}$, where the left hand side of the inequality is slope of the iso-profit curve and the right hand side is the slope of the line corresponding to the implementability condition $IM_{\{H,L\}}$.

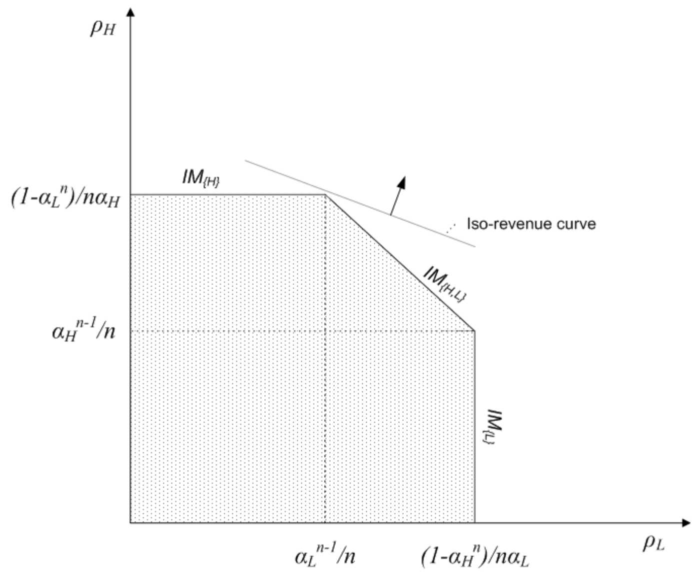

Figure 1: Single object - Optimal allocation probabilities are the same regardless of buyers' risk attitude.

case of losing the object, they incur a payment that is less than what they would pay if they win: $1 < y_L^l < y_L^w$.

Next, we show that the solution to the relaxed problem also solves the full problem that does not ignore IC_L.

Proposition 5 *Low-type buyers do not have the incentive to misrepresent their type. That is,* IC_L *is slack.*

The reduced form of the revelation scheme that we've constructed above is optimal within the class of schemes in which the *expected* payments contingent on winning and losing are deterministic. Finally, we establish that making t_i^w and t_i^l random has a negative effect on seller's revenue.

Proposition 6 *If buyer preferences are represented by CARA, then, in an optimal auction, the payments,* t_i^w *and* t_i^l*, must be deterministic.*

Remark 1 *Above proposition also implies that it is not profitable for the seller to condition the payments made by a buyer on the realizations of his opponents' types.*

3 Optimal Multi-object Auctions

3.1 Description of the Problem

Now, there are two nonidentical objects, denoted A and B, to be sold to $n \geq 2$ buyers. The seller's valuation for both objects is zero, whereas buyer valuations are random and described by a pair (v^A, v^B), where v^o denotes the buyer's valuation for object o. Suppose that $v^o \in \{v^o_H, v^o_L\}$, where the subscripts denote whether the buyer is of high-type (H) or low-type (L). Thus, we assume $v^o_H - v^o_L > 0$. There are four types of buyers corresponding to the four possibilities $(v^A_H, v^B_H), (v^A_H, v^B_L), (v^A_L, v^B_H)$ and (v^A_L, v^B_L). Using a slightly shorter notation, we define the set of possible types as $\Theta = \{HH, HL, LH, LL\}$. A typical element of this set is denoted with ij, where i represents a buyer's valuation for object A and j represents his valuation for object B. Types are independently and identically distributed across buyers according to a probability measure α over Θ, so that the probability that a buyer is of type ij is represented by α_{ij}. The marginal probability that a buyer has a high value for object A is denoted with $\alpha^A_H = \alpha_{HH} + \alpha_{HL}$. Similarly, $\alpha^A_L = \alpha_{LH} + \alpha_{LL}$ denotes the marginal probability that the buyer has a low value for object A. In the same fashion, we define $\alpha^B_H = \alpha_{HH} + \alpha_{LH}$ and $\alpha^B_L = \alpha_{HL} + \alpha_{LL}$ to be the marginal probabilities that the buyer has a high and low value for object B, respectively.

Each buyer is risk-averse and has preferences represented by the common CARA utility function of the form $u(\omega) = -\frac{e^{-r\omega}}{r}$, where $r > 0$. In the event that a buyer wins object(s) of a (total) value v and incurs a net payment τ, his utility will be equal to $u(v - \tau)$. For example, if a buyer wins only object A when his valuation for that object is v^A_L and incurs a net payment τ^A then his utility is equal to $u(v^A_L - \tau^A)$. Similarly, if a buyer of type HL wins both objects and incurs a net payment τ^{AB} then his utility will be $u(v^A_H + v^B_L - \tau^{AB})$. Both the seller and the buyers are expected utility maximizers.[34]

The seller's problem is to design a selling scheme that maximizes her revenue. In view of the Revelation Principle, we solve this problem within the class of revelation schemes which satisfy incentive compatibility and individual rationality constraints.[35] Furthermore, as justified in the

[34] We assume that there are no economies of scope in the production of the bundle nor are there complementarities in the consumption of the bundle. We make this assumption so as to isolate the role that bundling has on the seller's ability to extract the consumer surplus.

[35] Remember that in a revelation scheme, buyers are asked to report their types.

previous section, among the revelation schemes, we focus only on the symmetric ones in which the buyers of the same type are treated the same.

Let n_{ij} be the number of buyers of type ij and $\eta = (n_{HH}, n_{HL}, n_{LH}, n_{LL})$ be the vector representing the profile of reports where $\sum_{ij\in\Theta} n_{ij} = n$. Then, a symmetric revelation scheme can most generally be described with two sets of rules:

- a *decision rule*, $p_{ij}^{k}(\eta)$, that assigns each type $ij \in \Theta$ probabilities of realizing possible events $k = A, B, AB, O$, for each profile of reports η. Given η, the decision rule must satisfy

$$\begin{aligned} \sum\nolimits_{ij\in\Theta} n_{ij}[p_{ij}^{A}(\eta) + p_{ij}^{AB}(\eta)] &\leq 1 \quad (10)\\ \sum\nolimits_{ij\in\Theta} n_{ij}[p_{ij}^{B}(\eta) + p_{ij}^{AB}(\eta)] &\leq 1 \quad (11)\\ p_{ij}^{A}(\eta) + p_{ij}^{B}(\eta) + p_{ij}^{AB}(\eta) + p_{ij}^{O}(\eta) &= 1 \quad \forall ij \in \Theta \quad (12) \end{aligned}$$

- a *payment rule*, $\tilde{t}_{ij}^{k}(\eta)$, that, for each profile of reports η, assigns each type $ij \in \Theta$ possibly random payments to be made to the seller at each possible event $k = A, B, AB, O$.

The decision rule specifies the probability that a buyer b of type ij realizes the valuations v_i^A, v_j^B, $v_i^A + v_j^B$ or 0. We abuse the notation and list these four events respectively as:

Event A	- winning *only* object A
Event B	- winning *only* object B
Event AB	- winning *both* object A and object B
Event O	- winning neither object.

Remember from Armstrong [1] that the *risk-neutral* buyers are only interested in the marginal probabilities of winning the objects. For risk-averse buyers, on the other hand, the correlation between the events of winning object A and object B matters. The decision rule in the above specification takes this into consideration.

Note that, $p_{ij}^{A}(\eta) + p_{ij}^{AB}(\eta)$, in 10, represents the marginal probability of winning object A which we shortly denote with $\hat{p}_{ij}^{A}(\eta)$. Similarly, $p_{ij}^{B}(\eta) + p_{ij}^{AB}(\eta)$, in 11, represents the marginal probability of obtaining object B which is denoted with $\hat{p}_{ij}^{B}(\eta)$. Thus, conditions 10 and 11 are the resource constraints representing the fact that there is only one unit of each object. Condition 12 states that the events A, B, AB and O are all inclusive.

Although the payment rule allows the seller impose random payments, when we solve the seller's problem, we assume $\tilde{t}^k_{ij}(\eta) = \tau^k_{ij}$ where $\tau^k_{ij} \in \mathbb{R}$ for all $ij \in \Theta$ and $k = A, B, AB, O$, and characterize the optimal scheme within the class of schemes that assign deterministic payments. We will show later that imposing random payments to each type ij under each event k cannot improve the seller's revenue.

Now, define an ij type buyer's *expected* probability of realizing the event $k = A, B, AB, O$ as

$$\rho^k_{ij} = \sum_{n_{HH}=0}^{n} \sum_{n_{HL}=0}^{n-n_{HH}} \sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} p^k_{ij}(n_{HH}, n_{HL}, n_{LH}, n_{LL}) \Psi \frac{n_{ij}}{\alpha_{ij}} \tag{13}$$

where $\Psi = \frac{(n-1)! \alpha_{HH}^{n_{HH}} \alpha_{HL}^{n_{HL}} \alpha_{LH}^{n_{LH}} \alpha_{LL}^{n_{LL}}}{n_{HH}! n_{HL}! n_{LH}! n_{LL}!}$. For any $n_{ij} > 0$, $\Psi \frac{n_{ij}}{\alpha_{ij}}$ denotes the probability that the buyer profile is $\eta = (n_{HH}, n_{HL}, n_{LH}, n_{LL})$ given that there is one ij in that profile (of course, conditional on incentive constraints hold).[36]

The reduced form of a symmetric revelation scheme, then, can be represented with

$$\{\rho^A_{ij}, \rho^B_{ij}, \rho^{AB}_{ij}, \rho^O_{ij}, \tau^A_{ij}, \tau^B_{ij}, \tau^{AB}_{ij}, \tau^O_{ij}\}_{ij \in \Theta}.$$

ρ^A_{ij} and ρ^B_{ij} are type ij's *expected* probability of winning object A or B, *alone*; whereas ρ^{AB}_{ij} is his probability of winning *both* objects. Apparently, $\rho^O_{ij} = 1 - \rho^A_{ij} - \rho^B_{ij} - \rho^{AB}_{ij}$ represents the probability of winning neither object. τ^k_{ij} is the net deterministic payment that type ij must incur if event k occurs.

Then, the utility of a buyer of type ij who misrepresents his type as $i'j'$ is

$$\rho^A_{i'j'} u(v^A_i - \tau^A_{i'j'}) + \rho^B_{i'j'} u(v^B_j - \tau^B_{i'j'}) + \rho^{AB}_{i'j'} u(v^A_i + v^B_j - \tau^{AB}_{i'j'}) + \rho^O_{i'j'} u(-\tau^O_{i'j'}).$$

Let $c^o_i = e^{-rv^o_i}$ for $o = A, B$ and $i = H, L$ and $y^k_{ij} = e^{r\tau^k_{ij}}$ for $k \in K = \{A, B, AB, O\}$ and $ij \in \Theta$. Then a scheme is *individually rational* if, for each type $ij \in \Theta$,

$$D_{ij} \equiv \rho^A_{ij} c^A_i y^A_{ij} + \rho^B_{ij} c^B_j y^B_{ij} + \rho^{AB}_{ij} c^A_i c^B_j y^{AB}_{ij} + \rho^O_{ij} y^O_{ij} \leq 1.$$

[36] The multinomial distribution is used.

An auction is *incentive compatible* if, for any $ij \in \Theta$ and $i'j' \in \Theta \setminus \{\, ij\}$,

$$D_{ij} \leq \rho^A_{i'j'} c^A_i y^A_{i'j'} + \rho^B_{i'j'} c^B_j y^B_{i'j'} + \rho^{AB}_{i'j'} c^A_i c^B_j y^{AB}_{i'j'} + \rho^O_{i'j'} y^O_{i'j'} \equiv D^{i'j'}_{ij}.$$

The seller's revenue can, then, be written in terms of the expected payment of an arbitrary buyer, namely the term in brackets:

$$\pi = n[\sum_{ij\in\Theta}\{\alpha_{ij}\sum_{k\in K}\rho^k_{ij}\tau^k_{ij}\}]. \tag{14}$$

Note that, $\tau^k_{ij} = \frac{1}{r}\ln y^k_{ij}$. Then, if the reduced form probabilities are 'implementable' we can write the seller's problem in reduced form as

$$\max_{\{\rho^k_{ij}, y^k_{ij}\}_{ij\in\Theta, k\in K}} \frac{n}{r}\sum_{ij\in\Theta}\{\alpha_{ij}\sum_{k\in K}\rho^k_{ij}\ln y^k_{ij}\} \tag{SP}$$

subject to

$$D_{ij} \leq 1 \qquad ij \in \Theta \tag{15}$$

$$D_{ij} \leq D^{i'j'}_{ij} \quad ij \in \Theta,\ i'j' \in \Theta\setminus\{ij\} \tag{16}$$

Since the buyers are risk-averse, the correlation between the events of winning object A (namely, event $A \cup AB$) and object B (namely, event $B \cup AB$) matters for the buyers and also for the seller through 14. Thus, Border's [6] theorem does not apply to this problem.[37] As it is also mentioned in Armstrong [1], the conditions that we need to impose to ensure that the reduced form probabilities are implementable are not clear. For this reason, different from the previous section, we aim to construct the actual probabilities, $p^k_{ij}(\eta)$, $\forall ij \in \Theta$, $k = A, B, AB$ and $\forall\eta$.[38] Given a payment rule, the optimality of a decision rule will be analyzed as follows: For any modification of $p^k_{ij}(\eta)$, we will first describe how *expected* probabilities ρ^k_{ij} will be affected. Then, we figure out whether the incentive constraints in 16 and individual rationality constraints in 15 hold and whether the objective function (SP) increases after the modification. To demonstrate how this works, we borrow

[37] See footnote 28.

[38] Given ij and η, $p^O_{ij}(\eta)$ can be calculated using 13 and the values of $p^A_{ij}(\eta)$, $p^B_{ij}(\eta)$, and $p^{AB}_{ij}(\eta)$ are found.

the following example from Menicucci [22]:

Suppose for a given profile of reports with $n_{HH} \geq 1$ and $n_{LH} \geq 1$ each type wins object A with probability $\frac{1}{n_{HH}}$ and each type LH wins object B with probability $\frac{\beta}{n_{LH}}(0 < \beta \leq 1)$. Note that from 13, this generates a contribution to ρ^B_{LH} equal to

$$\frac{\beta}{n_{LH}}\Psi\frac{n_{LH}}{\alpha_{LH}}.$$

Consider reducing β by $\Delta\beta > 0$ while increasing by $\Delta\beta$ the probability that the same buyer of type HH winning object A will also win object B. Then,

$$\begin{aligned}
\Delta\rho^B_{LH} &= -\frac{\Delta\beta}{n_{LH}}\Psi\frac{n_{LH}}{\alpha_{LH}} \\
\Delta\rho^A_{HH} &= -\frac{\Delta\beta}{n_{HH}}\Psi\frac{n_{HH}}{\alpha_{HH}} = -\Delta\rho^{AB}_{HH}.
\end{aligned}$$

So, $\Delta\rho^{AB}_{HH} = -\Delta\rho^A_{HH} = -\frac{\alpha_{LH}}{\alpha_{HH}}\Delta\rho^B_{LH}$. We can then evaluate the profitability of reducing β since the seller's profit function and the constraints are linear with respect to the *expected* probabilities.

3.2 Solution to the problem

Before we attempt to solve problem SP, note that, since $0 < c_H < c_L$, incentive compatibility conditions imply that among the individual rationality conditions only the one corresponding to type LL matters.

3.2.1 The relaxed problem

Using the same approach as in Armstrong [1], we first solve the seller's problem considering only the five downward incentive constraints, that ensure that a buyer does not underreport his valuation for an object. We show *ex post* that the remaining constraints are satisfied (Propositions 14 and 15).

Thus, the seller solves

$$
\begin{aligned}
\max \quad & \alpha_{HH}\{\rho_{HH}^{A}\ln y_{HH}^{A}+\rho_{HH}^{B}\ln y_{HH}^{B}+\rho_{HH}^{AB}\ln y_{HH}^{AB}+\rho_{HH}^{O}\ln y_{HH}^{O}\} \\
& +\alpha_{HL}\{\rho_{HL}^{A}\ln y_{HL}^{A}+\rho_{HL}^{B}\ln y_{HL}^{B}+\rho_{HL}^{AB}\ln y_{HL}^{AB}+\rho_{HL}^{O}\ln y_{HL}^{O}\} \\
& +\alpha_{LH}\{\rho_{LH}^{A}\ln y_{LH}^{A}+\rho_{LH}^{B}\ln y_{LH}^{B}+\rho_{LH}^{AB}\ln y_{LH}^{AB}+\rho_{LH}^{O}\ln y_{LH}^{O}\} \\
& +\alpha_{LL}\{\rho_{LL}^{A}\ln y_{LL}^{A}+\rho_{LL}^{B}\ln y_{LL}^{B}+\rho_{LL}^{AB}\ln y_{LL}^{AB}+\rho_{LL}^{O}\ln y_{LL}^{O}\}
\end{aligned}
$$

subject to

$$
\rho_{LL}^{A}c_{L}^{A}y_{LL}^{A}+\rho_{LL}^{B}c_{L}^{B}y_{LL}^{B}+\rho_{LL}^{AB}c_{L}^{A}c_{L}^{B}y_{LL}^{AB}+\rho_{LL}^{O}y_{LL}^{O}\leq 1 \qquad (IR_{LL})
$$

$$
\begin{aligned}
& \rho_{LH}^{A}c_{L}^{A}y_{LH}^{A}+\rho_{LH}^{B}c_{H}^{B}y_{LH}^{B}+\rho_{LH}^{AB}c_{L}^{A}c_{H}^{B}y_{LH}^{AB}+\rho_{LH}^{O}y_{LH}^{O} \\
\leq \quad & \rho_{LL}^{A}c_{L}^{A}y_{LL}^{A}+\rho_{LL}^{B}c_{H}^{B}y_{LL}^{B}+\rho_{LL}^{AB}c_{L}^{A}c_{H}^{B}y_{LL}^{AB}+\rho_{LL}^{O}y_{LL}^{O} && (IC_{LH}^{LL}) \\
& \rho_{HL}^{A}c_{H}^{A}y_{HL}^{A}+\rho_{HL}^{B}c_{L}^{B}y_{HL}^{B}+\rho_{HL}^{AB}c_{H}^{A}c_{L}^{B}y_{HL}^{AB}+\rho_{HL}^{O}y_{HL}^{O} \\
\leq \quad & \rho_{LL}^{A}c_{H}^{A}y_{LL}^{A}+\rho_{LL}^{B}c_{L}^{B}y_{LL}^{B}+\rho_{LL}^{AB}c_{H}^{A}c_{L}^{B}y_{LL}^{AB}+\rho_{LL}^{O}y_{LL}^{O} && (IC_{HL}^{LL}) \\
& \rho_{HH}^{A}c_{H}^{A}y_{HH}^{A}+\rho_{HH}^{B}c_{H}^{B}y_{HH}^{B}+\rho_{HH}^{AB}c_{H}^{A}c_{H}^{B}y_{HH}^{AB}+\rho_{HH}^{O}y_{HH}^{O} \\
\leq \quad & \rho_{LL}^{A}c_{H}^{A}y_{LL}^{A}+\rho_{LL}^{B}c_{H}^{B}y_{LL}^{B}+\rho_{LL}^{AB}c_{H}^{A}c_{H}^{B}y_{LL}^{AB}+\rho_{LL}^{O}y_{LL}^{O} && (IC_{HH}^{LL}) \\
& \rho_{HH}^{A}c_{H}^{A}y_{HH}^{A}+\rho_{HH}^{B}c_{H}^{B}y_{HH}^{B}+\rho_{HH}^{AB}c_{H}^{A}c_{H}^{B}y_{HH}^{AB}+\rho_{HH}^{O}y_{HH}^{O} \\
\leq \quad & \rho_{LH}^{A}c_{H}^{A}y_{LH}^{A}+\rho_{LH}^{B}c_{H}^{B}y_{LH}^{B}+\rho_{LH}^{AB}c_{H}^{A}c_{H}^{B}y_{LH}^{AB}+\rho_{LH}^{O}y_{LH}^{O} && (IC_{HH}^{LH}) \\
& \rho_{HH}^{A}c_{H}^{A}y_{HH}^{A}+\rho_{HH}^{B}c_{H}^{B}y_{HH}^{B}+\rho_{HH}^{AB}c_{H}^{A}c_{H}^{B}y_{HH}^{AB}+\rho_{HH}^{O}y_{HH}^{O} \\
\leq \quad & \rho_{HL}^{A}c_{H}^{A}y_{HL}^{A}+\rho_{HL}^{B}c_{H}^{B}y_{HL}^{B}+\rho_{HL}^{AB}c_{H}^{A}c_{H}^{B}y_{HL}^{AB}+\rho_{HL}^{O}y_{HL}^{O}. && (IC_{HH}^{HL})
\end{aligned}
$$

We first establish that it is not optimal to make the expected payments, namely y_{ij}^{k}s, random. This is because if a y_{ij}^{k} is random for an ij and k, then the seller could replace it with its expected value without affecting the incentive conditions (because they are linear in y_{ij}^{k}) and increase her revenue (as the seller's revenue is a concave function of y_{ij}^{k}).

Proposition 7 *If the buyers' preferences are represented by CARA utility function then, in an optimal auction, the expected payments conditional on types and allocation must be deterministic.*

Now, we determine which of the six conditions in the relaxed problem are binding.

Lemma 3 *At the optimum of the relaxed problem, IR_{LL} must be binding.*

Lemma 4 *At the optimum of the relaxed problem, IC_{LH}^{LL} and IC_{HL}^{LL} must be binding.*

Lemma 5 *At the optimum of the relaxed problem, at least one of $IC_{HH}^{LL}, IC_{HH}^{LH}$ and IC_{HH}^{HL} must be binding.*

Using the above lemmata, we write the Lagrangian of the relaxed problem and derive its Kuhn-Tucker conditions with respect to the payments, namely y_{ij}^{k}s Then, we establish the relation among the payments using these Kuhn-Tucker conditions, the details of which we relegate to the appendix.

Similar to the single object case, when a buyer wins an object, say object i, for which he has high valuation, he pays v_H^i more than what he would have paid if he lost that object. The intuition for proposition 1 also applies here.

If the objects are not limited, the seller can make the high-type buyer's probability of obtaining the object(s) equal to one in order to reward him for revealing his true valuation(s). However, when the objects are limited, the same rewarding strategy does not work because each high-type buyer may face the risk of losing the object(s) to another high-type buyer and hence, the marginal utility of income may differ in the events of winning and losing. The resource constrained seller, however, can reward a high-type buyer by offering perfect insurance and increase her revenue. Note that, if buyers are risk neutral, there is no insurance issue. In other words, if the buyers are risk averse the seller has an additional tool to extract more revenue from them when compared to risk neutral environment.

Proposition 8 *Each buyer is perfectly insured against the risk of losing the object(s) for which he has high valuation.*

When it comes to the LL-type buyers, the seller faces the following predicament: to extract more revenue from the LL-type buyer by offering insurance and to exploit the risk-bearing of the buyers who have high-valuation for one or both of the objects to screen them. At the optimum, the marginal benefit of exploiting high-type buyers risk bearing exceeds the marginal cost of not offering insurance to LL-type buyers. Moreover, LL-type buyers pay penalty when he loses both objects which further deters high-type buyers from behaving as if they are LL-type.

Proposition 9 *Suppose that type LL is not excluded from the auction. Then, he incurs a positive payment if he loses both objects.*

With the help of the preceding results, the seller's problem can be written as

$$
\begin{aligned}
&[\alpha_{HH}\hat{\rho}^A_{HH} + \alpha_{HL}\hat{\rho}^A_{HL}]\ln\frac{1}{c^A_H} + [\alpha_{HH}\hat{\rho}^B_{HH} + \alpha_{LH}\hat{\rho}^B_{LH}]\ln\frac{1}{c^B_H} + \alpha_{HH}\ln y^O_{HH} \\
&+\alpha_{HL}[\hat{\rho}^B_{HL}\ln y^B_{HL} + (1-\hat{\rho}^B_{HL})\ln y^O_{HL}] + \alpha_{LH}[\hat{\rho}^A_{LH}\ln y^A_{LH} + (1-\hat{\rho}^A_{LH})\ln y^O_{LH}] \\
&+\alpha_{LL}[\rho^A_{LL}\ln y^A_{LL} + \rho^B_{LL}\ln y^B_{LL} + \rho^{AB}_{LL}\ln y^{AB}_{LL} + \rho^O_{LL}\ln y^O_{LL}]
\end{aligned}
$$

subject to

$$
\begin{aligned}
D_{LL} &= 1 \\
D^{LL}_{LH} &= \hat{\rho}^A_{LH}c^A_L y^A_{LH} + (1-\hat{\rho}^A_{LH})y^O_{LH} \\
D^{LL}_{HL} &= \hat{\rho}^B_{HL}c^B_L y^B_{HL} + (1-\hat{\rho}^B_{HL})y^O_{HL} \\
y^O_{HH} &= \min\begin{cases} D^{LL}_{HH} \\ \hat{\rho}^A_{LH}c^A_H y^A_{LH} + (1-\hat{\rho}^A_{LH})y^O_{LH} \\ \hat{\rho}^B_{HL}c^B_H y^B_{HL} + (1-\hat{\rho}^B_{HL})y^O_{HL} \end{cases}
\end{aligned}
$$

where $\hat{\rho}^A_{ij} = \rho^A_{ij} + \rho^{AB}_{ij}$ and $\hat{\rho}^B_{ij} = \rho^B_{ij} + \rho^{AB}_{ij}$. Let's call this problem SP'.

Thus, for the optimality of an auction only the following reduced form probabilities matter:

$$\{\hat{\rho}^A_{ij}, \hat{\rho}^B_{ij}\}_{ij=HH,HL,LH}, \{\rho^k_{LL}\}_{k=A,B,AB}$$

Consider a mechanism where, for a given profile, η, both objects are sold with probability one. Then, if the seller modifies the mechanism by increasing $p^k_{ij}(\eta)$ by $\frac{1}{n_{ij}}\varepsilon^k_{ij}$, the following condition must hold:

$$\sum_{ij\in S}(\varepsilon^k_{ij} + \varepsilon^{AB}_{ij}) \leq 0 \quad \text{for} \quad k = A, B.$$

After this modification, ρ^k_{ij} will increase by $\frac{1}{\alpha_{ij}}\varepsilon^k_{ij}\Psi$.

We now establish that the solution to the relaxed problem is weakly efficient. That is, if there is a buyer with high valuation for an object then that object is never sold to a buyer who has low valuation for that object.

Proposition 10 *Let* $\eta = (n_{HH}, n_{LH}, n_{HL}, n_{LL})$ *be the profile of the participating buyers. Then, the solution to the relaxed problem satisfies the following two rules:*

i) For any η *with* $n_{HH} + n_{HL} > 0$, $n_{HH}\hat{p}^A_{HH}(\eta) + n_{HL}\hat{p}^A_{HL}(\eta) = 1$

ii) For any η *with* $n_{HH} + n_{LH} > 0$, $n_{HH}\hat{p}^B_{HH}(\eta) + n_{HL}\hat{p}^B_{LH}(\eta) = 1$.

If there is a buyer who has a high value for object A (B) then with probability one it is given to a buyer who has a high value for it. While proposition 10 states this result in terms of actual probabilities, the following corollary does the same in terms of the expected probabilities.

Corollary 1 *At the optimum of the relaxed problem, reduced form probabilities satisfy*

i) $\alpha_{HH}\hat{\rho}^A_{HH} + \alpha_{HL}\hat{\rho}^A_{HL} = \frac{1}{n}(1 - (\alpha^B_L)^n)$ *and*

ii) $\alpha_{HH}\hat{\rho}^B_{HH} + \alpha_{LH}\hat{\rho}^B_{LH} = \frac{1}{n}(1 - (\alpha^A_L)^n)$.

The next lemma establishes that both objects are sold with probability one, if a buyer's payment contingent on winning an object for which he has low valuation is larger than his payment contingent on losing both objects.

Similar to the previous section, we assume that the seller never keeps the object. We have already established in proposition 10 that the seller does not keep an object whenever there is a buyer who has a high value for it. This requires the probability that an object is won by a buyer who has a low value for it to be equal to the probability that all buyers have low value for it.

$$\begin{aligned} \alpha_{LL}\hat{\rho}^A_{LL} + \alpha_{LH}\hat{\rho}^A_{LH} &= \frac{1}{n}(\alpha^A_L)^n \\ \alpha_{LL}\hat{\rho}^B_{LL} + \alpha_{HL}\hat{\rho}^B_{HL} &= \frac{1}{n}(\alpha^B_L)^n \end{aligned}$$

In terms of actual probabilities, we can write these conditions as

$$\text{For any } \eta \text{ with } n_{HH} + n_{HL} = 0,\ n_{LH}\hat{p}^A_{LH}(\eta) + n_{LL}\hat{p}^A_{LL}(\eta) = 1 \tag{17}$$

$$\text{For any } \eta \text{ with } n_{HH} + n_{LH} = 0,\ n_{HL}\hat{p}^B_{HL}(\eta) + n_{LL}\hat{p}^B_{LL}(\eta) = 1 \tag{18}$$

Proposition 11 *The necessary conditions for 17-18 are* $y_{LH}^A > y_{LH}^O, y_{HL}^B > y_{HL}^O$, *and* $y_{LL}^A, y_{LL}^B, y_{LL}^{AB} > y_{LL}^O$.

Since $D_{HH} = y_{HH}^O \leq 1$, when HH loses both objects he either does not pay anything (i.e. $y_{HH}^O = 1$) or he is compensated (i.e. $y_{HH}^O < 1$).

Proposition 12 *In any mechanism that solves the relaxed problem, if an* HH *type buyer loses both objects then he is compensated.*

This proposition results because the seller needs to provide insurance to type HH. This is a property that carries over from the single unit optimal auction. (Maskin and Riley [19]) They show that when the type space is continuous, the seller provides full insurance (and hence full compensation) only to the highest type but partial insurance to the types that are sufficiently high.

Proposition 13 *In any mechanism that solves the relaxed problem, if all the buyers are of type* LL *(i.e.* $n_{LL} = n$*) then the objects are bundled and each buyer wins the bundle with equal probability. (i.e.* $p_{LL}^{AB}(\eta) = \frac{1}{n}$*).*

An immediate implication of the proposition above is that it is not optimal to sell the goods independently in which case with positive probability the objects may end up in the hands of different LL type buyers. Yet, the proposition has further implications.

When the buyers are risk neutral (Armstrong [1]), depending on how buyers' valuations are correlated across objects, the optimal multi-object auction can take the form of independent auctions, mixed auction or bundling auction. But all of these auction forms allocate the two objects independently and randomly when all buyers are of type LL. This contradicts with the proposition. Therefore, none of these auction forms are optimal when the buyers are risk averse.

Theorem 1 *Whenever the parameter values are such that the relaxed method solves the full problem, the three auction formats that are optimal when the buyers are risk neutral do not maximize the seller's revenue if the buyers are risk averse.*

The main reason as to why we obtain this contradictory result is that the optimal auction forms for the risk neutral buyers do not impose the right amount of risk on type LL. The optimal auction for risk averse buyers, on the other hand, imposes two kinds of risk on this type. The first kind removes the possibility of winning a single object when all buyers are of type LL and the second kind assigns a positive payment if he doesn't win any objects. These two kinds of risk improve the sellers revenue in the following way. The former exploits the risk bearing of the buyers who have high valuation for one or both objects by facing them with even greater risk when imitating LL than the optimal auction for risk neutral buyers. The latter, on the other hand, help the seller collect the penalty fees from more people.

Since the seller probabilistically assesses the buyer valuations (i.e. only *ex ante* probabilities of the type distribution matter) and never keeps the objects by assumption, there always exists a probability that LL type buyers can obtain both objects. This can happen only if all buyers are of type LL. On the other hand, whenever there is a type HH or both HL and LH, then LL cannot win any objects. The following lemma states the conditions under which an LL can obtain a single object.

Lemma 6 *In any mechanism that solves the relaxed problem,*

i) if η is such that $n_{LH}, n_{LL} > 0$ and $n_{LH} + n_{LL} = n$, then object A is sold to an LH type buyer (i.e. $n_{LH}\hat{p}^A_{LH}(\eta) = 1$) if

$$\mu_{LH} < (\frac{\alpha_{HL}}{y^O_{HL}}\frac{y^O_{HH}}{\alpha_{HH}} + 1)(\frac{\alpha_{LL}}{y^O_{LL}}\frac{y^O_{LH}}{\alpha_{LH}} + 1)^{-1} \equiv \gamma_{LH}. \tag{†}$$

Otherwise, an LL type buyer gets object A (i.e. $n_{LL}\hat{p}^A_{LL}(\eta) = 1$).

ii) if η is such that $n_{HL}, n_{LL} > 0$ and $n_{HL} + n_{LL} = n$, then object B is sold to an HL type buyer (i.e. $n_{HL}\hat{p}^B_{HL}(\eta) = 1$) if

$$\mu_{HL} < (\frac{\alpha_{LH}}{y^O_{LH}}\frac{y^O_{HH}}{\alpha_{HH}} + 1)(\frac{\alpha_{LL}}{y^O_{LL}}\frac{y^O_{HL}}{\alpha_{HL}} + 1)^{-1} \equiv \gamma_{HL}. \tag{‡}$$

Otherwise, an LL type buyer gets object B (i.e. $n_{LL}\hat{p}^B_{LL}(\eta) = 1$).

According to the previous lemma, in the optimal auction, if the excess payment that LH makes for object A is larger than that of LL (namely, $t^A_{LH} - t^O_{LH} > t^A_{LL} - t^O_{LL}$), then LH wins object A.

By this lemma, the solution to the relaxed problem depends on the values of γ_{LH} and γ_{HL}. Note that, $\gamma_{LH} \geq 1$ if and only if $\gamma_{HL} \geq 1$. Thus, we can divide the rest of the analysis into three

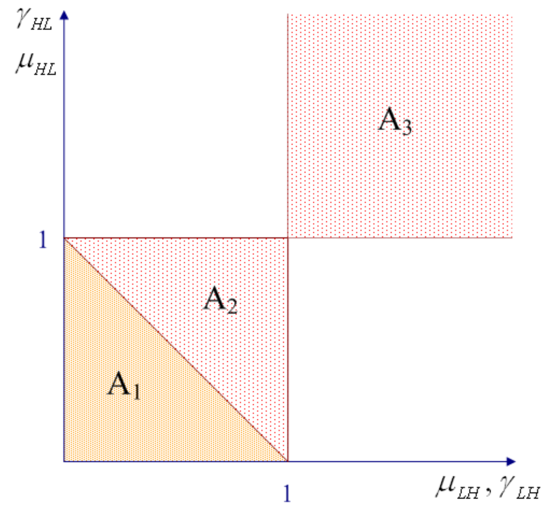

Figure 2: Allocation of each object when all buyers have low valuation for it.

cases (See Figure 2):

- $\gamma_{LH} + \gamma_{HL} \leq 1$ (Region A_1),
- $1 \leq \gamma_{LH} + \gamma_{HL} \leq 2$ (Region A_2),
- $2 \leq \gamma_{LH} + \gamma_{HL}$ (Region A_3).

Remark 2 *Readers should note that the three cases listed above are analogous to those mentioned in Lemma 2 of Armstrong [1]: strong positive correlation, weak positive correlation, and negative correlation, respectively.*

Whether object A (B) is given to an LL or LH (HL) type buyer depends on whether $(\gamma_{LH}, \gamma_{HL})$ falls in region A_1, A_2, or A_3.

3.2.2 Case A1 - Strong positive correlation:

$[\gamma_{LH} + \gamma_{HL} \leq 1]$ We can set

$$\mu_{LL} = 1 - \gamma_{LH} - \gamma_{HL},\ \mu_{LH} = \gamma_{LH},\ \mu_{HL} = \gamma_{HL} \tag{19}$$

In this case, all incentive constraints of type HH are binding. This also implies that the seller is indifferent between LH and LL for object A and between HL and LL for object B.

For any given allocation probabilities, the payments

$$\{y^A_{LL}, y^B_{LL}, y^{AB}_{LL}, y^O_{LL}, y^A_{LH}, y^O_{LH}, y^B_{HL}, y^O_{HL}, y^O_{HH}\}^{39} \tag{20}$$

solve

$$\begin{aligned} \max \qquad & \alpha_{HH} \ln y^O_{HH} + \alpha_{LH}(1-\hat{\rho}^A_{LH}) \ln y^O_{LH} + \alpha_{HL}(1-\hat{\rho}^B_{HL}) \ln y^O_{HL} + +\alpha_{LL}\rho^O_{LL} \ln y^O_{LL} \qquad (21) \\ & +\alpha_{LH}\hat{\rho}^A_{LH} \ln y^A_{LH} + \alpha_{HL}\hat{\rho}^B_{HL} \ln y^B_{HL} + \alpha_{LL}\rho^A_{LL} \ln y^A_{LL} + \alpha_{LL}\rho^B_{LL} \ln y^B_{LL} + \alpha_{LL}\rho^{AB}_{LL} \ln y^{AB}_{LL} \end{aligned}$$

subject to

$$\begin{aligned} \rho^A_{LL} c^A_L y^A_{LL} + \rho^B_{LL} c^B_L y^B_{LL} + \rho^{AB}_{LL} c^A_L c^B_L y^{AB}_{LL} + \rho^O_{LL} y^O_{LL} &= 1 & (22) \\ \rho^A_{LL} c^A_L y^A_{LL} + \rho^B_{LL} c^B_H y^B_{LL} + \rho^{AB}_{LL} c^A_L c^B_H y^{AB}_{LL} + \rho^O_{LL} y^O_{LL} &= \hat{\rho}^A_{LH} c^A_L y^A_{LH} + (1-\hat{\rho}^A_{LH}) y^O_{LH} & (23) \\ \rho^A_{LL} c^A_H y^A_{LL} + \rho^B_{LL} c^B_L y^B_{LL} + \rho^{AB}_{LL} c^A_H c^B_L y^{AB}_{LL} + \rho^O_{LL} y^O_{LL} &= \hat{\rho}^B_{HL} c^B_L y^B_{HL} + (1-\hat{\rho}^B_{HL}) y^O_{HL} & (24) \\ \rho^A_{LL} c^A_H y^A_{LL} + \rho^B_{LL} c^B_H y^B_{LL} + \rho^{AB}_{LL} c^A_H c^B_H y^{AB}_{LL} + \rho^O_{LL} y^O_{LL} &= y^O_{HH} & (25) \\ \hat{\rho}^A_{LH} c^A_H y^A_{LH} + (1-\hat{\rho}^A_{LH}) y^O_{LH} &= y^O_{HH} & (26) \\ \hat{\rho}^B_{HL} c^B_H y^B_{HL} + (1-\hat{\rho}^B_{HL}) y^O_{HL} &= y^O_{HH}. & (27) \end{aligned}$$

By 19 and lemma 6,

$$y^A_{LH} y^O_{LL} = y^A_{LL} y^O_{LH} \text{ and } y^B_{HL} y^O_{LL} = y^B_{LL} y^O_{HL} \tag{28}$$

must also be true. Using equations 22-27, and the two conditions in 28, we can solve for eight of the variables (say, except y^O_{HH}) listed in 20 in terms of y^O_{HH}, the parameters and the reduced form probabilities. After plugging these variables into the objective function 21 we can solve it for y^O_{HH}.

Now, we consider the conditions that we have omitted in the relaxed problem.

Proposition 14 *(Full problem - Case A1) The upward incentive conditions,* IC^{LH}_{LL}*,* IC^{HL}_{LL}*,* IC^{HH}_{LH}

[39] The payments that are not listed in (20), namely $y^B_{LH}, y^{AB}_{LH}, y^A_{HL}, y^{AB}_{HL}, y^A_{HH}, y^B_{HH}, y^{AB}_{HH}$, can be calculated using proposition 15.

and IC^{HH}_{HL} are not binding.

The above proposition states that type LL does not have incentive to imitate the types LH or HL. Moreover, neither type LH nor type HL has incentive to imitate HH.

The conditions IC^{HL}_{LH} and IC^{LH}_{HL} together imply

$$\begin{aligned} & y^O_{HH}\{\frac{\rho^A_{LH}}{\hat{\rho}^A_{LH}}\frac{\Delta^A}{c^A_H} + (\frac{\rho^{AB}_{LH}}{\hat{\rho}^A_{LH}} - \frac{\rho^{AB}_{HL}}{\hat{\rho}^B_{HL}})\frac{c^A_L c^B_H - c^A_H c^B_L}{c^A_H c^B_H} + \frac{\rho^B_{HL}}{\hat{\rho}^B_{HL}}\frac{\Delta^B}{c^B_H}\} \\ \leq \quad & y^O_{HL}\{\rho^A_{HL}\frac{\Delta^A}{c^A_H} - \frac{\rho^{AB}_{HL}}{\hat{\rho}^B_{HL}}(1-\hat{\rho}^B_{HL})\frac{c^A_L c^B_H - c^A_H c^B_L}{c^A_H c^B_H} + \frac{\rho^B_{HL}}{\hat{\rho}^B_{HL}}(1-\hat{\rho}^B_{HL})\frac{\Delta^B}{c^B_H}\} \\ + \quad & y^O_{LH}\{\rho^B_{LH}\frac{\Delta^B}{c^B_H} + \frac{\rho^{AB}_{LH}}{\hat{\rho}^A_{LH}}(1-\hat{\rho}^A_{LH})\frac{c^A_L c^B_H - c^A_H c^B_L}{c^A_H c^B_H} + \frac{\rho^A_{LH}}{\hat{\rho}^A_{LH}}(1-\hat{\rho}^A_{LH})\frac{\Delta^A}{c^A_H}\}. \end{aligned} \tag{29}$$

where $\Delta^i = c^i_H - c^i_L$.

IC^{HH}_{LL} takes the following form:

$$1 \leq y^O_{HH}\{\rho^A_{HH}\frac{c^A_L}{c^A_H} + \rho^B_{HH}\frac{c^B_L}{c^B_H} + \rho^{AB}_{HH}\frac{c^A_L c^B_L}{c^A_H c^B_H} + \rho^O_{HH}\} \tag{30}$$

and $\gamma_{HL} \leq 1$ can be written as

$$\frac{\alpha_{LH}\alpha_{HL}}{\alpha_{LL}\alpha_{HH}} \leq \frac{y^O_{LH}y^O_{HL}}{y^O_{LL}y^O_{HH}} \tag{31}$$

Proposition 15 *The optimal allocation probabilities satisfy the necessary condition 31. Moreover, 29 and 30 are not binding.*

4 Discussion and Concluding Remarks

In a binary model, we show that when the buyers are risk-averse, the optimal auction is weakly efficient. That is, with probability one each object is sold to a buyer who has high valuation for it, if such a buyer exists. Each buyer is perfectly insured against the risk of losing the object(s) for which he has high valuation. Buyers who are eager to win both objects are compensated if they lose both objects; whereas, buyers who have low value for both objects make a positive payment to the seller if they lose both objects. The optimal auction must bundle the two objects when all buyers have low value for both objects. This result allows us to conclude that the auction forms listed in Armstrong [1] are not optimal.

In a more general framework, it has been shown that among all mechanisms for allocating multiple objects that are strongly efficient, incentive compatible, and individually rational, the Vickrey-Clarke-Groves mechanism maximizes the expected revenue.[40] The optimal multi-object auction that we have constructed for risk averse buyers is incentive compatible and individually rational but is only weakly efficient.

While the inefficiency may result either because some types are *ex ante* excluded from participating in the auction, or because of a misallocation, in this paper, we confined ourselves to the first kind of inefficiency, and showed that the latter kind of inefficiency does not occur in an optimal auction. Yet, this result is very sensitive to relaxing the assumption of binary type distribution. Armstrong [1] shows that weak efficiency does not survive once the type space is made continuous.

Finally, we comment on the restrictions of our model. For tractability reasons, we focused only on the case where the buyers' utility function exhibits constant absolute risk aversion. Instead a buyer's utility may exhibit increasing or decreasing absolute risk aversion, in which case the answer to the optimal design problem is not clear. Alternatively, one can also consider the situations where the buyers have different risk attitudes with respect to each good, in addition to that with respect to the wealth level. In that case, one would have to consider a generalization of the Arrow-Pratt theory (Arrow [3] and Pratt [25]) which extends to the case of multi-dimensional risk attitudes. One such generalization is proposed by Kihlstrom and Mirman [13].

Gal-Or [11], considers the case where the risk-averse buyers worry about the possibility of breakdowns. She shows that running "sales" improves the revenue of the single-unit monopolist. This is because the risk-averse buyers tend to buy more frequently than necessary to avoid buying at the higher regular price and to avoid the cost of waiting for the next sales period. Since, in our model, the seller owns only one unit of each object and the objects are not related, our results would not change if the buyers worry about breakdowns. In this case, buyers' concerns can be easily embodied into their valuations.

[40] For a clear and concise discussion of VCG mechanisms see Krishna [14].

Appendix A: Optimal single object auction

The Lagrangian to the relaxed problem can be written as

$$\begin{aligned}
\mathcal{L} &= \pi - \lambda_L(D_L - 1) - \mu_H(D_H - D_H^L) \\
&\quad -\phi_{\{H,L\}}(n\alpha_L\rho_L + n\alpha_H\rho_H - 1) - \phi_{\{H\}}(n\alpha_H\rho_H - 1 + \alpha_L^n) \\
&\quad -\phi_{\{L\}}(n\alpha_L\rho_L - 1 + \alpha_H^n)
\end{aligned}$$

where λ_L and μ_H are the Lagrange multipliers on IR_L and IC_H, respectively, and $\phi_{\{H,L\}}$, $\phi_{\{H\}}$, and $\phi_{\{L\}}$ are the multipliers on the implementability conditions.

Proof of Lemma 1. Suppose first that IR_L is slack. Then, the seller can improve her revenue by increasing y_L^l by $\varepsilon = \frac{1-D_L}{2} > 0$. This would not violate any of the constraints of the relaxed problem. So, IR_L must be binding.

Suppose, next, that IC_H is slack. Then, again, the mechanism can be improved profitably, without violating any of the conditions considered in the relaxed problem. Namely, increasing y_H^l by $\varepsilon = \frac{D_H^L - D_H}{2} > 0$ improves the revenue. Hence, IC_H is also binding. ■

Proof of Lemma 2. Suppose, by contradiction, that IR_H is binding. Then, we have $1 = D_H = D_H^L = D_L$, where the equalities are due to IR_H, IC_H, and IR_L, respectively. Yet, since low-type buyers are not excluded, this would contradict with $D_L - D_H^L = \rho_L(c_L - c_H)y_L^w > 0$. Hence, IR_H is slack. ■

Proof of Proposition 1. Kuhn-Tucker conditions with respect to y_H^w and y_H^l yield

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y_H^w} &= \alpha_H\rho_H\frac{1}{y_H^w} - \mu_H\rho_H c_H = 0 \\
\frac{\partial \mathcal{L}}{\partial y_H^l} &= \alpha_H(1-\rho_H)\frac{1}{y_H^l} - \mu_H(1-\rho_H) = 0
\end{aligned}$$

These equations together imply that $y_H^l = c_H y_H^w$. ■

Proof of Proposition 2. Remember that IR_H is slack by Lemma 2. Using Proposition 1, we can rewrite this condition as

$$D_H = y_H^l < 1.$$

This is equivalent to $t_H^l < 0$, implying that, at the optimum, an high-type buyer is compensated when he loses the object. ■

Proof of Proposition 4. Armed with the optimal values of ρ_H, and ρ_L,(see 9) we will now calculate the payments made by each type of buyer. Using IC_H, IR_L, and proposition 1, we write the payments, y_L^w, y_L^l, and y_L^w, as

$$y_L^w = \tfrac{1-y_H^l}{\rho_L(c_L-c_H)}, \quad y_L^l = \tfrac{c_L y_H^l - c_H}{(1-\rho_L)(c_L-c_H)}, \quad y_H^w = \tfrac{y_H^l}{c_H}$$

where y_H^l is in

$$\arg\max_{y_H^l}\{\frac{n}{r}[\alpha_H(\rho_H \ln\frac{1}{c_H} + \ln y_H^l) + \alpha_L(\rho_L \ln(1-y_H^l) + (1-\rho_L)\ln(c_L y_H^l - c_H))]\}.$$

Equivalently, y_H^l solves the first-order condition of the form

$$\frac{\alpha_H}{y_H^l} + \frac{\alpha_H(1-\rho_L)c_L}{c_L y_H^l - c_H} - \frac{\alpha_L \rho_L}{1-y_H^l} = 0.$$

This equation can be rewritten as

$$c_L(y_H^l)^2 - \xi y_H^l + \alpha_H c_H = 0 \tag{32}$$

where $\xi = (1-\rho_L)(c_L + \alpha_H c_H) + \rho_L(c_H + \alpha_H c_L)$.

Since $0 < \rho_L < 1$ and $c_H < c_L$, $\xi > (c_H + \alpha_H c_L)$ must be true. Then, $\xi^2 - 4\alpha_H c_L c_H > (c_H + \alpha_H c_L)^2 - 4\alpha_H c_L c_H = (c_H - \alpha_H c_L)^2 \geq 0$. Thus, a solution to equation 32 exists. Furthermore, if a buyer of type H loses the object he pays

$$y_H^l = \frac{\xi + \sqrt{\xi^2 - 4\alpha_H c_L c_H}}{2c_L}.$$

■

Proof of Proposition 5. We have already established above that IR_L and IC_H are binding and IR_H is slack. We only need to show that IC_L is slack. Equivalently, we need to show that

$\rho_L y_L^w < \rho_H y_H^w$.[41] Plugging in the values of y_L^w and y_H^w gives

$$\frac{1-y_H^l}{(c_L-c_H)} < \frac{\rho_H y_H^l}{c_H} \Longleftrightarrow \frac{c_H}{\rho_H c_L + (1-\rho_H)c_H} < y_H^l.$$

We substitute in the value of y_H^l to get

$$c_L c_H + \alpha_H[\rho_H c_L + (1-\rho_H)c_H]^2 < \xi[\rho_H c_L + (1-\rho_H)c_H].$$

Substituting in the value of ξ and using $IM_{\{H,L\}}$ yields

$$0 < c_L^2 \rho_H (n-1) + c_H^2(1-\rho_H) + c_L c_H[(2-n)\rho_H - 1].$$

Now, we plug in the value of ρ_H and rewrite this condition as

$$0 < (1-\alpha_L^n)[c_L^2(n-1) - c_H^2 + c_L c_H(2-n)] + (1-\alpha_L)[c_H^2 n - c_L c_H n].$$

Since $c_H^2 n - c_L c_H n < 0$, we can replace $(1-\alpha_L)$ with $(1-\alpha_L^n)$ and get the following more restrictive condition

$$0 < (1-\alpha_L^n)(n-1)(c_L-c_H)^2,$$

which holds for any parameter values. Hence, IC_L must be slack. ■

Proof of Proposition 6. Suppose that t_i^w and t_i^l [hence y_i^w and y_i^l] are stochastic. Replacing y_i^w and y_i^l with their expected values would not affect any of the incentive compatibility and individual rationality conditions (because buyers' utilities are linear with respect to these variables), but would strictly improve the seller's revenue (as revenue is concave with respect to y_i^w and y_i^l), which is a contradiction. ■

Appendix B: Optimal multi-object auction

We can write the Lagrangian of the relaxed problem as

[41] We add up IC_H (binding) and IC_L (slack).

$$
\begin{aligned}
\mathcal{L} = \quad & \alpha_{HH}\{\rho^{A}_{HH}\ln y^{A}_{HH} + \rho^{B}_{HH}\ln y^{B}_{HH} + \rho^{AB}_{HH}\ln y^{AB}_{HH} + \rho^{O}_{HH}\ln y^{O}_{HH}\} \\
& +\alpha_{HL}\{\rho^{A}_{HL}\ln y^{A}_{HL} + \rho^{B}_{HL}\ln y^{B}_{HL} + \rho^{AB}_{HL}\ln y^{AB}_{HL} + \rho^{O}_{HL}\ln y^{O}_{HL}\} \\
& +\alpha_{LH}\{\rho^{A}_{LH}\ln y^{A}_{LH} + \rho^{B}_{LH}\ln y^{B}_{LH} + \rho^{AB}_{LH}\ln y^{AB}_{LH} + \rho^{O}_{LH}\ln y^{O}_{LH}\} \\
& +\alpha_{LL}\{\rho^{A}_{LL}\ln y^{A}_{LL} + \rho^{B}_{LL}\ln y^{B}_{LL} + \rho^{AB}_{LL}\ln y^{AB}_{LL} + \rho^{O}_{LL}\ln y^{O}_{LL}\} \\
& +\lambda_{LL}\{1 - \rho^{A}_{LL}c^{A}_{L}y^{A}_{LL} - \rho^{B}_{LL}c^{B}_{L}y^{B}_{LL} - \rho^{AB}_{LL}c^{A}_{L}c^{B}_{L}y^{AB}_{LL} - \rho^{O}_{LL}y^{O}_{LL}\} \\
& +\lambda_{LH}\{c^{A}_{L}[\rho^{A}_{LL}y^{A}_{LL} - \rho^{A}_{LH}y^{A}_{LH}] + c^{B}_{H}[\rho^{B}_{LL}y^{B}_{LL} - \rho^{B}_{LH}y^{B}_{LH}] \\
& \qquad + c^{A}_{L}c^{B}_{H}[\rho^{AB}_{LL}y^{AB}_{LL} - \rho^{AB}_{LH}y^{AB}_{LH}] + [\rho^{O}_{LL}y^{O}_{LL} - \rho^{O}_{LH}y^{O}_{LH}]\} \\
& +\lambda_{HL}\{c^{A}_{H}[\rho^{A}_{LL}y^{A}_{LL} - \rho^{A}_{HL}y^{A}_{HL}] + c^{B}_{L}[\rho^{B}_{LL}y^{B}_{LL} - \rho^{B}_{HL}y^{B}_{HL}] \\
& \qquad + c^{A}_{H}c^{B}_{L}[\rho^{AB}_{LL}y^{AB}_{LL} - \rho^{AB}_{HL}y^{AB}_{HL}] + [\rho^{O}_{LL}y^{O}_{LL} - \rho^{O}_{HL}y^{O}_{HL}]\} \\
& +\lambda_{HH}(\mu_{LL}\{c^{A}_{H}[\rho^{A}_{LL}y^{A}_{LL} - \rho^{A}_{HH}y^{A}_{HH}] + c^{B}_{H}[\rho^{B}_{LL}y^{B}_{LL} - \rho^{B}_{HH}y^{B}_{HH}] \\
& \qquad\qquad + c^{A}_{H}c^{B}_{H}[\rho^{AB}_{LL}y^{AB}_{LL} - \rho^{AB}_{HH}y^{AB}_{HH}] + [\rho^{O}_{LL}y^{O}_{LL} - \rho^{O}_{HH}y^{O}_{HH}]\} \\
& \qquad + \mu_{LH}\{c^{A}_{H}[\rho^{A}_{LH}y^{A}_{LH} - \rho^{A}_{HH}y^{A}_{HH}] + c^{B}_{H}[\rho^{B}_{LH}y^{B}_{LH} - \rho^{B}_{HH}y^{B}_{HH}] \\
& \qquad\qquad + c^{A}_{H}c^{B}_{H}[\rho^{AB}_{LH}y^{AB}_{LH} - \rho^{AB}_{HH}y^{AB}_{HH}] + [\rho^{O}_{LH}y^{O}_{LH} - \rho^{O}_{HH}y^{O}_{HH}]\} \\
& \qquad + \mu_{HL}\{c^{A}_{H}[\rho^{A}_{HL}y^{A}_{HL} - \rho^{A}_{HH}y^{A}_{HH}] + c^{B}_{H}[\rho^{B}_{HL}y^{B}_{HL} - \rho^{B}_{HH}y^{B}_{HH}] \\
& \qquad\qquad + c^{A}_{H}c^{B}_{H}[\rho^{AB}_{HL}y^{AB}_{HL} - \rho^{AB}_{HH}y^{AB}_{HH}] + [\rho^{O}_{HL}y^{O}_{HL} - \rho^{O}_{HH}y^{O}_{HH}]\})
\end{aligned}
$$

Since the number of buyers participating in the auction are assumed to be larger than three and since buyers of each type are treated the same in a symmetric auction, each type's probability of losing both objects is positive. That is, $\rho^{O}_{ij} > 0$ for all $ij \in S$. Thus, using the four Kuhn-Tucker conditions, $\frac{\partial \mathcal{L}}{\partial y^{O}_{ij}} = 0$,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y^O_{HH}} &= \rho^O_{HH}[\frac{\alpha_{HH}}{y^O_{HH}} - \lambda_{HH}] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^O_{LH}} &= \rho^O_{LH}[\frac{\alpha_{LH}}{y^O_{LH}} + \lambda_{HH}\mu_{LH} - \lambda_{LH}] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^O_{HL}} &= \rho^O_{HL}[\frac{\alpha_{HL}}{y^O_{HL}} + \lambda_{HH}\mu_{HL} - \lambda_{HL}] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^O_{LL}} &= \rho^O_{LL}[\frac{\alpha_{LL}}{y^O_{LL}} - \lambda_{LL} + \lambda_{LH} + \lambda_{HL} + \lambda_{HH}\mu_{LL}] = 0
\end{aligned}
$$

we can solve for λ_{ij}s:

$$
\begin{aligned}
\lambda_{HH} &= \frac{\alpha_{HH}}{y^O_{HH}} \\
\lambda_{HL} &= \frac{\alpha_{HL}}{y^O_{HL}} + \frac{\alpha_{HH}}{y^O_{HH}}\mu_{HL} \\
\lambda_{LH} &= \frac{\alpha_{LH}}{y^O_{LH}} + \frac{\alpha_{HH}}{y^O_{HH}}\mu_{LH} \\
\lambda_{LL} &= \frac{\alpha_{LL}}{y^O_{LL}} + \frac{\alpha_{LH}}{y^O_{LH}} + \frac{\alpha_{HL}}{y^O_{HL}} + \frac{\alpha_{HH}}{y^O_{HH}}.
\end{aligned}
$$

The remaining Kuhn-Tucker conditions are of the following form

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y^A_{HH}} &= \rho^A_{HH}[\frac{\alpha_{HH}}{y^A_{HH}} - \lambda_{HH}c^A_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{HH}} &= \rho^B_{HH}[\frac{\alpha_{HH}}{y^B_{HH}} - \lambda_{HH}c^B_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{HH}} &= \rho^{AB}_{HH}[\frac{\alpha_{HH}}{y^{AB}_{HH}} - \lambda_{HH}c^A_H c^B_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^A_{HL}} &= \rho^A_{HL}[\frac{\alpha_{HL}}{y^A_{HL}} - (\lambda_{HL} - \lambda_{HH}\mu_{HL})c^A_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{HL}} &= \rho^B_{HL}[\frac{\alpha_{HL}}{y^B_{HL}} - (\lambda_{HL}c^B_L - \lambda_{HH}\mu_{HL}c^B_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{HL}} &= \rho^{AB}_{HL}[\frac{\alpha_{HL}}{y^{AB}_{HL}} - c^A_H(\lambda_{HL}c^B_L - \lambda_{HH}\mu_{HL}c^B_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^A_{LH}} &= \rho^A_{LH}[\frac{\alpha_{LH}}{y^A_{LH}} - (\lambda_{LH}c^A_L - \lambda_{HH}\mu_{LH}c^A_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{LH}} &= \rho^B_{LH}[\frac{\alpha_{LH}}{y^B_{LH}} - (\lambda_{LH} - \lambda_{HH}\mu_{LH})c^B_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{LH}} &= \rho^{AB}_{LH}[\frac{\alpha_{LH}}{y^{AB}_{LH}} - c^B_H(\lambda_{LH}c^A_L - \lambda_{HH}\mu_{LH}c^A_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^A_{LL}} &= \rho^A_{LL}[\frac{\alpha_{LL}}{y^A_{LL}} - c^A_L(\lambda_{LL} - \lambda_{LH}) + c^A_H(\lambda_{HL} + \lambda_{HH}\mu_{LL})] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{LL}} &= \rho^B_{LL}[\frac{\alpha_{LL}}{y^B_{LL}} - c^B_L(\lambda_{LL} - \lambda_{HL}) + c^B_H(\lambda_{LH} + \lambda_{HH}\mu_{LL})] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{LL}} &= \rho^{AB}_{LL}[\frac{\alpha_{LL}}{y^{AB}_{LL}} - c^A_L(\lambda_{LL}c^B_L - \lambda_{LH}c^B_H) + c^A_H(\lambda_{HL}c^B_L + \lambda_{HH}\mu_{LL}c^B_H)] = 0.
\end{aligned}
$$

Proof of Lemma 3. Suppose that IR_{LL} is slack. Then, we have

$$D_{LL} \equiv \rho^A_{LL}c^A_L y^A_{LL} + \rho^B_{LL}c^B_L y^B_{LL} + \rho^{AB}_{LL}c^A_L c^B_L y^{AB}_{LL} + \rho^O_{LL}y^O_{LL} < 1.$$

Since number of buyers are larger than three and since buyers are treated symmetrically, each type's probability of losing both objects is positive. So, $\rho^O_{LL} > 0$. Thus, an increase in y^O_{LL} by ε/ρ^O_{LL} for $\varepsilon = (1 - D_{LL})/2 > 0$ strictly improves seller's payoff. Note that, this modification on y^O_{LL} does not violate any of the constraints, yielding a contradiction.

Hence, IR_{LL} must be binding. ■

Proof of Lemma 4. Suppose first that IC_{LH}^{LL} is slack. Then, we have

$$\begin{aligned} D_{LH} &\equiv \rho_{LH}^{A} c_{L}^{A} y_{LH}^{A} + \rho_{LH}^{B} c_{H}^{B} y_{LH}^{B} + \rho_{LH}^{AB} c_{L}^{A} c_{H}^{B} y_{LH}^{AB} + \rho_{LH}^{O} y_{LH}^{O} \\ &< \rho_{LL}^{A} c_{L}^{A} y_{LL}^{A} + \rho_{LL}^{B} c_{H}^{B} y_{LL}^{B} + \rho_{LL}^{AB} c_{L}^{A} c_{H}^{B} y_{LL}^{AB} + \rho_{LL}^{O} y_{LL}^{O} \equiv D_{LH}^{LL} \end{aligned}$$

Let $\varepsilon = (D_{LH}^{LL} - D_{LH})/2$. Since $\rho_{LH}^{O} > 0$, if we increase y_{LH}^{O} by $\varepsilon/\rho_{LH}^{O}$, seller's payoff will improve and none of the constraints are violated. This is a contradiction. So, IC_{LH}^{LL} must be binding.

Along the same lines, we can easily show that IC_{HL}^{LL} is binding, too. ■

Proof of Lemma 5. Suppose that all three conditions are slack. Then, we have $D_{HH} < \min\{D_{HH}^{LL}, D_{HH}^{LH}, D_{HH}^{HL}\}$. Define $\varepsilon = (\min\{D_{HH}^{LL}, D_{HH}^{LH}, D_{HH}^{HL}\} - D_{HH})/2$. An increase in y_{HH}^{O} in the amount of $\varepsilon/\rho_{HH}^{O}$, improves seller's payoff and does not violate any of the conditions. This is a contradiction. So, at least one of these three conditions must be binding. ■

Proof of Proposition 8. Since $D_{HH} = \min\{D_{HH}^{LL}, D_{HH}^{LH}, D_{HH}^{HL}\}$, we can replace the last three incentive compatibility conditions with $D_{HH} = \mu_{LL} D_{HH}^{LL} + \mu_{LH} D_{HH}^{LH} + \mu_{HL} D_{HH}^{HL}$ where $\mu_{LL}, \mu_{LH}, \mu_{HL} \geq 0$ and $\mu_{LL} + \mu_{LH} + \mu_{HL} = 1$ provided that $\mu_{ij} = 0$ if and only if $D_{HH} < D_{HH}^{ij}$ (or equivalently, $\mu_{ij} > 0$ if and only if $D_{HH} = D_{HH}^{ij}$).

The Kuhn-Tucker conditions with respect to y_{ij}^{k} for $k = A, B, AB$ and $ij \in S$ can be written as

$$\rho^{A}_{HH}\alpha_{HH}[y^{O}_{HH}-y^{A}_{HH}c^{A}_{H}] = 0 \quad \text{(a)}$$

$$\rho^{B}_{HH}\alpha_{HH}[y^{O}_{HH}-y^{B}_{HH}c^{B}_{H}] = 0 \quad \text{(b)}$$

$$\rho^{AB}_{HH}\alpha_{HH}[y^{O}_{HH}-y^{AB}_{HH}c^{A}_{H}c^{B}_{H}] = 0 \quad \text{(c)}$$

$$\rho^{A}_{HL}\alpha_{HL}[y^{O}_{HL}-y^{A}_{HL}c^{A}_{H}] = 0 \quad \text{(d)}$$

$$\rho^{B}_{HL}[\frac{\alpha_{HL}}{y^{B}_{HL}}-\frac{\alpha_{HH}}{y^{O}_{HH}}\mu_{HL}(c^{B}_{L}-c^{B}_{H})-\frac{\alpha_{HL}}{y^{O}_{HL}}c^{B}_{L}] = 0 \quad \text{(e)}$$

$$\rho^{AB}_{HL}[\frac{\alpha_{HL}}{y^{AB}_{HL}}-c^{A}_{H}(\frac{\alpha_{HH}}{y^{O}_{HH}}\mu_{HL}(c^{B}_{L}-c^{B}_{H})+\frac{\alpha_{HL}}{y^{O}_{HL}}c^{B}_{L})] = 0 \quad \text{(f)}$$

$$\rho^{A}_{LH}[\frac{\alpha_{LH}}{y^{A}_{LH}}-\frac{\alpha_{HH}}{y^{O}_{HH}}\mu_{LH}(c^{A}_{L}-c^{A}_{H})-\frac{\alpha_{LH}}{y^{O}_{LH}}c^{A}_{L}] = 0 \quad \text{(g)}$$

$$\rho^{B}_{LH}\alpha_{LH}[y^{O}_{LH}-y^{B}_{LH}c^{B}_{H}] = 0 \quad \text{(h)}$$

$$\rho^{AB}_{LH}[\frac{\alpha_{LH}}{y^{AB}_{LH}}-c^{B}_{H}(\frac{\alpha_{HH}}{y^{O}_{HH}}\mu_{LH}(c^{A}_{L}-c^{A}_{H})+\frac{\alpha_{LH}}{y^{O}_{LH}}c^{A}_{L})] = 0 \quad \text{(i)}$$

$$\rho^{A}_{LL}[\frac{\alpha_{LL}}{y^{A}_{LL}}-\frac{\alpha_{LL}}{y^{O}_{LL}}c^{A}_{L}-\{\frac{\alpha_{HL}}{y^{O}_{HL}}+\frac{\alpha_{HH}}{y^{O}_{HH}}(\mu_{HL}+\mu_{LL})\}(c^{A}_{L}-c^{A}_{H})] = 0 \quad \text{(j)}$$

$$\rho^{B}_{LL}[\frac{\alpha_{LL}}{y^{B}_{LL}}-\frac{\alpha_{LL}}{y^{O}_{LL}}c^{B}_{L}-\{\frac{\alpha_{LH}}{y^{O}_{LH}}+\frac{\alpha_{HH}}{y^{O}_{HH}}(\mu_{LH}+\mu_{LL})\}(c^{B}_{L}-c^{B}_{H})] = 0 \quad \text{(k)}$$

$$\begin{aligned}\rho^{AB}_{LL}[\frac{\alpha_{LL}}{y^{AB}_{LL}}-\frac{\alpha_{LL}}{y^{O}_{LL}}c^{A}_{L}c^{B}_{L}-\frac{\alpha_{LH}}{y^{O}_{LH}}c^{A}_{L}(c^{B}_{L}-c^{B}_{H})-\frac{\alpha_{HL}}{y^{O}_{HL}}c^{B}_{L}(c^{A}_{L}-c^{A}_{H})&\\ -\frac{\alpha_{HH}}{y^{O}_{HH}}(c^{A}_{L}c^{B}_{L}-\mu_{LH}c^{A}_{L}c^{B}_{H}-\mu_{HL}c^{A}_{H}c^{B}_{L}-\mu_{LL}c^{A}_{H}c^{B}_{H})] &= 0. \quad \text{(l)}\end{aligned}$$

Note that, these equations are of the form $\rho^{k}_{ij}\Omega=0$. We can use them to solve for y^{k}_{ij} for $ij\in S$ and $k=A,B,AB$, by implicitly assuming that $\rho^{k}_{ij}=0$. This is without loss of generality, because each of these y^{k}_{ij}'s appears with the corresponding ρ^{k}_{ij} everywhere in the problem. Thus, if $\rho^{k}_{ij}=0$ for a type ij and for an event k, then the value of y^{k}_{ij} will not matter in the solution, if $\rho^{k}_{ij}>0$, on the other hand, then $\Omega=0$ must be true.

Thus, equations (a)-(d) and (h) respectively yield

$$y^{A}_{HH}=\frac{y^{O}_{HH}}{c^{A}_{H}};\quad y^{B}_{HH}=\frac{y^{O}_{HH}}{c^{B}_{H}};\quad y^{AB}_{HH}=\frac{y^{O}_{HH}}{c^{A}_{H}c^{B}_{H}};$$
$$y^{A}_{HL}=\frac{y^{O}_{HL}}{c^{A}_{H}};\quad y^{B}_{LH}=\frac{y^{O}_{LH}}{c^{B}_{H}};$$

and the pairs '(e),(f)' and '(g),(i)' respectively give

$$y^{AB}_{HL} = \frac{y^B_{HL}}{c^A_H}; \quad y^{AB}_{LH} = \frac{y^A_{LH}}{c^B_H}.$$

These two sets of equations imply that the excess payment that a buyer makes for an object for which he has high valuation is equal to his valuation for that object. In other words, each buyer is perfectly insured against the risk of losing the object(s) for which he has high valuation. ■

Proof of Proposition 9. Similarly, equations (e),(g),(j),(k) and (l) can be used to solve for $y^B_{HL}, y^A_{LH}, y^A_{LL}, y^B_{LL}$ and y^{AB}_{LL}, respectively.

$$\begin{aligned}
\frac{\alpha_{LH}}{y^A_{LH}} &= \frac{\alpha_{LH}}{y^O_{LH}} c^A_L + \frac{\alpha_{HH}}{y^O_{HH}} \mu_{LH}(c^A_L - c^A_H) \\
\frac{\alpha_{HL}}{y^B_{HL}} &= \frac{\alpha_{HL}}{y^O_{HL}} c^B_L + \frac{\alpha_{HH}}{y^O_{HH}} \mu_{HL}(c^B_L - c^B_H) \\
\frac{\alpha_{LL}}{y^A_{LL}} &= \frac{\alpha_{LL}}{y^O_{LL}} c^A_L + \frac{\alpha_{HL}}{y^O_{HL}} (c^A_L - c^A_H) + \frac{\alpha_{HH}}{y^O_{HH}} (\mu_{HL} + \mu_{LL})(c^A_L - c^A_H) \\
\frac{\alpha_{LL}}{y^B_{LL}} &= \frac{\alpha_{LL}}{y^O_{LL}} c^B_L + \frac{\alpha_{LH}}{y^O_{LH}} (c^B_L - c^B_H) + \frac{\alpha_{HH}}{y^O_{HH}} (\mu_{LH} + \mu_{LL})(c^B_L - c^B_H) \\
\frac{\alpha_{LL}}{y^{AB}_{LL}} &= \frac{\alpha_{LL}}{y^O_{LL}} c^A_L c^B_L + \frac{\alpha_{LH}}{y^O_{LH}} c^A_L (c^B_L - c^B_H) + \frac{\alpha_{HL}}{y^O_{HL}} c^B_L (c^A_L - c^A_H) \\
&+ \frac{\alpha_{HH}}{y^O_{HH}} (c^A_L c^B_L - \mu_{LH} c^A_L c^B_H - \mu_{HL} c^A_H c^B_L - \mu_{LL} c^A_H c^B_H)
\end{aligned}$$

Remember from first section that a low-type buyer has to make a payment if he cannot win the object. Using the last three of the above equations we get a similar result for type LL.

Using the last three equations, one can write

$$y^A_{LL} = \frac{y^O_{LL}}{c^A_L + \varepsilon_1}; \quad y^B_{LL} = \frac{y^O_{LL}}{c^B_L + \varepsilon_2}; \quad y^{AB}_{LL} = \frac{y^O_{LL}}{c^A_L c^B_L + \varepsilon_3}$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. We plug these values into LL's individual rationality constraint to get

$$y^O_{LL}(1 - \rho^A_{LL} \frac{\varepsilon_1}{c^A_L + \varepsilon_1} - \rho^B_{LL} \frac{\varepsilon_2}{c^B_L + \varepsilon_2} - \rho^{AB}_{LL} \frac{\varepsilon_3}{c^A_L c^B_L + \varepsilon_3}) = 1.$$

Note that, the term in the parenthesis is less than one if LL gets either or both objects. Thus, if $\rho^O_{LL} \neq 1$, then $y^O_{LL} > 1$ (hence, $t^O_{LL} > 0$) must be true. ■

Proof of Proposition 10. *i)* Let η be such that $n_{HH} + n_{HL} > 0$ and without loss of generality assume that $n_{HH} > 0$. Now, suppose by contradiction, that $n_{HH}\hat{p}^A_{HH}(\eta) + n_{HL}\hat{p}^A_{HL}(\eta) < 1$. Let $\varepsilon \leq 1 - n_{HH}\hat{p}^A_{HH}(\eta) - n_{HL}\hat{p}^A_{HL}(\eta)$.

There are three possibilities that we need to consider:

- $n_{LH} + n_{LL} = 0$:

In this case, modify the mechanism by increasing $p^A_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$. This would increase $\hat{\rho}^A_{HH}$ by $\Psi\frac{\varepsilon}{\alpha_{HH}}$. Change in the Lagrangian can be calculated as $\Psi\varepsilon \ln \frac{1}{c_H} > 0$. This is a contradiction.

- $n_{LH}\hat{p}^A_{LH}(\eta) > 0$:

We will now show that for some $\varepsilon < n_{LH}\hat{p}^A_{LH}(\eta)$, decreasing $\hat{p}^A_{LH}(\eta)$ by $\frac{\varepsilon}{n_{LH}}$, and increasing $\hat{p}^A_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$ is profitable. After this modification, $\hat{\rho}^A_{LH}$ decreases by $\Psi\frac{\varepsilon}{\alpha_{LH}}$ and $\hat{\rho}^A_{HH}$ increases by $\Psi\frac{\varepsilon}{\alpha_{HH}}$.[42] We calculate the change in the Lagrangian as

$$\begin{aligned}
\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{1}{c^A_H} - \ln\frac{y^A_{LH}}{y^O_{LH}} + \lambda_{LH}[c^A_L\frac{y^A_{LH}}{\alpha_{LH}} - \frac{y^O_{LH}}{\alpha_{LH}}] - \lambda_{HH}\mu_{LH}[c^A_H\frac{y^A_{LH}}{\alpha_{LH}} - \frac{y^O_{LH}}{\alpha_{LH}}]\} \\
&= \Psi\varepsilon\ln\frac{y^O_{LH}}{c^A_H y^A_{LH}}
\end{aligned}$$

which is positive since $y^O_{LH} > c_H y^A_{LH}$.

- $n_{LL}\hat{p}^A_{LL}(\eta) > 0$ and $n_{LH}\hat{p}^A_{LH}(\eta) = 0$:

Suppose first that $n_{LL}p^A_{LL}(\eta) > 0$. Then consider modifying the mechanism by decreasing $p^A_{LL}(\eta)$ by $\frac{\varepsilon}{n_{LL}}$ and increasing $p^A_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$ for some $\varepsilon < n_{LL}p^A_{LL}(\eta)$. This would decrease ρ^A_{LL} by $\Psi\frac{\varepsilon}{\alpha_{LL}}$ and increase $\hat{\rho}^A_{HH}$ by $\Psi\frac{\varepsilon}{\alpha_{HH}}$. Lagrangian then changes by

$$\begin{aligned}
\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{1}{c^A_H} - \ln\frac{y^A_{LL}}{y^O_{LL}} + (\lambda_{LL} - \lambda_{LH})[\frac{c^A_L y^A_{LL}}{\alpha_{LL}} - \frac{y^O_{LL}}{\alpha_{LL}}] - (\lambda_{HL} + \lambda_{HH}\mu_{LL})[\frac{c^A_H y^A_{LL}}{\alpha_{LL}} - \frac{y^O_{LL}}{\alpha_{LL}}]\} \\
&= \Psi\varepsilon\ln\frac{y^O_{LL}}{c^A_H y^A_{LL}} > 0
\end{aligned}$$

Suppose now that $n_{LL}p^A_{LL}(\eta) = 0$. Then, $n_{LL}p^{AB}_{LL}(\eta) > 0$ must be true. We will show that the following modification is profitable: For some $\varepsilon < n_{LL}p^{AB}_{LL}(\eta)$, decrease $p^{AB}_{LL}(\eta)$ by $\frac{\varepsilon}{n_{LL}}$ and increasing $p^{AB}_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$. This would decrease ρ^{AB}_{LL} by $\Psi\frac{\varepsilon}{\alpha_{LL}}$ and increase $\hat{\rho}^A_{HH}$ and $\hat{\rho}^B_{HH}$ by

[42] $\hat{p}^A_{LH}(\eta)$ can be decreased either by decreasing $p^A_{LH}(\eta)$ or $p^{AB}_{LH}(\eta)$. If the former, is positive then we decrease $p^A_{LH}(\eta)$ (and increase $p^A_{HH}(\eta)$). If the former is zero, however, $p^{AB}_{LH}(\eta)$ should be decreased (and in response $p^{AB}_{HH}(\eta)$ should be increased) In this case, marginal probabilities of winning A and B are affected for both types HH and LH. Yet, either modification, have the same effect on the Lagrangian.

$\Psi\frac{\varepsilon}{\alpha_{HH}}$. As a result, Lagrangian will increase by

$$\begin{aligned}\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{1}{c_H^A}+\ln\frac{1}{c_H^B}-\ln\frac{y_{LL}^{AB}}{y_{LL}^O}+\lambda_{LL}[\frac{c_L^Ac_L^By_{LL}^{AB}}{\alpha_{LL}}-\frac{y_{LL}^O}{\alpha_{LL}}] \\ &\quad -\lambda_{LH}[\frac{c_L^Ac_H^By_{LL}^{AB}}{\alpha_{LL}}-\frac{y_{LL}^O}{\alpha_{LL}}]-\lambda_{LH}[\frac{c_H^Ac_L^By_{LL}^{AB}}{\alpha_{LL}}-\frac{y_{LL}^O}{\alpha_{LL}}] \\ &\quad -\lambda_{HH}\mu_{LL}[\frac{c_H^Ac_H^By_{LL}^{AB}}{\alpha_{LL}}-\frac{y_{LL}^O}{\alpha_{LL}}]\} \\ &= \Psi\varepsilon\ln\frac{y_{LL}^O}{c_H^Ac_H^By_{LL}^{AB}}>0\end{aligned}$$

Thus, we conclude that if η is such that $n_{HH}+n_{HL}>0$, then $n_{HH}\hat{p}_{HH}^A(\eta)+n_{HL}\hat{p}_{HL}^A(\eta)=1$.

We can prove part *ii*) of the Lemma along the same lines. ■

Proof of Corollary 1. We will prove only part *i*). Proof of part *ii*) is similar. (*5) implies that

$$\begin{aligned}\alpha_{HH}\hat{\rho}_{HH}^A &= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} n_{HH}\hat{p}_{HH}^A(\eta)\Psi \\ \alpha_{HL}\hat{\rho}_{HL}^A &= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} n_{HL}\hat{p}_{HL}^A(\eta)\Psi.\end{aligned}$$

Adding these two equalities and multiplying both sides with n gives

$$\begin{aligned}n[\alpha_{HH}\hat{\rho}_{HH}^A+\alpha_{HL}\hat{\rho}_{HL}^A] &= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}}[n_{HH}\hat{p}_{HH}^A(\eta)+n_{HL}\hat{p}_{HL}^A(\eta)]n\Psi \\ &= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} n\Psi-\sum_{n_{LH}=0}^{n}\frac{n!\alpha_{LH}^{n_{LH}}\alpha_{LL}^{n-n_{LH}}}{n_{LH}!(n-n_{LH})!} \\ &= 1-(\alpha_{LH}+\alpha_{LL})^n.\end{aligned}$$

The second equality follows from the part i of proposition 9. ■

Proof of Proposition 11. *i*) Suppose that the profile is such that $n_{HH}+n_{HL}=0$, but $n_{LH}\hat{p}_{LH}^A(\eta)+n_{LL}\hat{p}_{LL}^A(\eta)<1$. Let $\varepsilon<1-n_{LH}\hat{p}_{LH}^A(\eta)-n_{LL}\hat{p}_{LL}^A(\eta)$. There are two cases that we need to consider:

- $n_{LH}>0$: Let's increase $\hat{p}_{LH}^A(\eta)$ by $\frac{\varepsilon}{n_{LH}}$, which would increase $\hat{\rho}_{LH}^A$ by $\Psi\frac{\varepsilon}{\alpha_{LH}}$. Change in the

Lagrangian is calculated as

$$\begin{aligned}\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{y^A_{LH}}{y^O_{LH}} + \lambda_{LH}[-c^A_L\frac{y^A_{LH}}{\alpha_{LH}} + \frac{y^O_{LH}}{\alpha_{LH}}] + \lambda_{HH}\mu_{LH}[c^A_H\frac{y^A_{LH}}{\alpha_{LH}} - \frac{y^O_{LH}}{\alpha_{LH}}]\} \\ &= \Psi\varepsilon\ln\frac{y^A_{LH}}{y^O_{LH}}\end{aligned}$$

which is positive if $y^A_{LH} > y^O_{LH}$, or $\frac{c^A_L - c^A_H}{1-c^A_L} < \frac{\alpha_{LH}}{y^O_{LH}}\frac{y^O_{HH}}{\alpha_{HH}}\frac{1}{\mu_{LH}}$.

- $n_{LH} = 0$: A profitable modification would be to increase $p^A_{LL}(\eta)$ by $\frac{\varepsilon}{n_{LL}}$ and hence ρ^A_{LL} by $\Psi\frac{\varepsilon}{\alpha_{LL}}$. Lagrangian will increase by

$$\begin{aligned}\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{y^A_{LL}}{y^O_{LL}} - (\lambda_{LL} - \lambda_{LH})[\frac{c^A_L y^A_{LL}}{\alpha_{LL}} - \frac{y^O_{LL}}{\alpha_{LL}}] + (\lambda_{HL} + \lambda_{HH}\mu_{LL})[\frac{c^A_H y^A_{LL}}{\alpha_{LL}} - \frac{y^O_{LL}}{\alpha_{LL}}]\} \\ &= \Psi\varepsilon\ln\frac{y^A_{LL}}{y^O_{LL}}\end{aligned}$$

which is positive if $y^A_{LL} > y^O_{LL}$, or $\frac{c^A_L - c^A_H}{1-c^A_L} < \frac{\alpha_{LL}}{y^O_{LL}}(\frac{\alpha_{HL}}{y^O_{HL}} + \frac{\alpha_{HH}}{y^O_{HH}}(1-\mu_{LH}))^{-1}$.

ii) Along the same lines of the previous part, we can easily show that this part holds, too, if $y^B_{HL} > y^O_{HL}$ and $y^B_{LL} > y^O_{LL}$, or equivalently if

$$\frac{c^B_L - c^B_H}{1-c^B_L} < \min\{\frac{\alpha_{HL}}{y^O_{HL}}\frac{y^O_{HH}}{\alpha_{HH}}\frac{1}{\mu_{HL}}, \frac{\alpha_{LL}}{y^O_{LL}}(\frac{\alpha_{LH}}{y^O_{LH}} + \frac{\alpha_{HH}}{y^O_{HH}}(1-\mu_{HL}))^{-1}\}.$$

■

Proof of Proposition 12. Suppose, for now, that HH is not compensated. Then $y^O_{HH} = 1$. Since $c^A_H < c^A_L$ and $c^B_H < c^B_L$, we have $1 = y^O_{HH} \leq D^{ij}_{HH} \leq D_{ij} \leq 1$ for $ij = LL, LH, HL$ where the first inequality is due to IC^{ij}_{HH}, and the last inequality is the individual rationality constraint. So, all individual rationality constraints are binding and $D_{ij} = D^{ij}_{HH} = 1$ for $ij = LL, LH, HL$. Moreover, since $D_{ij} - D^{ij}_{HH} = 0$, we have

$$\begin{aligned}\rho^A_{LL}(c^A_L - c^A_H)y^A_{LL} + \rho^B_{LL}(c^B_L - c^B_H)y^B_{LL} + \rho^{AB}_{LL}(c^A_L c^B_L - c^A_H c^B_H)y^{AB}_{LL} &= 0 \\ \hat{\rho}^A_{LH}(c^A_L - c^A_H)y^A_{LH} &= 0 \\ \hat{\rho}^B_{HL}(c^B_L - c^B_H)y^B_{HL} &= 0\end{aligned}$$

Each term in these equations are nonnegative, therefore $\rho^A_{LL} = \rho^B_{LL} = \rho^{AB}_{LL} = \hat{\rho}^A_{LH} = \hat{\rho}^B_{HL} = 0$ must be true. This contradicts with the previous Corollary because $\alpha_{LL}\hat{\rho}^A_{LL} + \alpha_{LH}\hat{\rho}^A_{LH} > 0$. ■

Proof of Proposition 13. Suppose, by contradiction, that for some profile η with $n_{LL} = n$, $p^{AB}_{LL}(\eta) < \frac{1}{n}$. Since both objects are sold with probability one, this implies that $p^A_{LL}(\eta) = p^B_{LL}(\eta) > 0$. Let $\varepsilon < 1 - np^{AB}_{LL}(\eta)$. Consider modifying the mechanism by decreasing $p^A_{LL}(\eta)$ and $p^B_{LL}(\eta)$ both by $\frac{\varepsilon}{n}$ and increasing $p^{AB}_{LL}(\eta)$ by $\frac{\varepsilon}{n}$. This would imply $\Delta\rho^{AB}_{LL} = -\Delta\rho^A_{LL} = -\Delta\rho^B_{LL} = \Psi\frac{\varepsilon}{\alpha_{LL}}$. Now, we calculate the change in the Lagrangian:

$$\Delta\mathcal{L} = \Psi\varepsilon \ln \frac{y^O_{LL} y^{AB}_{LL}}{y^A_{LL} y^B_{LL}}$$

which is positive if $y^O_{LL} y^{AB}_{LL} > y^A_{LL} y^B_{LL}$ or, equivalently, if

$$\begin{aligned} & \frac{\alpha_{LL}}{y^A_{LL}} \frac{\alpha_{LL}}{y^B_{LL}} > \frac{\alpha_{LL}}{y^O_{LL}} \frac{\alpha_{LL}}{y^{AB}_{LL}} \\ \iff \quad & (\lambda_{LH}\lambda_{HL} + \lambda_{LL}\lambda_{HH}\mu_{LL})(c^A_L - c^A_H)(c^B_L - c^B_H) > 0. \end{aligned}$$

Since the last inequality holds for any parameter values, this modification is profitable. Thus, we conclude that if all the buyers are of type LL then the objects are bundled and each buyer gets the bundle with equal probability. ■

Proof of Theorem 1. Any of the three auction formats, namely independent auction, bundling auction and mixed auction, that are optimal when the buyers are risk neutral allocate the objects independently and randomly when all buyers report to be of type LL.

Yet, by proposition 21, when the buyers are risk averse, a necessary condition for the optimality of the auction is to give both object to the same buyer if all buyers are of type LL. ■

Proof of Lemma 6. *i)* Suppose that for some η with $n_{LH}, n_{LL} > 0$ and $n_{LH} + n_{LL} = n$, $n_{LH}\hat{p}^A_{LH}(\eta) < 1$. Then, since A is sold with probability one, $p^A_{LL}(\eta)$ must be positive. Let $\varepsilon < n_{LL}p^A_{LL}(\eta)$. Now, consider modifying the mechanism by decreasing $p^A_{LL}(\eta)$ by $\frac{\varepsilon}{n_{LL}}$ and increasing $\hat{p}^A_{LH}(\eta)$ by $\frac{\varepsilon}{n_{LH}}$. This, would decrease $\hat{\rho}^A_{LL}$ by $\frac{\Psi\varepsilon}{\alpha_{LL}}$ and increase $\hat{\rho}^A_{LH}$ by $\frac{\Psi\varepsilon}{\alpha_{LH}}$. As a result, the Lagrangian will change by

$$\Delta\mathcal{L} = \Psi\varepsilon \ln \frac{y^A_{LH} y^O_{LL}}{y^O_{LH} y^A_{LL}}.$$

This is positive if $y^A_{LH} y^O_{LL} > y^O_{LH} y^A_{LL}$, or equivalently if $\frac{\alpha_{LH}}{y^O_{LH}} \frac{\alpha_{LL}}{y^A_{LL}} > \frac{\alpha_{LH}}{y^A_{LH}} \frac{\alpha_{LL}}{y^O_{LL}}$. Using the Kuhn-Tucker conditions, we can rewrite this inequality as

$$\begin{aligned}(\lambda_{LH} - \lambda_{HH}\mu_{LH})[c_L^A(\lambda_{LL} - \lambda_{LH}) - c_H^A(\lambda_{HL} + \lambda_{HH}\mu_{LL})] &> \\ (c_L^A\lambda_{LH} - c_H^A\lambda_{HH}\mu_{LH})(\lambda_{LL} - \lambda_{LH} - \lambda_{HL} - \lambda_{HH}\mu_{LL}).\end{aligned}$$

After some manipulation, we get

$$\begin{aligned}\lambda_{LH}(\lambda_{HL} + \lambda_{HH}\mu_{LL}) &> \lambda_{HH}\mu_{LH}(\lambda_{LL} - \lambda_{LH}) \\ (\frac{\alpha_{HL}}{y_{HL}^O}\frac{y_{HH}^O}{\alpha_{HH}} + 1)(\frac{\alpha_{LL}}{y_{LL}^O}\frac{y_{LH}^O}{\alpha_{LH}} + 1)^{-1} &> \mu_{LH}.\end{aligned}$$

Proof of part *ii*) is similar. ■

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