Symmetric multi-person zero-sum game with two sets of strategic variables

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Abstract

We consider a symmetric multi-person zero-sum game with two sets of alternative strategic variables which are related by invertible functions. They are denoted by $(s_1, s_2, \dots, s_n)$ and $(t_1, t_2, \dots, t_n)$ for players 1, 2, $\dots$, n. The number of players is larger than two. We consider a symmetric game in the sense that all players have the same payoff functions. We do not postulate differentiability of the payoff functions of players. We will show that the following patterns of competition, 1) all players choose s_i, 2) all players choose t_i and 3) m players choose $t_i,\ i = 1, \dots, m$ and $n-m$ players choose $s_j,\ j = m+1, \dots, n$ where $1 \leq m \leq n-1$, are equivalent, that is, they yield the same outcome. However, in an asymmetric zero-sum game with more than two players the equivalence does not hold.

keywords multi-person zero-sum game, two strategic variables

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1 Introduction

We consider an n-person symmetric zero-sum game with two sets of strategic variables which are related by invertible functions. They are denoted by $(s_1, s_2, \dots, s_n)$ and $(t_1, t_2, \dots, t_n)$ for players 1, 2, ..., n. n is an integer number which is larger than 2. We do not postulate differentiability of the payoff functions of players.

We will show that the following patterns of competition are equivalent, that is, they yield the same outcome.

(1) All players choose s_i, $i \in N$. We call this competition s_i *competition.*

(2) All players choose t_i, $i \in N$. We call this competition t_i *competition.*

(3) Some players choose t_i and other players choose s_j. Specifically, m players choose t_i, $i = 1, 2, \dots, m$, and $n - m$ players choose s_j, $j = m + 1, m + 2, \dots, n$, where $1 \leq m \leq n - 1$. We call this competition $t_i - s_j$ *competition.*

We assume that the game is symmetric in the sense that all players have the same payoff functions, and consider symmetric equilibria where all players, whose strategic variables are s_i's, choose the same values, and also all players, whose strategic variables are t_i's, choose the same values.

Relative profit maximization in a symmetric oligopoly with differentiated goods is an example of symmetric n-person zero-sum game with two alternative strategic variables. Each firm chooses its output or price. The results of this paper imply that when firms in a symmetric oligopoly maximize their relative profits, Cournot and Bertrand equilibria are equivalent, and price-setting behavior and output-setting behavior are equivalent[1].

However, in an asymmetric n-person zero-sum game with more than two players the equivalence does not hold. In Section 7 we present an example that shows the non-equivalence of Cournot and Bertrand equilibria in an asymmetric oligopoly.

2 The model

Consider an n-person zero-sum game with $n \geq 3$ as follows. There are n players, $1, 2, \dots, n$. The set of players is denoted by N. They have two sets of alternative strategic variables, $(s_1, s_2, \dots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ and $(t_1, t_2, \dots, t_n) \in T_1 \times T_2 \times \cdots \times T_n$. S_i and T_i for $i \in N$ are compact sets in metric spaces. The relations of them are represented by

$$s_i = f_i(t_1, t_2, \dots, t_n),\ i \in N.$$

$(f_1, f_2, \dots, f_n)$ is a continuous invertible function, and so it is one-to-one and onto function. We denote

$$t_i = g_i(s_1, s_2, \dots, s_n),\ i \in N.$$

[1]About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997). An oligopoly is symmetric when demand functions are symmetric and all firms have the same cost functions.

$(g_1, g_2, \ldots, g_n)$ is also a continuous invertible function. The payoff functions of the players are $u_i(s_1, s_2, \ldots, s_n)$ for $i \in N$. They are continuous and quasi-concave. We do not postulate differentiability of the payoff functions[2]. All players have the same payoff functions. Since the game is zero-sum, we have

$$\sum_{i=1}^{n} u_i(s_1, s_2, \ldots, s_n) = 0, \tag{1}$$

for given $(s_1, s_2, \ldots, s_n)$.

3 s_i competition

First, consider competition by s_i, $i \in N$, for all players. Let s_i^*, $i \in N$, be the values of s_i's which, respectively, maximizes u_i, $i \in N$, given $s_j^*, j \neq i$, in a neighborhood around $(s_1^*, s_2^*, \ldots, s_n^*)$ in $S_1 \times S_2 \times \cdots \times S_n$. Then,

$$u_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_i, \ldots, s_n^*) \text{ for all } s_i \neq s_i^*,\ i \in N. \tag{2}$$

We assume that all s_i^*'s are equal at equilibria. Thus, $u_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*)$'s for all i are equal, and by the property of zero-sum game they are zero. By symmetry of the game we have

$$u_j(s_1^*, \ldots, s_i, \ldots, s_n^*) = u_k(s_1^*, \ldots, s_i, \ldots, s_n^*) \text{ for } j \neq i, k \neq i, j \neq k.$$

From this and (1)

$$-\sum_{j=1, j\neq i} u_j(s_1^*, \ldots, s_i, \ldots, s_n^*) = -(n-1)u_j(s_1^*, \ldots, s_i, \ldots, s_n^*) = u_i(s_1^*, \ldots, s_i, \ldots, s_n^*).$$

Therefore, from (2)

$$u_j(s_1^*, \ldots, s_i, \ldots, s_n^*) \geq u_j(s_1^*, \ldots, s_i^*, \ldots, s_n^*) \text{ for } j \neq i.$$

By symmetry

$$u_i(s_1^*, \ldots, s_j, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*) \text{ for } j \neq i.$$

Combining this and (2)

$$\begin{aligned} &u_i(s_1^*, \ldots, s_i, \ldots, s_n^*) \leq u_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*) \leq u_i(s_1^*, \ldots, s_j, \ldots, s_n^*) \\ &\text{for all } s_i \neq s_i^*, \text{ and all } s_j \neq s_j^*,\ j \neq i,\ i \in N. \end{aligned}$$

This is equivalent to

$$\begin{aligned} &u_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*) = \max_{s_i} u_i(s_1^*, \ldots, s_i, \ldots, s_n^*) = \min_{s_j} u_i(s_1^*, \ldots, s_j, \ldots, s_n^*), \\ &j \neq i \text{ given } s_k^*,\ k \neq i, j, \end{aligned}$$

[2] In Satoh and Tanaka (2016) we analyze maximin and minimax strategies in oligopoly when payoff functions of firms are differentiable.

$(s_1^*, s_2^*, \ldots, s_n^*)$ is a Nash equilibrium of the s_i competition game. By the Glicksberg's theorem (Glicksberg (1952)) there exists a Nash equilibrium.

Let $s_{-i,j}^*$ be a vector of s_k^* for $k \neq i, j$. We can show the following lemma.

Lemma 1. *The following three statements are equivalent.*

(1) *There exists a Nash equilibrium in the s_i competition game.*

(2) *Given s_k^* for all $k \neq i, j$, the following relation holds.*

$$\mathbf{v}_i^s \equiv \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) = \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) \equiv \mathbf{v}_j^s \text{ for any pair of } i \text{ and } j.$$

(3) *There exists a real number $\mathbf{v}_s$, s_i^m and s_j^m such that*

$$u_i(s_i^m, s_j, s_{-i,j}^*) \geq \mathbf{v}_s \text{ for any } s_j, \text{ and } u_i(s_i, s_j^m, s_{-i,j}^*) \leq \mathbf{v}_s \text{ for any } s_i, \tag{3}$$

for any pair of i and j.

Proof. $(1 \to 2)$

Let s_i^* and s_j^* be the equilibrium strategies of Player i and j. Then,

$$\begin{aligned}\mathbf{v}_j^s &= \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) \leq \max_{s_i} u_i(s_i, s_j^*, s_{-i,j}^*) = u_i(s_i^*, s_j^*, s_{-i,j}^*) \\ &= \min_{s_j} u_i(s_i^*, s_j, s_{-i,j}^*) \leq \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) = \mathbf{v}_i^s.\end{aligned}$$

On the other hand, $\min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \leq u_i(s_i, s_j, s_{-i,j}^*)$, then $\max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \leq \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*)$, and so $\max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \leq \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*)$. Thus, $\mathbf{v}_i^s \leq \mathbf{v}_j^s$, and we have $\mathbf{v}_i^s = \mathbf{v}_j^s$.

$(2 \to 3)$

Let $s_i^m = \arg\max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*)$ (the maximin strategy), $s_j^m = \arg\min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*)$ (the minimax strategy), and let $\mathbf{v}_s = \mathbf{v}_i^s = \mathbf{v}_j^s$. Then, we have

$$\begin{aligned}u_i(s_i^m, s_j, s_{-i,j}^*) &\geq \min_{s_j} u_i(s_i^m, s_j, s_{-i,j}^*) = \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) = \mathbf{v}_s \\ &= \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) = \max_{s_i} u_i(s_i, s_j^m, s_{-i,j}^*) \geq u_i(s_i, s_j^m, s_{-i,j}^*).\end{aligned}$$

$(3 \to 1)$

From (3)

$$u_i(s_i^m, s_j, s_{-i,j}^*) \geq \mathbf{v}_s \geq u_i(s_i, s_j^m, s_{-i,j}^*) \text{ for all } s_i \in S_i,\ s_j \in S_j.$$

Putting $s_i = s_i^m$ and $s_j = s_j^m$, we see $\mathbf{v}_s = u_i(s_i^m, s_j^m, s_{-i,j}^*)$ and $(s_i^m, s_j^m, s_{-i,j}^*)$ is an equilibrium. Thus, $s_i^m = s_i^*$ and $s_j^m = s_j^*$. □

Since at equilibria all u_i's are zero, we have $\mathbf{v}_i^s = \mathbf{v}_j^s = \mathbf{v}_s = 0$. Denote the values of t_i, $i \in N$, which are derived from the following equation;

$$(t_1, t_2, \ldots, t_n) = (g_i(s_1^*, s_2^*, \ldots, s_n^*), g_2(s_1^*, s_2^*, \ldots, s_n^*), \ldots, g_n(s_1^*, s_2^*, \ldots, s_n^*)),$$

by t_i^*, $i \in N$.

4 t_i competition

Next consider competition by $t_i,\ i \in N$, for all players. In this section we use the following notation.

$$v_i(t_1, \dots, t_n) = u_i(f_1(t_1, \dots, t_n), \dots, f_n(t_1, \dots, t_n)) \text{ for each } i \in N.$$

Let $\tilde{t}_i,\ i \in N$, be the values of t_i's which, respectively, maximizes $v_i,\ i \in N$, given $\tilde{t}_j,\ j \neq i$, in a neighborhood around $(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n)$ in $T_1 \times T_2 \times \cdots \times T_n$. Then,

$$v_i(\tilde{t}_1, \dots, \tilde{t}_i, \dots, \tilde{t}_n) \geq v_i(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) \text{ for all } t_i \neq \tilde{t}_i,\ i \in N, \tag{4}$$

We assume that all $\tilde{t}_i$'s are equal at equilibria. Thus, $v_i(\tilde{t}_1, \dots, \tilde{t}_i, \dots, \tilde{t}_n)$ for all i are equal, and by the property of zero-sum game all v_i's are zero. By symmetry of the model

$$v_j(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) = v_k(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) \text{ for } j \neq i, k \neq i, j \neq k.$$

From this and (1)

$$-\sum_{j=1, j\neq i} v_j(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) = -(n-1)v_j(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) = v_i(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n).$$

Therefore, from (4)

$$v_j(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) \geq v_j(\tilde{t}_1, \dots, \tilde{t}_i, \dots, \tilde{t}_n) \text{ for } j \neq i.$$

By symmetry we get

$$v_i(\tilde{t}_1, \dots, t_j, \dots, \tilde{t}_n) \geq v_i(\tilde{t}_1, \dots, \tilde{t}_i, \dots, \tilde{t}_n) \text{ for } j \neq i.$$

Combining this and (4)

$$\begin{aligned} &v_i(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) \leq v_i(\tilde{t}_1, \dots, \tilde{t}_i, \dots, \tilde{t}_n) \leq v_i(\tilde{t}_1, \dots, t_j, \dots, \tilde{t}_n) \\ &\text{for all } t_i \neq \tilde{t}_i, \text{ and all } t_j \neq \tilde{t}_j,\ j \neq i,\ i \in N. \end{aligned}$$

This is equivalent to

$$\begin{aligned} &v_i(\tilde{t}_1, \dots, \tilde{t}_i, \dots, \tilde{t}_n) = \max_{t_i} v_i(\tilde{t}_1, \dots, t_i, \dots, \tilde{t}_n) = \min_{t_j} v_i(\tilde{t}_1, \dots, t_j, \dots, \tilde{t}_n), \\ &j \neq i \text{ given } \tilde{t}_k,\ k \neq i, j. \end{aligned}$$

Let $\tilde{t}_{-i,j}$ be a vector of $\tilde{t}_k$ for $k \neq i, j$. Similarly to Lemma 1 we can show the following lemma.

Lemma 2. *The following three statements are equivalent.*

(1) *There exists a Nash equilibrium in the t_i competition game.*

(2) *Given $\tilde{t}_k$ for all $k \neq i, j$, the following relation holds.*

$$\mathbf{v}_i^t \equiv \max_{t_i} \min_{t_j} v_i(t_i, t_j, \tilde{t}_{-i,j}) = \min_{t_j} \max_{t_i} v_i(t_i, t_j, \tilde{t}_{-i,j}) \equiv \mathbf{v}_j^t \text{ for any pair of } i \text{ and } j.$$

(3) *There exists a real number $\mathbf{v}_t$, t_i^m and t_j^m such that*

$$v_i(t_i^m, t_j, \tilde{t}_{-i,j}) \geq \mathbf{v}_t \text{ for any } t_j, \text{ and } v_i(t_i, t_j^m, \tilde{t}_{-i,j}) \leq \mathbf{v}_t \text{ for any } t_i$$
for any pair of i and j.

Thus, $t_i^m = \tilde{t}_i$ and $t_j^m = \tilde{t}_j$. Since at equilibria all v_i's are zero, we have $\mathbf{v}_i^t = \mathbf{v}_j^t = \mathbf{v}_t = 0$. Denote the values of s_i, $i \in N$, which are derived from the following equation;

$$(s_1, s_2, \dots, s_n) = (f_i(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n), f_2(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n), \dots, f_n(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n)),$$

by $\tilde{s}_i$, $i \in N$.

5 $t_i - s_j$ competition

Next, consider $t_i - s_j$ competition. Assume that m players choose t_i, $i = 1, 2 \dots, m$, and the remaining $n - m$ players choose s_j, $j = m + 1, m + 2, \dots, n$. m is an integer such that $1 \leq m \leq n - 1$. At least one player chooses t_i, and at least one player chooses s_j. In this section we use the following notation.

$w_i(t_1, \dots, t_m, s_{m+1}, \dots, s_n)$
$= u_i(f_1(t_1, \dots, t_m, g_{m+1}(\dots), \dots, g_n(\dots)), \dots, f_m(t_1, \dots, t_m, g_{m+1}(\dots), \dots, g_n(\dots)), s_{m+1}, \dots, s_n)$
for each $i \in N$, where
$g_j(\dots) = g_j(s_1, \dots, s_m, s_{m+1}, \dots, s_n)$ for $j \in \{m + 1, \dots, n, \}$
with
$s_i = f_i(t_1, \dots, t_m, g_{m+1}(\dots), \dots, g_n(\dots))$ for $i \in \{1, \dots, m\}$.

Let $\bar{t}_i$, $i = 1, 2, \dots, m$, and $\bar{s}_j$, $j = m+1, \dots, n$, be the values of t_i and s_j which maximizes, respectively, w_i and w_j, in a neighborhood around the equilibrium point. Then,

$$w_i(\bar{t}_1, \dots, \bar{t}_i, \dots, \bar{t}_m, \bar{s}_{m+1}, \dots, \bar{s}_n) \geq w_i(\bar{t}_1, \dots, t_i, \dots, \bar{t}_m, \bar{s}_{m+1}, \dots, \bar{s}_n) \text{ for all } t_i \neq \bar{t}_i,\ i = 1, 2, \dots, m, \tag{5}$$

and

$$w_j(\bar{t}_1, \dots, \bar{t}_m, \bar{s}_{m+1}, \dots, \bar{s}_j, \dots, \bar{s}_n) \geq w_j(\bar{t}_1, \dots, \bar{t}_m, \bar{s}_{m+1}, \dots, s_j, \dots, \bar{s}_n) \text{ for all } s_j \neq \bar{s}_j,\ j = m + 1, m + 2, \dots, n, \tag{6}$$

We assume that at equilibria all $\bar{t}_i,\ i=1,2,\dots,m$, are equal, and all $\bar{s}_j,\ j=m+1,m+2,\dots,n$, are equal. Since all players have the same payoff functions, all $w_i,\ i=1,2,\dots,m$, are equal, and all $w_j,\ j=m+1,m+2,\dots,n$, are equal. Then, from (1) we obtain

$$mw_i(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n)+(n-m)w_j(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n)=0,$$

and so

$$w_j(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n)=-\frac{m}{n-m}w_i(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n).$$

If $w_i=0$ (or $w_j=0$), then $w_j=0$ (or $w_i=0$). (6) is rewritten as

$$\begin{aligned}&w_i(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n)\\&\le w_i(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,s_j,\dots,\bar{s}_n)\ \text{for all}\ s_j\neq\bar{s}_j,\ j=m+1,m+2,\dots,n,\end{aligned}$$

Combining this and (5),

$$\begin{aligned}&w_i(\bar{t}_1,\dots,t_i,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n)\le w_i(\bar{t}_1,\dots,\bar{t}_i,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n)\\&\le w_i(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,s_j,\dots,\bar{s}_n)\\&\text{for all}\ t_i\neq\bar{t}_i,\ i=1,2,\dots,m,\ \text{and all}\ s_j\neq\bar{s}_j,\ j=m+1,m+2,\dots,n.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&w_i(\bar{t}_1,\dots,\bar{t}_i,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_j,\dots,\bar{s}_n)=\max_{t_i}w_i(\bar{t}_1,\dots,t_i,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,\bar{s}_n)\\&=\min_{s_j}w_i(\bar{t}_1,\dots,\bar{t}_m,\bar{s}_{m+1},\dots,s_j,\dots,\bar{s}_n)\ \text{for any pair of}\ i,j.\end{aligned}$$

Let $\bar{t}_{-i}$ be a vector of $\bar{t}_k$ for $k\in\{1,\dots,m\},\ k\neq i$ and $\bar{s}_{-j}$ be a vector of $\bar{s}_l$ for $l\in\{m+1,\dots,n\},\ l\neq j$. Similarly to Lemma 1 we can show the following lemma.

Lemma 3. *The following three statements are equivalent.*

(1) *There exists a Nash equilibrium in the t_i-s_j competition game.*

(2) *Given $\bar{t}_k,\ k\neq i,\ k\in\{1,\dots,m\}$ and $\bar{s}_l,\ l\neq j,\ l\in\{m+1,\dots,n\}$, the following relation holds.*

$$\mathbf{v}_i^{ts}\equiv\max_{t_i}\min_{s_j}w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j})=\min_{s_j}\max_{t_i}w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j})\equiv\mathbf{v}_j^{ts}\ \text{for any pair of}\ i\ \text{and}\ j.$$

(3) *There exists a real number $\mathbf{v}_{ts}$, t_i^{ts} and s_j^{ts} such that*

$$w_i(t_i^{ts},\bar{t}_{-i},s_j,\bar{s}_{-j})\ge\mathbf{v}_{ts}\ \text{for any}\ s_j,\ \text{and}\ w_i(t_i,\bar{t}_{-i},s_j^{ts},\bar{s}_{-j})\le\mathbf{v}_{ts}\ \text{for any}\ t_i$$

for any pair of i and j,

Thus, $t_i^{ts}=\bar{t}_i$ and $s_j^{ts}=\bar{s}_j$. Denote the values of $s_i,\ i\in\{1,2,\dots,m\}$ and the values of $t_j,\ j\in\{m+1,m+2,\dots,n\}$, which are derived from the following equation;

$$\begin{aligned}&(s_1,s_2,\dots,s_m)=(f_1(\bar{t}_1,\bar{t}_2,\dots,\bar{t}_m,t_{m+1},t_{m+2},\dots,t_n),f_2(\dots),\dots,f_m(\dots)),\\&(t_{m+1},t_{m+2},\dots,t_n)=(g_1(s_1,s_2,\dots,s_m,\bar{s}_{m+1},\bar{s}_{m+2},\dots,s_n),g_2(\dots),\dots,g_m(\dots)),\end{aligned}$$

by $\bar{s}_i,\ i\in\{1,2,\dots,m\}$ and $\bar{t}_j,\ j\in\{m+1,m+2,\dots,n\}$.

6 Equivalence of three patterns of competition

First we show the following proposition.

Proposition 1. *s_i competition and $t_i - s_j$ competition where one player, Player 1, chooses t_1 are equivalent.*

Each player j in $\{2, \ldots, n\}$ chooses s_j as his/her strategic variable. To prove this proposition we need the following lemma.

Lemma 4.

$$\max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) = \max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}).$$

$\bar{s}_{-j}$ is a vector of $\bar{s}_l$ for $l \in \{2, \ldots, n\}$, $l \neq j$.

Proof. $\min_{s_j} w_1(t_1, s_j, \bar{s}_{-j})$ is the minimum of $w_1(= u_1)$ with respect to s_j given t_1 and $\bar{s}_{-j}$. Let $s_j(t_1) = \arg\min_{s_j} w_1(t_1, s_j, \bar{s}_{-j})$, and fix the value of s_1 at

$$s_1^0 = f_1(t_1, g_2(s_1^0, s_2, \ldots, s_j(t_1), \ldots, s_n), \ldots, g_n(s_1^0, s_2, \ldots, s_j(t_1), \ldots, s_n)).$$

Then, we have

$$\min_{s_j} u_1(s_1^0, s_j, \bar{s}_{-j}) \leq u_1(s_1^0, s_j(t_1), \bar{s}_{-j}) = \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}),$$

where $\min_{s_j} u_1(s_1^0, s_j, \bar{s}_{-j})$ is the minimum of u_1 with respect to s_j given the value of s_1 at s_1^0. We assume that $s_j(t_1) = \arg\min_{s_j} w_1(t_1, s_j, \bar{s}_{-j})$ is single-valued. By the maximum theorem and continuity of w_1, $s_j(t_1)$ is continuous. Then, any value of s_1^0 in some neighborhood around $(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)$ can be realized by appropriately choosing t_1 given s_j and $\bar{s}_{-j}$ as $s_1^0 = f_1(t_1, g_2(s_1^0, s_2, \ldots, s_j(t_1), \ldots, s_n), \ldots, g_n(s_1^0, s_2, \ldots, s_j(t_1), \ldots, s_n))$. Therefore,

$$\max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}) \leq \max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}). \tag{7}$$

On the other hand, $\min_{s_j} u_1(s_1, s_j, \bar{s}_{-j})$ is the minimum of u_1 with respect to s_j given s_1 and $\bar{s}_{-j}$. Let $s_j(s_1) = \arg\min_{s_j} u_1(s_1, s_j, \bar{s}_{-j})$, and fix the value of t_1 at $g_1(s_1, s_j(s_1), \bar{s}_{-j})$. Then, we have

$$\min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) = \min_{s_j} w_1(g_1(s_1, s_j(s_1), \bar{s}_{-j}), s_j, \bar{s}_{-j}) \leq u_1(s_1, s_j(s_1), \bar{s}_{-j}) = \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}),$$

where $\min_{s_j} w_1(g_1(s_1, s_j(s_1), \bar{s}_{-j}), s_j, \bar{s}_{-j})$ is the minimum of $w_1(= u_1)$ with respect to s_j given the value of t_1 at $g_1(s_1, s_j(s_1), \bar{s}_{-j})$. We assume that $s_j(s_1) = \arg\min_{s_j} u_1(s_1, s_j, \bar{s}_{-j})$ is single-valued. By the maximum theorem and continuity of u_1, $s_j(s_1)$ is continuous. Then, any value of t_1 in some neighborhood around $(\bar{t}_1, \bar{s}_2, \ldots, \bar{s}_n)$ can be realized by appropriately choosing s_1 given s_j and $\bar{s}_{-j}$ as $t_1 = g_1(s_1, s_j(s_1), \bar{s}_{-j})$. Therefore,

$$\max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) \leq \max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}). \tag{8}$$

Combining (7) and (8), we get

$$\max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) = \max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}).$$

□

Proof of Proposition 1. We show that the condition for $(\bar{s}_1, \ldots, \bar{s}_n)$ and the condition for $(s_1^*, \ldots s_n^*)$ are the same. From Lemma 3

$$\max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) = \min_{s_j} \max_{t_1} w_1(t_1, s_j, \bar{s}_{-j}).$$

Since any value of s_1 can be realized by appropriately choosing t_1 given s_j, $j \neq 1$, and $\bar{s}_{-j}$, we have $\max_{s_1} u_1(s_1, s_j, \bar{s}_{-j}) = \max_{t_1} w_1(t_1, s_j, \bar{s}_{-j})$ for any s_j. Thus,

$$\min_{s_j} \max_{s_1} u_1(s_1, s_j, \bar{s}_{-j}) = \min_{s_j} \max_{t_1} w_1(t_1, s_j, \bar{s}_{-j}).$$

With Lemma 4 we conclude

$$\max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}) = \min_{s_j} \max_{s_1} u_1(s_1, s_j, \bar{s}_{-j}) = u_1(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) = 0.$$

This is 2 of Lemma 1. The result of this proposition means that $w_1(\bar{t}_1, \bar{s}_j, \bar{s}_{-j}) = w_j(\bar{t}_1, \bar{s}_j, \bar{s}_{-j}) = 0$. □

Next we show the following proposition.

Proposition 2. *t_i competition and $t_i - s_j$ competition where one player, Player n, chooses s_n, are equivalent.*

To prove this proposition we need the following lemma.

Lemma 5.

$$\min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n) = \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n).$$

$\bar{t}_{-i}$ is a vector of $\bar{t}_k$ for $k \in \{1, \ldots, n-1\}$, $k \neq i$.

Proof. $\max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n)$ is the maximum of $w_i (= v_i)$ with respect to t_i given s_n and $\bar{t}_{-i}$. Let $t_i(s_n) = \arg\max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n)$, and fix the value of t_n at

$$t_n^0 = g_n(f_i(t_i(s_n), \bar{t}_{-i}, t_n), f_{-i}((t_i(s_n), \bar{t}_{-i}, t_n)), s_n),$$

where f_{-i} is a vector of f_k for $k \in \{2, \ldots, n-1\}$, $k \neq i$. Then, we have

$$\begin{aligned}&\max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n^0) = \max_{t_i} v_i(t_i, \bar{t}_{-i}, g_n(f_i(t_i(s_n), \bar{t}_{-i}, t_n), f_{-i}((t_i(s_n), \bar{t}_{-i}, t_n)), s_n)) \\ &\geq w_i(t_i(s_n), \bar{t}_{-i}, s_n) = \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n),\end{aligned}$$

where $\max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n^0)$ is the maximum of v_i with respect to t_i given the value of t_n at

$$g_n(f_i(t_i(s_n), \bar{t}_{-i}, t_n), f_{-i}((t_i(s_n), \bar{t}_{-i}, t_n)), s_n).$$

We assume that $t_i(s_n) = \arg\max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n)$ is single-valued. By the maximum theorem and continuity of w_i, $t_i(s_n)$ is continuous. Then, any value of t_n^0 in some neighborhood around $(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n)$ can be realized by appropriately choosing s_n given t_i and $\bar{t}_{-i}$ as

$$t_n^0 = g_n(f_i(t_i(s_n), \bar{t}_{-i}, t_n), f_{-i}((t_i(s_n), \bar{t}_{-i}, t_n)), s_n).$$

Therefore,

$$\min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n) \geq \min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n). \tag{9}$$

On the other hand, $\max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n)$ is the maximum of v_i with respect to t_i given t_n and $\bar{t}_{-i}$. Let $t_i(t_n) = \arg\max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n)$, and fix the value of s_n at $f_n(t_i(t_n), \bar{t}_{-i}, t_n)$. Then, we have

$$\max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n) = \max_{t_i} w_i(t_i, \bar{t}_{-i}, f_n(t_i(t_n), \bar{t}_{-i}, t_n)) \geq v_i(t_i(t_n), \bar{t}_{-i}, t_n) = \max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n),$$

where $\max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n)$ is the maximum of $w_i(= v_i)$ with respect to t_i given the value of s_n at $f_n(t_i(t_n), \bar{t}_{-i}, t_n)$.We assume that $t_i(t_n) = \arg\max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n)$ is single-valued. By the maximum theorem and continuity of v_i, $t_i(t_n)$ is continuous. Then, any value of s_n in some neighborhood around $(\bar{t}_1, \bar{t}_2, \ldots, \bar{s}_n)$ can be realized by appropriately choosing t_n given $\bar{t}_i$ and $\bar{t}_{-i}$ as $s_n = f_n(t_i(t_n), \bar{t}_{-i}, t_n)$. Therefore,

$$\min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n) \geq \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n). \tag{10}$$

Combining (9) and (10), we get

$$\min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n) = \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n).$$

□

Proof of Proposition 2. We show that the condition for $(\bar{t}_1, \ldots, \bar{t}_n)$ and the condition for $(\tilde{t}_1, \ldots, \tilde{t}_n)$ are the same. From Lemma 3

$$\max_{t_i} \min_{s_n} w_i(t_i, \bar{t}_{-i}, s_n) = \min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n).$$

Since any value of t_n can be realized by appropriately choosing s_n given t_i, $i \neq n$, and $\bar{t}_{-i}$, we have $\min_{s_n} w_i(t_i, \bar{t}_{-i}, s_n) = \min_{t_n} v_i(t_i, \bar{t}_{-i}, t_n)$ for any t_i. Thus,

$$\max_{t_i} \min_{t_n} v_i(t_i, \bar{t}_{-i}, t_n) = \max_{t_i} \min_{s_n} w_i(t_i, \bar{t}_{-i}, s_n).$$

From Lemma 5 we have $\min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_n) = \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n)$. Therefore, we obtain

$$\max_{t_i} \min_{t_n} v_i(t_i, \bar{t}_{-i}, t_n) = \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_{-i}, t_n) = v_i(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n) = 0.$$

This is 2 of Lemma 2. The result of this proposition means that $w_i(\bar{t}_i, \bar{t}_{-i}, \bar{s}_n) = w_n(\bar{t}_i, \bar{t}_{-i}, \bar{s}_n) = 0$. □

Finally we show the following proposition.

Proposition 3. *$t_i - s_j$ competition in which m players choose t_i's as their strategic variables, and $t_i - s_j$ competition in which $m-1$ players choose t_i's as their strategic variables are equivalent, where $2 \leq m \leq n-1$.*

To prove this proposition we need the following lemma.

Lemma 6.

$$\max_{t_i}\min_{s_j} w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j}) = \max_{s_i}\min_{s_j} w_i(s_i,\bar{t}_{-i},s_j,\bar{s}_{-j}).$$

$\bar{t}_{-i}$ is a vector of $\bar{t}_k$ for $k \in \{1,\dots,m\}$, $k \neq i$. $\bar{s}_{-j}$ is a vector of $\bar{s}_l$ for $l \in \{m+1,\dots,n\}$, $l \neq j$.

Proof. $\min_{s_j} w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j})$ is the minimum of $w_i(=u_i)$ with respect to s_j given t_i, $\bar{t}_{-i}$ and $\bar{s}_{-j}$. Let $s_j(t_i) = \arg\min_{s_j} w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j})$. The values of variables other than t_i, $s_j(t_i)$, $\bar{t}_{-i}$ and $\bar{s}_{-j}$ are determined by the following equations;

$$s_1 = f_1(\bar{t}_1,\dots,t_i,\dots,\bar{t}_m,t_{m+1},\dots,t_j,\dots,t_n),$$

$$\dots,$$

$$s_i = f_i(\bar{t}_1,\dots,t_i,\dots,\bar{t}_m,t_{m+1},\dots,t_j,\dots,t_n),$$

$$\dots,$$

$$s_m = f_m(\bar{t}_1,\dots,t_i,\dots,\bar{t}_m,t_{m+1},\dots,t_j,\dots,t_n),$$

$$t_{m+1} = g_{m+1}(s_1,\dots,s_i,\dots,s_m,\bar{s}_{m+1},\dots,s_j(t_i),\dots,\bar{s}_n),$$

$$\dots,$$

$$t_j = g_j(s_1,\dots,s_i,\dots,s_m,\bar{s}_{m+1},\dots,s_j(t_i),\dots,\bar{s}_n),$$

$$\dots,$$

$$t_n = g_n(s_1,\dots,s_i,\dots,s_m,\bar{s}_{m+1},\dots,s_j(t_i),\dots,\bar{s}_n).$$

Denote this s_i by s_i^0, and fix the value of s_i at s_i^0. Then, we have

$$\min_{s_j} w_i(s_i^0,\bar{t}_{-i},s_j,\bar{s}_{-j}) \leq w_i(t_i,\bar{t}_{-i},s_j(t_i),\bar{s}_{-j}) = \min_{s_j} w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j}),$$

where $\min_{s_j} w_i(s_i^0,\bar{t}_{-i},s_j,\bar{s}_{-j})$ is the minimum of $w_i(=u_i)$ with respect to s_j given the value of s_i at s_i^0. We assume that $s_j(t_i) = \arg\min_{s_j} w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j})$ is single-valued. By the maximum theorem and continuity of w_i, $s_j(t_i)$ is continuous. Then, any value of s_i^0 in some neighborhood around $(\bar{s}_i,\bar{t}_{-i},\bar{s}_j,\bar{s}_{-j})$ can be realized by appropriately choosing t_i given s_j, $\bar{t}_{-i}$ and $\bar{s}_{-j}$. Therefore,

$$\max_{s_i}\min_{s_j} w_i(s_i,\bar{t}_{-i},s_j,\bar{s}_{-j}) \leq \max_{t_i}\min_{s_j} w_i(t_i,\bar{t}_{-i},s_j,\bar{s}_{-j}). \tag{11}$$

On the other hand, $\min_{s_j} w_i(s_i,\bar{t}_{-i},s_j,\bar{s}_{-j})$ is the minimum of $w_i(=u_i)$ with respect to s_j given s_i, $\bar{t}_{-i}$ and $\bar{s}_{-j}$. Let $s_j(s_i) = \arg\min_{s_j} w_i(s_i,\bar{t}_{-i},s_j,\bar{s}_{-j})$. The values of variables other than s_i, $s_j(s_i)$, $\bar{t}_{-i}$ and $\bar{s}_{-j}$ are determined by the following equations;

$$s_1 = f_1(\bar{t}_1,\dots,t_i,\dots,\bar{t}_m,t_{m+1},\dots,t_j,\dots,t_n),$$

$$\ldots,$$

$$s_m = f_m(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n),$$

$$t_{m+1} = g_{m+1}(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(s_i), \ldots, \bar{s}_n),$$

$$\ldots,$$

$$t_i = g_i(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(s_i), \ldots, \bar{s}_n),$$

$$t_j = g_j(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(s_i), \ldots, \bar{s}_n),$$

$$\ldots,$$

$$t_n = g_n(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(s_i), \ldots, \bar{s}_n).$$

Denote this t_i by t_i^0, and fix the value of t_i at t_i^0. Then, we have

$$\min_{s_j} w_i(t_i^0, \bar{t}_{-i}, s_j, \bar{s}_{-j}) \leq w_i(s_i, \bar{t}_{-i}, s_j(s_i), \bar{s}_{-j}) = \min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}),$$

where $\min_{s_j} w_i(t_i^0, \bar{t}_{-i}, s_j, \bar{s}_{-j})$ is the minimum of $w_i (= u_i)$ with respect to s_j given the value of t_i at t_i^0. We assume that $s_j(s_i) = \arg\min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j})$ is single-valued. By the maximum theorem and continuity of w_i, $s_j(s_i)$ is continuous. Then, any value of t_i^0 in some neighborhood around $(\bar{t}_i, \bar{t}_{-i}, \bar{s}_j, \bar{s}_{-j})$ can be realized by appropriately choosing s_i given s_j, $\bar{s}_{-j}$ and $\bar{t}_{-i}$. Therefore,

$$\max_{t_i} \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) \leq \max_{s_i} \min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}). \tag{12}$$

Combining (11) and (12), we get

$$\max_{t_i} \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) = \max_{s_i} \min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}).$$

□

Proof of Proposition 3. We show that the condition for $(\bar{t}_i, \bar{t}_{-i}, \bar{s}_j, \bar{s}_{-j})$ and the condition for $(\bar{s}_i, \bar{t}_{-i}, \bar{s}_j, \bar{s}_{-j})$ are the same. From Lemma 3

$$\max_{t_i} \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) = \min_{s_j} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}).$$

Since any value of s_i can be realized by appropriately choosing t_i given s_j, $\bar{t}_{-i}$, $\bar{s}_{-j}$, we have $\max_{t_i} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) = \max_{s_i} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j})$ for any s_j. Thus,

$$\min_{s_j} \max_{t_i} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) = \min_{s_j} \max_{s_i} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}).$$

With Lemma 6 we conclude

$$\max_{s_i} \min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) = \max_{s_i} \min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}).$$

□

Summarizing these results we conclude.

Proposition 4. *s_i competition, t_i competition and $t_i - s_j$ competition with any number of players whose strategic variables are t_i's are equivalent, and payoffs of all players at any equilibrium are zero.*

Proof. From Proposition 1

$$w_1(\bar{t}_1, \bar{s}_j, \bar{s}_{-j}) = w_j(\bar{t}_1, \bar{s}_j, \bar{s}_{-j}) = 0,\ j \in \{2, 3, \dots, n\}.$$

This means that the payoffs of all players when only one player chooses t_i and all other players choose s_j's are zero. From Proposition 2

$$w_n(\bar{t}_i, \bar{t}_{-i}, \bar{s}_n) = w_i(\bar{t}_i, \bar{t}_{-i}, \bar{s}_n) = 0,\ i \in \{1, 2, \dots, n-1\}.$$

This means that the payoffs of all players when only one player chooses s_j and all other players choose t_i's are zero. From the result of Proposition 3

$$w_i(\bar{t}_1, \bar{t}_2, \bar{s}_j, \bar{s}_{-j}) = w_j(\bar{t}_1, \bar{t}_2, \bar{s}_j, \bar{s}_{-j}) = 0,\ i \in \{1, 2\},\ j \in \{3, 4, \dots, n\}.$$

This means that the payoffs of all players when two players choose t_i's and all other players choose s_j's are zero. Then, inductively we conclude that

$$w_i(\bar{t}_i, \bar{t}_{-i}, \bar{s}_j, \bar{s}_{-j}) = w_j(\bar{t}_i, \bar{t}_{-i}, \bar{s}_j, \bar{s}_{-j}) = 0,$$

in the game where m players choose t_i's as their strategic variables for any m such that $2 \le m \le n-1$. i denotes a player whose strategic variable is t_i, and j denotes a player whose strategic variable is s_j. Thus, payoffs of all players in any $t_i - s_j$ competition are zero. By the definitions of s_i competition and t_i competition payoffs of all players in the s_i competition and the t_i competition are zero. □

7 Example: relative profit maximization in oligopoly with differentiated goods

Consider an oligopoly with three firms producing differentiated goods. The firms are A, B and C. The inverse demand functions are

$$p_A = a - x_A - bx_B - bx_C,$$

$$p_B = a - x_B - bx_A - bx_C,$$

and

$$p_C = a - x_C - bx_A - bx_B,$$

where $0 < b < 1$. p_A, p_B and p_C are the prices of the goods of Firm A, B and C, and x_A, x_B and x_C are the outputs of them. From these inverse demand functions the direct demand functions are derived as follows;

$$x_A = \frac{(1-b)a - (1+b)p_A + b(p_A + p_C)}{(1-b)(1+2b)},$$

$$x_B = \frac{(1-b)a - (1+b)p_B + b(p_B + p_C)}{(1-b)(1+2b)},$$

and

$$x_C = \frac{(1-b)a - (1+b)p_C + b(p_A + p_B)}{(1-b)(1+2b)}.$$

The (absolute) profits of the firms are

$$\pi_A = p_A x_A - c_A x_A,$$

$$\pi_B = p_B x_B - c_B x_B,$$

and

$$\pi_C = p_C x_C - c_C x_C.$$

c_A, c_B and c_C are the constant marginal costs of Firm A, B and C. The relative profits of the firms are

$$\varphi_A = \pi_A - \frac{\pi_B + \pi_C}{2},$$

$$\varphi_B = \pi_B - \frac{\pi_A + \pi_C}{2},$$

and

$$\varphi_C = \pi_C - \frac{\pi_A + \pi_B}{2}.$$

We see

$$\varphi_A + \varphi_B + \varphi_C = 0,$$

so . the game is zero-sum. In a Cournot model the firms determine their outputs to maximize their relative profits. In a Bertrand model they determine the prices of their goods to maximize their relative profits. The Cournot equilibrium outputs are

$$x_A^C = \frac{bc_C + bc_B - bc_A - 4c_A - ab + 4a}{(4-b)(2+b)},$$

$$x_B^C = \frac{bc_C - bc_B - 4c_B + bc_A - ab + 4a}{(4-b)(2+b)},$$

$$x_C^C = \frac{bc_B - bc_C - 4c_C + bc_A - ab + 4a}{(4-b)(2+b)}.$$

The Bertrand equilibrium prices are

$$p_A^B = \frac{3b^2c_C + 3bc_C + 3b^2c_B + 3bc_B + 4b^2c_A + 7bc_A + 4c_A - 5ab^2 + ab + 4a}{(2+b)(4+5b)},$$

$$p_B^B = \frac{3b^2c_C + 3bc_C + 4b^2c_B + 7bc_B + 4c_B + 3b^2c_A + 3bc_A - 5ab^2 + ab + 4a}{(2+b)(4+5b)},$$

$$p_C^B = \frac{4b^2c_C + 7bc_C + 4c_C + 3b^2c_B + 3bc_B + 3b^2c_A + 3bc_A - 5ab^2 + ab + 4a}{(2+b)(4+5b)},$$

The difference between the relative profit in the Bertrand equilibrium and that in the Cournot equilibrium for each of Firm A, B, C is

$$\varphi_A^B - \varphi_A^C = \frac{9b^3(b+2)(c_C^2 - 4c_Bc_C + 2c_Ac_C + c_B^2 + 2c_Ac_B - 2c_A^2)}{(b-4)^2(b-1)(5b+4)^2},$$

$$\varphi_B^B - \varphi_B^C = \frac{9b^3(b+2)(c_C^2 + 2c_Bc_C - 4c_Ac_C - 2c_B^2 + 2c_Ac_B + c_A^2)}{(b-4)^2(b-1)(5b+4)^2},$$

and

$$\varphi_C^B - \varphi_C^C = -\frac{9b^3(b+2)(2c_C^2 - 2c_Bc_C - 2c_Ac_C - c_B^2 + 4c_Ac_B - c_A^2)}{(b-4)^2(b-1)(5b+4)^2}.$$

If and only if $c_A = c_B = c_C$, we have $\varphi_A^C = \varphi_A^B, \varphi_B^C = \varphi_B^B, \varphi_C^C = \varphi_C^B$. Thus, in a symmetric oligopoly Cournot and Bertrand equilibria coincide. However, in an asymmetric oligopoly they do not coincide. For example, if $c_B = c_C$ but $c_A > c_B$, the difference between the relative profit in the Bertrand equilibrium and the relative profit in the Cournot equilibrium for each firm is

$$\varphi_A^B - \varphi_A^C = -\frac{18b^3(b+2)(c_B - c_A)^2}{(b-4)^2(b-1)(5b+4)^2} < 0,$$

$$\varphi_B^B - \varphi_B^C = \frac{9b^3(b+2)(c_B - c_A)^2}{(b-4)^2(b-1)(5b+4)^2} > 0,$$

and

$$\varphi_C^B - \varphi_C^C = \frac{9b^3(b+2)(c_B - c_A)^2}{(b-4)^2(b-1)(5b+4)^2} > 0.$$

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