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Symmetric multi-person zero-sum game with two sets of strategic variables

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Abstract

We consider a symmetric multi-person zero-sum game with two sets of alternative strategic variables which are related by invertible functions. They are denoted by \((s_1, s_2, \ldots, s_n)\) and \((t_1, t_2, \ldots, t_n)\) for players 1, 2, \ldots, \(n\). The number of players is larger than two. We consider a symmetric game in the sense that all players have the same payoff functions. We do not postulate differentiability of the payoff functions of players. We will show that the following patterns of competition, 1) all players choose \(s_i\), 2) all players choose \(t_i\) and 3) \(m\) players choose \(t_i\), \(i = 1, \ldots, m\) and \(n-m\) players choose \(s_j\), \(j = m+1, \ldots, n\) where \(1 \leq m \leq n-1\), are equivalent, that is, they yield the same outcome. However, in an asymmetric zero-sum game with more than two players the equivalence does not hold.

keywords multi-person zero-sum game, two strategic variables

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1 Introduction

We consider an $n$-person symmetric zero-sum game with two sets of strategic variables which are related by invertible functions. They are denoted by $(s_1, s_2, \ldots, s_n)$ and $(t_1, t_2, \ldots, t_n)$ for players 1, 2, \ldots, $n$. $n$ is an integer number which is larger than 2. We do not postulate differentiability of the payoff functions of players.

We will show that the following patterns of competition are equivalent, that is, they yield the same outcome.

1. All players choose $s_i$, $i \in N$. We call this competition $s_i$ competition.
2. All players choose $t_i$, $i \in N$. We call this competition $t_i$ competition.
3. Some players choose $t_i$ and other players choose $s_j$. Specifically, $m$ players choose $t_i$, $i = 1, 2, \ldots, m$, and $n - m$ players choose $s_j$, $j = m + 1, m + 2, \ldots, n$, where $1 \leq m \leq n - 1$. We call this competition $t_i - s_j$ competition.

We assume that the game is symmetric in the sense that all players have the same payoff functions, and consider symmetric equilibria where all players, whose strategic variables are $s_i$’s, choose the same values, and also all players, whose strategic variables are $t_i$’s, choose the same values.

Relative profit maximization in a symmetric oligopoly with differentiated goods is an example of symmetric $n$-person zero-sum game with two alternative strategic variables. Each firm chooses its output or price. The results of this paper imply that when firms in a symmetric oligopoly maximize their relative profits, Cournot and Bertrand equilibria are equivalent, and price-setting behavior and output-setting behavior are equivalent\(^1\).

However, in an asymmetric $n$-person zero-sum game with more than two players the equivalence does not hold. In Section 7 we present an example that shows the non-equivalence of Cournot and Bertrand equilibria in an asymmetric oligopoly.

2 The model

Consider an $n$-person zero-sum game with $n \geq 3$ as follows. There are $n$ players, 1, 2, \ldots, $n$. The set of players is denoted by $N$. They have two sets of alternative strategic variables, $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ and $(t_1, t_2, \ldots, t_n) \in T_1 \times T_2 \times \cdots \times T_n$. $S_i$ and $T_i$ for $i \in N$ are compact sets in metric spaces. The relations of them are represented by

\[ s_i = f_i(t_1, t_2, \ldots, t_n), \quad i \in N. \]

$\{f_1, f_2, \ldots, f_n\}$ is a continuous invertible function, and so it is one-to-one and onto function. We denote

\[ t_i = g_i(s_1, s_2, \ldots, s_n), \quad i \in N. \]

\(^1\)About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997). An oligopoly is symmetric when demand functions are symmetric and all firms have the same cost functions.
(g₁, g₂, . . . , gₙ) is also a continuous invertible function. The payoff functions of the players are \( u_i(s_1, s_2, . . . , s_n) \) for \( i \in N \). They are continuous and quasi-concave. We do not postulate differentiability of the payoff functions\(^2\). All players have the same payoff functions. Since the game is zero-sum, we have

\[
\sum_{i=1}^{n} u_i(s_1, s_2, . . . , s_n) = 0, \tag{1}
\]

for given \((s_1, s_2, . . . , s_n)\).

### 3 \( s_i \) Competition

First, consider competition by \( s_i \), \( i \in N \), for all players. Let \( s_i^* \), \( i \in N \), be the values of \( s_i \)'s which, respectively, maximizes \( u_i \), \( i \in N \), given \( s_j^* \), \( j \neq i \), in a neighborhood around \((s_1^*, s_2^*, . . . , s_n^*)\) in \( S_1 \times S_2 \times \cdots \times S_n \). Then,

\[
u_i(s_1^* \ldots s_i^* \ldots s_n^*) \geq u_i(s_1^* \ldots s_i \ldots s_n^*) \quad \text{for all } s_i \neq s_i^*, \ i \in N. \tag{2}\]

We assume that all \( s_i^* \)'s are equal at equilibria. Thus, \( u_i(s_1^*, s_2^*, . . . , s_n^*) \)'s for all \( i \) are equal, and by the property of zero-sum game they are zero. By symmetry of the game we have

\[
u_j(s_1^* \ldots s_i \ldots s_n^*) = u_k(s_1^* \ldots s_i \ldots s_n^*) \quad \text{for } j \neq i, k \neq i, j \neq k. \]

From this and (1)

\[
- \sum_{j=1, j \neq i} u_j(s_1^* \ldots s_i \ldots s_n^*) = -(n-1)u_j(s_1^* \ldots s_i \ldots s_n^*) = u_i(s_1^* \ldots s_i \ldots s_n^*).
\]

Therefore, from (2)

\[
u_j(s_1^* \ldots s_i \ldots s_n^*) \geq u_j(s_1^* \ldots s_i^* \ldots s_n^*) \quad \text{for } j \neq i.
\]

By symmetry

\[
u_i(s_1^* \ldots s_j \ldots s_n^*) \geq u_i(s_1^* \ldots s_i \ldots s_n^*) \quad \text{for } j \neq i.
\]

Combining this and (2)

\[
u_i(s_1^* \ldots s_i \ldots s_n^*) \leq u_i(s_1^* \ldots s_i^* \ldots s_n^*) \leq u_i(s_1^* \ldots s_j \ldots s_n^*)
\]

for all \( s_i \neq s_i^* \), and all \( s_j \neq s_j^* \), \( j \neq i, i \in N \).

This is equivalent to

\[
u_i(s_1^* \ldots s_i^* \ldots s_n^*) = \max_{s_i} u_i(s_1^* \ldots s_i \ldots s_n^*) = \min_{s_j} u_i(s_1^* \ldots s_j \ldots s_n^*),
\]

\( j \neq i \) given \( s_k^*, k \neq i, j \).

\(^2\)In Satoh and Tanaka (2016) we analyze maximin and minimax strategies in oligopoly when payoff functions of firms are differentiable.
Lemma 1. The following three statements are equivalent.

1. There exists a Nash equilibrium in the $s_i$ competition game.

2. Given $s^*_k$ for all $k \neq i, j$, the following relation holds.
\[
v^i_j = \max_{s_i} \min_{s_j} u_i(s_i, s_j, s^{*}_{i,j}) = \max_{s_j} \min_{s_i} u_i(s_i, s_j, s^{*}_{i,j}) \equiv v^j_i \text{ for any pair of } i \text{ and } j.
\]

3. There exists a real number $v_s, s^m_i$ and $s^m_j$ such that
\[
u_i(s^m_i, s_j, s^{*}_{i,j}) \geq v_s \text{ for any } s_j, \text{ and } \nu_i(s_i, s^m_j, s^{*}_{i,j}) \leq v_s \text{ for any } s_i,
\]
for any pair of $i$ and $j$.

**Proof.** $(1 \rightarrow 2)$

Let $s^i_l$ and $s^j_l$ be the equilibrium strategies of Player $i$ and $j$. Then,
\[
v^i_j = \max_{s_i} \min_{s_j} u_i(s_i, s_j, s^{*}_{i,j}) \leq \max_{s_i} u_i(s_i, s^*_j, s^{*}_{i,j}) = u_i(s^*_i, s^*_j, s^{*}_{i,j})
\]
\[= \min_{s_j} \max_{s_i} u_i(s^*_i, s_j, s^{*}_{i,j}) \leq \min_{s_j} \max_{s_i} u_i(s_i, s_j, s^{*}_{i,j}) = v^j_i.
\]

On the other hand, $\min_{s_j} u_i(s_i, s_j, s^{*}_{i,j}) \leq u_i(s_i, s^*_j, s^{*}_{i,j})$, then $\max_{s_i} \min_{s_j} u_i(s_i, s_j, s^{*}_{i,j}) \leq \max_{s_i} u_i(s_i, s^*_j, s^{*}_{i,j})$, and so $\max_{s_i} \min_{s_j} u_i(s_i, s_j, s^{*}_{i,j}) \leq \min_{s_j} \max_{s_i} u_i(s_i, s_j, s^{*}_{i,j})$.

Thus, $v^i_j \leq v^j_i$, and we have $v^i_j = v^j_i$.

$(2 \rightarrow 3)$

Let $s^m_i = \arg \max_{s_i} \min_{s_j} u_i(s_i, s_j, s^{*}_{i,j})$ (the maximin strategy), $s^m_j = \arg \min_{s_j} \max_{s_i} u_i(s_i, s_j, s^{*}_{i,j})$ (the minimax strategy), and let $v_s = v^j_i = v^i_j$. Then, we have
\[
u_i(s^m_i, s_j, s^{*}_{i,j}) \geq \min_{s_j} \max_{s_i} u_i(s^m_i, s_j, s^{*}_{i,j}) = \max_{s_i} \min_{s_j} u_i(s_i, s^m_j, s^{*}_{i,j}) = v_s
\]
\[= \min_{s_j} \max_{s_i} u_i(s^m_i, s_j, s^{*}_{i,j}) = \max_{s_i} \min_{s_j} u_i(s^m_i, s^m_j, s^{*}_{i,j}) \geq u_i(s^m_i, s^m_j, s^{*}_{i,j}).
\]

$(3 \rightarrow 1)$

From $(3)$
\[
u_i(s^m_i, s_j, s^{*}_{i,j}) \geq v_s \geq u_i(s^m_i, s^m_j, s^{*}_{i,j}) \text{ for all } s_i \in S_i, s_j \in S_j.
\]

Putting $s_i = s^m_i$ and $s_j = s^m_j$, we see $v_s = u_i(s^m_i, s^m_j, s^{*}_{i,j})$ and $(s^m_i,s^m_j,s^{*}_{i,j})$ is an equilibrium. Thus, $s^m_i = s^*_i$ and $s^m_j = s^*_j$.

Since at equilibria all $u_i$'s are zero, we have $v^i_j = v^j_i = v_s = 0$. Denote the values of $t_i, i \in N$, which are derived from the following equation;
\[
(t_1, t_2, \ldots, t_n) = (g_1(s^*_1, s^*_2, \ldots, s^*_n), g_2(s^*_1, s^*_2, \ldots, s^*_n), \ldots, g_n(s^*_1, s^*_2, \ldots, s^*_n)),
\]
by $t^*_i, i \in N$.  

4 \textbf{t}_i \textbf{ competition}

Next consider competition by \( t_i, \ i \in N \), for all players. In this section we use the following notation.

\[ v_i(t_1, \ldots, t_n) = u_i(f_1(t_1, \ldots, t_n), \ldots, f_n(t_1, \ldots, t_n)) \text{ for each } i \in N. \]

Let \( \tilde{t}_i, \ i \in N \), be the values of \( t_i \)'s which, respectively, maximizes \( v_i, \ i \in N \), given \( \tilde{t}_j, \ j \neq i \), in a neighborhood around \((\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)\) in \( T_1 \times T_2 \times \cdots \times T_n \). Then,

\[ v_i(\tilde{t}_1, \ldots, \tilde{t}_i, \ldots, \tilde{t}_n) \geq v_i(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) \text{ for all } t_i \neq \tilde{t}_i, \ i \in N, \quad (4) \]

We assume that all \( \tilde{t}_i \)'s are equal at equilibria. Thus, \( v_i(\tilde{t}_1, \ldots, \tilde{t}_i, \ldots, \tilde{t}_n) \) for all \( i \) are equal, and by the property of zero-sum game all \( v_i \)'s are zero. By symmetry of the model

\[ v_j(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) = v_k(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) \text{ for } j \neq i, k \neq i, j \neq k. \]

From this and (1)

\[ - \sum_{j=1, j \neq i} v_j(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) = -(n-1)v_j(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) = v_i(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n). \]

Therefore, from (4)

\[ v_j(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) \geq v_j(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) \text{ for } j \neq i. \]

By symmetry we get

\[ v_i(\tilde{t}_1, \ldots, t_j, \ldots, \tilde{t}_n) \geq v_i(\tilde{t}_1, \ldots, t_j, \ldots, \tilde{t}_n) \text{ for } j \neq i. \]

Combining this and (4)

\[ v_i(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) \leq v_i(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) \leq v_i(\tilde{t}_1, \ldots, t_j, \ldots, \tilde{t}_n) \]

for all \( t_i \neq \tilde{t}_i \), and all \( t_j \neq \tilde{t}_j, \ j \neq i, \ i \in N \).

This is equivalent to

\[ v_i(\tilde{t}_1, \ldots, \tilde{t}_i, \ldots, \tilde{t}_n) = \max_{t_i} v_i(\tilde{t}_1, \ldots, t_i, \ldots, \tilde{t}_n) = \min_{t_j} v_i(\tilde{t}_1, \ldots, t_j, \ldots, \tilde{t}_n) \]

\[ j \neq i \text{ given } \tilde{t}_k, \ k \neq i, j. \]

Let \( \tilde{t}_{-i,j} \) be a vector of \( \tilde{t}_k \) for \( k \neq i, j \). Similarly to Lemma 1 we can show the following lemma.

\textbf{Lemma 2.} The following three statements are equivalent.

(1) There exists a Nash equilibrium in the \( t_i \) competition game.
(2) Given \( \bar{t}_k \) for all \( k \neq i, j \), the following relation holds.

\[
\forall_i^j = \max \min_{t_i, t_j} v_i(t_i, t_j, \bar{t}_i, \bar{t}_j) = \min \max_{t_i, t_j} v_i(t_i, t_j, \bar{t}_i, \bar{t}_j) = \forall_i^j \text{ for any pair of } i \text{ and } j.
\]

(3) There exists a real number \( v_i, t^m_i \) and \( t^m_j \) such that

\[
v_i(t^m_i, t_j, \bar{t}_i, \bar{t}_j) \geq v_i \text{ for any } t_j, \text{ and } v_i(t_i, t^m_j, \bar{t}_i, \bar{t}_j) \leq v_i \text{ for any } t_i \text{ for any pair of } i \text{ and } j.
\]

Thus, \( t^m_i = \bar{t}_i \) and \( t^m_j = \bar{t}_j \). Since at equilibria all \( v_i \)'s are zero, we have \( \forall_i^j = v^j = v_t = 0 \).

Denote the values of \( \bar{s}_i, i \in \mathbf{N} \), which are derived from the following equation;

\[
(s_1, s_2, \ldots, s_n) = (f_1(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n), f_2(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n), \ldots, f_n(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n)),
\]

by \( \bar{s}_i, i \in \mathbf{N} \).

5 \( t_i - s_j \) competition

Next, consider \( t_i - s_j \) competition. Assume that \( m \) players choose \( t_i, i = 1, 2, \ldots, m \), and the remaining \( n - m \) players choose \( s_j, j = m + 1, m + 2, \ldots, n \). \( m \) is an integer such that \( 1 \leq m \leq n - 1 \). At least one player chooses \( t_i \), and at least one player chooses \( s_j \). In this section we use the following notation.

\[
w_i(t_1, \ldots, t_m, s_{m+1}, \ldots, s_n)
= u_i(f_1(t_1, \ldots, t_m, g_{m+1}(\ldots), \ldots, g_n(\ldots)), \ldots, f_m(t_1, \ldots, t_m, g_{m+1}(\ldots), \ldots, g_n(\ldots)), s_{m+1}, \ldots, s_n)
\]

for each \( i \in \mathbf{N} \), where

\[
g_j(\ldots) = g_j(s_1, \ldots, s_m, s_{m+1}, \ldots, s_n) \text{ for } j \in \{m+1, \ldots, n\}
\]

with

\[
s_i = f_i(t_1, \ldots, t_m, g_{m+1}(\ldots), \ldots, g_n(\ldots)) \text{ for } i \in \{1, \ldots, m\}.
\]

Let \( \bar{t}_i, i = 1, 2, \ldots, m \), and \( \bar{s}_j, j = m + 1, \ldots, n \), be the values of \( t_i \) and \( s_j \) which maximizes, respectively, \( w_i \) and \( w_j \), in a neighborhood around the equilibrium point. Then,

\[
w_i(\bar{t}_1, \ldots, \bar{t}_i, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_n)
\geq w_i(\bar{t}_1, \ldots, \bar{t}_i, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_n) \text{ for all } t_i \neq \bar{t}_i, i = 1, 2, \ldots, m,
\]

and

\[
w_j(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n)
\geq w_j(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n) \text{ for all } s_j \neq \bar{s}_j, j = m + 1, m + 2, \ldots, n.
\]
Lemma 3. We assume that at equilibria all $\bar{t}_i$, $i = 1, 2, \ldots, m$, are equal, and all $\bar{s}_j$, $j = m + 1, m + 2, \ldots, n$, are equal. Since all players have the same payoff functions, all $w_i$, $i = 1, 2, \ldots, m$, are equal, and all $w_j$, $j = m + 1, m + 2, \ldots, n$, are equal. Then, from (1) we obtain

$$mw_i(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n) + (n - m)w_j(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n) = 0,$$

and so

$$w_j(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n) = -\frac{m}{n-m}w_i(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n).$$

If $w_i = 0$ (or $w_j = 0$), then $w_j = 0$ (or $w_i = 0$). (6) is rewritten as

$$w_i(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n) \leq w_i(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, s_j, \ldots, \bar{s}_n)$$

for all $s_j \neq \bar{s}_j$, $j = m + 1, m + 2, \ldots, n$.

Combining this and (5),

$$w_i(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n) \leq w_i(\bar{t}_1, \ldots, \bar{t}_i, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n)$$

for all $t_i \neq \bar{t}_i$, $i = 1, 2, \ldots, m$, and all $s_j \neq \bar{s}_j$, $j = m + 1, m + 2, \ldots, n$.

This is equivalent to

$$w_i(\bar{t}_1, \ldots, \bar{t}_i, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n) = \max_{\bar{t}_i} w_i(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_n)$$

and

$$= \min_{s_j} w_i(\bar{t}_1, \ldots, \bar{t}_m, \bar{s}_{m+1}, \ldots, \bar{s}_j, \ldots, \bar{s}_n)$$

for any pair of $i, j$.

Let $\bar{v}_{-i}$ be a vector of $\bar{v}_k$ for $k \in \{1, \ldots, m\}$, $k \neq i$ and $\bar{v}_{-j}$ be a vector of $\bar{v}_l$ for $l \in \{m + 1, \ldots, n\}$, $l \neq j$. Similarly to Lemma 1 we can show the following lemma.

Lemma 3. The following three statements are equivalent.

1. There exists a Nash equilibrium in the $t_i - s_j$ competition game.

2. Given $\bar{v}_k$, $k \neq i$, $k \in \{1, \ldots, m\}$ and $\bar{v}_l$, $l \neq j$, $l \in \{m + 1, \ldots, n\}$, the following relation holds.

$$v_i^{s_j} = \max_{t_i} \min_{s_j} w_i(t_i, \bar{v}_{-i}, s_j, \bar{v}_{-j}) = \min_{s_j} \max_{t_i} w_i(t_i, \bar{v}_{-i}, s_j, \bar{v}_{-j}) = v_j^{s_i}$$

for any pair of $i$ and $j$.

3. There exists a real number $v_{t_s}, t_i^{s_j}$ and $s_j^{s_i}$ such that

$$w_i(t_i^{s_j}, \bar{v}_{-i}, s_j, \bar{v}_{-j}) \geq v_{t_s}$$

for any $s_j$, and $w_i(t_i, \bar{v}_{-i}, s_j^{s_i}, \bar{v}_{-j}) \leq v_{t_s}$ for any $t_i$ for any pair of $i$ and $j$.

Thus, $t_i^{s_j} = \bar{t}_i$ and $s_j^{s_i} = \bar{s}_j$. Denote the values of $s_i$, $i \in \{1, 2, \ldots, m\}$ and the values of $t_j$, $j \in \{m + 1, m + 2, \ldots, n\}$, which are derived from the following equation:

$$(s_1, s_2, \ldots, s_m) = (f_1(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m, \bar{t}_{m+1}, \bar{t}_{m+2}, \ldots, \bar{t}_n), f_2(\ldots), \ldots, f_m(\ldots)),$$

$$(t_{m+1}, t_{m+2}, \ldots, t_n) = (g_1(s_1, s_2, \ldots, s_m, \bar{s}_{m+1}, \bar{s}_{m+2}, \ldots, s_n), g_2(\ldots), \ldots, g_m(\ldots)),$$

by $\bar{s}_i$, $i \in \{1, 2, \ldots, m\}$ and $\bar{t}_j$, $j \in \{m + 1, m + 2, \ldots, n\}$. 7
6 Equivalence of three patterns of competition

First we show the following proposition.

Proposition 1. $s_i$ competition and $t_i - s_j$ competition where one player, Player 1, chooses $t_1$ are equivalent.

Each player $j$ in \( \{2, \ldots, n\} \) chooses $s_j$ as his/her strategic variable. To prove this proposition we need the following lemma.

Lemma 4.

$$\max_{s_j} \min_{s_1} w_1(t_1, s_j, \bar{s}_{-j}) = \min_{s_1} u_1(s_1, s_j, \bar{s}_{-j}).$$

where $\bar{s}_{-j}$ is a vector of $\bar{s}_l$, for $l \in \{2, \ldots, n\}$, $l \neq j$.

Proof. $\min_{s_j} w_1(t_1, s_j, \bar{s}_{-j})$ is the minimum of $w_1 = u_1$ with respect to $s_j$ given $t_1$ and $\bar{s}_{-j}$. Let $s_j(t_1) = \arg \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j})$, and fix the value of $s_1$ at

$$s^0_1 = f_1(t_1, g_2(s^0_1, s_2, \ldots, s_j(t_1), \ldots, s_n), \ldots, g_n(s^0_1, s_2, \ldots, s_j(t_1), \ldots, s_n)).$$

Then, we have

$$\min_{s_j} u_1(s^0_1, s_j, \bar{s}_{-j}) \leq u_1(s^0_1, s_j(t_1), \bar{s}_{-j}) = \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}),$$

where $\min_{s_j} u_1(s^0_1, s_j, \bar{s}_{-j})$ is the minimum of $u_1$ with respect to $s_j$ given the value of $s_1$ at $s^0_1$. We assume that $s_j(t_1) = \arg \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j})$ is single-valued. By the maximum theorem and continuity of $w_1, s_j(t_1)$ is continuous. Then, any value of $s^0_1$ in some neighborhood around $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n$ can be realized by appropriately choosing $t_1$ given $s_j$ and $\bar{s}_{-j}$ as $s^0_1 = f_1(t_1, g_2(s^0_1, s_2, \ldots, s_j(t_1), \ldots, s_n), \ldots, g_n(s^0_1, s_2, \ldots, s_j(t_1), \ldots, s_n))$. Therefore,

$$\max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}) \leq \max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}). \tag{7}$$

On the other hand, $\min_{s_j} u_1(s_1, s_j, \bar{s}_{-j})$ is the minimum of $u_1$ with respect to $s_j$ given $s_1$ and $\bar{s}_{-j}$. Let $s_j(s_1) = \arg \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j})$, and fix the value of $t_1$ at $g_1(s_1, s_j(s_1), \bar{s}_{-j})$. Then, we have

$$\min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) = \min_{s_j} w_1(g_1(s_1, s_j(s_1), \bar{s}_{-j}), s_j, \bar{s}_{-j}) \leq u_1(s_1, s_j(s_1), \bar{s}_{-j}) = \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}),$$

where $\min_{s_j} u_1(g_1(s_1, s_j(s_1), \bar{s}_{-j}), s_j, \bar{s}_{-j})$ is the minimum of $u_1$ with respect to $s_j$ given the value of $t_1$ at $g_1(s_1, s_j(s_1), \bar{s}_{-j})$. We assume that $s_j(s_1) = \arg \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j})$ is single-valued. By the maximum theorem and continuity of $u_1, s_j(s_1)$ is continuous. Then, any value of $t_1$ in some neighborhood around $(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n)$ can be realized by appropriately choosing $s_1$ given $s_j$ and $\bar{s}_{-j}$ as $t_1 = g_1(s_1, s_j(s_1), \bar{s}_{-j})$. Therefore,

$$\max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) \leq \max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}). \tag{8}$$

Combining (7) and (8), we get

$$\max_{t_1} \min_{s_j} w_1(t_1, s_j, \bar{s}_{-j}) = \max_{s_1} \min_{s_j} u_1(s_1, s_j, \bar{s}_{-j}).$$
Proof of Proposition 1. We show that the condition for \((\tilde{s}_1, \ldots, \tilde{s}_n)\) and the condition for \((s^*_1, \ldots, s^*_n)\) are the same. From Lemma 3

\[
\max_{t_1} \min_{s_j} w_1(t_1, s_j, \tilde{s}_{-j}) = \min_{s_j} \max_{t_1} w_1(t_1, s_j, \tilde{s}_{-j}).
\]

Since any value of \(s_1\) can be realized by appropriately choosing \(t_1\) given \(s_j, j \neq 1, \) and \(\tilde{s}_{-j}, \) we have

\[
\max_{s_1} u_1(s_1, s_j, \tilde{s}_{-j}) = \max_{t_1} w_1(t_1, s_j, \tilde{s}_{-j}) \text{ for any } s_j.
\]

Thus,

\[
\min_{s_j} \max_{s_1} u_1(s_1, s_j, \tilde{s}_{-j}) = \min_{s_j} \max_{t_1} w_1(t_1, s_j, \tilde{s}_{-j}).
\]

With Lemma 4 we conclude

\[
\max_{s_1} \min_{s_j} u_1(s_1, s_j, \tilde{s}_{-j}) = \min_{s_j} \max_{s_1} u_1(s_1, s_j, \tilde{s}_{-j}) = u_1(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = 0.
\]

This is 2 of Lemma 1. The result of this proposition means that \(w_1(\tilde{t}_1, \tilde{s}_j, \tilde{s}_{-j}) = w_j(\tilde{t}_1, \tilde{s}_j, \tilde{s}_{-j}) = 0.\)

Next we show the following proposition.

Proposition 2. \(t_i\) competition and \(t_i - s_j\) competition where one player, Player \(n,\) chooses \(s_n,\) are equivalent.

To prove this proposition we need the following lemma.

Lemma 5.

\[
\min_{t_i} \max_{s_n} w_i(t_i, \tilde{t}_{-i}, s_n) = \min_{t_i} \max_{s_n} v_i(t_i, \tilde{t}_{-i}, t_n).
\]

\(\tilde{t}_{-i}\) is a vector of \(\tilde{t}_k\) for \(k \in \{1, \ldots, n-1\}, k \neq i.\)

Proof. \(\max_{t_i} w_i(t_i, \tilde{t}_{-i}, s_n)\) is the maximum of \(w_i(= v_i)\) with respect to \(t_i\) given \(s_n\) and \(\tilde{t}_{-i}.\) Let \(t_i(s_n) = \arg \max_{t_i} w_i(t_i, \tilde{t}_{-i}, s_n),\) and fix the value of \(t_n\) at

\[
t_n^0 = g_n(f_i(t_i(s_n), \tilde{t}_{-i}, t_n), f_{-i}((t_i(s_n), \tilde{t}_{-i}, t_n)), s_n).
\]

where \(f_{-i}\) is a vector of \(f_k\) for \(k \in \{2, \ldots, n-1\}, k \neq i.\) Then, we have

\[
\max_{t_i} v_i(t_i, \tilde{t}_{-i}, s_n) = \max_{t_i} v_i(t_i, \tilde{t}_{-i}, g_n(f_i(t_i(s_n), \tilde{t}_{-i}, t_n), f_{-i}((t_i(s_n), \tilde{t}_{-i}, t_n)), s_n))
\]

\[
\geq w_i(t_i(s_n), \tilde{t}_{-i}, s_n) = \max_{t_i} w_i(t_i, \tilde{t}_{-i}, s_n),
\]

where \(\max_{t_i} v_i(t_i, \tilde{t}_{-i}, s_n)\) is the maximum of \(v_i\) with respect to \(t_i\) given the value of \(t_n\) at

\[
g_n(f_i(t_i(s_n), \tilde{t}_{-i}, t_n), f_{-i}((t_i(s_n), \tilde{t}_{-i}, t_n)), s_n).
\]

We assume that \(t_i(s_n) = \arg \max_{t_i} w_i(t_i, \tilde{t}_{-i}, s_n)\) is single-valued. By the maximum theorem and continuity of \(w_i, t_i(s_n)\) is continuous. Then, any value of \(t_n^0\) in some neighborhood around \((\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)\) can be realized by appropriately choosing \(s_n\) given \(t_i\) and \(\tilde{t}_{-i}\) as

\[
t_n^0 = g_n(f_i(t_i(s_n), \tilde{t}_{-i}, t_n), f_{-i}((t_i(s_n), \tilde{t}_{-i}, t_n)), s_n).
\]
Therefore,
\[
\min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_i, t_n) \geq \min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_i, s_n). \tag{9}
\]

On the other hand, \(\max_{t_i} v_i(t_i, \bar{t}_i, t_n)\) is the maximum of \(v_i\) with respect to \(t_i\) given \(t_n\) and \(\bar{t}_i\). Let \(t_i(t_n) = \arg \max_{t_i} v_i(t_i, \bar{t}_i, t_n)\), and fix the value of \(s_n\) at \(f_n(t_i(t_n), \bar{t}_i, t_n)\). Then, we have
\[
\max_{t_i} w_i(t_i, \bar{t}_i, s_n) = \max_{t_i} w_i(t_i, \bar{t}_i, f_n(t_i(t_n), \bar{t}_i, t_n)) \geq v_i(t_i(t_n), \bar{t}_i, t_n) = \max_{t_i} v_i(t_i, \bar{t}_i, t_n),
\]
where \(\max_{t_i} w_i(t_i, \bar{t}_i, s_n)\) is the maximum of \(w_i(= v_i)\) with respect to \(t_i\) given the value of \(s_n\) at \(f_n(t_i(t_n), \bar{t}_i, t_n)\). We assume that \(t_i(t_n) = \arg \max_{t_i} v_i(t_i, \bar{t}_i, t_n)\) is single-valued. By the maximum theorem and continuity of \(v_i, t_i(t_n)\) is continuous. Then, any value of \(s_n\) in some neighborhood around \((\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n)\) can be realized by appropriately choosing \(t_n\) given \(\bar{t}_i\) and \(\bar{t}_i\) as \(s_n = f_n(t_i(t_n), \bar{t}_i, t_n)\). Therefore,
\[
\min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_i, s_n) \geq \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_i, t_n). \tag{10}
\]
Combining (9) and (10), we get
\[
\min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_i, s_n) = \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_i, t_n).
\]

**Proof of Proposition 2.** We show that the condition for \((\bar{t}_1, \ldots, \bar{t}_n)\) and the condition for \((\bar{t}_1, \ldots, \bar{t}_n)\) are the same. From Lemma 3
\[
\max_{t_i} \min_{s_n} w_i(t_i, \bar{t}_i, s_n) = \min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_i, s_n).
\]
Since any value of \(t_n\) can be realized by appropriately choosing \(s_n\) given \(t_i, i \neq n\), and \(\bar{t}_i\), we have \(\min_{s_n} w_i(t_i, \bar{t}_i, s_n) = \min_{s_n} v_i(t_i, \bar{t}_i, t_n)\) for any \(t_i\). Thus,
\[
\max_{t_i} \min_{t_n} v_i(t_i, \bar{t}_i, t_n) = \max \min_{t_i} w_i(t_i, \bar{t}_i, s_n).
\]
From Lemma 5 we have \(\min_{s_n} \max_{t_i} w_i(t_i, \bar{t}_i, s_n) = \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_i, t_n)\). Therefore, we obtain
\[
\max_{t_i} \min_{t_n} v_i(t_i, \bar{t}_i, t_n) = \min_{t_n} \max_{t_i} v_i(t_i, \bar{t}_i, t_n) = v_i(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n) = 0.
\]
This is 2 of Lemma 2. The result of this proposition means that \(w_i(\bar{t}_1, \bar{t}_i, \bar{s}_n) = w_n(\bar{t}_1, \bar{t}_i, \bar{s}_n) = 0\).

Finally we show the following proposition.

**Proposition 3.** \(t_i - s_j\) competition in which \(m\) players choose \(t_i\)'s as their strategic variables, and \(t_i - s_j\) competition in which \(m - 1\) players choose \(t_i\)'s as their strategic variables are equivalent, where \(2 \leq m \leq n - 1\).
Lemma 6.

$$\max_{t_i} \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) = \max_{s_j} w_i(s_j, \bar{t}_{-i}, s_j, \bar{s}_{-j}).$$

$\bar{t}_{-i}$ is a vector of $\bar{t}_k$ for $k \in \{1, \ldots, m\}, k \neq i$. $\bar{s}_{-j}$ is a vector of $\bar{s}_l$ for $l \in \{m + 1, \ldots, n\}, l \neq j$.

Proof: $\min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j})$ is the minimum of $w_i(= u_i)$ with respect to $s_j$ given $t_i$, $\bar{t}_{-i}$ and $\bar{s}_{-j}$. Let $s_j(t_i) = \arg \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j})$. The values of variables other than $t_i$, $s_j(t_i)$, $\bar{t}_{-i}$ and $\bar{s}_{-j}$ are determined by the following equations;

$$s_1 = f_1(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n),$$

$$\quad \vdots$$

$$s_i = f_i(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n),$$

$$\quad \vdots$$

$$s_m = f_m(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n),$$

$$t_{m+1} = g_{m+1}(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(t_i), \ldots, \bar{s}_n),$$

$$\quad \vdots$$

$$t_j = g_j(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(t_i), \ldots, \bar{s}_n),$$

$$\quad \vdots$$

$$t_n = g_n(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(t_i), \ldots, \bar{s}_n).$$

Denote this $s_i$ by $s_i^0$, and fix the value of $s_i$ at $s_i^0$. Then, we have

$$\min_{s_j} w_i(s_i^0, \bar{t}_{-i}, s_j, \bar{s}_{-j}) \leq w_i(t_i, \bar{t}_{-i}, s_j(t_i), \bar{s}_{-j}) = \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}),$$

where $\min_{s_j} w_i(s_i^0, \bar{t}_{-i}, s_j, \bar{s}_{-j})$ is the minimum of $w_i(= u_i)$ with respect to $s_j$ given the value of $s_i$ at $s_i^0$. We assume that $s_j(t_i) = \arg \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j})$ is single-valued. By the maximum theorem and continuity of $w_i$, $s_j(t_i)$ is continuous. Then, any value of $s_i^0$ in some neighborhood around $(\bar{s}_i, \bar{t}_{-i}, \bar{s}_j, \bar{s}_{-j})$ can be realized by appropriately choosing $t_i$ given $s_j(t_i)$, $\bar{t}_{-i}$ and $\bar{s}_j$. Therefore,

$$\max_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}) \leq \max_{t_i} \min_{s_j} w_i(t_i, \bar{t}_{-i}, s_j, \bar{s}_{-j}). \quad (11)$$

On the other hand, $\min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j})$ is the minimum of $w_i(= u_i)$ with respect to $s_j$ given $s_i$, $\bar{t}_{-i}$ and $\bar{s}_{-j}$. Let $s_j(s_i) = \arg \min_{s_j} w_i(s_i, \bar{t}_{-i}, s_j, \bar{s}_{-j})$. The values of variables other than $s_i$, $s_j(s_i)$, $\bar{t}_{-i}$ and $\bar{s}_{-j}$ are determined by the following equations;

$$s_1 = f_1(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n),$$

$$\vdots$$

$$s_i = f_i(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n),$$

$$\vdots$$

$$s_m = f_m(\bar{t}_1, \ldots, t_i, \ldots, \bar{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n),$$

$$t_{m+1} = g_{m+1}(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(s_i), \ldots, \bar{s}_n),$$

$$\vdots$$

$$t_j = g_j(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(s_i), \ldots, \bar{s}_n),$$

$$\vdots$$

$$t_n = g_n(s_1, \ldots, s_i, \ldots, s_m, \bar{s}_{m+1}, \ldots, s_j(s_i), \ldots, \bar{s}_n).$$
\begin{align*}
  s_m &= f_m(t_1, \ldots, t_i, \ldots, \tilde{t}_m, t_{m+1}, \ldots, t_j, \ldots, t_n), \\
  t_{m+1} &= g_{m+1}(s_1, \ldots, s_i, \ldots, s_m, \tilde{s}_{m+1}, \ldots, s_j(s_i), \ldots, \tilde{s}_n), \\
  &\ldots \\
  t_i &= g_i(s_1, \ldots, s_i, \ldots, s_m, \tilde{s}_{m+1}, \ldots, s_j(s_i), \ldots, \tilde{s}_n), \\
  t_j &= g_j(s_1, \ldots, s_i, \ldots, s_m, \tilde{s}_{m+1}, \ldots, s_j(s_i), \ldots, \tilde{s}_n), \\
  &\ldots \\
  t_n &= g_n(s_1, \ldots, s_i, \ldots, s_m, \tilde{s}_{m+1}, \ldots, s_j(s_i), \ldots, \tilde{s}_n).
\end{align*}

Denote this \( t_i \) by \( t_i^0 \), and fix the value of \( t_i \) at \( t_i^0 \). Then, we have
\[
\min_{\tilde{s}_j} w_i(t_i^0, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) \leq w_i(s_i, \tilde{t}_{i-1}, s_j(s_i), \tilde{s}_{-j}) = \min_{\tilde{s}_j} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}),
\]
where \( \min_{s_j} w_i(t_i^0, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) \) is the minimum of \( w_i(= u_i) \) with respect to \( s_j \) given the value of \( t_i \) at \( t_i^0 \). We assume that \( s_j(s_i) = \arg\min_{s_j} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) \) is single-valued. By the maximum theorem and continuity of \( w_i, s_j(s_i) \) is continuous. Then, any value of \( t_i^0 \) in some neighborhood around \((\tilde{t}_i, \tilde{t}_{i-1}, \tilde{s}_j, \tilde{s}_{-j})\) can be realized by appropriately choosing \( s_i \) given \( s_j, \tilde{s}_{-j} \) and \( \tilde{t}_{i-1} \). Therefore,
\[
\max_{t_i} \min_{s_j} w_i(t_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) \leq \max_{s_i} \min_{s_j} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}). \tag{12}
\]
Combining (11) and (12), we get
\[
\max_{t_i} \min_{s_j} w_i(t_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) = \max_{s_i} \min_{s_j} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}).
\]

\[\square\]

\textit{Proof of Proposition 3.} We show that the condition for \((\tilde{t}_i, \tilde{t}_{i-1}, \tilde{s}_j, \tilde{s}_{-j})\) and the condition for \((\tilde{s}_i, \tilde{t}_{i-1}, \tilde{s}_j, \tilde{s}_{-j})\) are the same. From Lemma 3
\[
\max_{t_i} \min_{s_j} w_i(t_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) = \min_{\tilde{s}_j} \max_{t_i} w_i(t_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}).
\]
Since any value of \( s_i \) can be realized by appropriately choosing \( t_i \) given \( s_j, \tilde{t}_{i-1}, \tilde{s}_{-j} \), we have
\[
\max_{t_i} w_i(t_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) = \max_{s_i} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) \text{ for any } s_j. \quad \text{Thus,}
\]
\[
\min_{s_j} \max_{t_i} w_i(t_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) = \min_{s_i} \max_{t_i} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}).
\]
With Lemma 6 we conclude
\[
\max_{s_i} \min_{s_j} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}) = \max_{s_i} \min_{s_j} w_i(s_i, \tilde{t}_{i-1}, s_j, \tilde{s}_{-j}). \quad \square
\]
Summarizing these results we conclude.

**Proposition 4.** \( s_i \) competition, \( t_i \) competition and \( t_i-s_j \) competition with any number of players whose strategic variables are \( t_i \)’s are equivalent, and payoffs of all players at any equilibrium are zero.

**Proof.** From Proposition 1

\[ w_1(\tilde{t}_1, \tilde{s}_j, \tilde{s}_{-j}) = w_j(\tilde{t}_1, \tilde{s}_j, \tilde{s}_{-j}) = 0, \ j \in \{2, 3, \ldots, n\}. \]

This means that the payoffs of all players when only one player chooses \( t_i \) and all other players choose \( s_j \)’s are equivalent, and payoffs of all players at any equilibrium are zero. From Proposition 2

\[ w_n(\tilde{t}_i, \tilde{t}_{-i}, \tilde{s}_n) = w_i(\tilde{t}_i, \tilde{t}_{-i}, \tilde{s}_n) = 0, \ i \in \{1, 2, \ldots, n-1\}. \]

This means that the payoffs of all players when only one player_choose \( s_j \) and all other players choose \( t_i \)’s are zero. From the result of Proposition 3

\[ w_i(\tilde{t}_1, \tilde{t}_2, \tilde{s}_j, \tilde{s}_{-j}) = w_j(\tilde{t}_1, \tilde{t}_2, \tilde{s}_j, \tilde{s}_{-j}) = 0, \ i \in \{1, 2\}, \ j \in \{3, 4, \ldots, n\}. \]

This means that the payoffs of all players when two players choose \( t_i \)’s and all other players choose \( s_j \)’s are zero. Then, inductively we conclude that

\[ w_i(\tilde{t}_1, \tilde{t}_{-i}, \tilde{s}_j, \tilde{s}_{-j}) = w_j(\tilde{t}_i, \tilde{t}_{-i}, \tilde{s}_j, \tilde{s}_{-j}) = 0, \]

in the game where \( m \) players choose \( t_i \)’s as their strategic variables for any \( m \) such that \( 2 \leq m \leq n-1 \). \( i \) denotes a player whose strategic variable is \( t_i \), and \( j \) denotes a player whose strategic variable is \( s_j \). Thus, payoffs of all players in any \( t_i-s_j \) competition are zero. By the definitions of \( s_i \) competition and \( t_i \) competition payoffs of all players in the \( s_i \) competition and the \( t_i \) competition are zero. \( \square \)

7 Example: relative profit maximization in oligopoly with differentiated goods

Consider an oligopoly with three firms producing differentiated goods. The firms are A, B and C. The inverse demand functions are

\[ p_A = a - x_A - bx_B - bx_C, \]
\[ p_B = a - x_B - bx_A - bx_C, \]
and
\[ p_C = a - x_C - bx_A - bx_B, \]

where \( 0 < b < 1 \). \( p_A, p_B \) and \( p_C \) are the prices of the goods of Firm A, B and C, and \( x_A, x_B \) and \( x_C \) are the outputs of them. From these inverse demand functions the direct demand functions are derived as follows;

\[ x_A = \frac{(1-b)a - (1 + b)p_A + b(p_A + p_C)}{(1-b)(1+2b)}, \]
The (absolute) profits of the firms are
\[ \pi_A = p_A x_A - c_A x_A, \]
\[ \pi_B = p_B x_B - c_B x_B, \]
\[ \pi_C = p_C x_C - c_C x_C. \]
c_A, c_B and c_C are the constant marginal costs of Firm A, B and C. The relative profits of the firms are
\[ \varphi_A = \pi_A - \frac{\pi_B + \pi_C}{2}, \]
\[ \varphi_B = \pi_B - \frac{\pi_A + \pi_C}{2}, \]
\[ \varphi_C = \pi_C - \frac{\pi_A + \pi_B}{2}. \]
We see
\[ \varphi_A + \varphi_B + \varphi_C = 0, \]
so the game is zero-sum. In a Cournot model the firms determine their outputs to maximize their relative profits. In a Bertrand model they determine the prices of their goods to maximize their relative profits. The Cournot equilibrium outputs are
\[ x_A^C = \frac{bc_C + bc_B - bc_A - 4c_A - ab + 4a}{(4 - b)(2 + b)}, \]
\[ x_B^C = \frac{bc_C - bc_B - 4c_B + bc_A - ab + 4a}{(4 - b)(2 + b)}, \]
\[ x_C^C = \frac{bc_B - bc_C - 4c_C + bc_A - ab + 4a}{(4 - b)(2 + b)}. \]
The Bertrand equilibrium prices are
\[ p_A^B = \frac{3b^2 c_C + 3b c_C + 3b^2 c_B + 3b c_B + 4b^2 c_A + 7bc_A + 4c_A - 5ab^2 + ab + 4a}{(2 + b)(4 + 5b)}, \]
\[ p_B^B = \frac{3b^2 c_C + 3b c_C + 4b^2 c_B + 7bc_B + 4c_B + 3b^2 c_A + 3bc_A - 5ab^2 + ab + 4a}{(2 + b)(4 + 5b)}, \]
\[ p_C^B = \frac{4b^2 c_C + 7bc_C + 4c_C + 3b^2 c_B + 3bc_B + 3b^2 c_A + 3bc_A - 5ab^2 + ab + 4a}{(2 + b)(4 + 5b)}. \]
The difference between the relative profit in the Bertrand equilibrium and that in the Cournot equilibrium for each of Firm A, B, C is

\[ \varphi_A^B - \varphi_A^C = \frac{9b^3 (b + 2)(c_B^2 - 4c_Bc_C + 2c_Ac_C + c_B^2 + 2c_Ac_B - 2c_A^2)}{(b - 4)^2(b - 1)(5b + 4)^2}, \]

\[ \varphi_B^B - \varphi_B^C = \frac{9b^3 (b + 2)(c_B^2 + 2c_Bc_C - 4c_Ac_C - 2c_B^2 + 2c_Ac_B + c_A^2)}{(b - 4)^2(b - 1)(5b + 4)^2}, \]

and

\[ \varphi_C^B - \varphi_C^C = \frac{-9b^3 (b + 2)(2c_B^2 - 2c_Bc_C - 2c_Ac_C - c_B^2 + 4c_Ac_B - c_A^2)}{(b - 4)^2(b - 1)(5b + 4)^2}. \]

If and only if \( c_A = c_B = c_C \), we have \( \varphi_A^C = \varphi_A^B, \varphi_B^C = \varphi_B^B, \varphi_C^C = \varphi_C^B \). Thus, in a symmetric oligopoly Cournot and Bertrand equilibria coincide. However, in an asymmetric oligopoly they do not coincide. For example, if \( c_B = c_C \) but \( c_A > c_B \), the difference between the relative profit in the Bertrand equilibrium and the relative profit in the Cournot equilibrium for each firm is

\[ \varphi_A^B - \varphi_A^C = -\frac{18b^3 (b + 2)(c_B - c_A)^2}{(b - 4)^2(b - 1)(5b + 4)^2} < 0, \]

\[ \varphi_B^B - \varphi_B^C = \frac{9b^3 (b + 2)(c_B - c_A)^2}{(b - 4)^2(b - 1)(5b + 4)^2} > 0, \]

and

\[ \varphi_C^B - \varphi_C^C = \frac{9b^3 (b + 2)(c_B - c_A)^2}{(b - 4)^2(b - 1)(5b + 4)^2} > 0. \]

References


