Generalized quasi-maximum likelihood inference for periodic conditionally heteroskedastic models

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Abstract

This paper establishes consistency and asymptotic normality of the generalized quasi-maximum likelihood estimate ($GQMLE$) for a general class of periodic conditionally heteroskedastic time series models (PCH). In this class of models, the volatility is expressed as a measurable function of the infinite past of the observed process with periodically time-varying parameters, while the innovation of the model is an independent and periodically distributed sequence. In contrast with the aperiodic case, the proposed $GQMLE$ is rather based on S instrumental density functions where S is the period of the model while the corresponding asymptotic variance is in a "*sandwich*" form. Application to the periodic $GARCH$ and the periodic asymmetric power $GARCH$ model is given. Moreover, we discuss how to apply the $GQMLE$ to the prediction of power problem in a one-step framework and to PCH models with complex periodic patterns such as high frequency seasonality and non-integer seasonality.

Keywords: Periodic conditionally heteroskedastic models, periodic asymmetric power $GARCH$, generalized QML estimation, consistency and asymptotic normality, prediction of powers, high frequency periodicity, non-integer periodicity.

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1. Introduction

Since the introduction of the AutoRegressive Conditionally Heteroskedastic ($ARCH$) model by Engle (1982) and its leading $GARCH$ generalization by Bollerslev (1986), conditional volatility models have continued to capture the interest of researchers in the statistical and financial econometric literature (e.g. Francq and Zakoïan, 2010). Among the numerous extensions of the original $ARCH$ formulation, which have been introduced, there is the periodic $GARCH$ ($PGARCH$) specification proposed by Bollerslev and Ghysels (1996). This model whose coefficients are periodic over time, aims at modeling time series volatility with periodic dynamic behavior (e.g. Franses and Paap, 2000-2004; Ghysels and Osborn, 2001; Taylor, 2006; Osborn et *al*, 2008; Regnard and Zakoïan, 2011; Sigauke and Chikobvu, 2011; Rossi and Fantazani, 2015; Ziel et al, 2016). Typical examples of time series for which periodic volatility models have proved useful are financial stock return series, which tend to show seasonal volatility. In particular, it is well documented that daily return series are characterized by the day-of-the-week effect (e.g. Franses and Paap, 2000; Balaban et *al*, 2001; Tsiakas, 2006; Berument et *al*, 2007; Osborn et *al*, 2008; Charles, 2010; Aknouche, 2016) while the month-of-the-year effect is present in monthly return series (e.g. Beller and Nofsinger, 1998; Tsiakas, 2006; Aknouche, 2016). Moreover, various intraday high frequency return series also exhibit periodicity in volatility (e.g. Andersen and Bollerslev, 1997; Bollerslev et *al*, 2000; Taylor, 2004; Taylor, 2006; Smith, 2010; Rossi and Fantazani, 2015). Other important examples of non-financial intraday series that may be affected by periodicity in volatility are half-hourly Net Imbalance Volume (NIV) series (Taylor, 2006) and hourly wind power and wind speed series (e.g. Ambach and Croonenbroeck, 2015; Ambach and Schmid, 2015; Ziel et al, 2016).

Statistical inference for $PGARCH$ models and their extensions has been mainly conducted using the standard Gaussian quasi-maximum likelihood estimate ($QMLE$). This estimate, which is calculated on the basis of the Gaussian likelihood, is consistent and asymptotically Normal (CAN) under quite mild assumptions (cf. Bollerslev and Ghysels, 1996; Franses and Paap, 2000, Aknouche and Bibi, 2009; Aknouche and Al-Eid, 2012; Ziel, 2015).

In particular, no moment condition on the observed process is required (Aknouche and Bibi, 2009; Aknouche and Al-Eid, 2012). However, asymptotic normality of the Gaussian $QMLE$ requires a fourth moment condition on the model innovation, which constitutes a serious limitation, especially for heavy-tailed innovations that are well-observed in practice (e.g. Boynton et *al*, 2009; Bidarkota et *al*, 2009).

For non-periodic conditionally heteroskedastic (CH) models, a large amount of research has been executed in latter decades to study the so-called Generalized $QMLE$ ($GQMLE$, Newey and Steigerwald, 1997; Berkes and Horvàth, 2004; Francq et *al*, 2011; Francq and Zakoïan, 2013; Fan et *al*, 2014; Zhu and Li, 2015). This estimate is calculated on the basis of a given instrumental distribution and reduces to the Gaussian $QMLE$ when the instrumental function is Gaussian. In fact, the $GQMLE$ has been partly introduced as a flexible alternative to the Gaussian $QMLE$ in reducing the inherent moment condition on innovation. An interesting application of the $GQMLE$ is the prediction of powers of return series in a fully parametric one-step framework (Francq and Zakoïan, 2013). Furthermore, the $GQMLE$ may be seen as a useful and flexible alternative to the Gaussian $QMLE$ in estimating some risk measures, like the Value at Risk (VaR), where the Gaussian $QMLE$ fails in the presence of heavy tailed series (Francq and Zakoïan, 2015-2016, El Ghourabi et *al*, 2016).

This paper establishes consistency and asymptotic normality of the $GQMLE$ for a general class of periodic conditionally heteroskedastic (PCH) models. In this class, the volatility is expressed as a measurable parametric function of the infinite past of the observed process, whereas the innovation of the model is an independent and periodically distributed sequence. Most earlier works on periodic conditionally heteroskedastic models assume independence and stationarity of the innovation of the model while the volatility coefficients are periodic over time (Bollerslev and Ghysels, 1996; Franses and Paap, 2000, Osborn et *al*, 2008; Aknouche and Bibi, 2009; Aknouche and Al-Eid, 2012; Rossi and Fantazani, 2015; Ziel, 2015-2016). Here, periodicity of the model is manifested via the volatility coefficients and also the distribution of the model innovation. This makes the model more flexible in representing seasonality in volatility with possible various identifiability assumptions on the marginal distributions of the periodic innovation. For example, it is well known that in daily return series certain trading days of the week may have different distributions to those in alternate trading days (Boynton et *al*, 2009; Bidarkota et *al*, 2009). These distributions may be light-tailed or heavy-tailed with different orders of magnitude (e.g. different Kurtoses). For certain trading days one even suspects that second moment do not exist. So, a PCH model with periodic innovation would be better in representing such situations than a PCH with stationary independent innovation.

In contrast with non-periodic CH models for which the $GQMLE$ only involves one instrumental density, our $GQMLE$ for PCH models is calculated on the basis of S instrumental functions corresponding to the different seasons, where S is the period of the model. This choice seems assorted with the independence and periodicity of the model innovation, which implies at most S different marginal distributions. It allows the proposed $GQMLE$ to reduce to the Maximum Likelihood estimate (MLE) when the S chosen instrumental functions coincide with the S marginal distributions of the innovation and hence to be asymptotically efficient. The assumptions of consistency and asymptotic normality of the proposed $GQMLE$ are quite mild. In addition, due to the periodicity of the model innovation, the asymptotic variance has an unusual "sandwich" form compared to non-periodic CH models (Francq and Zakoïan, 2013). As an application, we examine the asymptotic behavior of the $GQMLE$ for the particular periodic asymmetric power $GARCH$ (PAP-$GARCH\,(1,1)$) process that we define below. This model generalizes the well-known asymmetric power $GARCH$ (AP-$GARCH\,(1,1)$) model proposed by Deng et *al* (1993) to the case where the volatility coefficients, the power and the innovation of the model are periodic over time. It retains the main features of the AP-$GARCH$ model, which are asymmetry, correlation power and persistence in volatility, and is expected to account for periodicity in volatility. We also discuss application of the $GQMLE$ to the prediction of power problem as well as to PCH models with complex periodic patterns like high-frequency periodicity and non-integer periodicity.

The rest of this paper is structured as follows. In Section 2, the general PCH model is briefly described and some results that are needed in the subsequent Sections are provided. Then, the $GQMLE$ is defined in Section 3 and its consistency and asymptotic normality are established under mild assumptions. To illustrate the results, Section 4 shows asymptotic properties of the $GQMLE$ on some specific instrumental densities and also on the periodic asymmetric power $GARCH\,(1,1)$ model, where the general assumptions are made more explicit. Moreover, the applicability of the $GQMLE$ to the prediction of power problem and to PCH models with large and/or non-integer periods is discussed. Section 5 concludes while detailed proofs of the main results are left to Section 6.

2. A general class of periodic conditionally heteroskedastic models

A sequence of real-valued random variables $\{\eta_t, t \in \mathbb{Z}\}$ is said to be independent and S-periodically distributed (ipd_S in short) if $\{\eta_t, t \in \mathbb{Z}\}$ is independent and η_t has the same distribution as η_{nS+t} for all $t, n \in \mathbb{Z}$, where S, called the period, is the smallest positive integer satisfying the latter property. For $S = 1$ an idp_1 sequence is clearly independent and identically distributed (henceforth iid). Let $\{\eta_t, t \in \mathbb{Z}\}$ be an unobservable ipd_S sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ with unknown probability densities $\{f_v, 1 \leq v \leq S\}$, i.e. f_v is the density of η_{nS+v} $(n \in \mathbb{Z}, 1 \leq v \leq S)$. Consider a S-periodic sequence of unknown parameters $\{\theta_{0t}, t \in \mathbb{Z}\}$ satisfying $\theta_{0,nS+v} = \theta_{0v} = (\theta_{0v,1}, ...\theta_{0v,m_v})' \in \mathbb{R}^{m_v}$ with $m_v \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $(n \in \mathbb{Z}, 1 \leq v \leq S)$. A general periodic conditionally heteroskedastic (PCH) time series model is a stochastic difference equation of the form

$$\begin{cases} \epsilon_t = \sigma_t \eta_t, \\ \sigma_t = \varphi_t\left(\epsilon_{t-1}, \epsilon_{t-2}, ...; \theta_{0t}\right), \end{cases} \qquad t \in \mathbb{Z}, \tag{2.1a}$$

whose solution, $\{\epsilon_t, t \in \mathbb{Z}\}$, is an observable stochastic process on $(\Omega, \mathcal{F}, P)$. It is assumed that η_t is independent of $\{\epsilon_i, i < t\}$ and φ_v, which satisfies $\varphi_v = \varphi_{nS+v}$ $(1 \leq v \leq S, n \in \mathbb{Z})$, is a positive real-valued measurable function: $\mathbb{R}^{\infty} \times \mathbb{R}^{m_v} \rightarrow (0, \infty)$. To emphasize model periodicity, equation $(2.1a)$ may be rewritten as follows

$$\begin{cases} \epsilon_{nS+v} = \sigma_{nS+v}\eta_{nS+v}, \\ \sigma_{nS+v} = \varphi_v\left(\epsilon_{nS+v-1}, \epsilon_{nS+v-2}, ...; \theta_{0v}\right), \end{cases} \quad 1 \leq v \leq S,\ n \in \mathbb{Z}. \tag{2.1b}$$

where for all $1 \leq v \leq S$, the vth season (or channel) stands for the set $\{..., v-S, v, v+S, ...\}$. The true unknown parameter of the model, denoted by $\theta_0 = (\theta_{01}', ..., \theta_{0S}')' \in \mathbb{R}^m$, belongs to a compact parameter space $\Theta = \Theta_1 \times ... \times \Theta_S \subset \mathbb{R}^m$ with $m = \sum_{v=1}^{S} m_v$ and $\Theta_v \subset \mathbb{R}^{m_v}$ $(1 \leq v \leq S)$. Thus in (2.1) various specifications are allowed along seasons with possibly various parameter dimensions m_v $(1 \leq v \leq S)$. In the periodic $GARCH$ literature (e.g. Bollerslev and Ghysels, 1996; Aknouche and Bibi, 2009; Ziel, 2015), it is generally assumed, as in non-periodic CH models, that $\{\eta_t, t \in \mathbb{Z}\}$ is iid so periodicity of the model appears only through the sequence of parameters $\{\theta_{0t}, t \in \mathbb{Z}\}$. Here, $\{\eta_t, t \in \mathbb{Z}\}$ is rather ipd_S and in a more general framework periodicity of the model is expressed via both inputs $\{\theta_{0t}, t \in \mathbb{Z}\}$ and $\{\eta_t, t \in \mathbb{Z}\}$ of (2.1). In fact, for model (2.1) to be identifiable, a scaling assumption on $\{\eta_t, t \in \mathbb{Z}\}$ is needed. The standard identifiability assumption is the unit second moment condition $E\left(\eta_t^2\right) = 1$ (e.g. Bollerslev and Ghysels, 1996; Aknouche and Bibi, 2009; Ziel, 2015) but we do not need to make it in this paper. Instead, we will assume S general conditions on $\{\eta_t, t \in \mathbb{Z}\}$ ensuring consistency of the generalized $QMLE$ we propose below. It turns out that these conditions (see **A4** below) also allow to identify the model and replace in flexible manner the unit second moment assumption. Other identifiability assumptions on the $\{\eta_t, t \in \mathbb{Z}\}$ may be induced by some objectives of the model posterior to its building such as predicting the powers of $\{\varepsilon_t, t \in \mathbb{Z}\}$ (Francq and Zakoïan, 2013), estimating the conditional value at risk of the model (Francq and Zakoïan, 2015-2016), etc. We will see that the $GQMLE$ should be defined so that the implied sets of identifiability assumptions on $\{\eta_t, t \in \mathbb{Z}\}$ would be compatible for all distributions of the innovation (cf. Section 4.2).

Model (2.1) is quite general and important examples thereof are: the stable periodic $GARCH$ ($PGARCH$) proposed by Bollerslev and Ghysels (1996), the infinite periodic $ARCH$ model (Ziel, 2015-2016), the stable long memory periodic $EGARCH$ model (Rossi and Fantazani, 2015), and the stable periodic asymmetric power $GARCH$ (PAP-$GARCH$) model that we will define below (cf. Example 2.1-2.3 and Section 4.2). When $S = 1$, equation (2.1) reduces to the general conditionally heteroskedastic (CH) model studied by Francq and Zakoïan (2013, 2015, 2016) and El Ghourabi et *al* (2016). It is also a particular case of the general multivariate causal periodic time series model suggested by Ziel (2015). The following examples illustrate the model (2.1) via some specific subclasses of it.

Example 2.1 (The infinite periodic $ARCH\,(\infty)$ model)

An important example of (2.1) is the infinite periodic $ARCH$ ($PARCH\,(\infty)$) model, which is defined by

$$\begin{cases} \epsilon_t = \sigma_t \eta_t, \\ \sigma_t^2 = \alpha_{0t,0} + \alpha_{0t,1}\epsilon_{t-1}^2 + \alpha_{0t,2}\epsilon_{t-2}^2 + ... \end{cases} \qquad t \in \mathbb{Z}, \tag{2.2}$$

where $\{\eta_t, t \in \mathbb{Z}\}$ is ipd_S and the positive coefficients $(\alpha_{0t,j}, t \in \mathbb{Z})$ are S-periodic over t for all $j \in \mathbb{N}$, i.e. $\alpha_{0t,j} = \alpha_{0,t+kS,j}$, $k, t \in \mathbb{Z}$, $j \in \mathbb{N}$. A number of S identifiability conditions on $\{\eta_t, t \in \mathbb{Z}\}$ are required. They are induced by the instrumental functions used in calculating the $GQMLE$ we propose below and should also be compatible with other objectives of the model such as the prediction of powers of the observed process (cf. Francq and Zakoïan, 2013 in the CH case). For model (2.2), the function φ_t in (2.1) corresponds to $\varphi_t(x_1, x_2, ...) = \alpha_{0t,0} + \sum_{j=1}^{\infty} \alpha_{0t,j} x_j$ $(t \in \mathbb{Z})$ while the corresponding parameter vector $\theta_0 = (\theta_{01}', ..., \theta_{0S}')' \in \Theta$ is obtained by parametrizing the coefficients $(\alpha_{0t,j}, t \in \mathbb{Z}, j \in \mathbb{N})$. Specifically, for $1 \leq v \leq S$ we assume that

$$\alpha_{0v,j} = \alpha_{v,j}(\theta_{0v}),$$

with known functions $\alpha_{v,j}(.) : \Theta_v \to [0, \infty)$, for some $\Theta_v \subset \mathbb{R}^{m_v}$. For instance, a simple $PARCH\,(\infty)$ model is obtained for the functions

$$\alpha_{v,j}(\theta_{0v}) = \begin{cases} \frac{b_v}{j^{d_v+1}} & \text{if } j \geq 1 \\ 1 & \text{if } j = 0, \end{cases} \quad , 1 \leq v \leq S,$$

with $\theta_{0v} = (b_v, d_v)' \in \Theta_v = \left[\underline{b}_v, \overline{b}_v\right] \times \left[\underline{d}_v, \overline{d}_v\right]$ and $\Theta = \Theta_1 \times ... \times \Theta_S \subset (0, \infty)^{2S}$, where $0 < \underline{b}_v < \overline{b}_v$ and $0 < \underline{d}_v < \overline{d}_v$ $(1 \leq v \leq S)$ (see Francq and Zakoïan, 2010 in the $ARCH\,(\infty)$ case).

When $\{\eta_t, t \in \mathbb{Z}\}$ is iid, model (2.2) is a particular case of the periodic nonlinear $AR(\infty)$-$ARCH(\infty)$ model proposed by Ziel (2015-2016) where conditions on its existence are provided. It is a periodic version of the infinite $ARCH(\infty)$ introduced by Robinson (1991) and is recommended for representing strong persistence and periodicity in volatility. It is also a generalization of the most widely used periodic $GARCH$ model (Bollerslev and Ghysels, 1996). Indeed, consider the following $PGARCH(1,1)$ model given by

$$\begin{cases} \epsilon_{nS+v} = \sigma_{nS+v}\eta_{nS+v}, \\ \sigma^2_{nS+v} = \omega_{0v} + \alpha_{0v}\epsilon^2_{nS+v-1} + \beta_{0v}\sigma^2_{nS+v-1} \end{cases} \quad 1 \leq v \leq S, n \in \mathbb{Z}, \tag{2.3}$$

where $\{\eta_t, t \in \mathbb{Z}\}$ is ipd_S with $\sup_{1\leq v\leq S} E\left(\log\left(\eta_v^2\right)\right) < \infty$ and $\omega_{0v} > 0, \alpha_{0v} \geq 0, \beta_{0v} \geq 0$. Under the stability condition

$$\prod_{v=1}^{S} \beta_{0v} < 1, \tag{2.4}$$

which in turn is implied by the strict periodic stationarity condition

$$\sum_{v=1}^{S} E\left(\log\left(\alpha_{0v}\eta^2_{v-1} + \beta_{0v}\right)\right) < 0,$$

(cf. Aknouche and Bibi, 2009, Corollary 1) we have

$$\sigma_t^2 = \sum_{j=0}^{\infty}\prod_{i=0}^{j-1} \beta_{0,t-i}\left(\omega_{0,t-j} + \alpha_{0,t-j}\epsilon^2_{t-1-j}\right).$$

So (2.3) is a particular case of (2.2). $\square$

Example 2.2 (The periodic asymmetric power $GARCH(1,1)$ model)

Let $S = 5$ and consider the specific 5-periodic Asymmetric Power $GARCH(1,1)$ (PAP-$GARCH(1,1)$) given by

$$\begin{cases} \epsilon_{5n+v} = \sigma_{5n+v}\eta_{5n+v} \\ \sigma^{\delta_v}_{5n+v} = \omega_{0v} + \alpha_{0v+}(\epsilon^+_{5n+v-1})^{\delta_{v-1}} + \alpha_{0v-}(\epsilon^-_{5n+v-1})^{\delta_{v-1}} + \beta_{0v}\sigma^{\delta_{v-1}}_{5n+v-1} \end{cases} \quad 1 \leq v \leq 5,\ n \in \mathbb{Z}, \tag{2.5}$$

where $\omega_{0v} > 0$, $\alpha_{0v+} \geq 0$, $\alpha_{0v-} \geq 0$, $\beta_{0v} \geq 0$, $\delta_v > 0$ $(1 \leq v \leq 5)$, $\delta_0 := \delta_5$, $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$. Assuming δ_v known ($1 \leq v \leq 5$), the unknown parameter of the model is denoted by $\theta_0 = (\theta'_{01}, \cdots, \theta'_{05})' \in \Theta \subset (0,\infty)^{20}$ with $\theta'_{0v} = (\omega_{0v}, \alpha_{0v+}, \alpha_{0v-}, \beta_{0v}) \in \Theta_v \subset (0,\infty)^4$, $1 \leq v \leq 5$, where $\Theta = \Theta_1 \times \cdots \times \Theta_5$ is a compact parameter space. Moreover, parameter restriction on the model may be considered when having prior information on the marginal distributions of the returns. For example, model (2.5) may be used to represent daily stock returns where each trading day of the week $v \in \{1, 2, .., 5\}$ has a proper marginal distribution. If for a given trading day $w \in \{1, 2, .., 5\}$ we admit that the asymmetry (or leverage effect) of the model is insignificant then one may assume that $\alpha_{0w+} = \alpha_{0w-} := \alpha_{0w}$ so that the corresponding parameter reduces to $\theta_{0w} = (\omega_{0w}, \alpha_{0w}, \beta_{0w})' \in (0,\infty)^3$. It is also possible to consider the powers δ_v $(1 \leq v \leq 5)$ as unknown parameters to be jointly estimated with θ_0. In that case, the parameter vector of the model is denoted by $\psi_0 = (\delta', \theta'_0)'$ with $\delta = (\delta_1, \delta_2, ..., \delta_5)' \in (0,\infty)^5$. Note also that the powers may be considered constant, i.e. $\delta_1 = ... = \delta_5$. On the other hand, the sequence $\{\eta_t, t \in \mathbb{Z}\}$ is assumed to be ipd_S satisfying S identifiability conditions depending on the chosen estimation method as well as on the objective of the model (see Section 4.2).

For $S = 1$, model (2.5) reduces to the Asymmetric Power $GARCH$ (AP-$GARCH\,(1,1)$) model proposed by Ding et *al* (1993). It also reduces to the periodic $GARCH(1,1)$ when $\delta_v = 2$ and $\alpha_{0v+} = \alpha_{0v-}$ $(1 \leq v \leq S)$, to the periodic power $GARCH(1,1)$ corresponding to $\alpha_{0v+} = \alpha_{0v-}$ $(1 \leq v \leq S)$ and to the periodic threshold $GARCH(1,1)$ when $\delta_v = 1$ for all $1 \leq v \leq S$. Beside the stylized facts captured by the AP-$GARCH$ model such as the so-called "*leverage effect*" and the "*Taylor effect*" (e.g. Aknouche and Touche, 2015), model (2.5) might also account for periodicity, which is often observed in financial return data. Note finally that under the stability condition (2.4), σ_t given by (2.5) can be written in the form (2.1) and hence the PAP-$GARCH\,(1,1)$ model is a particular case of the PCH model (2.1). $\square$

Example 2.3 (Mixed specifications)

The PCH model (2.1) also allows different specifications along seasons. For $S = 5$, consider the following model

$$\begin{cases} \epsilon_{5n+v} = \sigma_{5n+v}\eta_{5n+v}, & 1 \leq v \leq 5, n \in \mathbb{N} \\ \sigma_{5n+1}^2 = \alpha_{1,0}\left(\theta_{01}\right) + \sum\limits_{j=1}^{\infty} \alpha_{1,j}\left(\theta_{01}\right)\epsilon_{5n+1-j}^2 \\ \log\left(\sigma_{5n+2}^2\right) = \omega_{02} + \alpha_{02}\epsilon_{5n+1}^2 + \beta_{02}\log\left(\sigma_{5n+1}^2\right) \\ \sigma_{5n+3} = \omega_{03} + \alpha_{03+}(\epsilon_{5n+2}^+) + \alpha_{03-}(\epsilon_{5n+2}^-) + \beta_{03}\sigma_{5n+2} \\ \sigma_{5n+4}^{\delta_4} = \omega_{04} + \alpha_{04}\left|\epsilon_{5n+3}\right| + \beta_{04}\sigma_{5n+3} \\ \sigma_{5n+5}^{\delta_5} = \omega_{05} + \alpha_{05+}(\epsilon_{5n+4}^+)^{\delta_4} + \alpha_{05-}(\epsilon_{5n+4}^-)^{\delta_4} + \beta_{05}\sigma_{5n+4}^{\delta_4} \end{cases}, \tag{2.6}$$

where the parameter of the model is denoted by $\theta_0 = (\theta_{01}', \cdots, \theta_{05}')' \in \Theta \subset (0,\infty)^{m_1+14}$ with $\theta_{01} \in \Theta_1 \subset \mathbb{R}^{m_1}$ for some $m_1 \in \mathbb{N}^*$, $\theta_{02} = (\omega_{02}, \alpha_{02}, \beta_{02})' \in \Theta_2 \subset (0,\infty)^3$, $\theta_{03} = (\omega_{03}, \alpha_{03+}, \alpha_{03-}, \beta_{03})' \in \Theta_3 \subset (0,\infty)^4$, $\theta_{04} = (\omega_{04}, \alpha_{04}, \beta_{04})' \in \Theta_4 \subset (0,\infty)^3$ and $\theta_{05} = (\omega_{05}, \alpha_{05+}, \alpha_{05-}, \beta_{05})' \in \Theta_5 \subset (0,\infty)^4$. All parameter spaces $\Theta_1, .., \Theta_5$ and $\Theta = \Theta_1 \times \cdots \times \Theta_5$ are assumed compact while the powers $\delta_4 > 0$ and $\delta_5 > 0$ are known. Note that θ_{01} is a parametrization of the coefficients $\alpha_{1,j}$ $(j \in \mathbb{N})$ in (2.6). As in the previous examples, the innovation sequence $\{\eta_t, t \in \mathbb{Z}\}$ satisfies certain identifiability assumptions depending on the chosen instrumental functions used in computing the $GQMLE$. For $v = 1$, it is clear that σ_{5n+1} has a similar form as (2.1). By successive replacement in (2.6), it can be seen that σ_{5n+v} $(2 \leq v \leq 5)$ may be cast in the form (2.1) with some conditions on the $\alpha_{1,j}\left(\theta_{01}\right)$ $(j \in \mathbb{N})$ for the volatility to exist, but without any requirement on β_{0v} $(2 \leq v \leq 5)$. So (2.6) is a particular case of (2.1). In fact, model (2.6) combines the infinite $ARCH\left(\infty\right)$ for $v = 1$, the Exponential $GARCH\left(1,1\right)$ $(EGARCH\left(1,1\right))$ for $v = 2$, the threshold $GARCH\left(1,1\right)$ for $v = 3$, the power $GARCH\left(1,1\right)$ for $v = 4$ and the asymmetric power $GARCH(1,1)$ for $v = 5$.

In this illustrative model, various specifications across seasons are permitted. In practice, many seasonal volatility series may show certain stylized facts on a given season and not on another. For example, in daily return series, which generally show the day-of-the-week effect (Tsiakas, 2006; Berument et *al*, 2007; Osborn et *al*, 2008), the "Monday" series may be characterized by a stronger persistence compared to other trading days. Also, a certain trading day may have a distribution with tails heaver than those of the other trading days (e.g. Boynton et *al*, 2009; Bidarkota et *al*, 2009). And this may be true for the asymmetry property (e.g. Balaban et *al*, 2001; Charles, 2010). Thus, one can consider a periodic volatility model with different specifications each of which is adapted to the specific stylized facts marking the season in question. Of course, this is only a schematic and hypothetical example and other mixed specifications are conceivable. However, they should be preceded by preliminary theories (financial for example) and confirmed and reinforced by applications. $\square$

Throughout this paper, we make on equation (2.1) a stability assumption, which implies the properties of *strict periodic stationarity* and *periodic ergodicity* that we recall here for convenience (see also Boyles and Gardener, 1983; Aknouche and Al-Eid, 2012). A real-valued stochastic process $\{Y_t,\ t \in \mathbb{Z}\}$ defined on $(\Omega, \mathcal{F}, P)$ is said to be *strictly periodically stationary* with period $S \in \mathbb{N}^*$ (henceforth sps_S) if its infinite-dimensional distribution is invariant under a shift multiple of S for all season v $(1 \leq v \leq S)$, i.e. the probability distribution of $(..., Y_v, Y_{v+1}, Y_{v+2}, ...)$ is the same as that of $(..., Y_{v+hS}, Y_{v+1+hS}, Y_{v+2+hS}, ...)$ for all $1 \leq v \leq S$ and $h \in \mathbb{Z}$. Here, S is the smallest positive integer verifying the latter property. Thus, a sps_1 process with $S = 1$ is strictly stationary and the simplest sps_S process is an ipd_S sequence. Strict periodic stationarity is intimately related to strict stationarity. Indeed, a process $\{Y_t,\ t \in \mathbb{Z}\}$ is sps_S if and only if all the S "sub-processes" $\{Y_{nS+v},\ n \in \mathbb{Z}\}$ $(1 \leq v \leq S)$ are strictly stationary. The periodic analog of the ergodic theorem for sps_S processes is the periodic ergodic theorem (e.g. Boyles and Gardener, 1983), which can be stated as follows. If $\{Y_t,\ t \in \mathbb{Z}\}$ is sps_S with $E(Y_v) < \infty$ for all $1 \leq v \leq S$ then

$$\frac{1}{n}\sum_{t=1}^{n} Y_t \underset{n\to\infty}{\overset{a.s.}{\to}} \frac{1}{S}\sum_{v=1}^{S} Y_v^*, \tag{2.7}$$

for some random variables Y_v^* $(1 \leq v \leq S)$ on $(\Omega, \mathcal{F}, P)$ satisfying $Y_v^* = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} Y_{kS+v}$, $a.s.$ Result (2.7) also extends to the case where $E(Y_v) \in \mathbb{R} \cup \{+\infty\}$ for some $v \in \{1, ..., S\}$. When for a given season $v_0 \in \{1, ..., S\}$ the corresponding strictly stationary sub-process $\{Y_{nS+v_0},\ n \in \mathbb{Z}\}$ is ergodic, then the limiting random variable $Y_{v_0}^*$ is almost surely constant and then

$$Y_{v_0}^* = E(Y_{v_0}), \quad a.s.$$

If all sub-processes $\{Y_{nS+v},\ n \in \mathbb{Z}\}$ $(v \in \{1, ..., S\})$ are ergodic, then the whole process $\{Y_t,\ t \in \mathbb{Z}\}$ is said to be *periodically ergodic*. In that case, the limiting variable in (2.7) simplifies to $\frac{1}{S}\sum_{v=1}^{S} E(Y_v)$, the mean of the seasonal means. Periodic ergodicity may also be defined more explicitly. Let $T : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ denote the shift transformation defined for any $\mathbf{x}_v = (..., x_v, x_{v+1}, x_{v+2}, ...) \in \mathbb{R}^{\mathbb{Z}}$ by $T\mathbf{x}_v = (..., x_{v+1}, x_{v+2}, x_{v+3}, ...)$ $(1 \leq v \leq S)$ and write T^S for the S-th power of T: $T^S = T \circ T \circ ... \circ T$, S times. A Borel set $D_v \subset \mathbb{R}^{\mathbb{Z}}$ of the form $D_v = \left\{\mathbf{x}_v \in \mathbb{R}^{\mathbb{Z}} : \mathbf{x}_v = (..., x_v, x_{v+S}, x_{v+2S}, ...)\right\}$ is called S-invariant along the season v $(1 \leq v \leq S)$ if $T^{-S}(D_v) = D_v$, where $T^{-S}(D_v) = \left\{\mathbf{x}_v \in \mathbb{R}^{\mathbb{Z}} : T^S\mathbf{x}_v \in D_v\right\}$. A sps_S process $\{Y_t,\ t \in \mathbb{Z}\}$ is said to be periodically ergodic if for any $v \in \{1, ..., S\}$, $P((..., Y_v, Y_{v+S}, Y_{v+2S}, ...) \in D_v) = 0$ or 1, for all S-invariant Borel set D_v over the season v. Similarly to strict periodic stationarity, the simplest periodically ergodic process is an ipd_S sequence. Like strict stationarity and ergodicity (see e.g. Billingsley, 1995, Theorem 36.4), strict periodic stationarity and periodic ergodicity are preserved under certain periodic transformations. Indeed, if $\{Y_t,\ t \in \mathbb{Z}\}$ is sps_S and periodically ergodic and if $\{Z_t, t \in \mathbb{Z}\}$ is given by $Z_t = f_t(..., Y_{t-1}, Y_t, Y_{t+1}, ...)$, where f_t is a function from $\mathbb{R}^{\mathbb{Z}}$ into $\mathbb{R}$, which is measurable, S-periodic over t ($f_t = f_{t+nS}$ for all n and t) and may depend on S-periodically time-varying parameters, then so is $\{Z_t, t \in \mathbb{Z}\}$.

Now consider the following assumption on model (2.1).

A1: $\{\epsilon_t, t \in \mathbb{Z}\}$ *is a strictly periodically stationary and periodically ergodic solution of equation* (2.1)*.*

For specific cases of (2.1), assumption **A1** may be expressed more explicitly in terms of the inputs of (2.1). See Section 4.3 for the periodic asymmetric power $GARCH(1,1)$.

3. Generalized $QMLE$ for periodic conditionally heteroskedastic models

Turn now to the statistical problem of estimating the parameter θ_0 using a series $\epsilon_1, \epsilon_2, ..., \epsilon_T$ generated from (2.1) with sample size $T = NS$ ($N \geq 1$). For every generic parameter $\theta = (\theta_1', ..., \theta_S')' \in \Theta$, if we set

$$\sigma_{nS+v}(\theta) = \varphi_v(\epsilon_{nS+v-1}, \epsilon_{nS+v-2}, ...; \theta_v), \tag{3.0}$$

then clearly $\sigma_{nS+v}(\theta_0) = \sigma_{nS+v}$ for all $1 \leq v \leq S$ and $n \in \mathbb{Z}$. Given any arbitrary fixed initial values $\widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, ...$, define

$$\widetilde{\sigma}_{nS+v}(\theta) = \varphi_v(\epsilon_{nS+v-1}, \epsilon_{nS+v-2}, ..., \epsilon_1, \widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, ...; \theta_v), \ 1 \leq v \leq S, \, n \geq 0, \tag{3.1}$$

as a proxy for $\sigma_{nS+v}(\theta)$. For some chosen measurable positive real-valued functions $h_1, ..., h_S$ that we call instrumental functions, define the generalized quasi-likelihood criterion relatively to $\underline{h} := (h_1, ..., h_S)'$ and for any $\theta \in \Theta$ to be

$$\widetilde{L}_{T,\underline{h}}(\theta) = \frac{1}{NS} \sum_{n=0}^{N-1} \sum_{v=1}^{S} g_v(\epsilon_{nS+v}, \widetilde{\sigma}_{nS+v}(\theta)) \tag{3.2}$$

with $g_v(x, \varsigma) = \log\left(\frac{1}{\varsigma} h_v\left(\frac{x}{\varsigma}\right)\right)$, $\varsigma > 0$, $x \in \mathbb{R}$, $1 \leq v \leq S$.

Then the generalized $QMLE$ (henceforth $GQMLE$) $\widehat{\theta}_{T,\underline{h}}$ of θ_0 is a solution to the problem

$$\widehat{\theta}_{T,\underline{h}} = \arg\max_{\theta \in \Theta} \widetilde{L}_{T,\underline{h}}(\theta), \tag{3.3}$$

for some compact space Θ.

Clearly, $\widehat{\theta}_{T,\underline{h}}$ reduces to the Gaussian $QMLE$ (cf. Aknouche and Bibi, 2009; Aknouche and Al-Eid, 2012 in the $PGARCH(p,q)$ case) when $h_1 = h_2 = ... = h_S = \phi$, ϕ being the standard Gaussian density. Moreover, $\widehat{\theta}_{T,\underline{h}}$ is the maximum likelihood estimate (MLE) when

$$h_1 = f_1, h_2 = f_2, ..., h_S = f_S,$$

where f_v is the density of η_{nS+v} ($1 \leq v \leq S$, $n \in \mathbb{Z}$).

As emphasized above, in calculating the $GQMLE$ we have used S instrumental densities $(h_1, ..., h_S)$ rather than just one density like in CH models (cf. Berkes and Horvàth, 2004; Francq and Zakoïan, 2013). The main motivation behind this choice stems from the fact that the innovation process $\{\eta_t, t \in \mathbb{Z}\}$ is assumed to be ipd_S rather than iid and hence it has S marginal distributions $(f_1, ..., f_S)$. Thus, our choice allows the $GQMLE$ to reduce to the MLE when the S chosen instrumental functions $(h_1, ..., h_S)$ coincide (in the appropriate order) with the S marginal densities $(f_1, ..., f_S)$ of $\{\eta_t, t \in \mathbb{Z}\}$. On the other hand, if only one instrumental density, say h_1, is used then the corresponding $GQMLE$ given by (3.3) cannot reduce to the MLE even when the S marginal distributions of the ipd_S innovation $\{\eta_t, t \in \mathbb{Z}\}$ are known. Therefore, it is likely that the $GQMLE$ cannot be asymptotically efficient.

Let $C > 0$ and $0 < \rho < 1$ be positive generic constants that are not necessarily the same when appearing in different terms. To study strong consistency of the $GQMLE$ consider the following assumptions.

A2 *For any* $\theta \in \Theta$, $\sigma_{nS+v}(\theta) > \underline{\omega}_v$ *a.s. for some* $\underline{\omega}_v > 0$ $(1 \leq v \leq S)$. *Moreover,* $\sigma_{nS+v}(\theta) = \sigma_{nS+v}(\theta_0)$ *a.s. if and only if* $\theta = \theta_0$.

A3 *The functions* $h_1, ..., h_S$ *are integrable and differentiable over* $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$. *Moreover, for all* $1 \leq v \leq S$, *there exist constants* $K_v > 0$ *and* $\delta_v > 0$ *such that:* $|x h'_v(x) / h_v(x)| \leq K_v \left(|x|^{\delta_v} + 1\right)$ *for all* $x \in \mathbb{R}^*$ *and* $E\left(|\eta_v|^{\delta_v}\right) > 0$.

A4 $E\left(\sum_{v=1}^{S} g_v(\eta_v, \varsigma_v)\right) \in [-\infty, +\infty)$ *and* $E\left(\sum_{v=1}^{S} g_v(\eta_v, 1)\right) < E\left(\sum_{v=1}^{S} g_v(\eta_v, \varsigma_v)\right)$ *for all* $\varsigma_v > 0$, $\varsigma_v \neq 1$ $(1 \leq v \leq S)$.

A5 *i) For any* $(x_1, x_2, ...) \in \mathbb{R}^\infty$ *the functions* $\theta \to \varphi_v(x_1, x_2, ...; \theta)$ *are a.s. continuous for all* $1 \leq v \leq S$. *ii) For some* $\tau_v > 0$, $E(|\epsilon_v|^{\tau_v}) < \infty$ *for all* $1 \leq v \leq S$. *iii)* $\sup_{\theta \in \Theta} |\widetilde{\sigma}_{nS+v}(\theta) - \sigma_{nS+v}(\theta)| \leq C\rho^n$ *a.s.*

Assumptions **A1**-**A5** are similar to those given for the non-periodic CH model (see Berkes and Horvàth (2004) for the $GARCH$ model and Francq and Zakoïan (2013) for the CH model) with an appropriate adaptation to the periodic case. Indeed, **A2** implies that the volatility is bounded from below *a.s.* Further, **A2** imposes an identifiability condition, which for the specific stable $PGARCH$ case is given in terms of the $PGARCH$ polynomials (cf. Aknouche and Bibi, 2009). On the other hand, the smoothness condition **A3** on h_v (cf. Berkes and Horvàth, 2004) is not so restrictive and is satisfied for a broad range of standard instrumental functions h_v. In addition, assumption **A4** naturally implies S moment conditions on the ipd_S sequence $\{\eta_t, t \in \mathbb{Z}\}$. These assumptions allow to identify the model and generalize the standard unit second moment assumption $E\left(\eta_t^2\right) = 1$. Finally, **A5** is a moment assumption on the observed process that may appear restrictive. However, most Markovian-like specifications ($PGARCH$, PAP-$GARCH$) can be cast in a recurrence equation of the form $Y_t = A_t Y_{t-1} + B_t$ with $\{(A_t, B_t), t \in \mathbb{Z}\}$ is ipd_S. For this equation the stability condition **A1** implies the finiteness of the moment $E\left(|\epsilon_v|^\tau\right) < \infty$ for some $\tau > 0$ ($1 \leq v \leq S$) and so part ii) of **A5** vanishes (see Aknouche and Bibi, 2009 in the specific $PGARCH\left(p,q\right)$ model and Berkes et *al*, 2003 for the standard $GARCH(p,q)$). Strong consistency of the $GQMLE$ given by (3.3) is now established.

Theorem 3.1 *Assume **A1**-**A5** hold for $K_v > 0$ and $\delta_v > 0$, $(1 \leq v \leq S)$. Then,*

$$\widehat{\theta}_{NS,\underline{h}} \underset{N\to\infty}{\overset{a.s.}{\to}} \theta_0. \tag{3.4}$$

To study asymptotic normality of $\widehat{\theta}_{T,\underline{h}}$, let

$$g_{v1}\left(x,\varsigma\right) = \tfrac{\partial g_v(x,\varsigma)}{\partial \varsigma},\; g_{v2}\left(x,\varsigma\right) = \tfrac{\partial^2 g_v(x,\varsigma)}{\partial \varsigma^2},\; x \in \mathbb{R},\; \varsigma > 0,\; 1 \leq v \leq S,$$

and define the matrices

$$\begin{aligned}
A_{\underline{h},\underline{f}}\left(\theta_0\right) &= \tfrac{1}{S}\sum_{v=1}^{S} E\left(g_{v2}\left(\eta_v,1\right)\right) E\left(\tfrac{1}{\sigma_v^4(\theta_0)}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta'}\right) && (3.5a)\\
B_{\underline{h},\underline{f}}\left(\theta_0\right) &= \tfrac{1}{S^2}\sum_{v=1}^{S} E\left(g_{v1}\left(\eta_v,1\right)^2\right) E\left(\tfrac{1}{\sigma_v^4(\theta_0)}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta'}\right) && (3.5b)\\
J_{\underline{h},\underline{f}}\left(\theta_0\right) &= A_{\underline{h},\underline{f}}^{-1}\left(\theta_0\right) B_{\underline{h},\underline{f}}\left(\theta_0\right) A_{\underline{h},\underline{f}}^{-1}\left(\theta_0\right), && (3.5c)
\end{aligned}$$

whose existence is guaranteed by **A1**-**A5** and the following assumptions:

A6 *θ_0 belongs to the interior of Θ.*

A7 *All h_v $(1 \leq v \leq S)$ are twice differentiable at all $x \in \mathbb{R}^*$ with $\left|x^2\left(h_v'\left(x\right)/h_v\left(x\right)\right)'\right| \leq K_v\left(|x|^{\delta_v} + 1\right)$ for all $x \in \mathbb{R}^*$ and $E\left(|\eta_v|^{2\delta_v}\right) < \infty$, $1 \leq v \leq S$.*

A8 *For all* $1 \leq v \leq S$, $0 < E\left(g_{v1}(\eta_v,1)^2\right) < \infty$, $E\left(g_{v2}(\eta_v,1)\right) < \infty$ *and* $E\left(g_{v2}(\eta_v,1)\right) \neq 0$. *Moreover,* $A_{\underline{h},\underline{f}}(\theta_0)$ *is nonsingular and for any* $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x}'\frac{\partial \sigma^2_{nS+v}(\theta_0)}{\partial \theta_i} = 0$ *(*$i = 1, ..., m$*) implies* $\mathbf{x} = 0$ *for all* $1 \leq v \leq S$.

A9 *For any* $(x_1, x_2, ...) \in \mathbb{R}^\infty$ *the functions* $\theta \to \varphi_v(x_1, x_2, ...; \theta)$ $(1 \leq v \leq S)$ *have continuous second-order derivatives. In addition, there is a neighborhood* $V(\theta_0)$ *of* θ_0 *such that*

$\sup_{\theta \in V(\theta_0)} \left\| \frac{\partial(\widetilde{\sigma}_{nS+v}(\theta) - \sigma_{nS+v}(\theta))}{\partial \theta} \right\| \leq C\rho^n$ *a.s.*

A10 The expectations $E\left(\sup\limits_{\theta \in V(\theta_0)} \left\| \frac{1}{\sigma_v(\theta)} \frac{\partial \sigma_v(\theta)}{\partial \theta} \right\|^4\right)$, $E\left(\sup\limits_{\theta \in V(\theta_0)} \left\| \frac{1}{\sigma_v(\theta)} \frac{\partial^2 \sigma_v(\theta)}{\partial \theta \partial \theta'} \right\|^2\right)$ and $E\left(\sup\limits_{\theta \in V(\theta_0)} \left|\sigma_v^{-1}(\theta)\, \sigma_v(\theta_0)\right|^{2\delta_v}\right)$ are finite for all $1 \leq v \leq S$.

Like consistency assumptions, **A6-A10** are also similar to standard assumptions made for the generalized $QMLE$ in non-periodic CH models (cf. Berkes and Horvàth, 2004; Francq and Zakoïan, 2013). Some of these assumptions simplify or vanish for certain specific cases (see Section 4.3 below). In particular, the last part of **A8**, which implies $B_{\underline{h},\underline{f}}(\theta_0)$ is non-singular, vanishes for the stable $PGARCH$ model with iid innovation. Now, we have the following asymptotic normality result.

Theorem 3.2 *Under* ***A1-A10***

$$\sqrt{N}\left(\widehat{\theta}_{NS,\underline{h}} - \theta_0\right) \underset{N \to \infty}{\xrightarrow{\mathcal{L}}} N\left(0, 4J_{\underline{h},\underline{f}}(\theta_0)\right). \tag{3.6}$$

Some remarks are in order:

i) When $S = 1$, result (3.6) reduces to Theorem 2 by Berkes and Horvàth (2004) for the $GARCH(p,q)$ and to Theorem 1 by Francq and Zakoïan (2013) in the CH case.

ii) When $h_1 = ... = h_S := h$ and $\{\eta_t, t \in \mathbb{Z}\}$ is iid so that $f_1 = ... = f_S := f$, then

$$g_1 = ... = g_S,\ g_{11} = ... = g_{S1},\ g_{12} = ... = g_{S2},$$

and $J_{\underline{h},\underline{f}}(\theta_0)$ given by (3.5c) reduces to

$$\begin{aligned} J_{\underline{h},f}(\theta_0) &= \tau^2_{h,f} J^{-1} \text{ with} \\ \tau^2_{h,f} &= \frac{E\left(g_{11}(\eta_0,1)^2\right)}{\left(E(g_{12}(\eta_0,1))\right)^2} \text{ and } J = \sum_{v=1}^{S} E\left(\frac{1}{\sigma_v^4(\theta_0)} \frac{\partial \sigma_v^2(\theta_0)}{\partial \theta} \frac{\partial \sigma_v^2(\theta_0)}{\partial \theta'}\right). \end{aligned}$$

In the latter case, the last part of **A8** implies that J is nonsingular (see e.g. Francq and Zakoïan, 2010) and the nonsingularity assumption on $A_{\underline{h},\underline{f}}(\theta_0)$ is unnecessary. In particular, for the specific stable $PGARCH(p,q)$ model with $h_1 = ... = h_S = \phi$ we find the asymptotic result by Aknouche and Bibi (2009, Theorem 4).

iii) When $(h_1, ..., h_S) = (f_1, ..., f_S)$, where f_v is the density of η_v $(1 \leq v \leq S)$, the $GQMLE$ reduces to the Maximum Likelihood Estimate (MLE), which is then asymptotically efficient. Furthermore,

$$\begin{aligned} E\left(g_{v2}\left(\eta_v, 1\right)\right) &= -E\left(g_{v1}\left(\eta_v, 1\right)^2\right) \\ &= -E\left(1 + \tfrac{f_v'(\eta_v)}{f_v(\eta_v)}\eta_v\right)^2, \quad 1 \leq v \leq S, \end{aligned}$$

and $J_{\underline{h},\underline{f}}(\theta_0)$ given by (3.5) simplifies when $\underline{h} = \underline{f}$ to

$$J_{\underline{f},\underline{f}}(\theta_0) = \left[\sum_{v=1}^{S} E\left(1 + \tfrac{f_v'(\eta_v)}{f_v(\eta_v)}\eta_v\right)^2 E\left(\tfrac{1}{\sigma_v^4(\theta_0)} \tfrac{\partial \sigma_v^2(\theta_0)}{\partial \theta} \tfrac{\partial \sigma_v^2(\theta_0)}{\partial \theta'}\right)\right]^{-1}.$$

iv) For some specific PCH models in which $\frac{\partial \sigma_{nS+v}^2(\theta_0)}{\partial \theta}$ does not depend on $\theta_{0v'}$ for all $v' \neq v$, as it often happens for finite pure $ARCH$-like models, the matrix $J_{\underline{h},\underline{f}}(\theta_0)$, which is in a "sandwich" form, may have a simpler expression as the inverse of a block-diagonal matrix (see (4.7) in Section 4.3 below).

4. Illustrations and applications

4.1. Examples of instrumental distributions

Example 4.1 (Gaussian $QMLE$)

For model (2.1), let $S = 5$ and consider the $GQMLE$ with the same instrumental density along seasons, which is the standard Gaussian distribution, i.e.

$$h_1(x) = ... = h_5(x) = \phi(x) = (2\pi)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}x^2\right), \quad x \in \mathbb{R}.$$

The resulting $GQMLE$ is called the Gaussian $QMLE$. The first derivatives of $g_v(x,\varsigma)$ $(1 \leq v \leq 5)$ are given by

$$g_{1v}(x,\varsigma) = \varsigma^{-1}\left(1 - x^2\varsigma^2\right), \ \ (1 \leq v \leq 5),$$

and the unique solution to the equation

$$\frac{\partial}{\partial\varsigma_v} E\left(\sum_{i=1}^{5} g_i(\eta_i, \varsigma_i)\right) = 0, \ 1 \leq v \leq 5,$$

is

$$\varsigma_1 = \frac{1}{E\left(\eta_1^2\right)}, \varsigma_2 = \frac{1}{E\left(\eta_2^2\right)}, ..., \varsigma_5 = \frac{1}{E\left(\eta_5^2\right)}.$$

Hence $\sum_{v=1}^{5} g_v(\eta_v, \varsigma_v)$ admits a unique maximum at $\left(\frac{1}{E\left(\eta_1^2\right)}, ..., \frac{1}{E\left(\eta_5^2\right)}\right)'$ so **A4** is satisfied if

$$E\left(\eta_1^2\right) = ... = E\left(\eta_5^2\right) = 1, \tag{4.1}$$

which is the standard unit second moment condition (cf. Bollerslev and Ghysels, 1996; Aknouche and Bibi, 2009; Ziel, 2015). On the other hand, **A8** holds if $E\left(\eta_v^4\right) < \infty$ $(1 \leq v \leq 5)$ and

$$\begin{array}{l} E\left(g_{1v}(\eta_v, 1)^2\right) = E\left(1 - \eta_v^2\right)^2 \\ \left(E\left(g_{2v}(\eta_v, 1)\right)\right)^2 = 4 \end{array}, \ 1 \leq v \leq 5.$$

Now if $\{\eta_t, t \in \mathbb{Z}\}$ is iid with marginal distribution f then $J_{\underline{h},\underline{f}}(\theta_0)$ given by (3.5c) reduces to

$$J_{\underline{\phi},f}(\theta_0) = \tau^2_{\underline{\phi},f} J^{-1},$$

with

$$\tau^2_{\underline{\phi},f} = \frac{E\left(1-\eta_v^2\right)^2}{4} = \frac{Var\left(\eta_v^2\right)}{4} \ \text{ and } \ J = \sum_{v=1}^{S} E\left(\frac{1}{\sigma_v^4(\theta_0)} \frac{\partial\sigma_v^2(\theta_0)}{\partial\theta} \frac{\partial\sigma_v^2(\theta_0)}{\partial\theta'}\right).$$

If, however, $\{\eta_t, t \in \mathbb{Z}\}$ is not iid, but ipd_S with marginal distributions $\underline{f} = (f_1, ..., f_S)$, then these distributions should be compatible with (4.1). Furthermore, $J_{\underline{h},\underline{f}}(\theta_0)$ has the sandwich form (3.5c)

$$J_{\underline{h},\underline{f}}(\theta_0) = A^{-1}_{\underline{h},\underline{f}}(\theta_0) B_{\underline{h},\underline{f}}(\theta_0) A^{-1}_{\underline{h},\underline{f}}(\theta_0),$$

with

$$\begin{aligned}
A_{\underline{h},\underline{f}}(\theta_0) &= \tfrac{2}{S}\sum_{v=1}^{S} E\left(\tfrac{1}{\sigma_v^4(\theta_0)}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta'}\right),\\
B_{\underline{h},\underline{f}}(\theta_0) &= \tfrac{1}{S^2}\sum_{v=1}^{S} Var\left(\eta_v^2\right) E\left(\tfrac{1}{\sigma_v^4(\theta_0)}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta}\tfrac{\partial\sigma_v^2(\theta_0)}{\partial\theta'}\right). \square
\end{aligned}$$

Example 4.2 (A mixed $QMLE$)

Let $S = 3$ and consider the $GQMLE$ of model (2.1) with instrumental densities given by: i) the standard Gaussian density $h_1(x) = \phi(x)$ for season 1, ii) the Laplace density

$$h_2(x) = \tfrac{1}{2}\exp(-|x|),\ x \in \mathbb{R},$$

for season 2 and iii) a particular case of the generalized Gaussian density

$$h_3(x) = \exp\left(-2|x|^{1/2}\right),\ x \in \mathbb{R},$$

for the third season. Then,

$$\begin{aligned}
g_{11}(x_1,\varsigma_1) &= \varsigma_1^{-1}\left(1 - x_1^2\varsigma_1^2\right),\\
g_{21}(x_2,\varsigma_2) &= -\varsigma_2^{-1}\left(1 - |x|\,\varsigma^{-1}\right),\\
g_{31}(x_3,\varsigma_3) &= -\varsigma_3^{-1}\left(1 - |x|^{\frac{1}{2}}\varsigma^{-\frac{1}{2}}\right).
\end{aligned}$$

so the unique solution to the equation

$$\tfrac{\partial}{\partial\varsigma_v}E\left(\sum_{i=1}^{3} g_i(\eta_i,\varsigma_i)\right) = 0,\ 1 \le v \le 3,$$

is

$$\varsigma_1 = \tfrac{1}{E\left(\eta_1^2\right)},\ \varsigma_2 = E(|\eta_2|),\ \varsigma_3 = \sqrt{E\left(|\eta_3|^{1/2}\right)}.$$

Therefore, $\sum\limits_{v=1}^{3} g_v(\eta_v,\varsigma_v)$ admits a unique maximum at $\left(\tfrac{1}{E\left(\eta_1^2\right)}, E(|\eta_2|), \sqrt{E\left(|\eta_3|^{1/2}\right)}\right)'$ and **A4** is satisfied if

$$E\left(\eta_1^2\right) = 1,\ E(|\eta_2|) = 1,\ E\left(|\eta_3|^{1/2}\right) = 1. \tag{4.2}$$

Moreover, **A8** holds if all the following conditions are fulfilled:

$$E\left(|\eta_1|^4\right) < \infty,\ E\left(\eta_2^2\right) < \infty,\ E\left(|\eta_3|\right) < \infty.$$

Note that the assumption of *iid* innovation $\{\eta_t, t \in \mathbb{Z}\}$ is not compatible with (4.2). Of course, this example is only illustrative and aims at showing that the $GQMLE$ may be given via various instrumental functions. However, the choice of these instrumental functions should be made carefully and depends on the adopted model and its objectives. $\square$

Example 4.3 (Another mixed $QMLE$)

Let $S = 5$ and consider the $GQMLE$ of model (2.1) with the following five instrumental functions:

i) $h_1(x) = \frac{r^{\left(1-\frac{1}{r}\right)}}{2\Gamma\left(\frac{1}{r}\right)} \exp\left(-\frac{1}{r}|x|^r\right)$, $r > 0$ (Generalized Gaussian density, Γ being the Gamma function).

ii) $h_2(x) = \frac{a}{2\Gamma(a)} |x|^{a-1} \exp(-a|x|)$, $a > 0$ (Double Gamma density).

iii) $h_3(x) = \frac{\delta}{2} |x|^{\delta-1} \exp\left(-|x|^\delta\right)$, $\delta > 0$ (Double Weibull density).

iv) $h_4(x) = \left(\frac{\vartheta-1}{2}\right)(1+|x|)^{-\vartheta}$, $\vartheta > 0$.

v) $h_5(x) = K\left(1+\left(\frac{x-\lambda}{b}\right)^2\right)^{-m} \exp\left(-\kappa \tan^{-1}\left(\frac{x-\lambda}{b}\right)\right)$, $K, b > 0$, $m \geq 1/2$, $\lambda, \kappa \in \mathbb{R}$ (Pearson's Type IV distribution, K being a normalizing constant (cf. Zhu et *al*, 2015)).

Then straightforward calculations similar to Example 4.1 and Example 4.2 show that **A4** is satisfied if

$$E\left(|\eta_1|^r\right) = 1,\ E\left(|\eta_2|\right) = 1,\ E\left(|\eta_3|^\delta\right) = 1,\ E\left(\frac{|\eta_4|}{1+|\eta_4|}\right) = \frac{1}{\vartheta},\ E\left(\frac{2m\eta_5^2+\kappa\eta_5}{1+\eta_5^2}\right) = 1.$$

Moreover, **A8** is satisfied if all the following conditions hold:

$$\begin{aligned} E\left(|\eta_1|^{2r}\right) &< \infty,\ E\left(\eta_2^2\right) < \infty,\ E\left(|\eta_3|^{2\delta}\right) < \infty, \\ E\left(\eta_v^{2\delta_*}\right) &< \infty \text{ for some } \delta_* > 0\ \ (4 \leq v \leq 5). \end{aligned}$$

4.2. Prediction of powers in PCH models: The one-step parametric approach

An important application of the $GQMLE$ in CH models is the prediction of the power of the observed process $\{\varepsilon_t, t \in \mathbb{Z}\}$ in a one-step setting (cf. Francq and Zakoïan, 2013). Though in CH models one usually considers prediction of the squared process $\{\varepsilon_t^2, t \in \mathbb{Z}\}$, Francq and Zakoïan (2013) pointed out the importance of predicting the powered term $|\varepsilon_t|^r$ when $r \in \mathbb{R}$ is rather a real number. This issue is particularly interesting i) for heavy-tailed distributions with infinite second moment when $0 \leq r < 2$, ii) for duration models when $r < 0$ and iii) for calculating the conditional variance of the prediction errors of the squares when $r > 2$ (see Francq and Zakoïan, 2013). Since the best prediction in the mean square sense of $|\varepsilon_t|^r$ $(\neq 0)$ is $\sigma_t^r(\theta_0)$ under $E(|\eta_1|^r) = 1$, Francq and Zakoïan (2013) used the $GQMLE$ to estimate the CH model under the latter assumption, getting that prediction without extra-calculation. They showed that their one-step approach has some advantages over the standard two-step approach, which consists in estimating the volatility $\sigma_t^r(\theta_0)$ by the Gaussian $QMLE$ in a first step, and then estimating $E(|\eta_1|^r)$ non-parametrically in a second step. Francq and Zakoïan (2013) also characterized a class of instrumental densities they called omnibus class, which makes the consistency assumptions of the $GQMLE$ compatible with the unit absolute power moment condition $E(|\eta_1|^r) = 1$.

In this subsection we show how the $GQMLE$ for the PCH model (2.1) can be applied to perform prediction of powers in a one-step parametric approach as in Francq and Zakoïan (2013). In contrast with non-periodic CH models, S different powers corresponding to seasons are considered in our PCH case.

For any non-null real numbers $r_1, ..., r_S$ such that $E(|\eta_v|^{r_v}) < \infty$ $(1 \leq v \leq S)$, the best predictor in the mean square sense of $|\varepsilon_{nS+v}|^{r_v}$ given its past history is

$$E\left(|\varepsilon_{nS+v}|^{r_v} / \mathcal{F}_{nS+v-1}\right) = \sigma_{nS+v}^{r_v} E\left(|\eta_v|^{r_v}\right), \, 1 \leq v \leq S. \tag{4.3}$$

Similarly, the best mean square predictor of $\log|\varepsilon_{nS+v}|$ given $\mathcal{F}_{nS+v-1}$ is $\log \sigma_{nS+v}^{r_v} + E(\log(|\eta_v|))$ provided that $E(\log(|\eta_v|)) < 0$. The latter case may be seen as a limit of (4.3) when $r_v \to 0$

for all $1 \leq v \leq S$. Thus, the one-step fully parametric method for predicting the powers of the PCH process (2.1) is described as follows.

i) Given a series generated from (2.1), estimate θ_0 by the $GQMLE$ $\widehat{\theta}_{NS,\underline{h}}$ under **A1**-**A10** and the following assumption:

A11 For all $1 \leq v \leq S$, $E\left(|\eta_v|^{r_v}\right) = 1$ if $r_v \neq 0$ and $E\left(\log|\eta_v|\right) = 0$ if $r_v \neq 0$.

ii) The best predictor in the mean square sense of $|\varepsilon_{nS+v}|^{r_v}$ given $\mathcal{F}_{nS+v-1}$ is estimated by

$$\begin{cases} \sigma_{nS+v}^{r_v}\left(\widehat{\theta}_{NS,\underline{h}}\right) & \text{if } r_v \neq 0 \\ \log\sigma_{nS+v}\left(\widehat{\theta}_{NS,\underline{h}}\right) & \text{if } r_v = 0 \end{cases}, \quad 1 \leq v \leq S.$$

Now the following corollary of Theorem 3.1 and Theorem 3.2 gives asymptotic properties of the $GQMLE$ in the framework of prediction of powers using the one-step parametric approach. It is a generalization of Theorem 1 by Francq and Zakoïan (2013) to the PCH case.

Corollary 4.1 *Under **A1-A11**, results* (3.4) *and* (3.6) *remain true.*

Note that **A11** is considered only in the framework of prediction of power problem in a one-step parametric approach. Apart from this problem, **A11** is unnecessary for the consistency and asymptotic normality of the $GQMLE$.

Note that depending on the choice of the instrumental densities $h_1, h_2, .., h_S$, assumption **A4** induces S moment conditions on $\{\eta_t, t \in \mathbb{Z}\}$ (cf. Examples 4.1-4.2), which may be inconsistent with **A11**. The functions $h_1, h_2, .., h_S$ are said to be omnibus for the prediction of power problem if the implied assumption **A4** is compatible with **A11** for all distributions of the innovations $\eta_1, ..., \eta_S$. For a given $r_v > 0$ ($1 \leq v \leq S$) let $\mathcal{C}(r_v)$ be the class of functions defined by (cf. Francq and Zakoïan, 2013)

$$\mathcal{C}(r_v) = \left\{ h : h(x) = \begin{cases} c_v |x|^{\lambda_v - 1} \exp\left(-\lambda \frac{|x|^{r_v}}{r_v}\right) & \text{if } r_v > 0 \\ c_v |x|^{-\lambda_v - 1} \exp\left(\lambda \frac{|x|^{r_v}}{r_v}\right) & \text{if } r_v < 0 \\ c_v |x|^{-\lambda_v - 1} \exp\left(\lambda \frac{|x|^{r_v}}{r_v}\right) & \text{if } r_v = 0 \end{cases} \right\}$$

for some $c_v, \lambda_v > 0$ ($1 \leq v \leq S$). The following result, which is a trivial generalization of Proposition 2 by Francq and Zakoïan (2013), shows that the class $\mathcal{C}_S(r_1, ..., r_S)$ defined by the following Cartesian product

$$\mathcal{C}_S(r_1,...,r_S) := \prod_{1\leq v\leq S} \mathcal{C}(r_v),$$

is the class of omnibus functions for the prediction of power problem in PCH models.

Proposition 4.1 *Let $h_1,...,h_S$ be instrumental functions satisfying **A3**. Then, **A4** holds for all distributions of $\eta_1,...,\eta_S$ satisfying **A11** if and only if $h_1 \in \mathcal{C}(r_1),...,h_S \in \mathcal{C}(r_S)$.*

The proof of Proposition 4.1 is very similar to that of Proposition 2 in Francq and Zakoïan (2.13) and hence is omitted. Thus assumption **A4** could be omitted in Corollary 4.1 if the instrumental functions $(h_1,...,h_S)'$ belong to the class of omnibus functions $\mathcal{C}_S(r_1,...,r_S)$.

4.3. $GQMLE$ of the Periodic Asymmetric Power $GARCH(1,1)$

We illustrate the $GQMLE$ asymptotics given in Section 3 on the following PAP-$GARCH(1,1)$ model with a general period $S \in \mathbb{N}^*$,

$$\begin{aligned} \epsilon_t &= \sigma_t\eta_t && (4.4a)\\ \sigma_t^{\delta_t} &= \omega_{0t} + \alpha_{0t+}(\epsilon_{t-1}^{+})^{\delta_{t-1}} + \alpha_{0t-}(\epsilon_{t-1}^{-})^{\delta_{t-1}} + \beta_{0t}\sigma_{t-1}^{\delta_{t-1}}, \quad t \in \mathbb{Z}, && (4.4b) \end{aligned}$$

where, as in Example 2.2, $\{\eta_t, t \in \mathbb{Z}\}$ is ipd_S and the volatility parameters $\omega_{0t} > 0$, $\alpha_{0t+} \geq 0$, $\alpha_{0t-} \geq 0$, $\beta_{0t} \geq 0$, $\delta_t > 0$ are S-periodic over t with δ_t is assumed known for all t. The parameter of the model is denoted by $\theta_0 = (\theta'_{01},...,\theta'_{0S})' \in \Theta \subset \mathbb{R}^{4S}$ with $\theta'_{0v} = (\omega_{0v}, \alpha_{0v+}, \alpha_{0v-}, \beta_{0v})$, $1 \leq v \leq S$ where Θ is a compact space. Letting

$$\begin{aligned} Y_t &= \sigma_t^{\delta_t},\\ A_t &= \alpha_{0t+}(\eta_{t-1}^{+})^{\delta_{t-1}} + \alpha_{0t-}(\eta_{t-1}^{-})^{\delta_{t-1}} + \beta_{0t},\\ B_t &= \omega_{0t}, \end{aligned}$$

model (4.4) may be written in the following stochastic recurrence equation

$$Y_t = A_tY_{t-1} + B_t, \qquad t \in \mathbb{Z}, \tag{4.5}$$

with ipd_S input $\{(A_t, B_t), t \in \mathbb{Z}\}$. From Brandt (1986), if we assume that

$$\sum_{v=1}^{S} E\left(\log^{+}(A_v)\right) < \infty,$$

then a sufficient condition for (4.4) to have a strictly periodically stationary and periodically ergodic solution is that

$$\gamma_0^S := \tfrac{1}{S}\sum_{v=1}^{S} E\left(\log(A_v)\right) < 0. \tag{4.6}$$

The latter condition may be interpreted as a stability condition in average among the different seasons. Following the same lines of Bougerol and Picard (1992) and Aknouche and Bibi (2009, Corollary 1), a necessary condition for (4.4) to have a strictly periodically stationary solution is that $\prod_{v=1}^{S} \beta_{0v} < 1$, which is the same condition as (2.4). Thus, concerning the $GQMLE$ for the specific model (4.4), several assumptions among **A1**-**A10** stated above can be made more explicit. Indeed, assumption **A1** for model (4.4) is satisfied if we assume (4.6) and the following condition:

B1 $\forall\theta \in \Theta : \prod_{v=1}^{S} \beta_v < 1$ and for all $1 \leq v \leq S$, $\omega_v > \underline{\omega}$ for some $\underline{\omega} > 0$.

On the other hand, from Berkes et *al* (2003), it is easy to show using equation (4.5) that under condition (4.6) there is $\tau > 0$ such that $E\left(|\epsilon_v|^{\tau}\right) < \infty$ for all $1 \leq v \leq S$ (see also Aknouche and Bibi, (2009, Theorem 2) in the $PGARCH(p, q)$ case). Hence, **A5** holds under (4.6) without any moment assumption on the process $\{\epsilon_t, t \in \mathbb{Z}\}$. Moreover, letting $\mathcal{A}_{\theta_{0v+}} = \alpha_{0v+}z$, $\mathcal{A}_{\theta_{0v-}} = \alpha_{0v-}z$ and $\mathcal{B}_{\theta_{0v}}(z) = 1 - \beta_{0v}z$ $(1 \leq v \leq S)$, the identifiability assumption **A2** can be replaced for model (4.4) by the following explicit condition:

B2 *For all* $1 \leq v \leq S$: $\mathcal{B}_{\theta_{0v}}(z)$ *has no common root with* $A_{\theta_{0v+}}(z)$ *and* $A_{\theta_{0v-}}(z)$, $A_{\theta_{0v+}}(1) + A_{\theta_{0v-}}(1) \neq 0$. *In addition,* $\alpha_{0v+} + \alpha_{0v-} + \beta_{0v} \neq 0$.

The latter condition also implies that $B_{\underline{h},\underline{f}}(\theta_0)$ given by (3.5) is nonsingular so the last part of **A8** holds. Finally, following Francq and Zakoïan (2013) (see also Hamadeh and Zakoïan (2011) for the Gaussian $QMLE$ with $S = 1$), we make on $\{\eta_t, t \in \mathbb{Z}\}$ the following assumption, which entails **A6**-**A10**.

B3 *For all* $1 \leq v \leq S$, *if* $P(\eta_v \in \Lambda_v) = 1$ *for a set* Λ_v *then* Λ_v *has a cardinal* $|\Lambda_v| > 2$. *Further,* $P(\eta_v > 0) \in (0, 1)$.

Consequently, we have the following asymptotic result for the $GQMLE$ of the PAP-$GARCH\,(1,1)$ model (4.4).

Corollary 4.2 *Under* (4.6)*,* ***A4****,* ***A8*** *and* ***B1****-****B3****, results* (3.4) *and* (3.6) *hold for the* $GQMLE$ *of model* (4.4).

It is worth noting that when the instrumental functions $(h_1, ..., h_S)'$ belong to the class of omnibus functions $\mathcal{C}_S\,(r_1, ..., r_S)$, for some $r_1, ..., r_S > 0$, then assumption **A4** may be replaced in Corollary 4.2 by the following more explicit moment condition on $\{\eta_t, t \in \mathbb{Z}\}$:

B4 $\forall v \in \{1, ..., S\}$, $E\,|\eta_v|^{r_v} = 1$ *and* $E\,|\eta_v|^{2r_v} < \infty$ *for some* $r_1, ..., r_S > 0$.

Now, consider the particular PAP-$ARCH\,(1)$ model, which corresponds to (4.4) with $\beta_v = 0$ for all $1 \leq v \leq S$. Then $\theta_0 = (\theta'_{01}, ..., \theta'_{0S})' \in \Theta \subset \mathbb{R}^{3S}$ with $\theta'_{0v} = (\omega_{0v}, \alpha_{0v+}, \alpha_{0v-})$. Moreover, the asymptotic variance $J_{\underline{h},\underline{f}}\,(\theta_0)$ in (3.5c) is block-diagonal and is explicitly given by

$$J_{\underline{h},\underline{f}}\,(\theta_0) = \begin{pmatrix} \tau^2_{h_1,f_1} J_1^{-1} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \tau^2_{h_2,f_2} J_2^{-1} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \ddots & \vdots \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \tau^2_{h_S,f_S} J_S^{-1} \end{pmatrix}, \tag{4.7}$$

with

$$J_v = E\left(\frac{1}{\sigma_v^4(\theta_0)} \frac{\partial \sigma_v^2(\theta_0)}{\partial \theta} \frac{\partial \sigma_v^2(\theta_0)}{\partial \theta'}\right) \text{ and } \tau^2_{h_v,f_v} = \frac{E\left(g_{v1}(\eta_v, 1)^2\right)}{\left(E\left(g_{v2}(\eta_v, 1)\right)\right)^2},\ \ 1 \leq v \leq S.$$

Note finally that Corollary 4.2 contains as a particular case asymptotics of the $GQMLE$ for: i) the periodic $GARCH(1,1)$ when $\delta_t = 2$ and $\alpha_{0t+} = \alpha_{0t-}$, ii) the periodic power $GARCH(1,1)$ corresponding to $\alpha_{0t+} = \alpha_{0t-}$ and iii) the periodic threshold $GARCH(1,1)$ when $\delta_t = 1$ $(1 \leq t \leq S)$.

4.4. $GQMLE$ for PCH models with complex periodic patterns

4.4.1. $GQMLE$ and reduction of the number of parameters in high frequency PCH models

Though periodic CH models have been successfully applied to low frequency seasonal series like daily series (e.g. Bollerslev and Ghysels, 1996; Franses and Paap, 2004; Osborn et *al*, 2008), a potential drawback with these models is that they involve a very large number of parameters when the period S tends to be large, like in intraday series. For instance, for half-hourly series (e.g. Taylor, 2006), which imply a period of $S = 48$, the unrestricted $PAP\text{-}GARCH(1,1)$ model (4.4) requires $4S = 192$ parameters, making their estimation and interpretation extremely challenging. To overcome this problem, several solutions have been suggested to reduce the number of implied parameters in high frequency periodic models. An *ad hoc* device is to restrict some parameters to reduce the parameter space. For example, in model (4.4) one might take $\beta_v = \beta_1$ $(2 \leq v \leq S)$ as already done by Franses and Paap (2000) for the $PGARCH$ model. However, the most usual approach is to use some basis functions like Fourier approximation (Jones and Brelsford, 1967; Bollerslev et *al*, 2000; Taylor, 2006; Anderson et *al*, 2007; Tesfaye et *al*, 2011; Franses and Paap, 2011; Rossi and Fantazani, 2015), periodic B-splines (Ziel et *al*, 2015) or periodic wavelets (see also Ziel et *al*, 2016; Ambach and Croonenbroeck, 2015; Ambach and Schmid, 2015). In this Subsection we will see how the $GQMLE$ may be adapted when model (2.1) is reparametrized to reduce the parameter space in high frequency PCH models. We follow here the approach of Jones and Brelsford (1967), which is based on the following reparametrization

$$\begin{cases} \theta_{0v} = (\theta_{0v,1}, \ldots \theta_{0v,m_v})' \\ \theta_{0v,j} = \theta^*_{0j} + \theta^*_{0j} \cos\left(\frac{2\pi v}{S} - \theta^*_{0j}\right), \ \ 1 \leq j \leq m_v \end{cases}, \ 1 \leq v \leq S. \tag{4.8}$$

where for identifiability reasons we assume that $\theta^*_{0j} \in (0,1)$ for all j as $\cos(x + n\pi) = (-1)^n \cos(x)$ (see also Rossi and Fantazani (2015) for the periodic long memory $EGARCH$ model and Franses and Paap (2011) for the periodic autoregression). In lieu of $m = \sum_{v=1}^{S} m_v$ parameters, the new reparametrization (4.8) only involves a number of $m^* = 3\max_{1 \leq v \leq S}(m_v)$ parameters to be estimated. For example, for the $PAP\text{-}GARCH(1,1)$ model (4.4), specification (4.8) reduces to

$$\begin{cases} \omega_{0v} = \omega^*_{01} + \omega^*_{02} \cos\left(\frac{2\pi v}{S} - \omega^*_{03}\right) \\ \alpha_{0v+} = \alpha^*_{01+} + \alpha^*_{02+} \cos\left(\frac{2\pi v}{S} - \alpha^*_{03+}\right) \\ \alpha_{0v-} = \alpha^*_{01-} + \alpha^*_{02-} \cos\left(\frac{2\pi v}{S} - \alpha^*_{03-}\right) \\ \beta_{0v} = \beta^*_{01} + \beta^*_{02} \cos\left(\frac{2\pi v}{S} - \beta^*_{03}\right) \end{cases}, \ 1 \leq v \leq S, \tag{4.9}$$

where the parameter of the model is now denoted by $\theta_0^* = \left(\omega_0^{*\prime}, \alpha_{0+}^{*\prime}, \alpha_{0-}^{*\prime}, \beta_0^{*\prime}\right)'$ with $\omega_0^* = (\omega_{01}^*, \omega_{02}^*, \omega_{03}^*)'$, $\alpha_{0+}^* = \left(\alpha_{01+}^*, \alpha_{01+}^*, \alpha_{01+}^*\right)'$, $\alpha_{0-}^* = \left(\alpha_{01-}^*, \alpha_{01-}^*, \alpha_{01-}^*\right)'$, $\beta_0^* = (\beta_{01}^*, \beta_{02}^*, \beta_{03}^*)'$. and $\left(\omega_{03}^*, \alpha_{03+}^*, \alpha_{03-}^*, \beta_{03}^*\right)' \in (0,1)^4$. Note that the number of parameters in (4.9) does not depend on S and is reduced for large S from $4S$ to 12.

Now with θ_0^* in place of θ_0, model (2.1) may be rewritten as follows

$$\begin{cases} \epsilon_{nS+v} = \sigma_{nS+v}\eta_{nS+v}, \\ \sigma_{nS+v} = \varphi_v^*\left(\epsilon_{nS+v-1}, \epsilon_{nS+v-2}, ...; \theta_{0v}^*\right) := \sigma_{nS+v}^*\left(\theta_0^*\right), \end{cases}, 1 \leq v \leq S, \tag{4.10}$$

where the function φ_v^* is obtained from φ_v by rearrangement while replacing θ_0 by θ_0^*. We assume that $\theta_0^* \in \Theta^* \subset \mathbb{R}^{m^*}$ for some compact parameter space Θ^*. Of course, the stability and positivity constraints on θ_0 in (2.1) are directly translated in terms of θ_0^* through (4.8). Like model (2.1), we define $\sigma_{nS+v}^*\left(\theta^*\right)$, $\widetilde{\sigma}_{nS+v}^*\left(\theta^*\right)$ and $\widetilde{L}_{T,\underline{h}}^*\left(\theta\right)$ as in (3.0), (3.1) and (3.2), respectively for some instrumental functions $\underline{h} := (h_1, ..., h_S)'$, i.e.

$$\begin{aligned} \sigma_{nS+v}^*\left(\theta^*\right) &= \varphi_v^*\left(\epsilon_{nS+v-1}, \epsilon_{nS+v-2}, ...; \theta_v^*\right), & 1 \leq v \leq S \\ \widetilde{\sigma}_{nS+v}^*\left(\theta^*\right) &= \varphi_v^*\left(\epsilon_{nS+v-1}, \epsilon_{nS+v-2}, ..., \widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, ...; \theta_v^*\right), & n \in \mathbb{Z}, \\ \widetilde{L}_{T,\underline{h}}^*\left(\theta^*\right) &= \tfrac{1}{NS}\sum_{n=0}^{N-1}\sum_{v=1}^{S} g_v\left(\epsilon_{nS+v}, \widetilde{\sigma}_{nS+v}^*\left(\theta^*\right)\right), \end{aligned}$$

where g_v $(1 \leq v \leq S)$ is defined as above and $\widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, ...$ are fixed initial values. The $GQMLE$ of θ^* is then given by

$$\widehat{\theta}_{T,\underline{h}}^* = \arg\max_{\theta^* \in \Theta^*} \widetilde{L}_{T,\underline{h}}^*\left(\theta^*\right).$$

Note finally that consistency and asymptotic normality of $\widehat{\theta}_{T,\underline{h}}^*$ are established in the same way as $\widehat{\theta}_{T,\underline{h}}$ under the same assumptions **A1**-**A10** with an appropriate adaptation considering θ^* in place of θ.

4.4.2. $GQMLE$ for PCH models when the period S is non-integer

Next to high frequency seasonality, another well-observed case of complex periodic patterns is seasonality with a non-integer period. For example, many weekly series have an annual seasonal pattern with period $365.25/7 \approx 52.179$ (e.g. De Livera et *al*, 2011). When a periodic model like (2.1) is fitted to a series characterized by a non-integer period $S \in (1, \infty)$, one usually takes (by simple approximation) the period to be the integer part of S, which is denoted by $[S]$, where $[S] = n \in \mathbb{N}^*$ with $n \leq S < n+1$. In doing so, the proposed $[S]$-periodic model in which

$$\theta_{0t} = \theta_{0,t+[S]}, \quad t \in \mathbb{Z},$$

will not reflect the actual S-periodicity of the series and will induce a kind of "shift" between the $[S]$-seasonal series it generates and the actual S-seasonal series to which it is devoted to represent. Thus, a $[S]$-periodic model will be inadequate. At first glance, it seems not possible to envisage a periodic model with non-integer S since the period actually represents the number of model parameters and hence it cannot take a priori non-integer values. However, we can exploit a variation of the trigonometric approximation (4.8) dealing with non-integer S. Indeed, in the framework of the PCH model (2.1) consider the following generalization of (4.8) given by

$$\begin{cases} \theta_{0v} = (\theta_{0v,1}, \ldots \theta_{0v,m_v})' \\ \theta_{0v,j} = \theta^*_{0j} + \theta^*_{0j} \cos\left(\frac{2\pi v}{S} - \theta^*_{0j}\right), \ \ 1 \leq j \leq m_v \end{cases}, \ 1 \leq v \leq [S], \tag{4.11}$$

where S is now assumed a positive real number. In particular, for the $PAP\text{-}GARCH\,(1,1)$ model (4.4), the corresponding "augmented" specification of (4.9) with non-integer period is

$$\begin{cases} \omega_{0v} = \omega^*_{01} + \omega^*_{02} \cos\left(\frac{2\pi v}{S} - \omega^*_{03}\right) \\ \alpha_{0v+} = \alpha^*_{01+} + \alpha^*_{02+} \cos\left(\frac{2\pi v}{S} - \alpha^*_{03+}\right) \\ \alpha_{0v-} = \alpha^*_{01-} + \alpha^*_{02-} \cos\left(\frac{2\pi v}{S} - \alpha^*_{03-}\right) \\ \beta_{0v} = \beta^*_{01} + \beta^*_{02} \cos\left(\frac{2\pi v}{S} - \beta^*_{03}\right) \end{cases}, \ 1 \leq v \leq [S]. \tag{4.12}$$

A similar approach has been introduced by De Livera et *al* (2011) in the case of seasonal (but non-periodic) exponential smoothing $TBATS$ models (The acronym $TBATS$ refers to: Trigonometric Box-Cox transform, $ARMA$ errors, Trend, and Seasonal components). But in contrast with seasonal models, the period S in a periodic model is generally interpreted as the number of model parameters, making the adaptation of periodic models to non-integer periods more challenging. Note that if S is non-integer then model (4.11) (and hence model

(4.12)) is not $[S]$-periodic over v, since for example

$$\begin{aligned}\omega_{0,v+[S]} &= \omega_{01}^* + \omega_{02}^* \cos\left(\tfrac{2\pi v}{S} - \omega_{03}^* + 2\pi \tfrac{[S]}{S}\right) \\ &\neq \omega_{0v},\end{aligned}$$

and so on. So specification (4.11) avoids inducing the aforementioned shift in modeling like model (2.1) and then it would be more suitable in representing non-integer periodicity.

Now with specification (4.11), model (2.1) may be reparametrized as in (4.10) to deal with non-integer periods, giving the following variation of (4.10) for a positive real period $S > 0$,

$$\begin{cases} \epsilon_{nS+v} = \sigma_{nS+v}\eta_{nS+v}, \\ \sigma_{nS+v} = \varphi_v^*\left(\epsilon_{nS+v-1}, \epsilon_{nS+v-2}, ...; \theta_{0v}^*\right) := \sigma_{nS+v}^*\left(\theta_0^*\right), \end{cases}, 1 \leq v \leq [S], \tag{4.13}$$

where the function φ_v^* and θ_0^* are defined as in (4.10). The $GQMLE$ of (4.13) is defined in the same way as (4.10) and its properties are established under similar assumptions.

Note finally that other trigonometric, or more generally other Fourier approximations, can be considered in place of (4.11) (see e.g. Tesfaye et al, 2011; Franses and Paap, 2011).

5. Conclusion

A few broad conclusions may be drawn. Firstly, the class of periodic conditional volatility PCH models considered here is quite general and covers most of the standard $ARCH$ formulations. Moreover, periodicity is expressed via the volatility coefficients as well as the innovation making the model more flexible in representing periodic series with different shapes of distribution along seasons. Secondly, the $GQMLE$ proposed for the PCH model is based on S instrumental functions and is then in accordance with the periodicity of the independent innovation, giving the possibility to the $GQMLE$ to reduce to the MLE, and then to be asymptotically efficient, when the instrumental functions coincide with the distributions of the innovation. Thirdly, the $GQMLE$ is consistent and asymptotically Gaussian under mild assumptions as in the non-periodic case. However, its asymptotic variance is in a "*sandwich*" form, which is unusual in CH models, and is reduced only for some special cases.

One useful application of the proposed $GQMLE$ is the prediction of powers of the PCH model in a one-step parametric framework, where S different powers along seasons are to be considered. In addition, a potential application of the $GQMLE$ of PCH models is the calculation of the corresponding Values at Risk (VaR's) for which the Gaussian $QMLE$ is generally inconsistent in the presence of heavy tailed distributions (see El Ghourabi et *al*, 2016; Francq and Zakoïan, 2015-2016 in the CH case). Another useful property of the $GQMLE$, is that it can be easily adapted to PCH models with complex periodic patterns such as high frequency periodicity and non-integer periodicity. Note finally that this work has been mainly considered in a theoretical perspective and applications of the proposed models and methods to real data are appealing.

6. Proofs

Proofs of Theorems 3.1 and Theorem 3.2 follow from similar arguments used in establishing asymptotics of the $GQMLE$ for non-periodic CH models (Berkes and Horvàth, 2004; Francq and Zakoïan, 2004-2013-2015, El Ghourabi et *al*, 2016).

6.1. Proof of Theorem 3.1

Result (3.4) follows while establishing the following three lemmas.

Lemma 6.1 *Under* ***A2***, ***A3*** *and* ***A5*** *we have*

$$\sup_{\theta\in\Theta}\left|\widetilde{L}_{NS,\underline{h}}(\theta)-L_{NS,\underline{h}}(\theta)\right| \underset{N\to\infty}{\overset{a.s.}{\to}} 0,$$

where $L_{T,\underline{h}}(\theta)=\frac{1}{T}\sum\limits_{n=0}^{N-1}\sum\limits_{v=1}^{S}g_v\left(\epsilon_{nS+v},\sigma_{nS+v}(\theta)\right)$.

Proof In view of **A2-A3** and (3.1)-(3.2), a Taylor expansion gives $a.s.$

$$\begin{aligned}\sup_{\theta\in\Theta}\left|\widetilde{L}_{NS,\underline{h}}(\theta)-L_{NS,\underline{h}}(\theta)\right| &\leq \frac{1}{T}\sum_{n=0}^{N-1}\sum_{v=1}^{S}\sup_{\theta\in\Theta}\left|g_{v1}\left(\epsilon_{nS+v},\sigma^*_{nS+v}(\theta)\right)\right|\left|\widetilde{\sigma}_{nS+v}(\theta)-\sigma_{nS+v}(\theta)\right| \\ &\leq \frac{1}{T}\sum_{n=0}^{N-1}\sum_{v=1}^{S}b_{nS+v}(\theta)\sup_{\theta\in\Theta}\left|\frac{\epsilon_t}{\sigma^{*2}_{nS+v}(\theta)}\frac{h'_{v1}}{h_{v1}}\left(\frac{\epsilon_{nS+v}}{\sigma^*_{nS+v}(\theta)}\right)\right|+\frac{1}{TC}\sum_{t=1}^{T}b_t(\theta)\end{aligned}$$

$$\leq \frac{1}{T}\sum_{n=0}^{N-1}\sum_{v=1}^{S}b_{nS+v}(\theta)\left|\epsilon_{nS+v}\right|^{\delta_v}\sup_{\theta\in\Theta}\left|\frac{1}{\sigma^*_{nS+v}(\theta)}\right|^{1+\delta_v}+\frac{1}{TC}\sum_{t=1}^{T}b_t(\theta), \tag{6.1}$$

where $\sigma^*_{nS+v}(\theta)$ is between $\widetilde{\sigma}_{nS+v}(\theta)$ and $\sigma_{nS+v}(\theta)$ and

$$b_t(\theta)=\sup_{\theta\in\Theta}\left|\widetilde{\sigma}_t(\theta)-\sigma_t(\theta)\right|.$$

Now from **A5** and the Markov inequality it follows that for all $1\leq v\leq S$ and $\xi_v>0$

$$\sum_{n=0}^{\infty}P\left(b_{nS+v}(\theta)\left|\epsilon_{nS+v}\right|^{\delta_v}>\xi_v\right)\leq\sum_{n=0}^{\infty}\frac{CE(|\epsilon|^{\tau_v})\rho^{\frac{\tau_v}{\delta_v}n}}{\xi_v^{\frac{\tau_v}{\delta_v}}},$$

so by the Borel-Cantelli lemma

$$b_{nS+v}(\theta)\left|\epsilon_{nS+v}\right|^{\delta_v}\underset{n\to\infty}{\overset{a.s.}{\to}}0\text{ for all }1\leq v\leq S.$$

Thus, Lemma 6.1 follows from (6.1) and the Césaro lemma. □

Lemma 6.2 *Under **A1**, **A2** and **A4***

$$E\left(\sum_{v=1}^{S}g_v\left(\epsilon_v,\sigma_v(\theta)\right)\right)<E\left(\sum_{v=1}^{S}g_v\left(\epsilon_v,\sigma_v(\theta_0)\right)\right)\text{ for all }\theta\neq\theta_0. \tag{6.2}$$

Proof Using **A1**, the fact that

$$g_v\left(\epsilon_{nS+v},\sigma_{nS+v}(\theta)\right)=g_v\left(\eta_{nS+v},\frac{\sigma_{nS+v}(\theta)}{\sigma_{nS+v}(\theta_0)}\right)-\log\left(\sigma_{nS+v}(\theta_0)\right),$$

and **A4** we have

$$\begin{aligned}&E\left(\sum_{v=1}^{S}g_v\left(\epsilon_{nS+v},\sigma_{nS+v}(\theta)\right)-\sum_{v=1}^{S}g_v\left(\epsilon_{nS+v},\sigma_{nS+v}(\theta_0)\right)\right)= \\ &\qquad\sum_{v=1}^{S}E\left(g_v\left(\eta_v,\frac{\sigma_v(\theta)}{\sigma_v(\theta_0)}\right)-g_v\left(\eta_v,1\right)\right)<0,\end{aligned}$$

with equality if and only if $\sigma_{nS+v}(\theta) = \sigma_{nS+v}(\theta_0)$ and by **A2** if and only if $\theta = \theta_0$. $\square$

Lemma 6.3 *Under **A1**-**A5**, for all $\theta \neq \theta_0$ there is a neighborhood $V(\theta)$ such that*

$$\limsup_{N\to\infty} \sup_{\theta^*\in V(\theta)} \widetilde{L}_{NS,\underline{h}}(\theta^*) < \limsup_{N\to\infty} \widetilde{L}_{NS,\underline{h}}(\theta_0) \quad a.s. \tag{6.3}$$

Proof For any $\theta \in \Theta$ and any positive integer k, let $V_k(\theta)$ be the open ball of center θ and radius $1/k$. Using Lemma 6.1 we have

$$\begin{aligned}
&\limsup_{N\to\infty} \sup_{\theta^*\in V_k(\theta)\cap\Theta} \widetilde{L}_{NS,\underline{h}}(\theta^*) \leq \\
&\limsup_{N\to\infty} \sup_{\theta^*\in V_k(\theta)\cap\Theta} L_{NS,\underline{h}}(\theta^*) - \limsup_{N\to\infty} \sup_{\theta\in\Theta} \left| L_{NS,\underline{h}}(\theta) - \widetilde{L}_{NS,\underline{h}}(\theta) \right| \\
\leq \quad &\limsup_{N\to\infty} \left(S^{-1} \sum_{v=1}^{S} N^{-1} \sum_{n=0}^{N-1} \sup_{\theta\in V_k(\theta)\cap\Theta} g_v\left(\epsilon_{nS+v}, \sigma_{nS+v}(\theta^*)\right) \right), \quad a.s.
\end{aligned}$$

As the instrumental functions $(h_1, ..., h_S)$ are by **A3** integrable and differentiable, they are bounded. Therefore, by **A2**

$$S^{-1} \sum_{v=1}^{S} E\left(\sup_{\theta^*\in V_k(\theta)\cap\Theta} g_v\left(\epsilon_{nS+v}, \sigma_{nS+v}(\theta^*)\right) \right) < S^{-1} \sum_{v=1}^{S} \left(\frac{1}{\underline{\omega}_v} + C \right) < \infty. \tag{6.4}$$

Now since by **A1** $\{\epsilon_t, t \in \mathbb{Z}\}$ is strictly periodically stationary and periodically ergodic, it follows that for all $1 \leq v \leq S$, the sub-process $\{\epsilon_{nS+v}, n \in \mathbb{Z}\}$ is strictly stationary and ergodic. Hence, as

$$\sup_{\theta^*\in V_k(\theta)\cap\Theta} \left(g_v\left(\epsilon_{nS+v}, \sigma_{nS+v}(\theta^*)\right) \right),$$

is a measurable function of the terms of $\{\epsilon_{nS+v}, n \in \mathbb{Z}\}$, it follows that the sequence

$$\left\{ \sup_{\theta^*\in V_k(\theta)\cap\Theta} g_v\left(\epsilon_{nS+v}, \sigma_{nS+v}(\theta^*)\right), n \in \mathbb{Z} \right\}, \tag{6.5}$$

is strictly stationary and ergodic with

$$E\left(\sup_{\theta^*\in V_k(\theta)\cap\Theta} g_v\left(\epsilon_{nS+v}, \sigma_{nS+v}(\theta^*)\right) \right) \in [-\infty, +\infty).$$

For the process given by (6.5), applying the ergodic theorem for strictly stationary and ergodic sequences with a possibly infinite mean (cf. Billingsley 1995, p. 284, 495) and using

$E_{\theta_0}\left(g_v^-\left(\epsilon_{nS+v},\sigma_{nS+v}\left(\theta\right)\right)\right)<\infty$ we get

$$\limsup_{N\to\infty}\sup_{\theta^*\in V_k(\theta)\cap\Theta}\widetilde{L}_{NS,\underline{h}}\left(\theta^*\right)\leq\tfrac{1}{S}\sum_{v=1}^{S}E\left(\sup_{\theta^*\in V_k(\theta)\cap\Theta}g_v\left(\epsilon_{nS+v},\sigma_{nS+v}\left(\theta^*\right)\right)\right).$$

By the Beppo-Levi theorem (e.g. Billingsley, 1995 p. 219) and using (6.4), the sequence

$$\left(\tfrac{1}{S}\sum_{v=1}^{S}E\left(\sup_{\theta^*\in V_k(\theta)\cap\Theta}g_v\left(\epsilon_{nS+v},\sigma_{nS+v}\left(\theta^*\right)\right)\right)\right)_{k\in\mathbb{N}^*},$$

converges while decreasing to

$$\tfrac{1}{S}\sum_{v=1}^{S}E_{\theta_0}\left(g_v\left(\epsilon_{nS+v},\sigma_{nS+v}\left(\theta\right)\right)\right),$$

as $k\to\infty$. Thus, (6.3) follows from (6.2). □

Proof of Theorem 3.1

To complete the proof of the theorem, we use a standard compactness argument and Lemmas 6.1-6.3. Note that we have shown from Lemma 6.1-6.3 that for any neighborhood $V\left(\theta_0\right)$ of θ_0,

$$\limsup_{N\to\infty}\sup_{\theta^*\in V(\theta_0)}\widetilde{L}_{NS,\underline{h}}\left(\theta^*\right)\leq\lim_{N\to\infty}\widetilde{L}_{NS,\underline{h}}\left(\theta_0\right)=\lim_{N\to\infty}L_{NS,\underline{h}}\left(\theta_0\right)=\tfrac{1}{S}\sum_{v=1}^{S}E_{\theta_0}\left(g_v\left(\epsilon_v,\sigma_v\left(\theta_0\right)\right)\right).\tag{6.6}$$

The compact Θ is recovered by the union of any neighborhood $V\left(\theta_0\right)$ of θ_0 and a set of neighborhoods $V\left(\theta\right)$, $\theta\in\Theta\backslash V\left(\theta_0\right)$, where $V\left(\theta\right)$ fulfills Lemma 6.3. Therefore, there exists a finite sub-covering of Θ by $V\left(\theta_0\right),V\left(\theta_1\right),...,V\left(\theta_k\right)$ such that

$$\sup_{\theta\in\Theta}\widetilde{L}_{NS,\underline{h}}\left(\theta\right)=\min_{i\in\{1,2,...,k\}}\sup_{\theta\in\Theta\cap V(\theta_i)}\left(\widetilde{L}_{NS,\underline{h}}\left(\theta\right)\right).$$

From Lemma 6.3 and (6.6), the latter equality shows that $\widehat{\theta}_{NS,\underline{h}}\in V\left(\theta_0\right)$ for N sufficiently large, which completes the proof of the theorem. □

6.2. Proof of Theorem 3.2

From **A6** and the strong consistency of $\widehat{\theta}_{NS,\underline{h}}$, a Taylor expansion gives

$$\begin{aligned}
\sqrt{N}\tfrac{\partial}{\partial\theta}\widetilde{L}_{T,\underline{h}}\left(\widehat{\theta}_{NS,\underline{h}}\right) &= 0 \\
&= \sqrt{N}\tfrac{\partial}{\partial\theta}L_{T,\underline{h}}\left(\widehat{\theta}_{NS,\underline{h}}\right)+\sqrt{N}\tfrac{\partial}{\partial\theta}\widetilde{L}_{T,\underline{h}}\left(\widehat{\theta}_{NS,\underline{h}}\right)-\sqrt{N}\tfrac{\partial}{\partial\theta}L_{T,\underline{h}}\left(\widehat{\theta}_{NS,\underline{h}}\right) \\
&= \sqrt{N}\tfrac{\partial}{\partial\theta}L_{T,\underline{h}}\left(\theta_0\right)+\tfrac{\partial^2}{\partial\theta\partial\theta'}L_{T,\underline{h}}\left(\theta^*\right)\sqrt{N}\left(\widehat{\theta}_{NS,\underline{h}}-\theta_0\right) \\
&\quad +\sqrt{N}\left(\tfrac{\partial}{\partial\theta}\widetilde{L}_{T,\underline{h}}\left(\widehat{\theta}_{NS,\underline{h}}\right)-\tfrac{\partial}{\partial\theta}L_{T,\underline{h}}\left(\widehat{\theta}_{NS,\underline{h}}\right)\right),
\end{aligned}$$

where θ^* is between $\widehat{\theta}_{NS,\underline{h}}$ and θ_0. Therefore, the asymptotic normality result (3.6) follows while the three following lemmas are proved.

Lemma 6.4 *Under* ***A1-A5****,* ***A7*** *and* ***A9-A10****,*

$$\sup_{\theta\in V(\theta_0)}\sqrt{N}\left\|\tfrac{\partial}{\partial\theta}L_{T,\underline{h}}\left(\theta\right)-\tfrac{\partial}{\partial\theta}\widetilde{L}_{T,\underline{h}}\left(\theta\right)\right\|\underset{N\to\infty}{\overset{p}{\to}}0.$$

for some neighborhood $V\left(\theta_0\right)$ of θ_0.

Proof We have

$$\begin{aligned}
\sup_{\theta\in V(\theta_0)}\sqrt{N}\left\|\tfrac{\partial}{\partial\theta}L_{T,\underline{h}}\left(\theta\right)-\tfrac{\partial}{\partial\theta}\widetilde{L}_{T,\underline{h}}\left(\theta\right)\right\| &= \sup_{\theta\in V(\theta_0)}\tfrac{1}{S\sqrt{N}}\left\|\sum_{v=1}^{S}\sum_{n=0}^{N-1}\left[g_{v1}\left(\varepsilon_{nS+v},\sigma_{nS+v}\left(\theta\right)\right)\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta}\right.\right. \\
&\quad \left.\left.-g_{v1}\left(\varepsilon_{nS+v},\widetilde{\sigma}_{nS+v}\left(\theta\right)\right)\tfrac{\partial\widetilde{\sigma}_{nS+v}(\theta)}{\partial\theta}\right]\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\theta\in V(\theta_0)}\tfrac{1}{S\sqrt{N}}\sum_{v=1}^{S}\sum_{n=0}^{N-1}\left|g_{v1}\left(\varepsilon_{nS+v},\sigma_{nS+v}\left(\theta\right)\right)-g_{v1}\left(\varepsilon_{nS+v},\widetilde{\sigma}_{nS+v}\left(\theta\right)\right)\right|\left\|\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta}\right\| \\
&\quad +\sup_{\theta\in V(\theta_0)}\tfrac{1}{S}\sum_{v=1}^{S}\tfrac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\left|g_{v1}\left(\varepsilon_{nS+v},\widetilde{\sigma}_{nS+v}\left(\theta\right)\right)\right|\left\|\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta}-\tfrac{\partial\widetilde{\sigma}_{nS+v}(\theta)}{\partial\theta}\right\|. \qquad (6.7)
\end{aligned}$$

From **A3** and **A9**, the second term in the right hand side of (6.7) is bounded by

$$\tfrac{C}{S\sqrt{N}}\sum_{v=1}^{S}\sum_{n=0}^{N-1}\rho^n\tfrac{1}{\underline{\omega}_v}\left(1+K_v\left|\tfrac{\varepsilon_{nS+v}}{\underline{\omega}_v}\right|^{\delta_v}\right),$$

which is of order $O\left(T^{-1/2}\right)$ *a.s.* For the first term in (6.7), using a Taylor expansion, assumptions **A3**, **A5**, **A7**, **A10** and the Cauchy-Shwartz inequality, it follows that this term is bounded by

$$
\begin{aligned}
&\tfrac{C}{S\sqrt{N}}\sum_{v=1}^{S}\sum_{n=0}^{N-1}\rho^{n}\tfrac{1}{\underline{\omega}_v}\left|g_{v2}\left(\varepsilon_{nS+v},\sigma^{*}_{nS+v}\left(\theta\right)\right)\right|\left\|\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta}\right\| \\
\leq\ &\tfrac{C}{S\sqrt{N}}\sum_{v=1}^{S}\sum_{n=0}^{N-1}\rho^{n}\tfrac{1}{\underline{\omega}_v}\left|1+3K_v\left(1+\left|\tfrac{\varepsilon_{nS+v}}{\underline{\omega}_v}\right|^{\delta_v}\right)\right|\sup_{\theta\in V(\theta_0)}\left\|\tfrac{1}{\sigma_{nS+v}(\theta)}\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta}\right\| \\
=\ &o\left(1\right)\ a.s.,
\end{aligned}
$$

where $\sigma^{*}_{nS+v}\left(\theta\right)$ is between $\widetilde{\sigma}_{nS+v}\left(\theta\right)$ and $\sigma_{nS+v}\left(\theta\right)$. This completes the proof of the lemma. □

Lemma 6.5 *Under **A1-A10**, for any θ^{*} between $\widehat{\theta}_{T,\underline{h}}$ and θ_0,*

$$\tfrac{\partial^2 L_{T,\underline{h}}(\theta^{*})}{\partial\theta\partial\theta'}\ \underset{N\to\infty}{\overset{p}{\to}}\ \tfrac{1}{4}A_{\underline{h},\underline{f}}\left(\theta_0\right).$$

Proof From **A3** and **A7** we have

$$
\begin{aligned}
&\left\|\tfrac{\partial^2 L_{T,\underline{h}}(\theta)}{\partial\theta\partial\theta'}\right\| = \\
&\left\|\tfrac{1}{NS}\sum_{n=0}^{N-1}\sum_{v=1}^{S}g_{v2}\left(\varepsilon_{nS+v},\sigma_{nS+v}\left(\theta\right)\right)\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta}\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta'}+g_{v1}\left(\varepsilon_{nS+v},\sigma_{nS+v}\left(\theta\right)\right)\tfrac{\partial^2\sigma_{nS+v}(\theta)}{\partial\theta\partial\theta'}\right\|\leq \\
&\tfrac{C}{NS}\sum_{n=0}^{N-1}\sum_{v=1}^{S}\left(1+\left|\tfrac{\sigma_{nS+v}(\theta_0)\eta_{nS+v}}{\sigma_{nS+v}(\theta)}\right|^{\delta_v}\right)\left(\left\|\tfrac{1}{\sigma_{nS+v}(\theta)}\tfrac{\partial^2\sigma_{nS+v}(\theta)}{\partial\theta\partial\theta'}\right\|+\left\|\tfrac{1}{\sigma^2_{nS+v}(\theta)}\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta}\tfrac{\partial\sigma_{nS+v}(\theta)}{\partial\theta'}\right\|\right).
\end{aligned}
$$

By the Hölder inequality, **A7** and **A10** it follows that

$$E\left(\sup_{\theta\in V(\theta_0)}\left\|\tfrac{\partial^2 L_{T,\underline{h}}(\theta)}{\partial\theta\partial\theta'}\right\|\right)<\infty,$$

so the ergodic theorem implies that

$$\lim_{N\to\infty}\sup_{\theta\in V(\theta_0)}\left\|\tfrac{\partial^2 L_{T,\underline{h}}(\theta)}{\partial\theta\partial\theta'}-\tfrac{\partial^2 L_{T,\underline{h}}(\theta_0)}{\partial\theta\partial\theta'}\right\|\leq E\left(\sup_{\theta\in V(\theta_0)}\left\|\sum_{v=1}^{S}\left(\tfrac{\partial^2 g_{v2}(\varepsilon_v,\sigma_v(\theta))}{\partial\theta\partial\theta'}-\tfrac{\partial^2 g_{v2}(\varepsilon_v,\sigma_v(\theta_0))}{\partial\theta\partial\theta'}\right)\right\|\right),\ a.s.$$

From the dominated convergence theorem, the latter expectation tends to zero when $V\left(\theta_0\right)$ tends to the singleton $\{\theta_0\}$. Now since $\widehat{\theta}_{T,\underline{h}}$ is consistent then

$$\left\|\tfrac{\partial^2 L_{T,\underline{h}}(\theta^{*})}{\partial\theta\partial\theta'}-\tfrac{\partial^2 L_{T,\underline{h}}(\theta_0)}{\partial\theta\partial\theta'}\right\|\ \underset{N\to\infty}{\overset{a.s.}{\to}}\ 0.$$

On the other hand since by **A3**

$$g_{v1}\left(x,\varsigma\right)=\frac{\partial g_v(x,\varsigma)}{\partial\varsigma}=-\frac{1}{\varsigma}-\frac{h_v'\left(\frac{x}{\varsigma}\right)}{h_v\left(\frac{x}{\varsigma}\right)}\frac{x}{\varsigma^2},\quad 1\leq v\leq S,$$

exists for all $\varsigma>0$ and $E\left(\sup_{\varsigma\in V(1)}\left|g_v\left(\eta_v,\varsigma\right)\right|\right)<\infty$, then **A4** and the dominated convergence theorem entail the following moment conditions

$$E\left(\frac{h_v'(\eta_v)}{h_v(\eta_v)}\eta_v\right)=-1,\ \text{for all } 1\leq v\leq S,$$

which in turn imply that

$$E\left(g_{v1}\left(\varepsilon_v,\sigma_v\left(\theta_0\right)\right)\frac{\partial^2\sigma_v(\theta_0)}{\partial\theta\partial\theta'}\right)=0.$$

Note that the following equality

$$g_{v2}\left(x,\varsigma\right)=\frac{\partial g_{v1}(x,\varsigma)}{\partial\varsigma}=\frac{1}{\varsigma^2}\left[1+\frac{x}{\varsigma}\left(2\frac{h_v'\left(\frac{x}{\varsigma}\right)}{h_v\left(\frac{x}{\varsigma}\right)}+\frac{x}{\varsigma}\left(\frac{h_v'\left(\frac{x}{\varsigma}\right)}{h_v\left(\frac{x}{\varsigma}\right)}\right)'\right)\right],$$

gives

$$g_{v2}\left(\varepsilon_{nS+v},\sigma_{nS+v}\left(\theta_0\right)\right)=g_{v2}\left(\eta_{nS+v},1\right)\frac{\partial^2\sigma_v(\theta_0)}{\partial\theta\partial\theta'}.$$

Therefore, by the periodic ergodic theorem we finally get

$$\frac{\partial^2 L_{T,\underline{h}}(\theta_0)}{\partial\theta\partial\theta'}\underset{N\to\infty}{\overset{a.s.}{\longrightarrow}}\frac{1}{4}A_{\underline{h},\underline{f}}\left(\theta_0\right),$$

which proves the lemma. □

Lemma 6.6 *Under* ***A1-A10***

$$\sqrt{N}\frac{\partial L_{T,\underline{h}}(\theta_0)}{\partial\theta}\underset{N\to\infty}{\overset{\mathcal{L}}{\longrightarrow}}N\left(0,\frac{1}{4}B_{\underline{h},\underline{f}}\left(\theta_0\right)\right),$$

where $B_{\underline{h},\underline{f}}\left(\theta_0\right)$ *given by* $(3.5b)$ *is invertible under* ***A8***.

Proof Note that

$$\begin{aligned}\sqrt{N}\frac{\partial L_{T,\underline{h}}(\theta_0)}{\partial\theta} &= \frac{1}{S\sqrt{N}}\sum_{v=1}^{S}\sum_{n=0}^{N-1}\frac{\partial g_v(\epsilon_{nS+v},\sigma_{nS+v}(\theta_0))}{\partial\theta}\\ &= \frac{1}{S}\sum_{v=1}^{S}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}g_{v1}\left(\eta_{nS+v},1\right)\frac{1}{2\sigma_{nS+v}^2(\theta_0)}\frac{\partial\sigma_{nS+v}^2(\theta_0)}{\partial\theta}.\end{aligned}$$

Since by the periodic ergodic theorem we have

$$\begin{aligned}\sum_{v=1}^{S}\sum_{n=0}^{N-1}\left(\frac{1}{S\sqrt{N}}\right)^{2}\frac{\partial g_{v}(\epsilon_{nS+v},\sigma_{nS+v}(\theta_{0}))}{\partial\theta}\frac{\partial g_{v}(\epsilon_{nS+v},\sigma_{nS+v}(\theta_{0}))}{\partial\theta'} &= \sum_{v=1}^{S}\frac{1}{N}\sum_{n=0}^{N-1}\frac{g_{v1}^{2}\left(\eta_{nS+v},1\right)}{4S^{2}\sigma_{nS+v}^{2}(\theta_{0})}\frac{\partial\sigma_{nS+v}^{2}(\theta_{0})}{\partial\theta}\frac{\partial\sigma_{nS+v}^{2}(\theta_{0})}{\partial\theta'}\\ &\underset{N\to\infty}{\overset{a.s.}{\to}}\frac{1}{4}B_{\underline{h},\underline{f}}\left(\theta_{0}\right),\end{aligned}$$

then by the martingale central limit theorem (Billingsley, 1961) we get (6.8).

Now we prove that $B_{\underline{h},\underline{f}}\left(\theta_{0}\right)$ is nonsingular under **A8**. If $B_{\underline{h},\underline{f}}\left(\theta_{0}\right)$ is singular, then there exists a non-null vector $\mathbf{x}\in\mathbb{R}^{m}$ such that $\mathbf{x}'B_{\underline{h},\underline{f}}\left(\theta_{0}\right)\mathbf{x}=0$. Note that

$$\begin{aligned}\mathbf{x}'B_{\underline{h},\underline{f}}\left(\theta_{0}\right)\mathbf{x} &= \frac{1}{S^{2}}\sum_{v=1}^{S}E\left(g_{v1}\left(\eta_{v},1\right)^{2}\right)E\left(\mathbf{x}'\frac{1}{\sigma_{v}^{4}(\theta_{0})}\frac{\partial\sigma_{v}^{2}(\theta_{0})}{\partial\theta}\frac{\partial\sigma_{v}^{2}(\theta_{0})}{\partial\theta'}\mathbf{x}\right)\\ &= \frac{1}{S^{2}}\sum_{v=1}^{S}E\left(g_{v1}\left(\eta_{v},1\right)^{2}\right)E\left(\frac{1}{\sigma_{v}^{4}(\theta_{0})}\left(\mathbf{x}'\frac{\partial\sigma_{v}^{2}(\theta_{0})}{\partial\theta}\right)^{2}\right).\end{aligned}$$

Since by **A8**, $E\left(g_{v1}\left(\eta_{v},1\right)^{2}\right)>0$ for any $v\in\{1,...,S\}$, it follows that $\mathbf{x}'B_{\underline{h},\underline{f}}\left(\theta_{0}\right)\mathbf{x}=0$ if and only if $E\left(\frac{1}{\sigma_{v}^{4}(\theta_{0})}\left(\mathbf{x}'\frac{\partial\sigma_{v}^{2}(\theta_{0})}{\partial\theta}\right)^{2}\right)=0$, $\forall v\in\{1,...,S\}$, which holds if and only if

$$\frac{1}{\sigma_{v}^{2}(\theta_{0})}\left(\mathbf{x}'\frac{\partial\sigma_{v}^{2}(\theta_{0})}{\partial\theta}\right)^{2}=0\quad a.s.\ \forall v\in\{1,...,S\}\Leftrightarrow\mathbf{x}'\frac{\partial\sigma_{v}^{2}(\theta_{0})}{\partial\theta}=0\quad a.s.\ \forall v\in\{1,...,S\}.$$

By the last part of **A8** this implies that $\mathbf{x}=0$**,** which contradicts the fact that $\mathbf{x}\neq0$**.** Hence, $B_{\underline{h},\underline{f}}\left(\theta_{0}\right)$ is nonsingular. □

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References

[1] Aknouche, A. (2016). Periodic autoregressive stochastic volatility. *Statistical Inference for Stochastic Processes*, forthcoming.

[2] Aknouche, A. and Al-Eid, E. (2012). Asymptotic inference of unstable periodic $ARCH$ processes. *Statistical Inference for Stochastic Processes*, **15**, 61-79.

[3] Aknouche, A. and Bibi, A. (2009). Quasi-maximum likelihood estimation of periodic $GARCH$ and periodic $ARMA$-$GARCH$ processes. *Journal of Time Series Analysis*, **30**, 19-46.

[4] Aknouche, A. and Touche, N. (2015). Weighted least squares-based inference for stable and unstable threshold power $ARCH$ processes. *Statistics & Probability Letters*, **97**, 108-115.

[5] Ambach, D. and Croonenbroeck, C. (2015). Obtaining superior wind power predictions from a periodic and heteroscedastic wind power prediction tool. *In Stochastic Models, Statistics and Their Applications, edt*, 225-232.

[6] Ambach, D. and Schmid, W. (2015). Periodic and long range dependent models for high frequency wind speed data. *Energy*, **82**, 277-293.

[7] Andersen, T. and Bollerslev, T. (1997). Intraday periodicity and volatility persistence in financial markets. *Journal of Empirical Finance*, **4**, 115-158.

[8] Anderson, P.L., Tesfaye, Y.G. and Meerschaert, M.M. (2007). Fourier-$PARMA$ models and their application to river flows. *Journal of Hydrologic Engineering*, **12**, 462-472.

[9] Balaban, E., Bayar, A. and Kan, O.B. (2001). Stock returns, seasonality and asymmetric conditional volatility in world equity markets. *Applied Economics Letters*, **8**, 263-268.

[10] Beller, K. and Nofsinger, J.R. (1998). On stock return seasonality and conditional heteroskedasticity. *The Journal of Financial Research*, **21**, 229-246.

[11] Berument, H., Coskunb, N.M. and Sahinc, A. (2007). Day of the week effect on foreign exchange market volatility: Evidence from Turkey. Research in International Business and Finance, 21, 87-97.

[12] Berkes, I. and Horvàth, L. (2004). The efficiency of the estimators of the parameters in $GARCH$ processes. *Annals of Statistics*, **32**, 633-655.

[13] Berkes, I., Horvàth, L. and Kokoskza, P. (2003). *GARCH* processes: structure and estimation. *Bernoulli*, **9**, 201-227.

[14] Bidarkota, P.V., Dupoyet, B.V. and McCulloch, J.H. (2009). Asset pricing with incomplete information and fat tails. *Journal of Economic Dynamics and Control*, **33**, 1314-1331.

[15] Billingsley, P. (1961). The Lindeberg-Levy theorem for martingales. *Proceedings of the American Mathematical Society*, **12**, 788-792.

[16] Billingsley, P. (1995). *Probability and measure*. 3rd edition, Wiley, New York.

[17] Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics,* **31**, 307-327.

[18] Bollerslev, T. and Ghysels, E. (1996). Periodic autoregressive conditional heteroskedasticity. *Journal of Business & Economic Statistics*, **14**, 139-152.

[19] Bollerslev, T., Cai, J. and Song, F.M. (2000). Intraday periodicity, long memory volatility, and macroeconomic announcement effects in the US Treasury bond market. *Journal of Empirical Finance*, **7**, 37-55.

[20] Bougerol, P. and Picard, N. (1992). Stationarity of *GARCH* processes and some nonnegative time series. *Journal of Econometrics*, **52**, 115-127.

[21] Boyles, R.A. and Gardner, W.A. (1983). Cycloergodic properties of discrete-parameter nonstationary stochastic processes. *IEEE, Trans. Infor. Theory.*, **29**, 105-114.

[22] Boynton, W., Oppenheimer, H.R. and Reid, S.F. (2009). Japanese day-of-the-week return patterns: New results. *Global Finance Journal*, **20**, 1-12.

[23] Brandt, A. (1986). The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients. *Advances in Applied Probability*, **18**, 211-220.

[24] Charles, A. (2010). The day-of-the-week effects on the volatility: The role of the asymmetry. *European Journal of Operational Research*, **202**, 143-152.

[25] De Livera, A.M., Hyndman, R.J. and Snyder, R.D. (2011). Forecasting time series with complex seasonal patterns using exponential smoothing. *Journal of the American Statistical Association*, **106**, 1513-1527.

[26] Ding, Z., Granger, C.W.J. and Engle, R.F. (1993). A long memory property of stock market returns and a new model. *Journal of Empirical Finance*, **1**, 83-106.

[27] El Ghourabi, M., Francq, C. and Telmoudi, F. (2016). Consistent estimation of the Value-at-Risk when the error distribution of the volatility model is misspecified. *Journal of Time Series Analysis*, **37**, 46-76.

[28] Engle, R.F. (1982). Autoregressive Conditional Heteroskedasticity with estimates of variance of U.K. Inflation. *Econometrica*, **50**, 987-1008.

[29] Fan, J., Qi, L. and Xiu, D. (2014). Quasi maximum likelihood estimation of $GARCH$ models with heavy-tailed likelihoods. *Journal of Business and Economic Statistics*, **32**, 178-191.

[30] Francq, C. and Zakoïan, J.M. (2004). Maximum likelihood estimation of pure $GARCH$ and $ARMA$-$GARCH$ processes. *Bernoulli*, **10**, 605-637.

[31] Francq, C. and Zakoïan, J.M. (2010). *$GARCH$ models: Structure, statistical inference and financial applications*. John Wiley.

[32] Francq, C. and Zakoïan, J.M. (2013). Optimal predictions of powers of conditionally heteroskedastic processes. *Journal of the Royal Statistical Society*, **B75**, 345-367.

[33] Francq, C. and Zakoïan, J-M. (2015). Risk-parameter estimation in volatility models. *Journal of Econometrics*, **184**, 158-173.

[34] Francq, C. and Zakoïan, J-M. (2016). Looking for efficient QML estimation of conditional VaRs at multiple risk levels. *Annals of Economics and Statistics*, to appear.

[35] Francq, C., Lepage, G. and Zakoïan, J.M. (2011). Two-stage non Gaussian QML estimation of $GARCH$ models and testing the efficiency of the Gaussian $QMLE$. *Journal of Econometrics*, **165**, 246-257.

[36] Franses, P.H. and Paap. R. (2000). Modeling day-of-the-week seasonality in the S&P 500 Index. *Applied Financial Economics*, **10**, 483-488.

[37] Franses, P. and Paap, R. (2004). *Periodic time series models*. Oxford University Press.

[38] Franses, P. and Paap, R. (2011). Random-coefficient periodic autoregressions. *Statistica Neerlandica*, **65**, 101-115.

[39] Ghysels, E. and Osborn, D. (2001). *The econometric analysis of seasonal time series*. Cambridge University Press.

[40] Hamadeh, T. and Zakoïan, J.M. (2011). Asymptotic properties of LS and QML estimators for a class of nonlinear $GARCH$ processes. *Journal of Statistical Planning and Inference*, **141**, 488-507.

[41] Jones, R.H. and Brelsford, W.M. (1967). Time series with periodic structure. *Biometrika*, **54**, 403-408.

[42] Newey, W.K. and Steigerwald, D.G. (1997). Asymptotic bias for quasi-maximum likelihood estimators in conditional heteroskedasticity models. *Econometrica*, **65**, 587-599.

[43] Osborn, D.R., Savva, C.S. and Gill, L. (2008). Periodic dynamic conditional correlations between stock markets in Europe and the US. *Journal of Financial Econometrics*, **6**, 307-325.

[44] Regnard, N. and Zakoïan, J.M. (2011). A conditionally heteroskedastic model with time-varying coefficients for daily gas spot prices. *Energy Economics*, **33**, 1240-1251.

[45] Robinson, P.M. (1991). Testing for strong correlation and dynamic conditional heteroskedasticity in multiple regression. *Journal of Econometrics*, **47**, 67-84.

[46] Rossi, E. and Fantazani, D. (2015). Long memory and periodicity in intraday volatility. *Journal of Financial Econometrics*, **13**, 922-961.

[47] Sigauke, C. and Chikobvu, D. (2011). Prediction of daily peak electricity demand in South Africa using volatility forecasting models. *Energy Economics*, **33**, 882-888.

[48] Smith, M.S. (2010). Bayesian inference for a periodic stochastic volatility model of intraday electricity prices. *In Statistical Modelling and Regression Structures*, edt, 353-376.

[49] Taylor, N. (2004). Modeling discontinuous periodic conditional volatility: Evidence from the commodity futures market. *Journal of Futures Markets*, **24**, 805-834.

[50] Taylor, J.W. (2006). Density forecasting for the efficient balancing of the generation and consumption of electricity. *International Journal of Forecasting*, **22**, 707-724.

[51] Tesfaye, Y.G., Anderson, P.L. and Meerschaert, M.M. (2011). Asymptotic results for Fourier-$PARMA$ time series. *Journal of Time Series Analysis*, **32**, 157-174.

[52] Tsiakas, I. (2006). Periodic stochastic volatility and fat tails. *Journal of Financial Econometrics*, **4**, 90-135.

[53] Zhu, K. and Li, W.K. (2015). A new Pearson-type $QMLE$ for conditionally heteroskedastic models. *Journal of Business & Economic Statistics*, **33**, 552-565.

[54] Ziel, F. (2015). Quasi-maximum likelihood estimation of periodic autoregressive, conditionally heteroskedastic time series. *In Stochastic Models, Statistics and Their Applications, edt*, 207-214.

[55] Ziel, F. (2016). Iteratively reweighted adaptive lasso for conditional heteroscedastic time series with applications to AR-$ARCH$ type processes. *Computational Statistics and Data Analysis*, **100**, 773-793.

[56] Ziel, F., Steinert, R. and Husmann, S. (2015). Efficient modeling and forecasting of electricity spot prices. *Energy Economics*, **47**, 98-111.

[57] Ziel, F., Croonenbroeck, C. and Ambach, D. (2016). Forecasting wind power - Modeling periodic and non-linear effects under conditional heteroscedasticity. *Applied Energy*, **177**, 285-297.