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Abstract

Mechanization (or automation) − the replacement by machines of humans (and animals) engaged in production tasks− has proceeded continuously since the Industrial Revolution and seems to have accelerated recently due to the rapid advancement of information technology. This paper examines interactions among long-run trends of mechanization, shifts of tasks humans perform, and earnings levels and inequality. Specifically, the paper develops a Ricardian model of task assignment and analyzes how improvements of productivities of machines and an increase in the relative supply of skilled workers affect task assignment (which factor performs which task), earnings levels and inequality, and aggregate output. The model succeeds in capturing the great majority of the long-run trends. The paper also explores possible future trends of the variables when information technology continues to grow rapidly.

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1 Introduction

Mechanization (or automation) − the replacement by machines of humans (and animals) engaged in production tasks− has proceeded continuously since the Industrial Revolution and seems to have accelerated recently due to the rapid advancement of information technology. This paper examines interactions among long-run trends of mechanization, shifts of tasks humans perform, and earnings levels and inequality. Specifically, the paper develops a Ricardian model of task assignment and analyzes how improvements of productivities of machines and an increase in the relative supply of skilled workers affect task assignment (which factor performs which task), earnings levels and inequality, and aggregate output. The model succeeds in capturing the great majority of the long-run trends. The paper also explores possible future trends of the variables when information technology continues to grow rapidly.

Facts. The long-run trends the paper focuses on are as follows.

Mechanization: During the Industrial Revolution, mechanization progressed in tasks intensive in manual labor: in manufacturing (particularly, textile and metal working), machines and factory workers replaced artisans and farmers engaged in side jobs; in transportation, railroads and steamboats supplanted wagons and sailboats; and in agriculture, threshing machines and reapers reduced labor input.[1] During the Second Industrial Revolution (from the second half of the 19th century to World War I), with the utilization of electric power and internal combustion engines, mechanization proceeded further in manual tasks: in manufacturing, broader sectors and production processes were mechanized with the introduction of mass production system; a wider range of tasks were mechanized with tractors in agriculture and with automobiles and trucks in transportation. Some analytical (cognitive) tasks too were mechanized: tabulating machines substituted data-processing workers at large organizations. In the post World War II era, especially since the 1970s, analytical tasks in much wider areas have been mechanized because of the progress of information technology: computers replaced clerical workers engaged in information processing tasks; sensors mechanized inspection processes in manufacturing and services; and simple troubleshooting tasks were automated with the construction of databases of known troubles.[2]

Task shifts: As a result of mechanization, humans have shifted to tasks machines cannot perform efficiently. The general trend until about the 1960s is the shift from manual tasks to analytical tasks: initially, humans shifted from manual tasks at farms, cottages, and workshops to manual tasks at factories and analytical tasks at offices and factories (generally associated with clerical, management, and technical jobs); after mechanization deepened in manufacturing, they shifted from manual tasks at factories as well as at farms to analytical tasks (Katz and Margo, 2013).[3] Since the 1970s, they have shifted from routine analytical

[1] Works on the two revolutions by economic historians include Landes (2003) and Mokyr (1985, 1999).

[2] Case studies of effects of information technology on the workplace include Autor, Levy, and Murnane (2002) on a commercial bank and Bartel, Ichniowski, and Shaw (2007) on a bulb manufacturing factory.

[3] Although it has been widely thought that technical change during the 19th century is unskill-biased, Katz and Margo (2013) show that this is *not* the case for the U.S.: while the share of middle-skill workers (artisans and agricultural operators) fell and shares of low-skill workers (unskilled workers and laborers) and high-skill workers (white collar) rose in manufacturing, *in the whole economy*, shares of low-skill and middle-skill workers fell and high-skill workers rose from 1850 to 1910. (The share of middle-skill workers

tasks (e.g. simple information processing tasks performed by clerks) as well as manual tasks toward non-routine analytical tasks (mainly associated with professional and technical jobs) and non-routine manual tasks in services (e.g. personal care and protective service) owing to the growth of information technology (Autor, Levy, and Murnane, 2003; Acemoglu and Autor, 2011).[4,5] Since the 1990s, due to the large shift from routine analytical tasks, the growth of middle-wage jobs has been weak relative to both low-wage and high-wage jobs, i.e. job polarization has been observed (Autor, 2015; Goos, Manning, and Salomons, 2014).

Earnings levels and inequality: Mechanization has affected relative demands for workers of different skill levels and thus earnings levels and inequality. In the early stage of industrialization, earnings of unskilled workers grew very moderately and the inequality between skilled and unskilled workers enlarged (Feinstein, 1998; Katz and Margo, 2013).[6] In later periods, unskilled workers have benefited more from mechanization, while, as before, the rising inequality has been the norm in economies with lightly regulated labor markets, except in periods of rapid growth of the relative supply of skilled workers and in the 1940s, when the inequality fell (Goldin and Katz, 1998, 2008).[7] Since the 1990s, associated with job polarization, wage polarization (the slower wage growth of middle-wage jobs relative to low-wage and high-wage jobs) has occurred in the U.S., although the evidence for Europe is mixed (Autor, 2015; Böhm, 2015; Naticchioni, Massari, and Ragusa, 2014).[8]

The model. The model economy is a static small-open competitive economy where three kinds of factors of production− skilled workers, unskilled workers, and machines− are available. Each factor is characterized by *analytical ability* and *manual ability.* Skilled workers have a higher level of analytical ability than unskilled workers, while both types of workers have the same level of manual ability, reflecting the fact that there is no strong

changed little if clerical/sales workers are classified as middle-skilled.) They also find that the same pattern is observed for the whole economy from 1920 to 1980 and the declining share of low-skill workers is driven by farm laborers until around 1950 and by unskilled workers and laborers (largely in manufacturing) thereafter.

[4]Similarly to Autor, Levy, and Murnane (2003), routine tasks refer to tasks whose procedures are organized so that they can be performed by machines after relevant technologies are developed.

[5]Autor, Levy, and Murnane (2003) examine changes in the composition of tasks in the U.S. from 1960 to 1998 and find that the growth of information technology is important in explaining the changes after the 1970s. Acemoglu and Autor (2011) explore changes in occupational composition for 1959−2007.

[6]Feinstein (1998) finds that real wages of British manual workers rose very moderately from the 1770s to the 1850s (stagnated until the 1830s), implying a large rise in the disparity with skilled workers. Katz and Margo (2013) find a secular rise in the wage premium for white-collar workers for 1820−80 in the U.S..

[7]Goldin and Katz (1998), using data for 1909−40, show that the introduction of mass production methods raised the relative demand for skilled workers in U.S. manufacturing. Goldin and Katz (2008) document that, after plummeting in the 1940s, the return to college education in the U.S. kept rising except in the 1970s when the relative supply of college graduates grew rapidly. As for the return to high school education, which is a good measure of inequality between skilled and unskilled workers until the 1940s (judging from a low elasticity of substitution between high school graduates and dropouts), it fell greatly from 1914 to 1939, when high school enrollment rates rose dramatically (from 20% to over 70%) and in the 1940s.

[8]For the U.S., Autor (2015) finds the evidence of wage polarization for 1989−1999, but rising inequality for 1999−2007 and falling inequality for 2007−2012. While Autor (2015) does not control for time-varying skill composition of workers engaged in each job, Böhm (2015) controls for the composition change and finds the robust evidence of wage polarization for 1989−1999 and some evidence of polarization for 1999−2007.

For Europe, Naticchioni, Massari, and Ragusa (2014) find no evidence of unconditional polarization, weak evidence of conditional polarization (conditional on technology variables) in individual-level data for 1996−2007, and no evidence in industry-level data for 1980−2005.

correlation between the two abilities, except in poorest countries.

The final good is produced from inputs of a continuum of *tasks* that are different in *the importance of analytical ability,* a, and *the ease of codification* (*routinization*), c, using a Leontief technology.[9] In the real economy, low a and high c tasks are those involving repetitive motions such as assembling or sorting objects and typical in production jobs; low a and low c tasks are those entailing non-repetitive motions such as driving vehicles and caring for the elderly and usual in low-wage service jobs; high a and high c tasks entail simple information processing such as calculation and recording information and are typical in clerical jobs; and high a and low c tasks involve complex analysis and judgement mainly associated with management, professional, and technical jobs.

The three factors are perfectly substitutable at each task. Both abilities contribute to production at each task (except the most manual and the most analytical tasks), but the relative contribution of analytical ability is higher in tasks of the greater importance of the ability. Given the ability's importance, machines are more productive in tasks of the greater ease of codification, while workers' productivities do not depend on the ease of codification.

A competitive equilibrium determines task assignment, factor prices, task prices, and output etc. Comparative advantages of factors determine task assignment: unskilled (skilled) workers are assigned to relatively manual (analytical) tasks and machines are assigned to tasks that are easier to codify. Among tasks a given factor is employed, it is employed heavily in tasks in which its productivities are low.

Main results. Based on the model, the paper examines how task assignment, earnings, earnings inequality, and output change over time, when analytical and manual abilities of machines and the relative supply of skilled workers grow exogenously over time.

Section 4 analyzes a simpler case in which the two abilities grow proportionately and machines have comparative advantages in relatively manual tasks. The analysis shows that tasks and workers strongly affected by mechanization and effects of the productivity growth on earnings and the inequality change over time. Mechanization starts from tasks that are highly manual and easy to routinize, and gradually spreads to tasks that are more analytical and difficult to routinize. Eventually, mechanization proceeds in highly analytical tasks previously performed by skilled workers too. Accordingly, unskilled workers shift to tasks that are more difficult to codify, so do skilled workers in later stages of mechanization, and both types shift to more analytical tasks except at the final stage. Skilled workers always benefit from the productivity growth, whereas the effect on earnings of unskilled workers is ambiguous while mechanization mainly affects them and the effect turns positive afterwards. Earnings inequality rises except at the final stage, where it does not change. The output of the final good always increases. In contrast, an increase in the relative supply of skilled workers raises (lowers) earnings of unskilled (skilled) workers and lowers the inequality, countervailing the inequality-enhancing effect of productivity growth (it also raises output).

The results are consistent with the long-run trends of task shifts, earnings, and the inequality described earlier, except job polarization after the 1990s and the development of earnings and the inequality after the 1980s and in the wartime 1940s. However, the assumption that the two abilities grow proportionately, which makes the analysis simple, is

[9] In this paper, the term codify/routinize means "organize procedures of tasks systematically so that tasks can be performed by machines after relevant technologies are developed".

rather restrictive, considering that the growth of manual ability was faster than analytical ability most of the time, while analytical ability seems to have been growing faster recently.

Hence, Section 5 analyzes the general case in which the two abilities may grow at different rates. Under realistic productivity growth, the model does much better jobs in explaining the development after the 1980s than under the special case (it is still inconsistent with the development in the 1940s). In particular, the model predicts that skilled workers shift from *non-routine analytical* tasks to manual tasks when the growth of analytical ability is fast, consistent with the development after around the year 2000 in the U.S. (Beaudry, Green, and Sand, 2016).[10] Although the job and wage polarization is beyond the scope of the model with two types of workers, the falling inequality predicted by the model captures a part of the development, the falling inequality between low-skill and middle-skill workers observed at least in the U.S..

Finally, the model is used to examine possible future trends of the variables when information technology and thus the analytical ability of machines continue to grow rapidly. It is found that earnings of both skilled and unskilled workers increase and earnings inequality falls over time, although the analysis based on the model with two types of workers may not capture the whole picture, considering the recent widening inequality between moderately and extremely high-skill workers (Alvaredo et al., 2013).

Related literature. The paper belongs to the literature on task (job) assignment model, which has been developed to analyze the distribution of earnings in labor economics (see Sattinger, 1993, for a review), and recently is used to examine broad issues, such as effects of technology on the labor market (Acemoglu and Autor, 2011; Acemoglu and Restrepo, 2016), on cross-country productivity differences (Acemoglu and Zilibotti, 2001), and on organizational structure and wages (Garicano and Rossi-Hansberg, 2006), effects of international trade and offshoring on the labor market (Grossman and Rossi-Hansberg, 2008; Costinot and Vogel, 2010, and Grossman, Helpman, and Kircher, 2015), and inter-industry wage differentials and the effect of trade on wages (Sampson, 2016).[11]

The most closely related is Acemoglu and Autor (2011), who argue that the conventional non-assignment model cannot examine shifts in tasks workers with a given skill level perform and fails to capture a large part of recent trends of task shifts, earnings, and earnings inequality, particularly job and wage polarization and stagnant or negative earnings growth of less-educated workers in the U.S.,[12] and develop a task assignment model with three types

[10] Beaudry, Green, and Sand (2016) find that the employment growth of non-routine analytical jobs stalled after around 2000, while the supply of high-skill workers continued to grow, suggesting a decrease in the demand for such jobs. Further, they show that the average intensity of non-routine analytical tasks for college graduates increased from the early 1980s until around 2000 but decreased thereafter.

[11] See also Nakamura and Nakamura (2015), who develop a model of mechanization through capital accumulation that provides a microfoundation for the neoclassical production function.

[12] Limitations of the conventional model, in which workers with different skill levels are imperfect substitutes in a macro production function, pointed out by them and relevant to this paper are: (i) technical change is factor-augmenting, thus it does not model mechanization through technical change, which is also pointed out in the literature on growth models with mechanization reviewed below, (ii) the model cannot explain stagnant or negative earnings growth of particular groups in a growing economy, (iii) since all workers with a given skill level have the same 'job', shifts in jobs and tasks performed by particular groups cannot be examined, (iv) systematic changes in the composition of employment by job (task) cannot be analyzed, (v) typically, workers are two type and thus it cannot examine job and wage polarization.

of workers (high-skill, middle-skill, low-skill), which is a generalization of the Acemoglu and Zilibotti (2001) model with two types of workers. The final good is produced from inputs of a continuum of tasks that are different in the degree of 'complexity' using a Cobb-Douglas technology. High (middle) skill workers have comparative advantages in more complex tasks against middle (low) skill workers. After examining the model economy without capital, they analyze the situation where a part of tasks initially performed by middle-skill workers come to be mechanized exogenously, and show that a fraction of them shift to tasks previously performed by the other types of workers and relative earnings of high-skill workers to middle-skill workers rise and those of middle-skill workers to low-skill workers fall, reproducing job and wage polarization.[13,14]

The present paper builds on their work, particularly in the modeling, but there are several important differences. First, the paper is interested in the long-run trends of task shifts, earnings, and earnings inequality since the Industrial Revolution, while they focus on the recent development, especially job and wage polarization after the 1990s. Second, the paper examines how tasks and workers strongly affected by mechanization and its effects on earnings and the inequality *change endogenously over time* with improvements of manual and analytical abilities of machines, whereas, because of their focus on job and wage polarization, they assume that mechanization occurs at tasks previously performed by middle-skill workers. Third, in order to examine the long-run trends, in particular, the changing impact of productivity growth on particular tasks and workers, the present model assumes that tasks are different in two dimensions, the importance of analytical ability and the ease of codification (routinization), while, in their model, tasks are different in one dimension, the degree of 'complexity'.

The paper is also related to the literature that examines the interaction between mechanization and economic growth, such as Zeira (1998, 2010), Boldrin and Levine (2002), Givon (2006), Zuleta (2008), Acemoglu (2010), and Peretto and Seater (2013). The literature is mainly interested in whether persistent growth is possible in models where economies grow through mechanization and whether the dynamics are consistent with stylized facts of growth. While the standard model assumes labor-augmenting technical change, which is labor-saving but *not* capital-using (thus does not capture mechanization), these papers (except Zeira, 2010) consider technical change that is labor-saving *and* capital-using.[15] Such

[13] They also examine the situation where a part of tasks initially performed by middle-skill workers come to be offshored exogenously. Further, they analyze the effect of changes in factor supplies on technical change using a version of the model with endogenous factor-augmenting technical change.

[14] Acemoglu and Restrepo (2016) develop a dynamic task assignment model with two types of technological changes, the automation of tasks replacing labor with capital and the development of new tasks replacing the least 'complex' existing tasks. Their main interests are to characterize conditions for asymptotically stable balanced growth for a version of the model with endogenous technological changes and one type of labor (and capital and intermediates embodying technologies) and to examine the effect of shocks to technologies on factor prices and factor shares in employment and income. In an extension, they also consider a version of the model with exogenous technological changes and two types of labor (skilled labor has a comparative advantage in more 'complex' tasks) and examine the effect of technological changes on wage inequality. In particular, they show that automation raises wage inequality.

[15] Acemoglu (2010) examines whether labor scarcity encourages technological advances and shows that it does if technology is strongly labor saving. He also shows that models with mechanization-type technological change have a tendency for strongly labor-saving technology, based on the Zeira (1998) model.

technical change typically implies the decreasing (increasing) share of labor (capital) income over time, which is consistent with the declining labor share of the real economy (Bentolila and Saint-Paul, 2003; Neiman, 2013).[16] By contrast, given technologies, Zeira (2010) examines interactions among capital accumulation, changes in factor prices, and mechanization. His model can be interpreted as a dynamic task assignment model after a slight modification of the production technology. However, the model assumes homogenous labor and constant productivity of machines and thus cannot examine the issue this paper focuses on.

Organization of the paper. The paper is organized as follows. Section 2 presents the model and Section 3 derives equilibrium allocations, given machine abilities. Section 4 examines effects of improved machine abilities and increased relative supply of skilled workers on task assignment, earnings levels and inequality, and aggregate output, when the two abilities improve proportionately. Section 5 examines the general case in which the abilities may improve at different rates, and Section 6 concludes. Appendix A presents lemmas, and Appendix B contains proofs of lemmas and propositions, except Propositions 4−7 whose proofs are very lengthy and are posted on the author's web site.[17]

2 Model

Consider a small open economy where three types of factors of production− skilled workers, unskilled workers, and machines− are available. All markets are perfectly competitive.

Factors of production and Tasks: Each factor is characterized by *analytical ability* and *manual ability*. Denote analytical abilities of a skilled worker, an unskilled worker, and a machine by h, l_a, and k_a, respectively, where $h > l_a$, and their manual abilities by l_m, l_m, and k_m, respectively. Two types of workers have the same level of manual ability, reflecting the fact that there is no strong correlation between the two abilities, except in poorest countries. The final good is produced from inputs of a continuum of *tasks* that are different in *the importance of analytical ability*, $a \in [0, 1]$, and *the ease of codification (routinization)*, $c \in [0, 1]$. In the real economy, low a and high c tasks are those involving repetitive motions such as assembling or sorting objects and typical in production jobs; low a and low c tasks are those entailing non-repetitive motions such as driving vehicles and caring for the elderly and usual in low-wage service jobs; high a and high c tasks entail simple information processing such as calculation and recording information and are typical in clerical jobs; and high a and low c tasks involve complex analysis and judgement mainly associated with management, professional, and technical jobs.

Tasks are uniformly distributed over the (a, c) space, and productivities of a skilled worker, an unskilled worker, and a machine in task (a, c) are given by:

$$A_h(a) = ah + (1 - a)l_m, \tag{1}$$

$$A_l(a) = al_a + (1 - a)l_m, \tag{2}$$

$$cA_k(a) = c[ak_a + (1 - a)k_m]. \tag{3}$$

[16] Neiman (2013) finds that the labor share declines significantly in the large majority of 59 countries and industries based on data between 1975 and 2012. Bentolila and Saint-Paul (2003) find that the labor share declines between 1970 and 1990 for most of 12 OECD countries.

[17] The address is http://www.econ.kyoto-u.ac.jp/~yuki/english.html.

Except the most manual tasks ($a = 0$) and the most analytical tasks ($a = 1$), both abilities contribute to production in each task, but the relative contribution of analytical ability is greater in tasks with higher a.[18] For given a, machines are more productive in tasks with higher c, while workers are assumed to be equally productive for any c. Since $h > l_a$, skilled workers have comparative advantages in more analytical tasks relative to unskilled workers.

Production: At each task, the three factors are perfectly substitutable and thus the production function of task (a, c) is expressed as:

$$y(a,c) = A_h(a)n_h(a,c) + A_l(a)n_l(a,c) + cA_k(a)n_k(a,c), \tag{4}$$

where $n_i(a,c)$ $(i = h, l, k)$ is the measure of factor i engaged in the task. The output of the task, $y(a,c)$, may be interpreted as either an intermediate good or a direct input in final good production, which is produced by either final good producers or separate entities.

The final good production function is Leontief with equal weights on all tasks, that is, all tasks are equally essential in the production:

$$Y = \min_{a,c}\{y(a,c)\}. \tag{5}$$

The Leontief specification is assumed for simplicity. Similar results would be obtained as long as different types of tasks are complementary in the production, although more general specifications seem to be analytically intractable.[19]

Factor markets: A unit of each factor supplies a unit of time inelastically. Let the final good be a numeraire and let the relative price of (the output of) task (a, c) be $p(a, c)$. Then, from cost minimization problems of intermediate producers,

$$p(a,c) = \min\left\{\frac{w_h}{A_h(a)}, \frac{w_l}{A_l(a)}, \frac{r}{cA_k(a)}\right\}, \tag{6}$$

where w_h (w_l) is earnings of a skilled (unskilled) worker and r is exogenous (and constant) interest rate.[20] That is, producers choose a factor(s) so that a unit cost of task production becomes lowest.

From the equation, the basic pattern of *task assignment* can be derived (details are explained later). Since the relative productivity of skilled to unskilled workers $\frac{A_h(a)}{A_l(a)}$ increases with a, there exists unique $a^* \in (0, 1)$ satisfying $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$ and unskilled (skilled) workers are chosen over skilled (unskilled) workers for $a < (>) a^*$. That is, unskilled (skilled) workers are assigned to relatively manual (analytical) tasks. Of course, which factor is employed in a given task depends on the relative profitability of workers to machines as well. For $a < a^*$, unskilled workers (machines) are assigned to tasks (a, c) with $\frac{A_l(a)}{cA_k(a)} > (<) \frac{w_l}{r}$, and for $a > a^*$,

[18] One interpretation of the specification is that a task with certain a is composed of the proportion a of analytical subtasks, where only analytical ability is useful, and the proportion $1 - a$ of manual ones, and the two types of subtasks requiring different abilities are perfectly substitutable in the production of the task. (Due to indivisibility of subtasks and economies of scope, one needs to perform both types of subtasks.)

[19] The model with a Cobb-Douglas technology seems to be quite difficult to analyze. An advantage of the Leontief specification over the Cobb-Douglas is that, as shown below, it yields a realistic result that, among tasks a certain factor is employed, it is employed heavily in tasks in which their productivities are low.

[20] The closed economy model is analytically intractable. Considering that the real interest rate has been stable in the U.K. and the U.S. over the long-run, main results would not be affected much by the assumption of the small open economy.

skilled workers (machines) are assigned to tasks (a,c) with $\frac{A_h(a)}{cA_k(a)} > (<) \frac{w_h}{r}$. Comparative advantages of factors and relative factor prices determine task assignment.

Task (intermediate) markets: Because each task (intermediate good) is equally essential in final good production, $y(a,c) = Y$ must hold for any (a,c). Thus, the following is true for any (a,c) with $n_h(a,c) > 0$, any (a',c') with $n_l(a',c') > 0$, and any (a'',c'') with $n_k(a'',c'') > 0$, except for the set of measure 0 tasks in which multiple factors are employed:

$$A_h(a)n_h(a,c) = A_l(a')n_l(a',c') = c''A_k(a'')n_k(a'',c'') = Y. \tag{7}$$

Given the task assignment, factors are employed heavily in low productivity tasks.

Denote the measure of total supply of factor i $(i = h,l,k)$ by N_i (N_k is endogenous). Then, by substituting (7) into $\iint_{n_i(a,c)>0} n_i(a,c)dadc = N_i$,

$$\frac{N_h}{\iint_{n_h(a,c)>0} \frac{1}{A_h(a)}dadc} = \frac{N_l}{\iint_{n_l(a,c)>0} \frac{1}{A_l(a)}dadc} = \frac{N_k}{\iint_{n_k(a,c)>0} \frac{1}{cA_k(a)}dadc} = Y. \tag{8}$$

The first equality of the equation is one of the two key equations, which states that task assignment must be determined so that demands for two types of workers satisfy the equality.

Since the final good is a numeraire and a unit of the final good is produced from inputs of a unit of every task,

$$\iint p(a,c)dadc = 1 \tag{9}$$

$$\Leftrightarrow w_l \iint_{n_l(a,c)>0} \frac{1}{A_l(a)}dadc + w_h \iint_{n_h(a,c)>0} \frac{1}{A_h(a)}dadc + r\iint_{n_k(a,c)>0} \frac{1}{cA_k(a)}dadc = 1, \tag{10}$$

where the second equation is from (6). (10) is the second key equation, which states that task assignment is determined so that the unit production cost of the final good equals 1.

Equilibrium: A competitive equilibrium is defined by (6)−(8), (10), and the task assignment conditions ($\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$, $\frac{A_l(a)}{cA_k(a)} = \frac{w_l}{r}$, and $\frac{A_h(a)}{cA_k(a)} = \frac{w_h}{r}$). By using the task assignment conditions, the first equality of (8) and (10) are expressed as equations of w_h and w_l. Once the factor prices and thus task assignment are determined from these equations, N_k and Y $(= y(a,c))$ are determined from the second and third equalities of (8), respectively; $n_i(a,c)$ $(i = h,l,k)$ is determined from (7); and $p(a,c)$ is determined from (6).

3 Analysis

This section derives task assignment and earnings explicitly, given machine abilities k_a and k_m. So far, no assumptions are imposed on comparative advantages of machines to labor. Until Section 5, it is assumed that $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$, that is, machines have comparative advantages in relatively manual tasks. Then, $\frac{A_l(a)}{A_k(a)}$ and $\frac{A_h(a)}{A_k(a)}$ increase with a. With this assumption, the task assignment conditions can be stated more explicitly.

3.1 Task assignment conditions

Remember that, for $a < a^*$, unskilled workers (machines) perform tasks (a,c) with $\frac{A_l(a)}{cA_k(a)} > (<) \frac{w_l}{r}$, and for $a > a^*$, skilled workers (machines) perform tasks (a,c) with $\frac{A_h(a)}{cA_k(a)} > (<) \frac{w_h}{r}$, where a^* is defined by $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$. Further, since $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$, humans (machines)

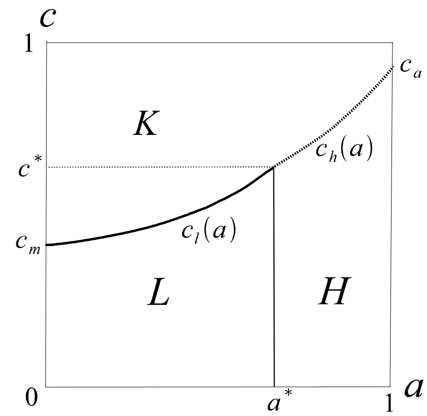


Figure 1: An example of task assignment when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ and $c_m < c^* < c_a < 1$

perform tasks with relatively high (low) a and low (high) c, and, for given c, machines perform tasks with $a > a^*$ only if they perform all tasks with $a \leq a^*$. Based on this pattern of assignment, critical variables and functions determining task assignment, $c_m, c^*, c_a, c_l(a)$, and $c_h(a)$, are defined next. (Figure 1 may be useful in understanding the following.)

Unskilled workers vs. machines: From the above discussion, whenever $n_k(a,c) > 0$ for some (a,c), $n_k(0,1) > 0$, i.e. whenever machines are used, they perform the most manual and easiest-to-codify task. Define c_m as $\frac{A_l(0)}{c_m A_k(0)} = \frac{l_m}{c_m k_m} = \frac{w_l}{r}$, i.e. c_m is the value of c such that hiring a machine and hiring an unskilled worker are equally profitable at task $(0, c_m)$ (see Figure 1).[21] Then, other (a,c)s satisfying $\frac{A_l(a)}{cA_k(a)} = \frac{w_l}{r}$ is given by $\frac{A_l(a)}{cA_k(a)} = \frac{l_m}{c_m k_m}$. Let $c_l(a) \equiv \frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m$. Given a, a machine and an unskilled worker are equally profitable at $c = c_l(a)$ and the former (latter) is hired for $c > (<) c_l(a)$. If there exists $c < 1$ such that they are equally profitable at $a = a^*$, i.e. $c_l(a^*) = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m < 1$, machines perform some tasks with $a > a^*$ (see Figure 1). If $c_l(a^*) \geq 1$, machines do not perform any tasks with $a > a^*$. Let $c^* \equiv \min\{c_l(a^*), 1\}$.

Skilled workers vs. machines: When $c^* < 1$, the choice between a machine and a skilled worker arises. Since $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$, (a,c)s satisfying $\frac{A_h(a)}{cA_k(a)} = \frac{w_h}{r}$ is given by $\frac{A_h(a)}{cA_k(a)} = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}$ and let $c_h(a) \equiv \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m$. Given a, hiring either factor is equally profitable at $c = c_h(a)$. If $c < 1$ exists such that either choice is equally profitable at $a = 1$, i.e. $c_h(1) = \frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m < 1$, machines perform some tasks with $a = 1$. Let $c_a \equiv \min\{c_h(1), 1\}$.

Figure 1 illustrates task assignment on the (a,c) space, assuming $c_m < c^* < c_a < 1$. Given a, machines perform tasks with higher c. From the assumption that machines have comparative advantages at relatively manual tasks, for given c, they perform tasks with lower a and the proportion of tasks performed by machines decreases with a, i.e. $c_l(a)$ and $c_h(a)$ are upward sloping. These properties hold when $c_m < c^* < c_a < 1$ is not true too.

[21] When such $c_m \leq 1$ does not exist, c_m is set to be equal 1. Under the assumption $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, as illustrated in Figure 1, c_m is the lowest c satisfying $n_k(a,c) > 0$.

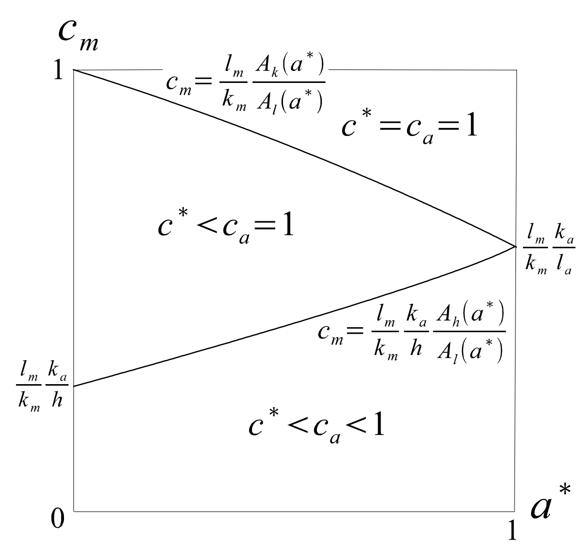


Figure 2: Values of c^* and c_a on the (a^*, c_m) space when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$

3.2 Key equations determining equilibrium

From their definitions, $c_l(a)$, $c_h(a)$, c^*, and c_a are functions of c_m and a^*:

$$c_l(a) = \frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m, \quad c_h(a) = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m, \tag{11}$$

$$c^* \equiv \min\{c_l(a^*), 1\}, \quad c_a \equiv \min\{c_h(1), 1\}. \tag{12}$$

From the equations defining a^* and c_m, earnings too are functions of c_m and a^*:

$$w_l = \frac{l_m}{k_m}\frac{r}{c_m}, \quad w_h = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{r}{c_m}. \tag{13}$$

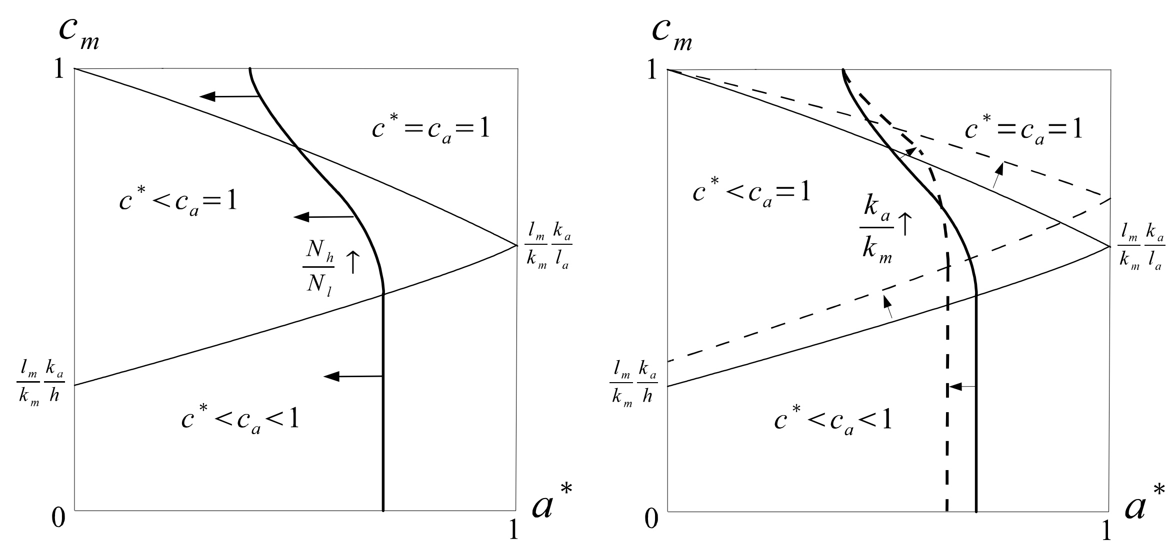
Hence, the two key equations determining equilibrium, the first equality of (8) and (10), can be expressed as equations of c_m and a^* (refer to Figure 1 for the derivation):

$$\frac{N_h}{N_l}\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{1}{A_l(a)}dcda = \int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{1}{A_h(a)}dcda, \tag{HL}$$

$$\frac{l_m}{k_m}\frac{r}{c_m}\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{dcda}{A_l(a)} + \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{r}{c_m}\int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}$$
$$+r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{dcda}{cA_k(a)} + \int_{a^*}^1\int_{\min\{c_h(a),1\}}^1\frac{dadc}{cA_k(a)}\right] = 1, \tag{P}$$

Once a^* and c_m are determined from (HL) and (P), $c^*, c_a, c_l(a), c_h(a)$ and thus task assignment are determined. Then, earnings are determined from (13), and the remaining variables are determined as stated in the definition of equilibrium.

The determination of equilibrium $a^*$and c_m can be illustrated using a figure depicting graphs of (HL) and (P) on the (a^*, c_m) space. Since, as explained below, the shape of (HL) differs depending on whether c^* and c_a equal 1 or not, from (11) and (12), the (a^*, c_m) space is divided into three regions based on values of c^* and c_a, as illustrated in Figure 2.



(a) Relation of (HL) with $\frac{N_h}{N_l}$ (b) Relation of (HL) with $\frac{k_a}{k_m}$

Figure 3: Shape of (HL) and its relations with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$

In the figure, when $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow \frac{A_l(a^*)}{1\times A_k(a^*)} \geq \frac{l_m}{c_m k_m} = \frac{w_l}{r}$ (from eq. 13), that is, when an unskilled worker is weakly chosen over a machine at task $(a,c)=(a^*,1)$, machines are not used in any tasks with $a > a^*$ and thus $c^* = c_a = 1$ holds.[22] When $c_m \geq \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow \frac{h}{1\times k_a} \geq \frac{l_m}{c_m k_m}\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{r}$ (from eq. 13) and $c_m < \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$, that is, when a skilled worker is weakly chosen over a machine at task $(a,c)=(1,1)$ and a machine is strictly chosen over an unskilled worker at task $(a,c)=(a^*,1)$, machines are employed in some tasks with $a > a^*$ but not in tasks with $a = 1$ and $c < 1$, thus $c^* < c_a = 1$ holds.[23] Finally, when $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, machines are employed in some tasks with $a = 1$ and $c < 1$ and thus $c^* < c_a < 1$ holds.

3.3 Shape of (HL) and its relations with exogenous variables

The shape of (HL) and its relations with exogenous variables, $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$, are illustrated in Figure 3, based on Lemmas 1−3 in Appendix A. Note that the shape and the relations do *not* depend on the assumption $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, except that the case $c^*=c_a=1$ (the upper region in the figure) does not arise when $\frac{k_a}{k_m} \geq \frac{l_a}{l_m}$ and the case $c^* < c_a = 1$ (the middle region) does not arise when $\frac{k_a}{k_m} \geq \frac{h}{l_m}$.

The left figure shows that (HL) is negatively sloped when $c_a = 1$ and is vertical when $c_a < 1$ on the (a^*, c_m) space. The shape can be explained intuitively as follows. A decrease in c_m lowers $c_l(a)$ and $c_h(a)$ from (11) and raises the proportion of tasks performed by machines (see Figure 1). When $c_a = 1$, i.e. machines do not perform any tasks with $a = 1$ and $c < 1$, the mechanization mainly affects unskilled workers engaged in relatively manual tasks and thus they shift to more analytical tasks, i.e. a^* increases. By contrast, when $c_a < 1$, both

[22] In this case, unlike Figure 1, $c_l(a)$ intersects with $c = 1$ at $a \leq a^*$ on the (a,c) plane.

[23] In this case, unlike Figure 1, $c_h(a)$ intersects with $c = 1$ at $a \in (a^*, 1]$ on the (a,c) plane.

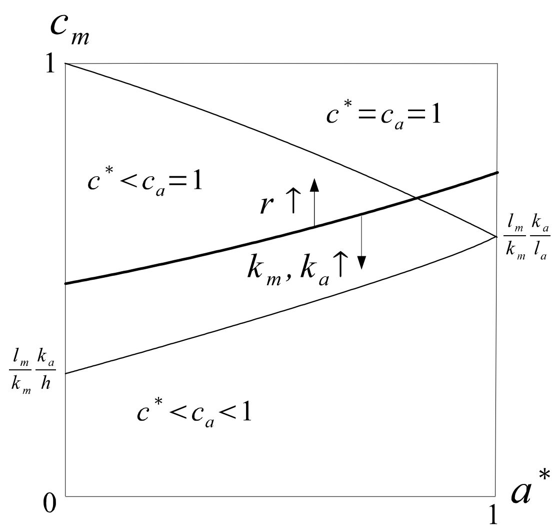


Figure 4: Shape of (P) and its relations with k_m, k_a, and r

types of workers are equally affected and thus a^* remains unchanged.

The left and right figures illustrate the relations of (HL) with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$, respectively. An increase in $\frac{N_h}{N_l}$ implies that a higher portion of tasks must be engaged by skilled workers and thus (HL) shifts to the left (given c_m, a^* decreases). Less straightforward is the effect of an increase in $\frac{k_a}{k_m}$, which shifts the locus to the right (left) when c_m is high (low), definitely so when $c^*=1$ (when $c_a<1$). An increase in $\frac{k_a}{k_m}$ weakens comparative advantages of humans in analytical tasks and thus lowers, particularly for high a, $c_l(a)$, $c_h(a)$, and the portion of tasks performed by humans (see Figure 1). When c_m (thus c^* and c_a) is high, such mechanization mainly affects unskilled workers and thus a^* must increase,[24] while the opposite is true when c_m is low.

3.4 Shape of (P) and its relations with exogenous variables

Figure 4 illustrates the shape of (P) and its relations with exogenous variables, k_m k_a, and r, based on Lemma 4 in Appendix A. Remember that, for (P) to hold, task assignment must be determined so that the unit production cost of the final good equals 1. When c_m increases, a^* must increase, that is, (P) is upward-sloping on the (a^*, c_m) plane, because, otherwise, both $w_l = \frac{l_m}{k_m}\frac{r}{c_m}$ and $w_h = \frac{A_h(a^*)}{A_l(a^*)} w_l$ fall and thus the unit production cost decreases. An increase in r raises the cost of hiring machines and thus a higher portion of tasks are assigned to humans, i.e. the locus shifts upward (given a^*, c_m increases, implying that $c_l(a)$ and $c_h(a)$ increase), while the opposite holds when abilities of machines, k_m and k_a, increase. The locus never intersects with $c_m = 0$, because machines are completely useless and thus hiring machines are prohibitively expensive at the hardest-to-codify tasks.

As Figure 5 illustrates, equilibrium (a^*, c_m) is determined at the intersection of the two

[24] For example, when $c^* = c_a = 1$, $c_l(a)$ intersects with $c = 1$ at $a \leq a^*$ on the (a, c) plane. In this case, it would be clear that the mechanization mainly affects unskilled workers.

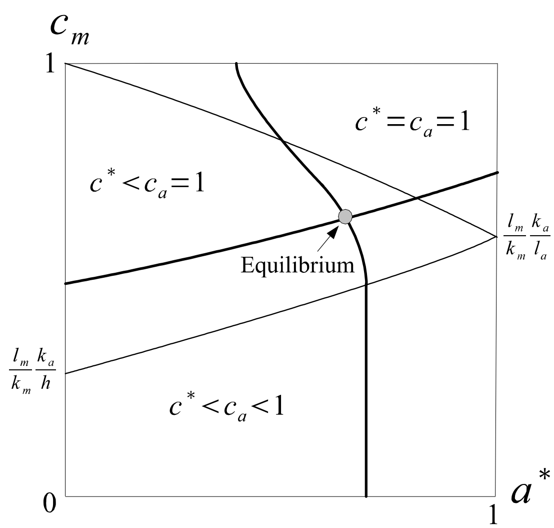


Figure 5: Determination of equilibrium a^* and c_m

loci. Of course, the position of the intersection depends on exogenous variables such as k_m and k_a. The next two sections examine how increases in k_m, k_a, and $\frac{N_h}{N_l}$ affect the equilibrium, particularly, task assignment, earnings, earnings inequality, and aggregate output.

4 Mechanization with constant $\frac{k_a}{k_m}$

Suppose that abilities of machines, k_m and k_a, improve exogenously over time. This section examines effects of such productivity growth and of an increase in $\frac{N_h}{N_l}$ on task assignment, earnings levels and inequality, and output, when k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow proportionately. Since (HL) does not shift under constant $\frac{k_a}{k_m}$ (Figure 3 (a)), the analysis is much simpler than the general case analyzed in the next section.

The next proposition presents the dynamics of the critical variables and functions determining task assignment of an economy undergoing the productivity growth.

Proposition 1 *Suppose that k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow over time with $\frac{k_a}{k_m}$ constant.*

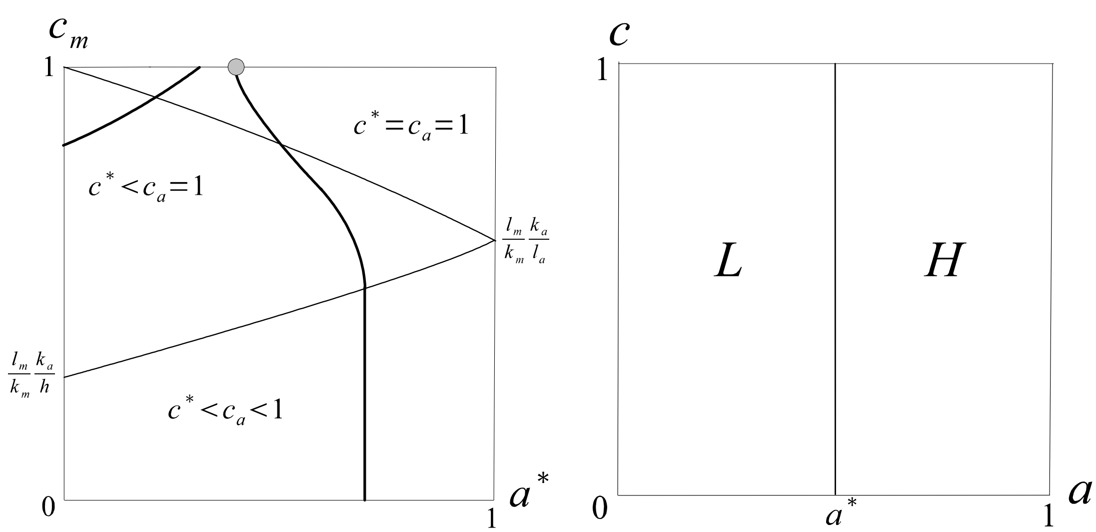
(i) When k_m is very low initially, $c_m = c^* = c_a = 1$ is satisfied at first;[25] *at some point, $c_m < c^* = c_a = 1$ holds and thereafter c_m falls over time; then, $c_m < c^* < c_a = 1$ and c^* too falls; finally, $c_m < c^* < c_a < 1$ and c_a falls as well.*

(ii) a^* increases over time when $c_m < c_a = 1$, while a^* is time-invariant when $c_a < 1$ (and when $c_m = 1$).

(iii) $c_l(a)$ and $c_h(a)$ (when $c^* < 1$) decrease over time when $c_m < 1$.

The results of this proposition can be understood using figures similar to Figure 5. When the level of k_m is very low, there are no (a^*, c_m) satisfying (P), or (P) is located at the left

[25] As noted in footnote 21, the value of c_m when all tasks are performed by humans is set to be equal 1.



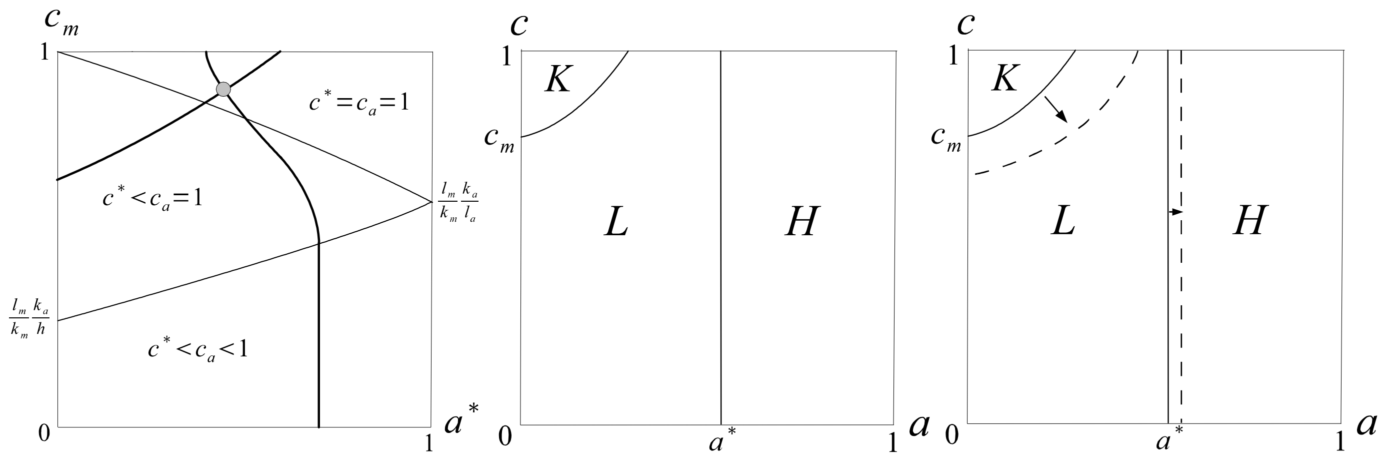
(a) Equilibrium (b) Task assignment

Figure 6: Equilibrium and task assignment when $c_m = c^* = c_a = 1$

side of (HL) on the (a^*, c_m) plane (see Figure 6 (a)). Hence, the two loci do not intersect and an equilibrium with $c_m < 1$ does not exist. Because the manual ability of machines is very low, hiring machines is not profitable at all and thus all tasks are performed by humans. Figure 6 (a) illustrates an example of the determination of equilibrium c_m and a^* in this case. Equilibrium a^* is determined at the intersection of (HL) with $c_m = 1$. Figure 6 (b) illustrates the corresponding task assignment on the (a, c) plane, which shows that unskilled (skilled) workers perform all tasks with $a < (>) a^*$.

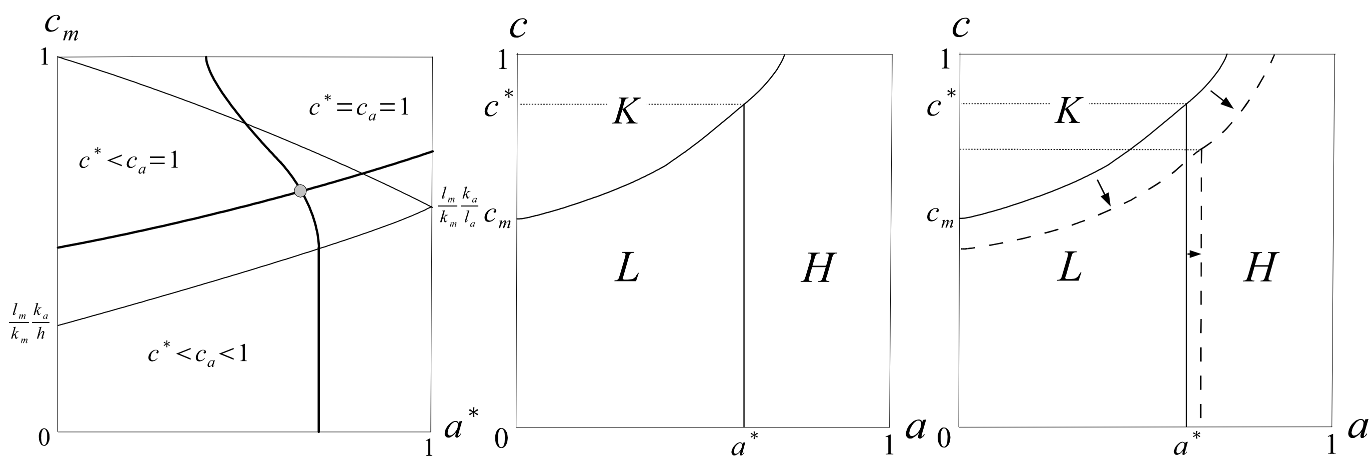
When k_m becomes high enough that (P) is located at the right side of (HL) at $c_m = 1$, the two loci intersect and thus machines begin to be used, i.e. $c_m < 1$. Note that k_a is not important for the first step of mechanization, because mechanization starts from the most manual tasks in which analytical ability is of no use. Because of low machine productivities, they perform only highly manual and easy-to-codify tasks that were previously performed by unskilled workers, i.e. $c^* = c_a = 1$ holds. Indeed, large-scale mechanization originated in tasks associated with simple repetitive motions in textile during the Industrial Revolution. Figure 7 (a) and (b) respectively illustrate the determination of equilibrium c_m and a^* and task assignment. Figure 7 (c) presents the effect of small increases in k_m and k_a on the task assignment. Since machines come to perform a greater portion of highly manual and easy-to-codify tasks, a^* increases and $c_l(a)$ decreases, that is, workers shift to more analytical and, for unskilled workers, harder-to-routinize tasks. Consistent with the model, during early stages of industrialization, humans shifted from manual tasks at farms, cottages, and workshops toward manual tasks at factories and analytical tasks at offices and factories (generally associated with clerical, management, and technical jobs), and manual workers shifted to tasks involving more complex motions machines were not good at.

As k_m and k_a grow over time, mechanization spreads to relatively analytical tasks, and eventually, machines come to perform highly analytical tasks, those previously performed by



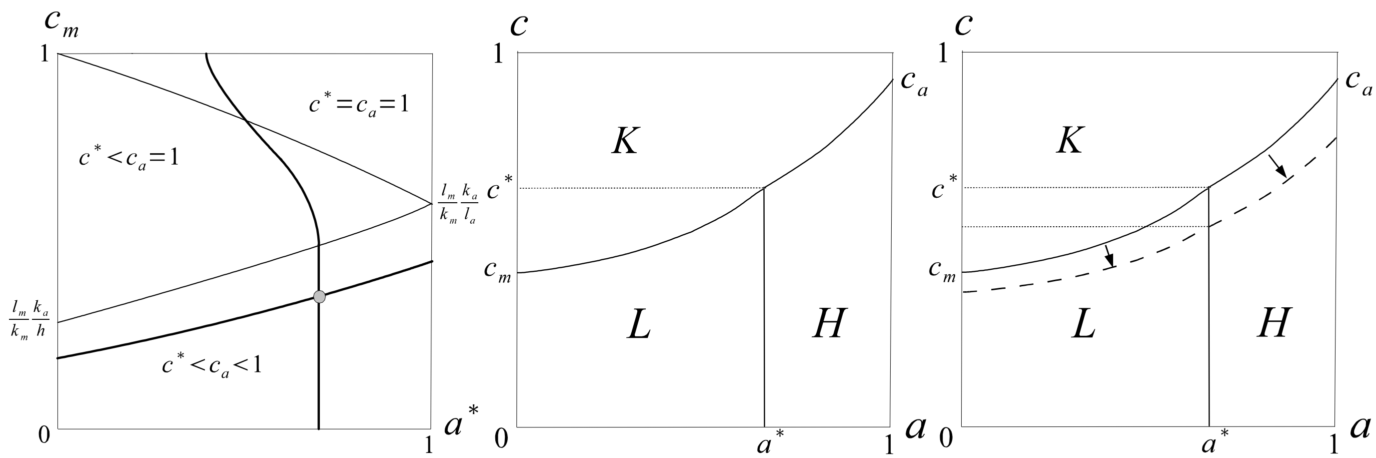
(a) Equilibrium (b) Task assignment (c) Effect of productivity growth with constant $\frac{k_a}{k_m}$

Figure 7: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* = c_a = 1$



(a) Equilibrium (b) Task assignment (c) Effect of productivity growth with constant $\frac{k_a}{k_m}$

Figure 8: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* < c_a = 1$



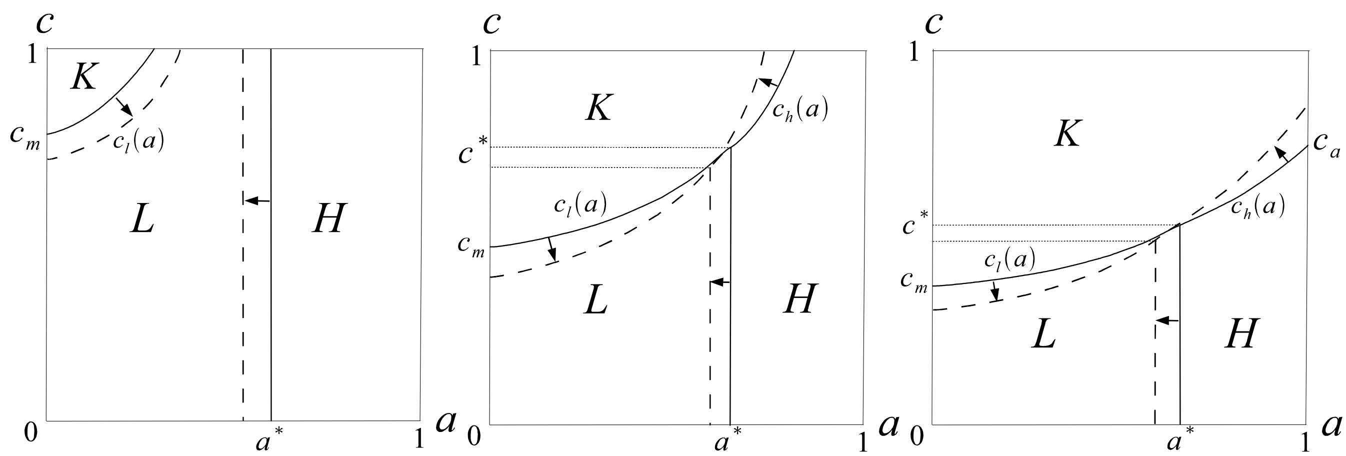
(a) Equilibrium (b) Task assignment (c) Effect of productivity growth with constant $\frac{k_a}{k_m}$

Figure 9: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* < c_a < 1$

skilled workers. In the real economy, the new phase of mechanization started during the Second Industrial Revolution – e.g. teleprinters replaced Morse code operators and tabulating machines substituted data-processing workers at large organizations– and has progressed on a large scale in the post World War II era, especially since the 1970s, because of the growth of information technology. Figure 8 (a) and (b) respectively illustrate the determination of equilibrium c_m and a^* and task assignment when $c_m < c^* < c_a = 1$. Machines perform some tasks with $a > a^*$ but not the most analytical ones, i.e. $c^* < c_a = 1$. Productivity growth lowers $c_h(a)$ as well as $c_l(a)$ (and raises a^*), thus skilled workers too shift to more difficult-to-codify tasks (Figure 8 (c)). Congruent with the model, since the 1970s, humans have shifted from routine analytical tasks (such as simple information processing tasks typical in clerical jobs) as well as manual tasks toward non-routine analytical tasks mainly associated with professional and technical jobs and non-routine manual tasks in services.

Finally, the economy reaches the case $c_m < c^* < c_a < 1$, which is illustrated in Figure 9. Machines perform a portion of the most analytical tasks, i.e. $c_a < 1$. In fact, currently, machines are engaged in some tasks involving analysis and decision-making, such as automated trading in financial markets. Unlike the previous cases, productivity growth affects two type of workers equally and thus a^* does not change, while $c_h(a)$ and $c_l(a)$ decrease and thus workers shift to more difficult-to-codify tasks.

In sum, when the two abilities of machines with $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ improve proportionally over time, mechanization starts from highly manual and easy-to-codify tasks and gradually spreads to more analytical and harder-to-codify tasks. Eventually, machines come to perform highly analytical tasks previously performed by skilled workers. Accordingly, unskilled workers shift to tasks that are more difficult to codify, so do skilled workers in later stages of mechanization, and both types shift to more analytical tasks except at the final stage.



(a) when $c_m < c^* = c_a = 1$ (b) when $c_m < c^* < c_a = 1$ (c) when $c_m < c^* < c_a < 1$

Figure 10: Effect of an increase in $\frac{N_h}{N_l}$ on task assignment when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$

The dynamics of task assignment accord with the long-run trends of mechanization and of shifts in tasks performed by humans, except job polarization after the 1990s, which is detailed in the introduction and is summarized as: initially, mechanization proceeded in tasks intensive in manual labor, while mechanization of tasks intensive in analytical labor started during the Second Industrial Revolution and has progressed on a large scale in the post World War II era, especially since the 1970s, because of the growth of information technology; humans shifted from manual tasks to analytical tasks until about the 1960s, whereas, thereafter, they have shifted away from routine analytical tasks as well as routine manual tasks toward non-routine analytical tasks and non-routine manual tasks in services.

Effects of the productivity growth on earnings levels and inequality, and aggregate output are examined in the next proposition.

Proposition 2 *Suppose that k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow proportionately over time when $c_m < 1$.*

(i) Earnings of skilled workers increase over time. When $c^* < c_a < 1$, earnings of unskilled workers too increase.

(ii) Earnings inequality, $\frac{w_h}{w_l}$, rises over time when $c_a = 1$ and is time-invariant when $c_a < 1$.

(iii) The output of the final good, Y, increases over time.

The proposition shows that, while skilled workers *always* benefit from mechanization, the effect on earnings of unskilled workers is ambiguous when mechanization mainly affects them, i.e. when $c_a = 1$, and the effect turns positive when $c_a < 1$. Mechanization worsens earnings inequality, $\frac{w_h}{w_l}$, when $c_a = 1$, while it has no effect when $c_a < 1$. The output of the final good *always* increases, even if $l_a < h < l_m$ and thus workers' productivities, $A_h(a)$ and $A_l(a)$, fall as they shift to more analytical tasks.

So far, the proportion of skilled workers to unskilled workers, $\frac{N_h}{N_l}$, is held constant, which has increased over time in the actual economy. Thus, the next proposition examines effects of the growth of $\frac{N_h}{N_l}$ under constant machine qualities.

Proposition 3 *Suppose that $\frac{N_h}{N_l}$ grows over time when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ and $c_m < 1$.*

(i) c_m, a^*, c^* (when $c^* < 1$), and $c_l(a)$ decrease, while c_a (when $c_a < 1$) and $c_h(a)$ (when $c^* < 1$) increase over time.

(ii) w_l (w_h) rises (falls) and earnings inequality, $\frac{w_h}{w_l}$, shrinks over time.

(iii) Y increases over time under constant $N_h + N_l$.

Figure 10 illustrates the effect of an increase in $\frac{N_h}{N_l}$ on task assignment. Since skilled workers become abundant relative to unskilled workers, they take over a portion of tasks previously performed by unskilled workers, i.e. a^* decreases. Further, earnings of unskilled workers rise and those of skilled workers fall, thus some tasks previously performed by unskilled workers are mechanized, i.e. $c_l(a)$ decreases, while, when $c^* < 1$, skilled workers take over some tasks performed by machines before, i.e. $c_h(a)$ increases. That is, skilled workers shift to more manual tasks, and unskilled workers shift to harder-to-routinize tasks. The output of the final good increases even when the total population is constant, mainly because skilled workers are more productive than unskilled workers at any tasks with $a > 0$.

By combining the results on effects of an increase in $\frac{N_h}{N_l}$ with those of the productivity growth, the model can explain the long-run trends of earnings and earnings inequality until the 1970s (except the wartime 1940s) detailed in the introduction, which is: in early stages of industrialization when mechanization directly affected unskilled workers only and the relative supply of skilled workers grew slowly, earnings of unskilled workers grew very moderately and earnings inequality rose; in later periods when skilled workers too were directly affected by mechanization and the relative supply of skilled workers grew faster, unskilled workers benefited more from mechanization, while, as before, the rising inequality was the norm in economies with lightly regulated labor markets (such as the U.S.), except in periods of a rapid increase in the level of education and in the 1940s, when the inequality fell.[26]

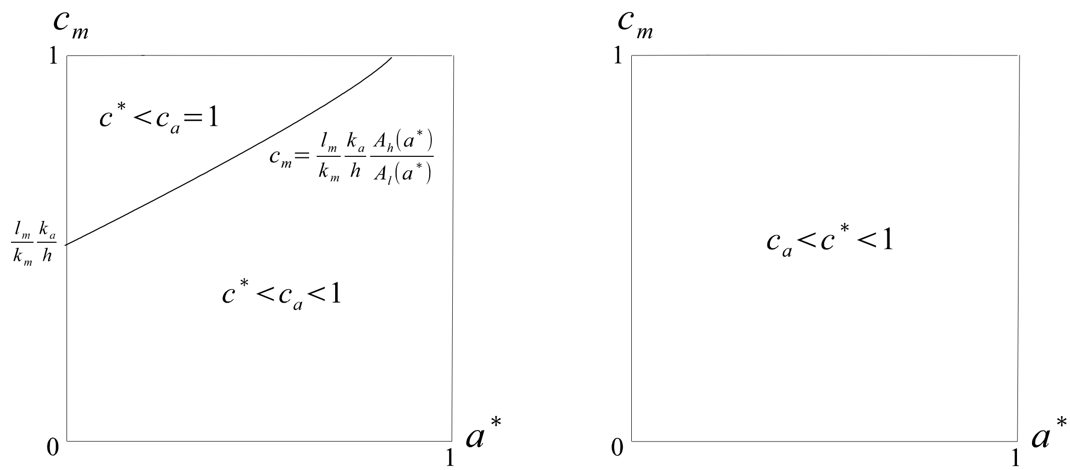
The model, however, fails to capture the trends after the 1980s, which is: earnings of unskilled workers stagnated and those of skilled workers rose until the mid 1990s in the U.S.;[27] the inequality rose greatly after the 1980s (after the 1990s in many European economies, OECD, 2008); and wage polarization has proceeded since the 1990s at least in the U.S. By contrast, the model predicts that earnings of unskilled workers increase and the inequality shrinks when highly analytical tasks are affected by mechanization, i.e. when $c_a < 1$, and the relative supply of skilled workers rises.

5 Mechanization with time-varying $\frac{k_a}{k_m}$

The previous section has examined the case in which k_m and k_a grow proportionately. This special case has been taken up first for analytical simplicity. However, the assumption of the proportionate growth is rather restrictive, because, according to the trend of mechanization

[26] Combined effects of an increase in $\frac{N_h}{N_l}$ and improvements of machine qualities on task assignment accord with the trend of task shifts in the real economy when $c^* = 1$. When $c^* < 1$, they are consistent with the fact, unless the negative effect of an increase in $\frac{N_h}{N_l}$ on $c_h(a)$ is very strong (see Figure 10).

[27] According to Acemoglu and Autor (2011), real wages of full-time male workers without college degrees are lower in 1995 than in 1980, while wages of those with more than college education are higher. As for female workers, real wages rose during the period except for high school dropouts, but the rise was moderate for those without college degrees.



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

Figure 11: c^* and c_a on the (a^*, c_m) space when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

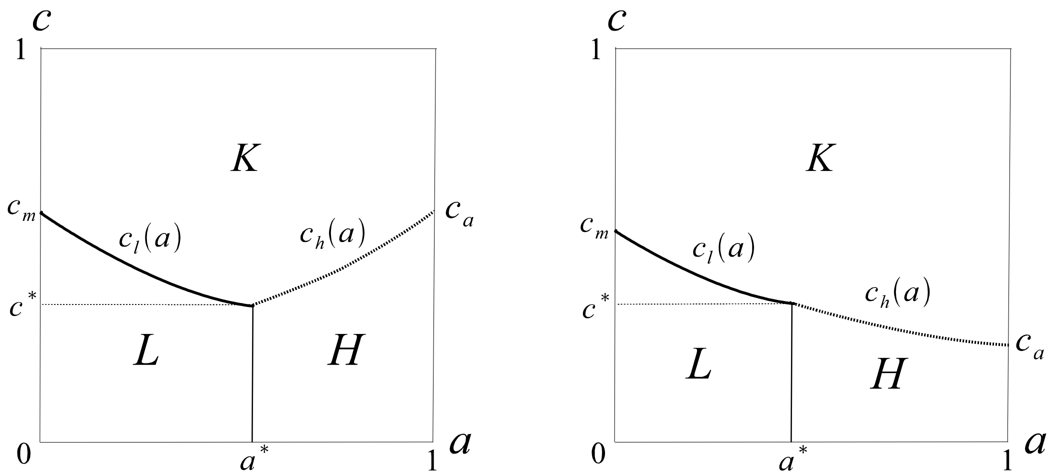
described in the introduction, the growth of k_m was apparently faster than that of k_a in most periods of time, while k_a seems to have been growing faster than k_m recently.[28]

This section examines the general case in which they may grow at different rates. This case is much more difficult to analyze because a change in $\frac{k_a}{k_m}$ shifts the graph of (HL) as well as that of (P) (see Figures 3 and 4 in Section 3). Under realistic productivity growth, the model does much better jobs in explaining the development after the 1980s than in the constant $\frac{k_a}{k_m}$ case.

Unlike the previous case, shapes of graphs in Figures 1 and 2 may change qualitatively with productivity growth. Starting from the situation where $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$ holds, if k_a keeps growing faster than k_m, i.e. the rapid growth of information technology is long-lasting, $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, then $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$ come to be satisfied. That is, comparative advantages of machines to two type of workers change over time. As illustrated in Figure 11, when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, $c^* < 1$ always holds, and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$, $c_a < c^* < 1$ always holds from $c^* = \min \left\{ \frac{k_m}{l_m} \frac{A_l(a^*)}{A_k(a^*)} c_m, 1 \right\}$ and $c_a = \min \left\{ \frac{h}{k_a} \frac{k_m}{l_m} \frac{A_l(a^*)}{A_h(a^*)} c_m, 1 \right\}$.

Figure 12 illustrates $c_l(a)$ and $c_h(a)$ and task assignment on the (a, c) space when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (the figure is drawn assuming $c_a < 1$) and when $\frac{k_a}{k_m} > \frac{h}{l_m}$. Unlike the original case $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, $c_l(a)$ is downward sloping and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$, $c_h(c)$ too is downward sloping. Hence, when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, for given c, machines tend to perform tasks with *intermediate* a and the proportion of tasks performed by machines is *highest at* $a = a^*$. When $\frac{k_a}{k_m} > \frac{h}{l_m}$, for given c, machines tend to perform relatively *analytical* tasks and the proportion of tasks performed by machines *increases* with a.

[28] Note that k_a seems to have been positive even before the Industrial Revolution. Various machines had automatic control systems whose major examples are: float valve regulators used in ancient Greece and in the medieval Arab world to control the level of water in tanks and devices such as water clocks and oil lamps; temperature regulators of furnaces invented in early 17th century Europe.



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 12: $c_l(a)$ and $c_h(a)$ when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ ($c_a < 1$ is assumed) and when $\frac{k_a}{k_m} > \frac{h}{l_m}$

5.1 Effects of changes in k_m, k_a, and $\frac{N_h}{N_l}$

Now, effects of changes in k_m and k_a on task assignment, earnings levels and inequality, and output are examined. Since results are different depending on the shape of (HL) (Figure 3), they are presented in three separate propositions.[29,30] The next proposition analyzes the case $c^* = c_a = 1$, which arises only when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$.

Proposition 4 *When* $c_m \geq \frac{l_m}{k_m} \frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_a = 1$ *(possible only when* $\frac{k_a}{k_m} < \frac{l_a}{l_m}$*),*

(i) c_m *decreases and* a^* *increases with* k_m *and* k_a ($\lim_{c_m \to 1} \frac{da^*}{dk_m} = \lim_{c_m \to 1} \frac{da^*}{dk_a} = 0$).

(ii) $c_l(a)$ *decreases with* k_m *and* k_a*.*

(iii) w_h, $\frac{w_h}{w_l}$, *and* Y *increase with* k_m *and* k_a*.* w_l *increases with* k_a*.*

The only difference from the constant $\frac{k_a}{k_m}$ case is that w_l increases when k_a rises with k_m unchanged. As before, with improved machine qualities, c_m and $c_l(a)$ decrease and a^* increases, i.e. workers shift to more analytical and, for unskilled workers, harder-to-codify tasks (see Figure 7 (c) in Section 4), and earnings of skilled workers, earnings inequality $\frac{w_h}{w_l}$, and output rise.

The next proposition examines the case $c^* < c_a = 1$, which is possible only when $\frac{k_a}{k_m} < \frac{h}{l_m}$.

Proposition 5 *When* $c_m \in \left[\frac{l_m}{k_m} \frac{k_a}{h} \frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m} \frac{A_k(a^*)}{A_l(a^*)}\right) \Leftrightarrow c^* < c_a = 1$ *(possible only when* $\frac{k_a}{k_m} < \frac{h}{l_m}$*),*

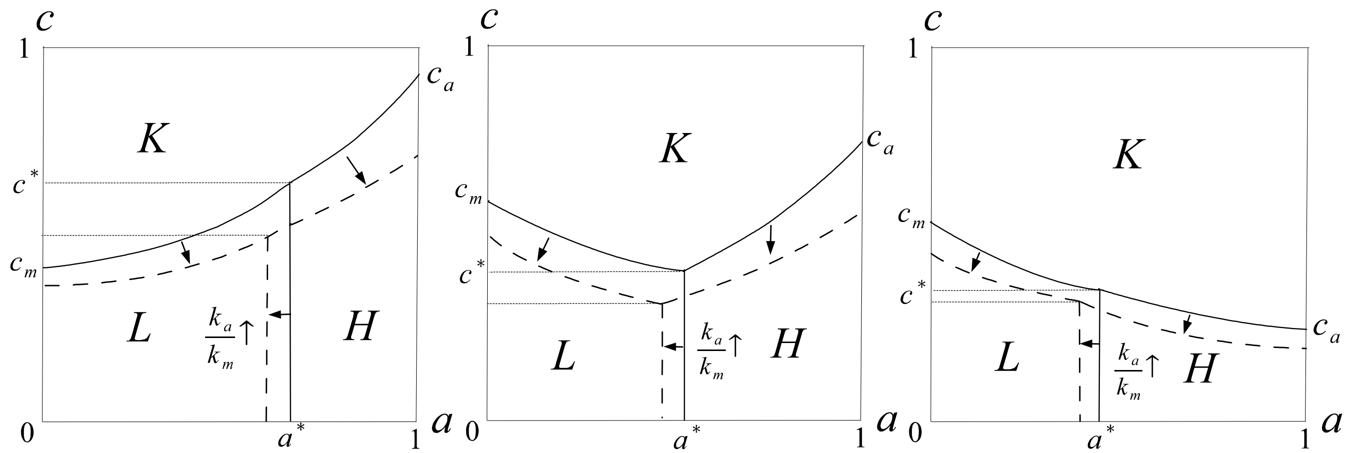
(i) c_m *decreases with* k_m *and* k_a*.* a^* *increases when* $\frac{k_a}{k_m}$ *non-increases.*

(ii) $c_l(a)$ *and* $c_h(a)$ *decrease with* k_m *and* k_a*.*

(iii) w_h *and* Y *increase with* k_m *and* k_a*, while* w_l *increases with* k_a*.* $\frac{w_h}{w_l}$ *increases when* $\frac{k_a}{k_m}$ *non-increases.*

[29]When $\frac{k_a}{k_m} > \frac{l_a}{l_m}$, $c_m = 1$ is possible with c^* or $c_a < 1$. However, such situation −the most manual and easy-to-codify task is not mechanized while some of other tasks are − is unrealistic and thus is not examined.

[30]Proofs of these propositions and Proposition 7 are very lengthy and thus are posted on the author's web site (http://www.econ.kyoto-u.ac.jp/~yuki/english.html).



(a) when $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (c) when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 13: Effect of productivity growth with increasing $\frac{k_a}{k_m}$ when $c^*, c_a < 1$

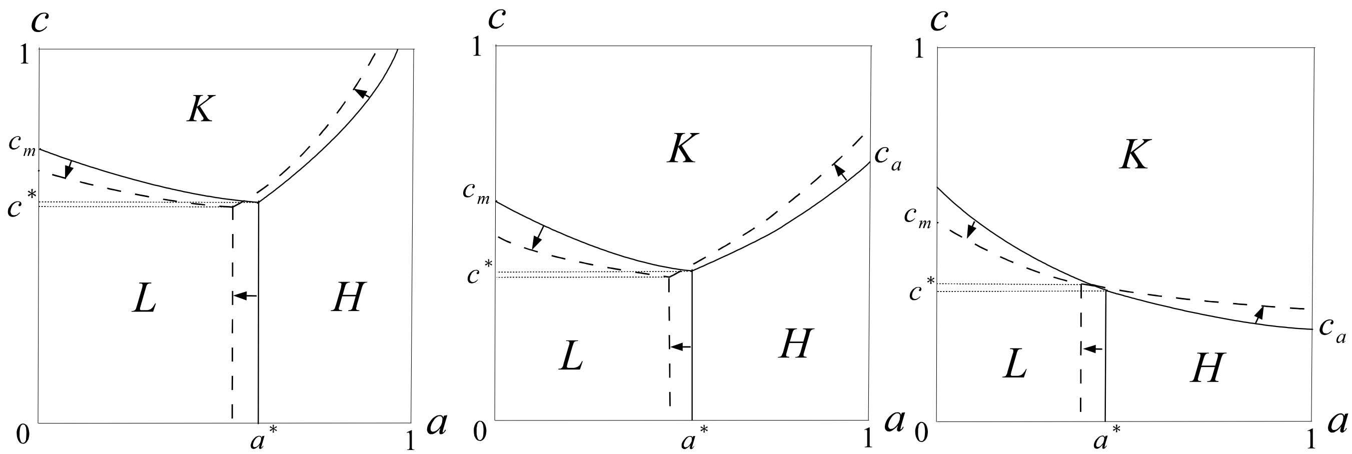
There are several differences from the constant $\frac{k_a}{k_m}$ case. First, effects of productivity growth with increasing $\frac{k_a}{k_m}$ on a^* and earnings inequality are ambiguous, and w_l increases with k_a. Second, although $c_l(a)$ (thus c_m) and $c_h(a)$ decrease and thus workers shift to harder-to-routinize tasks as in the original case, workers may not shift to more analytical tasks when a^* decreases (possible only when $\frac{k_a}{k_m}$ increases) and when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (see Figure 12 (a)). Remaining results are same as before, that is, when $\frac{k_a}{k_m}$ non-increases, a^* and earnings inequality increase; when $\frac{k_a}{k_m} \leq \frac{l_a}{l_m}$ too holds, workers shift to more analytical tasks; and earnings of skilled workers and output always increase.

Proposition 6 examines the case $c^*, c_a < 1$ ($c^* < (>) c_a$ when $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$).

Proposition 6 *When* $c_m < \frac{l_m}{k_m} \frac{k_a}{h} \frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c^*, c_a < 1$,
(i) c_m *and* c_a *decrease with* k_m *and* k_a*, and* a^* *decreases with* $\frac{k_a}{k_m}$.
(ii) $c_l(a)$ *and* $c_h(a)$ *decrease with* k_m *and* k_a.
(iii) w_h *and* Y *increase with* k_m *and* k_a*, while* w_l *increases when* $\frac{k_a}{k_m}$ *non-decreases.* $\frac{w_h}{w_l}$ *decreases with* $\frac{k_a}{k_m}$.

Unlike the constant $\frac{k_a}{k_m}$ case, in which a^* and thus $\frac{w_h}{w_l}$ are constant and w_l increases over time, a^* and $\frac{w_h}{w_l}$ decrease with $\frac{k_a}{k_m}$, and the effect on w_l is ambiguous when $\frac{k_a}{k_m}$ decreases. As for task assignment, while $c_l(a)$ (thus c_m) and $c_h(a)$ decrease as in the original case (thus workers shift to harder-to-routinize tasks), tasks performed by humans change in the skill dimension as well. In particular, when $\frac{k_a}{k_m}$ rises (falls), that is, when productivity growth is such that comparative advantages of machines to humans in analytical (manual) tasks rise, unskilled workers shift to more *manual* (analytical) tasks under $\frac{k_a}{k_m} > (<) \frac{l_a}{l_m}$, and skilled workers too shift to such tasks under $\frac{k_a}{k_m} > (<) \frac{h}{l_m}$.[31] Figure 13 illustrates the effect

[31] When $\frac{k_a}{k_m}$ rises (falls) under $\frac{k_a}{k_m} < (>) \frac{l_a}{l_m}$, unskilled workers shift to more manual (analytical) tasks at low c. The same is true for skilled workers under $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$. (See Figure 13.) Hence, at low c, workers always shift to more manual (analytical) tasks when $\frac{k_a}{k_m}$ rises (falls).



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and $c_a = 1$ (b) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and $c_a < 1$ (c) when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 14: Effect of an increase in $\frac{N_h}{N_l}$ when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m}$

of productivity growth with increasing $\frac{k_a}{k_m}$ on task assignment for this case. (The effect of productivity growth with decreasing $\frac{k_a}{k_m}$ can be illustrated by similar figures with increasing a^*.) Earnings of skilled workers and output rise as before.

Finally, Proposition 7 examines effects of an increase in $\frac{N_h}{N_l}$ when $\frac{k_a}{k_m} \geq \frac{l_a}{l_m}$ is allowed.

Proposition 7 *Suppose that $\frac{N_h}{N_l}$ grows over time when $c_m < 1$.*

(i) c_m, a^*, and $c_l(a)$ decrease, while c_a (when $c_a < 1$) and $c_h(a)$ (when $c^* < 1$) increase over time. c^* (when $c^* < 1$) falls (rises) when $\frac{k_a}{k_m} \leq \frac{l_a}{l_m}$ ($\frac{k_a}{k_m} \geq \frac{h}{l_m}$).

(ii) w_l (w_h) rises (falls) and $\frac{w_h}{w_l}$ shrinks over time.

(iii) Y increases over time under constant $N_h + N_l$.

Figure 14 illustrates the effect of an increase in $\frac{N_h}{N_l}$ on task assignment when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m}$. (Note that $c^* = c_a = 1$ does not occur in these cases and $c^* < c_a = 1$ does not occur when $\frac{k_a}{k_m} > \frac{h}{l_m}$.) As in the original case of $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, skilled workers take over some tasks previously performed by unskilled workers, i.e. a^* decreases, and machines (skilled workers) come to perform a portion of tasks performed by unskilled workers (machines) before, i.e. $c_l(a)$ decreases ($c_h(a)$ increases). However, unlike before, $c_l(a)$ is downward-sloping on the (a, c) plane, and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$, $c_h(a)$ too is downward-sloping. Thus, unskilled workers shift to harder-to-routinize *and* more manual tasks, and skilled workers may not shift to more manual tasks when $\frac{k_a}{k_m} > \frac{h}{l_m}$ (see Figure 14 (c)). As in the original case, earnings of unskilled (skilled) workers rise (fall), earnings inequality shrinks, and output increases.

5.2 Contrasting the model with facts

Based on the propositions, it is examined whether the model with realistic productivity growth can explain the long-run trends of task shifts, earnings, and earnings inequality in the real economy.

Two assumptions are imposed on comparative advantage of machines against humans and the relative growth of the two abilities of machines. First, it would be plausible to suppose that $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ has continued to hold until now (thus $c_l(a)$ and $c_h(a)$ are downward-sloping on the (a, c) plane), since the proportion of tasks performed by machines seems to have been and be higher in more manual tasks: consider the fact that the vast majority of non-routine analytical tasks generally associated with management, professional, and technical jobs and of non-routine "middle a" tasks typical in occupations such as mechanics and nurses are yet to be mechanized.

Second, the history of mechanization and task shifts described in the introduction suggests that k_m seems to have grown faster than k_a until sometime in the 1990s, after which the growth of k_a appears to be faster because of the growing application of information technology in many fields.[32] The supposed turning point would be not be far off the mark considering that a decrease in the employment share of production occupations, which are intensive in manual tasks, is greatest in the 1980s and slowed down considerably after the 1990s, while a decrease in the share of clerical occupations intensive in routine analytical tasks accelerated after the 1990s, according to Acemoglu and Autor (2011). Note also that information technology seems to have contributed to the growth of k_m more than the growth of k_a initially: CNC [Computer Numerical Control] machines and industrial robots, widely used since the 1970s and the 1980s respectively, raised productivities of machines to perform manual and relatively non-routine tasks considerably. Hence, suppose that $\frac{k_a}{k_m}$ falls over time when $c_a = 1$, i.e. none of the most analytical tasks are mechanized, while, when $c_a < 1$, $\frac{k_a}{k_m}$ falls initially, then rises.

Now, the dynamics of earnings and earnings inequality are examined. Since the result when $c^* = c_a = 1$ is almost the same as the constant $\frac{k_a}{k_m}$ case (Proposition 4), the model is consistent with the actual trends in the early stage of mechanization. The model accords with the trends in the intermediate stage as well (except a decline of the inequality in the wartime 1940s), because the result when $c^* < c_a = 1$ holds and $\frac{k_a}{k_m}$ falls is same as before (Proposition 5). Further, unlike the constant $\frac{k_a}{k_m}$ case, the model is congruent with stagnant earnings of U.S. unskilled workers in the 1980s and the early 1990s and the large inequality rise after the 1980s (after the 1990s in many European nations). This is because the effect of productivity growth with decreasing $\frac{k_a}{k_m}$ on their earnings is ambiguous and the effect on the inequality is positive when $c^* < c_a < 1$ (Proposition 6), and the growth of $\frac{N_h}{N_l}$, which contributes to raising their earnings and lowering the inequality (Proposition 7), greatly slowed down during the period. When $\frac{k_a}{k_m}$ rises under $c^* < c_a < 1$, earnings of unskilled workers too grow, which is consistent with the development in the late 1990s and the early 2000s.[33] Although the model with two types of workers cannot explain wage polarization

[32] It is true that several components of the composite analytical ability k_a, such as numerical ability, seems to have been growing faster than the composite manual ability k_m for much longer periods. But remaining components, such as analysis and decision-making abilities, seem to have grown slowly until recently.

[33] According to Acemoglu and Autor (2011), real wages of full-time workers of all education groups exhibited sound growth in the late 1990s and in the early 2000s in the U.S. Earnings growth of low education groups are stronger for females, probably because a higher proportion of them are in growing service occupations. After around the year 2004, however, earnings of all groups except male workers with post-college education have stagnated.

after the 1990s observed at least in the U.S., the falling inequality predicted by the model captures a part of the development, the shrinking inequality between low-skill and middle-skill workers.

As for the dynamics of task shifts, the result under $c^* = c_a = 1$ is same as the constant $\frac{k_a}{k_m}$ case, and so is the result under $c^* < c_a = 1$ when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ holds and $\frac{k_a}{k_m}$ falls (Propositions 4 and 5): $c_l(a)$ and $c_h(a)$ decrease and a^* increase over time, unless $\frac{N_h}{N_l}$ grows rapidly. Hence, the dynamics accord with the long-run trend until recently, i.e. workers shift to more analytical and harder-to-routinize tasks over time. By contrast, when $c^* < c_a < 1$, while $c_l(a)$ and $c_h(a)$ decrease over time (unless $\frac{N_h}{N_l}$ grows rapidly) as before, unlike the constant $\frac{k_a}{k_m}$ case, a^* increases (decreases) when $\frac{k_a}{k_m}$ falls (rises) (Proposition 6). Hence, workers shift to more analytical and harder-to-routinize tasks while $\frac{k_a}{k_m}$ falls, whereas after $\frac{k_a}{k_m}$ starts to rise, they shift to harder-to-codify tasks overall and shift to more *manual* tasks at low c (footnote 31). This is consistent with the shift *from non-routine analytical* tasks as well as routine tasks to non-routine manual tasks after around the year 2000 in the U.S. (Beaudry, Green, and Sand, 2016; see footnote 10 in the introduction for details).

In sum, unlike the proportionate growth case, the model with realistic productivity growth is consistent with a large part of the development after the 1980s, including several aspects of job and wage polarization after the 1990s. The result suggests that mechanization driven by the rising productivity of machines and the increased proportion of skilled workers are important in understanding the long-term evolution of task shifts, earnings levels and inequality from the era of the Industrial Revolution until the present. Of course, other factors, such as increased trade with and increased offshoring to developing countries after the 1990s (Firpo, Fortin, and Lemieux, 2013; Ebenstein et al., 2014), too have significant effects,[34] but only the two changes considered in the paper seem to have influenced the evolution continuously.

If the rapid progress of information technology continues and $\frac{k_a}{k_m}$ keeps rising, comparative advantages of machines to two type of workers could change over time, i.e. first, from $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ to $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, then to $\frac{k_a}{k_m} > \frac{h}{l_m}$. The model predicts what will happen to task assignment, earnings, and earnings inequality under such situations. As before, both types of workers shift to tasks that are more difficult to routinize (unless $\frac{N_h}{N_l}$ rises greatly, which is very unlikely). By contrast, unlike before, unskilled workers shift to more *manual* tasks (even at high c), and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$, skilled workers too shift to such tasks (see Figure 13). That is, workers will shift to relatively manual and difficult-to-codify tasks: the recent shift to low-wage service occupations such as personal care and protective service may continue into the future. Earnings of unskilled workers as well as those of skilled workers will rise, and earnings inequality will shrink over time. The analysis based on the model with two types of workers may not capture the whole picture, considering the recent widening inequality between moderately and extremely high-skill workers (Alvaredo et al., 2013). And, the extended model with more than two types of workers would not be sufficient to understand the evolution of the right tail of the distribution at which, Alvaredo et al. (2013), based

[34] Firpo, Fortin, and Lemieux (2013) find that the effect of offshoring or trade on wage inequality is important after the 1990s and strong in the 2000s for the U.S. economy. Ebenstein et al. (2014) find that the effect on real wages and employment is large after 1997 (until the end of the sample period, 2002).

on international evidence, argue that institutional and policy changes play important roles. However, episodes such as declining newspaper industry, burgeoning online education, and the increasing use of "big data" in marketing, trading, management and other decisions (such as the diagnosis of diseases) suggest that machines would replace a large number of tasks presently performed by highly skilled workers in the not-distant future and thus possible effects on a great majority of the population might be captured by the present model.

6 Conclusion

Since the Industrial Revolution, mechanization (or automation) has strongly affected types of tasks humans perform, relative demands for workers of different skill levels, earnings levels and inequality, and aggregate output. This paper has developed a Ricardian model of task assignment and examined how improvements of qualities of machines and an increase in the relative supply of skilled workers affect these variables. The analysis has shown that tasks and workers strongly affected by the productivity growth and the effects on earnings and the inequality change over time. The model is consistent with long-run trends of these variables in the real economy, except a sharp decline of the inequality in the wartime 1940s and job and wage polarization after the 1990s, which is beyond the scope of the model with two types of workers, although the model does capture an important part of the latter development. The model has also been employed to examine possible future trends of these variables when the rapid growth of information technology continues. It is found that earnings of both skilled and unskilled workers increase and earnings inequality falls over time, although the analysis based on the model with two types of workers may not capture the whole picture.

Several extensions of the model would be fruitful for analyzing the recent evolution of the labor market quantitatively. First, in order to understand the job and wage polarization more accurately, the model with more than two type of workers, who differ in levels of analytical ability or ability to perform non-routine tasks, could be developed. Second, empirical works find that international trade and offshoring have important effects on earnings inequality after the 1990s, thus it may be interesting to examine effects of these factors and productivity growth jointly.

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Appendix A: Lemmas

This appendix presents lemmas that examine the shape of (HL) and its relations with exogenous variables illustrated in Figure 3 of Section 3, and a lemma examining the shape of (P) and its relations with exogenous variables illustrated in Figure 4. The next lemma presents the result when $c^*, c_a < 1$ ($c^* < (>) c_a$ when $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$), the area below $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ of Figure 2. Note that *no* assumptions are imposed on magnitude relations of analytical abilities to manual abilities, although presentations in the lemmas appear to suppose $h > l_m$, $l_m > l_a$, and $k_m > k_a$.

Lemma 1 *When* $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c^*, c_a < 1$, *(HL) is expressed as*

$$\frac{N_h}{N_l}\ln\left(\frac{k_m}{A_k(a^*)}\right) = \frac{A_l(a^*)}{A_h(a^*)}\ln\left(\frac{A_k(a^*)}{k_a}\right), \quad \textit{when } \frac{k_a}{k_m} \neq 1, \tag{14}$$

$$\frac{N_h}{N_l}a^* = \frac{A_l(a^*)}{A_h(a^*)}(1-a^*), \quad \textit{when } \frac{k_a}{k_m} = 1. \tag{15}$$

a^* satisfying the equation decreases with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$.

Unlike the cases below, (HL) is independent of c_m. a^* satisfying the equation decreases with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$. The next lemma presents the result when $c^* < c_a = 1$, the area below $c_m = \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ and on or above $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ of Figure 2. This case arises only when $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} > \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow \frac{k_a}{k_m} < \frac{h}{l_m}$.

Lemma 2 *When* $c_m \in \left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right) \Leftrightarrow c^* < c_a = 1$, *which arises only when* $\frac{k_a}{k_m} < \frac{h}{l_m}$, *(HL) is expressed as*

$$\textit{when } \frac{k_a}{k_m} \neq 1, \quad \frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right)$$
$$= \frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right] + \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right], \tag{16}$$

$$\textit{when } \frac{k_a}{k_m} = 1, \quad \frac{N_h}{N_l}\frac{c_m a^*}{l_m} = \frac{1}{h-l_m}\left\{\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right] - \frac{A_l(a^*)}{l_m}c_m + 1\right\}. \tag{17}$$

a^* satisfying the equation decreases with c_m and $\frac{N_h}{N_l}$ ($\frac{\partial a^*}{\partial c_m} = 0$ at $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$), and decreases (increases) with $\frac{k_a}{k_m}$ for small (large) c_m.

Unlike the previous case, a^* satisfying (HL) decreases with c_m (except at $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, where $\frac{\partial a^*}{\partial c_m} = 0$), and it increases with $\frac{k_a}{k_m}$ when c_m is large. Finally, the next lemma presents the result when $c^* = c_a = 1$, the area on or above $c_m = \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ of Figure 2. This case arises only when $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} < 1 \Leftrightarrow \frac{k_a}{k_m} < \frac{l_a}{l_m}$.

Lemma 3 *When $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_a = 1$, which arises only when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, (HL) is expressed as*

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_a k_m - l_m k_a}{(k_m-k_a)l_m-(l_m-l_a)k_m c_m}\frac{l_m}{A_l(a^*)}\right]+\frac{k_m c_m}{(k_m-k_a)l_m}\ln\left[\frac{(k_m-k_a)l_m-(l_m-l_a)k_m c_m}{(l_a k_m - l_m k_a)c_m}\right]\right\}$$
$$=\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right),\ \ when\ \frac{k_a}{k_m}\neq 1, \tag{18}$$
$$\frac{N_h}{N_l}\frac{1}{l_a-l_m}\left\{\ln\left[\frac{c_m A_l(a^*)}{l_m}\right]+1-c_m\right\} = \frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right),\ \ when\ \frac{k_a}{k_m}=1, \tag{19}$$

where $a^* \in (0,1)$ holds for any c_m. a^* satisfying the equation decreases with c_m and $\frac{N_h}{N_l}$, and it increases with $\frac{k_a}{k_m}$ ($\lim_{c_m\to 1}\frac{\partial a^*}{\partial c_m}=\lim_{c_m\to 1}\frac{\partial a^*}{\partial \frac{k_a}{k_m}}=0$).

a^* satisfying (HL) decreases with c_m as in the previous case, while it increases with $\frac{k_a}{k_m}$ ($\lim_{c_m\to 1}\frac{\partial a^*}{\partial c_m}=\lim_{c_m\to 1}\frac{\partial a^*}{\partial \frac{k_a}{k_m}}=0$, though).

Finally, the next lemma presents the shape of (P) and its relations with k_m, k_a, and r.

Lemma 4 *c_m satisfying (P), which is positive, increases with a^* and r, and decreases with k_m and k_a.*

7 Appendix B: Proofs of Lemmas and Propositions

Proof of Lemma 1. [Derivation of the LHS of the equation]: When $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ and thus $c_m < \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_l(a^*) < 1$, the LHS of (HL) equals $\frac{N_h}{N_l}$ times

$$\int_0^{a^*}\int_0^{c_l(a)}\frac{1}{A_l(a)}dcda=\int_0^{a^*}\frac{c_l(a)}{A_l(a)}da=\frac{k_m}{l_m}c_m\int_0^{a^*}\frac{da}{A_k(a)}. \tag{20}$$

Hence, when $\frac{k_a}{k_m}\neq 1$, the LHS of (HL) equals

$$\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right). \tag{21}$$

Applying l'Hôpital's rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m}=1$ equals

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right) = \frac{N_h}{N_l}\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}\right)$$
$$=\frac{N_h}{N_l}\frac{c_m a^*}{l_m}. \tag{22}$$

[Derivation of the RHS of the equation]: When $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c_a = c_h(1) < 1$, the RHS of (HL) is expressed as

$$\int_{a^*}^1\int_0^{c_h(a)}\frac{1}{A_h(a)}dcda=\int_{a^*}^1\frac{c_h(a)}{A_h(a)}da=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\int_{a^*}^1\frac{da}{A_k(a)}. \tag{23}$$

Hence, when $\frac{k_a}{k_m}\neq 1$, the RHS of (HL) equals

$$\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left(\frac{A_k(a^*)}{k_a}\right). \tag{24}$$

By applying l'Hôpital's rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m}=1$ equals

$$\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{l_m}\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[a^*+(1-a^*)\frac{k_m}{k_a}\right]=-\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{-(1-a^*)(\frac{k_a}{k_m})^{-2}}{a^*+(1-a^*)\frac{k_m}{k_a}}\right)$$
$$=\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{l_m}(1-a^*). \tag{25}$$

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$]: Clearly, a^* satisfying the equation decreases with $\frac{N_h}{N_l}$. Noting that, from (21) and (24), (HL) when $\frac{k_a}{k_m}\neq 1$ can be expressed as

$$\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left[-\frac{N_h}{N_l}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)-\frac{A_l(a^*)}{A_h(a^*)}\ln\left(a^*+(1-a^*)\frac{k_m}{k_a}\right)\right]=0, \tag{26}$$

the derivative of the above equation with respect to $\frac{k_a}{k_m}$ equals

$$\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left(-\frac{N_h}{N_l}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}-\frac{A_l(a^*)}{A_h(a^*)}\frac{-(1-a^*)(\frac{k_a}{k_m})^{-2}}{a^*+(1-a^*)\frac{k_m}{k_a}}\right)$$
$$=\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{k_m}{A_k(a^*)}\left[-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right], \tag{27}$$

where the expression inside the large bracket can be rewritten as

$$-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}=\left[\ln\left(\frac{A_k(a^*)}{k_a}\right)\right]^{-1}\frac{N_h}{N_l}\left[-a^*\ln\left(a^*+(1-a^*)\frac{k_m}{k_a}\right)-(1-a^*)\frac{k_m}{k_a}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)\right]$$
$$=\left[\ln\left(\frac{A_k(a^*)}{k_a}\right)\right]^{-1}\frac{N_h}{N_l}\frac{k_m}{k_a}\left[a^*\frac{k_a}{k_m}\ln\left(\frac{k_a}{k_m}\right)-\left(a^*\frac{k_a}{k_m}+1-a^*\right)\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)\right]. \tag{28}$$

The expression inside the large bracket of the above equation is positive, because the expression equals 0 at $\frac{k_a}{k_m}=1$ and its derivative with respect to $\frac{k_a}{k_m}$ equals

$$a^*\left[\ln\left(\frac{k_a}{k_m}\right)-\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)\right], \tag{29}$$

which is negative (positive) for $\frac{k_a}{k_m}<(>)1$. Thus, noting that $\ln\left(\frac{A_k(a^*)}{k_a}\right)>(<)0$ for $\frac{k_a}{k_m}<(>)1$, (27) is positive. The derivative of (26) with respect to a^* is positive from $\partial\frac{A_l(a^*)}{A_h(a^*)}/\partial a^*<0$. Hence, a^* satisfying (14) decreases with $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m}\neq 1$. When $\frac{k_a}{k_m}\to 1$, (27) equals

$$\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{1}{a^*\frac{k_a}{k_m}+1-a^*}\left[-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right]\right\}$$
$$=-\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{-\left(a^*\frac{k_a}{k_m}+1-a^*\right)\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)(\frac{k_a}{k_m})^{-2}-\left(-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right)a^*}{\left(a^*\frac{k_a}{k_m}+1-a^*\right)^2}\right\}$$
$$=\frac{c_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)>0. \tag{30}$$

where (15) is used to derived the last equality. Hence, the same result holds when $\frac{k_a}{k_m}=1$ as well. ■

Proof of Lemma 2. [Derivation of the equation]: Since $c^* < 1$, the LHS of (HL) equals (21) (when $\frac{k_a}{k_m}\neq 1$) and (22) (when $\frac{k_a}{k_m}=1$) in the proof of Lemma 1.

The RHS of (HL) when $c_a = 1 \Leftrightarrow c_h(1)\geq 1$, $c^* < 1 \Leftrightarrow c_h(a^*)<1$, and $\frac{k_a}{k_m}\neq 1$ is expressed as

$$\begin{aligned}\int_{a^*}^{c_h^{-1}(1)}\int_0^{c_h(a)}\frac{dcda}{A_h(a)}+\int_{c_h^{-1}(1)}^1\int_0^1\frac{dcda}{A_h(a)} &= \int_{a^*}^{c_h^{-1}(1)}\frac{c_h(a)}{A_h(a)}da+\int_{c_h^{-1}(1)}^1\frac{da}{A_h(a)}\\ &=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\int_{a^*}^{c_h^{-1}(1)}\frac{da}{A_k(a)}+\int_{c_h^{-1}(1)}^1\frac{da}{A_h(a)}\\ &=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left(\frac{A_k(a^*)}{A_k(c_h^{-1}(1))}\right)+\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(c_h^{-1}(1))}\right),\end{aligned} \tag{31}$$

where $c_h^{-1}(1)$, i.e. the value of a when $c_h(a)=1$, equals, from (1) and (3),

$$\begin{aligned}\frac{A_h(a)}{A_k(a)} = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m} &\Leftrightarrow a(h-l_m)+l_m = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}[-a(k_m-k_a)+k_m]\\ &\Leftrightarrow a = \frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}.\end{aligned} \tag{32}$$

Hence, from (31) and

$$\begin{aligned}A_k(c_h^{-1}(1)) &= \frac{-l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)(k_m-k_a)+k_m\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\\ &= \frac{(hk_m-l_mk_a)c_m}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m},\end{aligned} \tag{33}$$

$$\begin{aligned}A_h(c_h^{-1}(1)) &= \frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)(h-l_m)+l_m\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\\ &= \frac{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m},\end{aligned} \tag{34}$$

the RHS of (HL) when $\frac{k_a}{k_m}\neq 1$, equals

$$\frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]+\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]. \tag{35}$$

By applying l'Hôpital's rule to the above equation, the RHS when $\frac{k_a}{k_m}=1$ equals

$$\frac{1}{h-l_m}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]+\frac{\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(h-l_m\frac{k_a}{k_m})c_m}{a^*\frac{k_a}{k_m}+(1-a^*)}}\right]$$

$$=\frac{1}{h-l_m}\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+(1-a^*)}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]$$

$$=\frac{1}{h-l_m}\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\left[\frac{A_h(a^*)}{h-l_m}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(h-l_m)c_m}\right]$$

$$=\frac{1}{h-l_m}\left\{\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{A_l(a^*)}{l_m}c_m+1\right\}. \tag{36}$$

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$ and c_m]: When $\frac{k_a}{k_m}\neq 1$, the derivative of the LHS−RHS of (16) with respect to a^* equals

$$\frac{N_h}{N_l}\frac{k_m}{l_m}c_m\frac{1}{A_k(a^*)}+\frac{1}{h-l_m}\left[\frac{1}{\frac{A_h(a^*)}{A_l(a^*)}}-\frac{(k_m-k_a)\frac{l_m}{k_m}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}+\frac{k_m}{l_m}c_m\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{A_k(a^*)}$$
$$-\frac{c_m\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}-\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]$$

$$=c_m\left[\begin{array}{c}\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{1}{A_k(a^*)}+\frac{\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}+\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{A_k(a^*)}\\ -\frac{\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}-\frac{k_m}{l_m}\frac{1}{k_m-k_a}\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\end{array}\right]$$

$$=\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{k_m-k_a}{A_k(a^*)}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(k_m-k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}-c_m\right]}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\right\}>0, \tag{37}$$

where the last equality is derived by using

$$\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}=\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m-\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}+\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}$$

$$=1+\frac{(k_m-k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}-c_m\right]}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}>(<)1 \text{ when } \frac{k_a}{k_m}<(>)1 \quad (\because c_m<\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}). \tag{38}$$

The derivative of the LHS-RHS of (16) with respect to c_m when $\frac{k_a}{k_m}\neq 1$ equals

$$\frac{1}{(h-l_m)c_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]-\frac{1+\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{A_l(a^*)}{A_h(a^*)}(h-l_m)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}+\frac{k_m}{l_m}\frac{1}{k_m-k_a}\frac{A_l(a^*)}{A_h(a^*)}$$

$$=\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)hk_m\left[c_m-\frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}\right]\geq 0 \quad (\because c_m\geq\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}), \tag{39}$$

where the last equality is derived by using

$$\frac{(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h=\frac{\left[(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m\right]h-l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)+l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}$$

$$=1+\frac{(h-l_m)hk_m\left[c_m-\frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}. \tag{40}$$

Hence, when $\frac{k_a}{k_m}\neq 1$, a^* satisfying (16) decreases with $\frac{N_h}{N_l}$ and c_m ($\frac{\partial a^*}{\partial c_m}=0$ at $c_m=\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$).

The corresponding derivatives when $\frac{k_a}{k_m}\to 1$ are

$$a^*:\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{1-\frac{k_a}{k_m}}{a^*\frac{k_a}{k_m}+1-a^*}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})c_m}\right]\right\}\right)$$

$$=-\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left\{\begin{aligned}&\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{-(a^*\frac{k_a}{k_m}+1-a^*)-(1-\frac{k_a}{k_m})(1-a^*)}{(a^*\frac{k_a}{k_m}+1-a^*)^2}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\left[1+\frac{(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})c_m}\right]^{-1}\\&\times\frac{(h-l_m\frac{k_a}{k_m})A_h(a^*)\left[-\left((a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right)+(1-\frac{k_a}{k_m})\frac{(1-a^*)l_m}{A_l(a^*)}\right]+l_m(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})^2c_m}\end{aligned}\right\}$$

$$=\frac{c_m}{l_m}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\frac{A_h(a^*)\left(\frac{l_m}{A_l(a^*)}-c_m\right)}{(h-l_m)c_m}\right\}>0, \tag{41}$$

$$c_m:\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)h\left[c_m-\frac{l_m}{h}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m)}\right]\geq 0. \tag{42}$$

Therefore, the same results hold when $\frac{k_a}{k_m}=1$ as well.

[**Relations of a^* satisfying the equation with $\frac{k_a}{k_m}$**]**:** Since (16) can be expressed as

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right) \tag{43}$$

$$=\frac{1}{h-l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]+\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\ln\left[\frac{a^*\frac{k_a}{k_m}+1-a^*}{h-l_m\frac{k_a}{k_m}}\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{c_m}\right],$$

the derivative of the LHS$-$RHS of (16) with respect to $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m}\neq 1$ equals

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\left[\frac{\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)}{1-\frac{k_a}{k_m}}+\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}\right]-\frac{1}{l_m}\frac{c_m}{(1-\frac{k_a}{k_m})^2}\frac{A_l(a^*)}{A_h(a^*)}\ln\left[\frac{a^*\frac{k_a}{k_m}+1-a^*}{h-l_m\frac{k_a}{k_m}}\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{c_m}\right]$$

$$+\frac{l_m}{h-l_m}\left[\frac{\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}-\frac{1}{h-l_m\frac{k_a}{k_m}}\right]-\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]$$

$$=\frac{1}{(h-l_m)(1-\frac{k_a}{k_m})}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}+\frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(h-l_m\frac{k_a}{k_m})\left[(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}$$

$$-\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}-\frac{l_m(h-l_m)\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(h-l_m\frac{k_a}{k_m})\left[(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}\right]$$

$$=\frac{k_m}{k_m-k_a}\left\{-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{k_m}{l_m}\frac{a^*}{A_k(a^*)}c_m+\frac{k_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{hk_m-l_mk_a}+\frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]\right\}. \tag{44}$$

Since the derivative on (HL) is examined, by substituting (16) into the above equation

$$\frac{k_m}{k_m-k_a}\left\{\begin{array}{c}-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{k_m}{l_m}\frac{a^*}{A_k(a^*)}c_m+\frac{k_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{hk_m-l_mk_a}+\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right)\\ -\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\end{array}\right\}$$

$$=\frac{k_mc_m}{(k_m-k_a)^2}\frac{k_m}{l_m}\left\{\begin{array}{c}\frac{N_h}{N_l}\left[\ln\left(\frac{k_m}{A_k(a^*)}\right)+1-\frac{k_m}{A_k(a^*)}\right]\\ -\frac{A_l(a^*)}{A_h(a^*)}\left[(k_m-k_a)\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}\frac{A_l(a^*)k_mc_m-l_mA_k(a^*)}{A_k(a^*)(hk_m-l_mk_a)}+\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\right]\end{array}\right\}. \tag{45}$$

The above expression is positive at $c_m=\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from (27) in the proof of Lemma 1 and is negative at $c_m=\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ from (56) in the proof of Lemma 3. Further, the derivative of the expression inside the big bracket of the above equation with respect to c_m equals

$$-(k_m-k_a)\frac{1}{c_m^2}\frac{l_m}{hk_m-l_mk_a}-\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{h-l_m}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}-\frac{1}{c_m}\right]$$

$$=\frac{l_m}{k_m}\frac{k_m-k_a}{c_m}\left[-\frac{1}{c_m}\frac{k_m}{hk_m-l_mk_a}+\frac{1}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]=-\frac{l_m^2}{k_m}\frac{1}{c_m^2}\frac{(k_m-k_a)^2\left[\frac{A_h(a^*)}{A_l(a^*)}-c_m\right]}{(hk_m-l_mk_a)\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}, \tag{46}$$

which is negative for $c_m\in\left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)},\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right]$ from $\frac{A_h(a^*)}{A_l(a^*)}-c_m\geq\frac{A_h(a^*)k_m-l_mA_k(a^*)}{A_l(a^*)k_m}=\frac{(hk_m-l_mk_a)a^*}{A_l(a^*)k_m}>0$ ($\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}>\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}\Leftrightarrow\frac{k_a}{k_m}<\frac{h}{l_m}$). Hence, there exists a unique $c_m\in(\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)},\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)})$ such that (44) is positive (negative) for smaller (greater) c_m.

When $\frac{k_a}{k_m}\to 1$, (44) equals

$$\lim_{\frac{k_a}{k_m}\to 1}\frac{1}{1-\frac{k_a}{k_m}}\left\{-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{1}{l_m}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}c_m+\frac{1-c_m\frac{A_l(a^*)}{A_h(a^*)}}{h-l_m\frac{k_a}{k_m}}+\frac{1}{h-l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]\right\}$$

$$=-\lim_{\frac{k_a}{k_m}\to 1}\left\{\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{1}{l_m}\frac{a^{*2}c_m}{(a^*\frac{k_a}{k_m}+1-a^*)^2}+\frac{l_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{(h-l_m\frac{k_a}{k_m})^2}+\frac{1}{h-l_m}\left[\frac{-l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}\right]\right\}$$

$$=-\left\{\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{a^{*2}c_m}{l_m}-\frac{l_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)\left(\frac{1}{c_m}\frac{A_h(a^*)}{A_l(a^*)}-1\right)}{(h-l_m)^2}\right\}. \tag{47}$$

The above expression is positive at $c_m=\frac{l_m}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from (30) in the proof of Lemma 1 and is negative at $c_m=\frac{l_m}{A_l(a^*)}$ from (58) in the proof of Lemma 3. Further, the derivative of the expression with respect to c_m is negative. Hence, the same result holds when $\frac{k_a}{k_m}=1$ as well.

■

Proof of Lemma 3. [Derivation of the equation]: The LHS of (HL) when $c^*=1\Leftrightarrow c_l(a^*)\geq 1$ and $\frac{k_a}{k_m}\neq 1$ equals $\frac{N_h}{N_l}$ times

$$\int_0^{c_l^{-1}(1)}\int_0^{c_l(a)}\frac{dcda}{A_l(a)}+\int_{c_l^{-1}(1)}^{a^*}\int_0^1\frac{dcda}{A_l(a)}=\int_0^{c_l^{-1}(1)}\frac{c_l(a)}{A_l(a)}da+\int_{c_l^{-1}(1)}^{a^*}\frac{da}{A_l(a)}$$
$$=\frac{k_m}{l_m}c_m\int_0^{c_l^{-1}(1)}\frac{da}{A_k(a)}+\int_{c_l^{-1}(1)}^{a^*}\frac{da}{A_l(a)}$$
$$=\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(c_l^{-1}(1))}\right)+\frac{1}{l_m-l_a}\ln\left(\frac{A_l(c_l^{-1}(1))}{A_l(a^*)}\right),\tag{48}$$

where the value of $c_l^{-1}(1)$, i.e. a when $c_l(a)=1$, equals, from (2) and (3),

$$\frac{A_l(a)}{A_k(a)}=\frac{l_m}{k_m}\frac{1}{c_m}\Leftrightarrow -a(l_m-l_a)+l_m=\frac{l_m}{k_m}\frac{1}{c_m}[-a(k_m-k_a)+k_m]$$
$$\Leftrightarrow a=\frac{l_m(1-c_m)}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m}.\tag{49}$$

Hence, from (48) and

$$A_k(c_l^{-1}(1))=\frac{-l_m(1-c_m)(k_m-k_a)+k_m\left[(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m}$$
$$=\frac{(l_ak_m-l_mk_a)c_m}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m},\tag{50}$$
$$A_l(c_l^{-1}(1))=\frac{-l_m(1-c_m)(l_m-l_a)+l_m\left[(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m}$$
$$=\frac{\frac{l_m}{k_m}(l_ak_m-l_mk_a)}{(k_m-k_a)\frac{l_m}{k_m}-(l_m-l_a)c_m},\tag{51}$$

the LHS of (HL) when $\frac{k_a}{k_m}\neq 1$ equals

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_ak_m-l_mk_a}{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}\frac{l_m}{A_l(a^*)}\right]+\frac{k_mc_m}{(k_m-k_a)l_m}\ln\left[\frac{(k_m-k_a)l_m-(l_m-l_a)k_mc_m}{(l_ak_m-l_mk_a)c_m}\right]\right\}.\tag{52}$$

Applying l'Hôpital's rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m}=1$ equals

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{l_a-l_m\frac{k_a}{k_m}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}\frac{l_m}{A_l(a^*)}\right]+\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})l_m}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}$$
$$=\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_m}{c_mA_l(a^*)}\right]+c_m\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{1}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}-\frac{1}{l_a-l_m\frac{k_a}{k_m}}\right)\right\}$$
$$=\frac{N_h}{N_l}\frac{1}{l_a-l_m}\left\{\ln\left[\frac{c_mA_l(a^*)}{l_m}\right]+1-c_m\right\}.\tag{53}$$

[$a^*\in(0,1)$ **for any** c_m]**:** $a^*<1$ is obvious from the equation. Since $c_m\geq\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$, $a^*=0$ is possible only at $c_m=1$. However, at $c_m=1$, the equation becomes $\frac{N_h}{N_l}\frac{1}{l_m-l_a}\ln\left(\frac{l_m}{A_l(a^*)}\right)=\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right)$ and thus $a^*>0$.

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$, c_m, and $\frac{k_a}{k_m}$]: Since the derivative of the LHS−RHS of (18) and (19) with respect to a^* equals $\frac{N_h}{N_l}\frac{1}{A_l(a^*)}+\frac{1}{A_h(a^*)}>0$, a^* satisfying the equation decreases with $\frac{N_h}{N_l}$.

When $\frac{k_a}{k_m}\neq 1$, a^* satisfying (18) decreases with c_m, because the derivative of the expression inside the large curly bracket of (18) with respect to c_m equals

$$\left(1-\frac{(l_m-l_a)k_m c_m}{(k_m-k_a)l_m}\right)\frac{k_m}{(k_m-k_a)l_m-(l_m-l_a)k_m c_m}-\frac{k_m}{(k_m-k_a)l_m}+\frac{k_m}{(k_m-k_a)l_m}\ln\left[\frac{(k_m-k_a)l_m-(l_m-l_a)k_m c_m}{(l_a k_m-l_m k_a)c_m}\right]$$
$$=\frac{1}{(1-\frac{k_a}{k_m})l_m}\ln\left[1+\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}\right]>0. \tag{54}$$

$\lim_{c_m\to 1}\frac{\partial a^*}{\partial c_m}=0$ is clear from the above equation.

Since (18) can be expressed as

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_a-l_m\frac{k_a}{k_m}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}\frac{l_m}{A_l(a^*)}\right]+\frac{c_m}{(1-\frac{k_a}{k_m})l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}$$
$$=\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right), \tag{55}$$

when $\frac{k_a}{k_m}\neq 1$, the derivative of the expression inside the large curly bracket of (18) with respect to $\frac{k_a}{k_m}$ equals

$$\frac{\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}-\frac{\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}}{l_a-l_m\frac{k_a}{k_m}}+\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]$$
$$=-\left(\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}\right)\frac{(l_m-l_a)(1-c_m)}{[(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m](l_a-l_m\frac{k_a}{k_m})}+\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]$$
$$=-\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\left(\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}-\ln\left[1+\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}\right]\right)<0. \tag{56}$$

The derivative is negative because the expression inside the large parenthesis of (56) equals 0 at $c_m=1$ and, when $\frac{k_a}{k_m}<(>)1$, it increases (decreases) with $\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}$ and thus decreases with c_m. Hence, a^* satisfying (18) increases with $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m}\neq 1$. $\lim_{c_m\to 1}\frac{\partial a^*}{\partial\frac{k_a}{k_m}}=0$ is clear from the above equation.

The corresponding derivatives when $\frac{k_a}{k_m}\to 1$ are

$$c_m:\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{1}{(1-\frac{k_a}{k_m})l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}=\frac{-1}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{-l_m}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}+\frac{l_m}{l_a-l_m\frac{k_a}{k_m}}\right]$$
$$=\frac{1}{l_a-l_m}\frac{1-c_m}{c_m}>0. \tag{57}$$

$$\frac{k_a}{k_m}:\lim_{\frac{k_a}{k_m}\to 1}\left\{-\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\left(\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}-\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right)\right\}$$
$$=\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{c_m}{2(1-\frac{k_a}{k_m})l_m}\left(\frac{1-c_m}{c_m}\frac{-(l_a-l_m\frac{k_a}{k_m})l_m+(1-\frac{k_a}{k_m})l_m^2}{(l_a-l_m\frac{k_a}{k_m})^2}-\left[\frac{-l_m}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}+\frac{l_m}{l_a-l_m\frac{k_a}{k_m}}\right]\right)\right\}$$
$$=\frac{c_m}{2l_m}\left(\frac{1-c_m}{c_m}\lim_{\frac{k_a}{k_m}\to 1}\frac{2l_m^2(l_a-l_m)}{(l_a-l_m\frac{k_a}{k_m})^3}+\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{-l_m^2}{[(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m]^2}+\frac{l_m^2}{(l_a-l_m\frac{k_a}{k_m})^2}\right]\right)$$
$$=\frac{c_m}{2l_m}\frac{l_m^2}{(l_a-l_m)^2}\left[2\frac{1-c_m}{c_m}+\left(1-\frac{1}{c_m^2}\right)\right]=-\frac{1}{2}\frac{l_m}{(l_a-l_m)^2}\frac{(1-c_m)^2}{c_m}<0, \tag{58}$$

where $l_a-l_m>0$ from $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}<1 \Leftrightarrow 1<\frac{l_a}{l_m}$. Therefore, the same results hold when $\frac{k_a}{k_m}=1$ as well. ■

Proof of Lemma 4. [Relations of c_m satisfying (P) **with** a^*, k_m, k_a, **and** r]: Derivatives of the LHS of (P) with respect to a^*, c_m, k_m, and k_a equal

$$a^*:\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}\frac{l_m}{k_m}\frac{r}{c_m}\int_{a^*}^{1}\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}>0, \tag{59}$$
$$c_m:-\frac{l_m}{k_m}\frac{r}{c_m^2}\left\{\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{dcda}{A_l(a)}+\frac{A_h(a^*)}{A_l(a^*)}\int_{a^*}^{1}\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}\right\}<0, \tag{60}$$
$$k_m:\quad -\frac{1}{k_m}\left\{1-r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{dcda}{cA_k(a)}+\int_{a^*}^{1}\int_{\min\{c_h(a),1\}}^{1}\frac{dadc}{cA_k(a)}\right]\right\}$$
$$-r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{(1-a)dcda}{c(A_k(a))^2}+\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{(1-a)dcda}{c(A_k(a))^2}\right]<0, \tag{61}$$
$$k_a:\quad -r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{adcda}{c(A_k(a))^2}+\int_0^{a^*}\int_{\min\{c_l(a),1\}}^{1}\frac{adcda}{c(A_k(a))^2}\right]<0, \tag{62}$$

where $c_l(a^*)=c_h(a^*)=c^*$, $\frac{1}{c_l(a)A_k(a)}=\frac{l_m}{k_m}\frac{1}{c_m}\frac{1}{A_l(a)}$, and $\frac{1}{c_h(a)A_k(a)}=\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}\frac{1}{A_h(a)}$ are used to derive the equations. The results are straightforward from the equations.

[**(P) does not hold at** $c_m=0$]**:** Noting that $c_l(a)=\frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m$ and $c_h(a)=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m$, when $c_m\to 0$, the LHS of (P) becomes

$$r\int_0^{a^*}\frac{da}{A_k(a)}+r\int_{a^*}^{1}\frac{da}{A_k(a)}+r\int_0^1\int_0^1\frac{dadc}{cA_k(a)}=r\int_0^1\frac{da}{A_k(a)}-\frac{r}{k_m-k_a}\ln\left(\frac{k_m}{k_a}\right)\lim_{c\to 0}\ln c=+\infty>1. \tag{63}$$

Hence, (P) does not hold at $c_m=0$. ■

Proof of Proposition 1. At $c_m=1$, $c_l(a)$, $c_h(a)>1$ from (11), thus (P) equals

$$\frac{l_m}{k_m}r\int_0^{a^*}\frac{da}{A_l(a)}+\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}r\int_{a^*}^{1}\frac{da}{A_h(a)}=1. \tag{64}$$

When k_m is very small, the LHS of the above equation is strictly greater than 1 for any $a^*\in[0,1]$ (thus, (P) does not hold for any c_m and a^* from Lemma 4), or a^* satisfying the equation is weakly smaller than $a^* \in (0,1)$ satisfying (HL) at $c_m = 1$ ($a^* \in (0,1)$ holds on (HL) from Lemma 3). In such case, there is no $a^* \in (0,1)$ and $c_m < 1$ satisfying both (HL) and (P), and thus machines are not employed, i.e. $c_m = 1$, in equilibrium, where equilibrium a^* is determined from (HL) with $c_m = 1$.

When k_m becomes large enough that a^* satisfying (64) is greater than $a^* \in (0,1)$ satisfying (HL) at $c_m = 1$, an equilibrium with $c_m < 1$ exists from shapes of (HL) and (P). The dynamics of c_m and a^* are straightforward from shapes of the two loci. The dynamics of c^* and c_a are from $c^* = \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$, $c_a = \min\left\{\frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m, 1\right\}$, and the assumptions that $\frac{k_a}{k_m}$ is time-invariant and satisfies $\frac{k_a}{k_m} < \frac{l_a}{l_m}$. The dynamics of $c_l(a)$ and $c_h(a)$ are from those of the other variables. ■

Proof of Proposition 2. (i) When $c_m \geq \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, earnings of skilled workers increase over time from Propositions 4 (iii) and 5 (iii) below. Earnings of both types of workers increase when $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from Proposition 6 (iii) below. (ii) is straightforward from Proposition 1 and the earnings equations (eq. 13).

(iii) Y decreases with the LHS and the RHS of (HL) from (8). When $c^* = c_a = 1$ and $\frac{k_a}{k_m} \neq 1$, the RHS of (HL) equals $\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right)$ from Lemma 3, which decreases with the growth of k_m and k_a with constant $\frac{k_a}{k_m}$ from Proposition 1. When $c^* < c_a < 1$ and $\frac{k_a}{k_m} \neq 1$, the RHS of (HL) equals $\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m - k_a}\ln\left(\frac{A_k(a^*)}{k_a}\right)$ from (24) in the proof of Lemma 1, which decreases with the productivity growth from Proposition 1. When $c^* < c_a = 1$ and $\frac{k_a}{k_m} \neq 1$, the derivative of the RHS of (HL) with respect to c_m equals, from (39) in the proof of Lemma 2 and (16),

$$
\begin{aligned}
&-\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)hk_m\left[c_m - \frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m - l_m k_a)}\right] + \frac{N_h}{N_l}\frac{k_m}{l_m}\frac{1}{k_m - k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right) \\
&= \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{k_m - k_a}\ln\left[\frac{(k_m - k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)} + (h-l_m)c_m}{\frac{(hk_m - l_m k_a)c_m}{A_k(a^*)}}\right] > 0, \qquad (65)
\end{aligned}
$$

and the derivative with respect to a^* equals, from (37) in the proof of Lemma 2 ,

$$
-\frac{k_m}{l_m}\frac{c_m}{k_m - k_a}\left\{\frac{A_l(a^*)}{A_h(a^*)}\frac{k_m - k_a}{A_k(a^*)} - \frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(k_m - k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} - c_m\right]}{\frac{(hk_m - l_m k_a)c_m}{A_k(a^*)}}\right]\right\} < 0. \qquad (66)
$$

From signs of the derivatives and Proposition 1, the RHS of (HL) decreases with the productivity growth. Hence, Y increases over time when $\frac{k_a}{k_m} \neq 1$. The result when $\frac{k_a}{k_m} = 1$ can be proved similarly. ■

Proof of Proposition 3. Since an increase in $\frac{N_h}{N_l}$ shifts (HL) to the left on the (a^*, c_m) space from Lemmas 1–3, the result that c_m and a^* decrease is straightforward from Figures 7–9. Then, $w_l = \frac{l_m}{k_m}\frac{r}{c_m}$ rises and $\frac{w_h}{w_l} = \frac{A_h(a^*)}{A_l(a^*)}$ falls. Since $c^* \equiv \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$, c^* falls when $c^* < 1$ from $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, $\frac{da^*}{d\frac{N_h}{N_l}} < 0$, and $\frac{dc_m}{d\frac{N_h}{N_l}} < 0$. $c_l(a)$ decreases from $\frac{dc_m}{d\frac{N_h}{N_l}} < 0$. Proofs of the results for $c_h(a)$, c_a, w_h, and Y are in the proof of Proposition 7. ■