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28 December 2016

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December 28, 2016

Abstract

Philip Reny's approach to games with discontinuous utility functions can work outside its original context. The existence of Nash equilibrium, as well as the possibility to approach an equilibrium with a finite individual improvement path, are established, under a condition slightly weaker than the better reply security, for three classes of strategic games: potential games, games with strategic complementarities, and aggregative games with appropriate monotonicity conditions. *MSC2010* Classification: 91A10; *JEL* Classification: C 72.

Key words: better reply security; Nash equilibrium; potential game; game with strategic complementarities; aggregative game.

1 Introduction

Reny (1999) made a significant step in the development of sufficient conditions for Nash equilibrium existence in games with discontinuous utility functions. A feature common to games considered by Reny and most of his followers, see, e.g., McLennan et al. (2011) or Prokopovych (2013), is that the strategy sets are convex and each utility function is quasiconcave in own argument. Bich (2009) relaxes the quasiconcavity, but not at all radically.

In this paper, we apply Reny's approach to three different classes of strategic games: potential games; games with strategic complementarities; aggregative games with appropriate monotonicity conditions. What unites them is that the existence of a Nash equilibrium in none of them has anything to do with convexity. Moreover, it is much easier to prove and understand in the case of a *finite* game; in an infinite game, there may be no equilibrium at all without *some* continuity-like assumptions. And for each class of games we obtain the weakest set of such assumptions known as of today.

Following Reny (2016), we consider games with purely ordinal preferences, i.e., where utility functions take values from arbitrary chains rather than the real line. Inevitably, we only consider pure strategies. Our (i.e., essentially, Reny's) topological assumptions do not ensure the existence of the best responses; therefore, the standard fixed point theorems cannot be applied directly. Instead, we analyze the behavior of individual improvement paths and hence obtain more than the mere existence

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of a Nash equilibrium, viz. the possibility to come arbitrarily close to the set of Nash equilibria after a finite number of individual improvements starting from an arbitrary strategy profile.

We understand potential games in a much broader sense than Monderer and Shapley (1996), viz. we consider games where individual improvements are acyclic. Thus, our Theorem 1 generalizes the main result of Kukushkin (2011), which in its turn generalized the good old "acyclicity plus open lower contour sets" theorem (Bergstrom, 1975; Walker, 1977).

Strategic complementarities are also understood in a more general, ordinal sense, as in Milgrom and Shannon (1994), rather than in the cardinal one, as in Vives (1990). Our Theorem 2 extends the main result of Kukushkin et al. (2005) to infinite games, even with some strengthening.

The observation that aggregation helps Nash equilibrium to exist can be traced back to, at least, Novshek (1985), see also Kukushkin (1994). The assumptions we impose on aggregation rules in Theorem 3 are taken from Jensen (2010).

Section 2 contains basic definitions and notations associated with a strategic game. In Section 3, we reproduce Reny's basic notions and a bit more general topological condition, which, via a technical Proposition 3, plays the key role in the rest of the paper. In Sections 4, 5, and 6, we consecutively apply Proposition 3 to potential games, games with strategic complementarities, and aggregative games. Several related questions of secondary importance are discussed in Section 7. More complicated (or just tedious) proofs (of Proposition 1, Theorem 2 and Theorem 3) are deferred to Appendix.

2 Basic definitions

A *strategic game* Γ is defined by a finite set of players N and, for each $i \in N$, a *strategy set* X_i, a chain $\mathcal{C}_i$ (a *utility scale*), and a "generalized" *utility function* $u_i\colon X_N \to \mathcal{C}_i$, where $X_N := \prod_{i\in N} X_i$ is the set of *strategy profiles*. For each $i \in N$, we denote $X_{-i} := \prod_{j\in N\setminus\{i\}} X_j$, and often use notation like $(x_i, x_{-i}) \in X_N$. Viewing functions u_i as components of a mapping $u_N\colon X_N \to \mathcal{C}_N$, where $\mathcal{C}_N := \prod_{i\in N} \mathcal{C}_i$, we denote G the graph of the mapping, i.e., the set of pairs $\langle x_N, u_N(x_N)\rangle \in X_N \times \mathcal{C}_N$ for all $x_N \in X_N$.

With every strategic game, we associate this *individual improvement relation* $\rhd^{\mathrm{Ind}}$ on X_N ($i \in N$, y_N, $x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{Ind}} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; u_i(y_N) > u_i(x_N)];$$
$$y_N \rhd^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{Ind}} x_N].$$

By definition, a Nash equilibrium is a *maximizer* of the relation $\rhd^{\mathrm{Ind}}$ on X_N, i.e., a strategy profile $x_N \in X_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ holds for no $y_N \in X_N$. The set of Nash equilibria is denoted $\mathrm{E}(\Gamma) \subseteq X_N$.

A *subgame* Γ' of Γ is a strategic game defined by subsets $X'_i \subseteq X_i$ for all $i \in N$ and the restriction of the utility mapping u_N to $X'_N := \prod_{i\in N} X'_i$; we will use the notation $\Gamma' \le \Gamma$. The individual improvement relation in a subgame is the restriction of $\rhd^{\mathrm{Ind}}$ to X'_N. If $x_N \in \mathrm{E}(\Gamma)\cap X'_N$, then $x_N \in \mathrm{E}(\Gamma')$; if $x_N \in \mathrm{E}(\Gamma')$, it need not belong to $\mathrm{E}(\Gamma)$.

An (*individual*) *improvement path* is a (finite or infinite) sequence $\langle x_N^k\rangle_{k=0,1,\dots}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ whenever $k \ge 0$ and x_N^{k+1} is defined. A strategic game Γ has the *weak finite improvement property* (*weak FIP*) iff, for every strategy profile $x_N^0 \in X_N$, there is a finite improvement path $x_N^0, \dots, x_N^m$ such that $x_N^m \in \mathrm{E}(\Gamma)$. Γ has the *quasi weak FIP* (*QwFIP*) iff, for every finite subgame Γ' of Γ, there is Γ'' such that $\Gamma' \le \Gamma'' \le \Gamma$ and Γ'' has the weak FIP.

Henceforth, the strategy sets X_i are assumed to be topological spaces; each chain $\mathcal{C}_i$ is endowed with its order interval topology; the sets X_N, $\mathcal{C}_N$, X_{-i}, and $X_N \times \mathcal{C}_N$ are endowed with their product topologies. For every $x_N \in X_N$, we denote $\bar{G}(x_N) := \{v_N \in \mathcal{C}_N \mid (x_N, v_N) \in \operatorname{cl} G\}$ (where $\operatorname{cl} G$ denotes the topological closure) and perceive $\bar{G}$ as a correspondence from X_N to $\mathcal{C}_N$. We say that Γ has the *very weak FIP* iff, for every open neighborhood O of $\mathrm{E}(\Gamma)$ and every $x_N^0 \in X_N$, there is a finite improvement path $x_N^0, \dots, x_N^m$ such that $x_N^m \in O$. If Γ has the very weak FIP, then $\mathrm{E}(\Gamma) \neq \emptyset$: otherwise, the definition should be applicable to $O = \emptyset$, and $x_N^m \in \emptyset$ is impossible.

Remark. Kukushkin (2011) defined the very weak FIP in a slightly different way: given a strategy profile, there should exist *a* Nash equilibrium every open neighborhood of which can be reached after a finite number of improvements. It remains unclear whether Proposition 3 would be valid under that definition of the property; however, the question does not look pressing.

3 Better-reply security and finite deviations

First, we reproduce Reny's (1999, 2016) definitions. Player $i \in N$ *can secure a payoff of* $\alpha \in \mathcal{C}_i$ *at* $x_N^* \in X_N$ iff there exists $y_i \in X_i$ such that $u_i(y_i, x_{-i}) \geq \alpha$ for all x_{-i} in some open neighborhood of x_{-i}^*. A game Γ is *better-reply secure* iff, whenever x_N is not a Nash equilibrium and $v_N \in \bar{G}(x_N)$, some player i can secure a payoff strictly above v_i at x_N.

Somewhat modifying Prokopovych's (2013) definition, we say that a subset $Y \subseteq X_N$ has the *finite deviations property* iff there is a finite set of pairs $\{\langle i(h) \in N, y_{i(h)}^h \in X_{i(h)}\rangle\}_{h \in H}$ such that for every $x_N \in Y \setminus \mathrm{E}(\Gamma)$ there holds $(y_{i(h)}^h, x_{-i(h)}) \rhd_{i(h)}^{\mathrm{Ind}} x_N$ for (at least) one h. A game Γ is said to have the *local finite deviations property* iff, for every $\bar{x}_N \in X_N \setminus \mathrm{E}(\Gamma)$, there is an open neighborhood of $\bar{x}_N$ having the finite deviations property.

Proposition 1. *If a game Γ is better-reply secure and $\operatorname{cl} u_N(X_N)$ is compact, then Γ has the local finite deviations property.*

The statement is almost indistinguishable from Lemma 2 of Prokopovych (2013). Since our assumptions are much broader, a complete proof is given in Appendix, Section A.

Proposition 2. *Let a game Γ have the local finite deviations property, and let $Y \subseteq (X_N \setminus \mathrm{E}(\Gamma))$ be compact. Then Y has the finite deviations property.*

Proof. By our assumption, there is an open neighborhood $O(x_N)$ with the finite deviations property of every $x_N \in Y$. Since Y is compact, it is covered by a finite number of those open neighborhoods. Taking the union of the appropriate sets $\{\langle i(h) \in N, y_{i(h)}^h \in X_{i(h)}\rangle\}_{h \in H}$, we see that Y has the finite deviations property indeed. $\square$

Proposition 3. *Let a game Γ have both local finite deviations and QwFIP properties, and let X_N be compact. Then Γ has the very weak FIP property and hence possesses a Nash equilibrium.*

Proof. Let $O \supseteq \mathrm{E}(\Gamma)$ be open and let $x_N^0 \in X_N \setminus O$. Since $X_N \setminus O$ is compact, it has the finite deviations property by Proposition 2. Let $\{\langle i(h) \in N, y_{i(h)}^h \in X_{i(h)}\rangle\}_{h \in H}$ be an appropriate finite set of pairs. For each $i \in N$, we define $X_i' := \{x_i^0\} \cup \{y_{i(h)}^h \mid h \in H \;\&\; i(h) = i\} \subseteq X_i$. The sets X_i' define a finite subgame

Γ' of Γ; by our assumption, there is Γ'' such that $\Gamma' \le \Gamma'' \le \Gamma$ and Γ'' has the weak FIP. Therefore, there is a finite improvement path $x_N^0, \dots, x_N^m$ in Γ'' such that $x_N^m \in \mathrm{E}(\Gamma'')$. Now, we have either $x_N^m \in O$ or $x_N^m \notin O$. In the first case, we are home because $x_N^0, \dots, x_N^m$ remains a finite improvement path in Γ. In the second case, we would have $x_N^m \in X_N \setminus O \subseteq X_N \setminus \mathrm{E}(\Gamma)$ and hence there would be $h \in H$ such that $(y^h_{i(h)}, x^m_{-i(h)}) \triangleright^{\mathrm{Ind}}_{i(h)} x_N^m$, which is incompatible with $x_N^m \in \mathrm{E}(\Gamma'')$.

Since $O \supseteq \mathrm{E}(\Gamma)$ and $x_N^0 \in X_N$ were arbitrary, we are home. □

4 Potential games

The relation $\triangleright^{\mathrm{Ind}}$ is *acyclic* iff there is no *finite improvement cycle*, i.e., no improvement path for which $x_N^0 = x_N^m$ with $m > 0$. A sufficient condition for that is the existence of a *generalized ordinal potential* (Monderer and Shapley, 1996), i.e., a function $P\colon X_N \to \mathbb{R}$ such that $P(y_N) > P(x_N)$ whenever $y_N \triangleright^{\mathrm{Ind}} x_N$. (For a finite game, that condition is also necessary.)

Theorem 1. *Let Γ be a strategic game with compact strategy sets X_i. Let $\triangleright^{\mathrm{Ind}}$ in Γ be acyclic. Let Γ have the local finite deviations property. Then Γ has the very weak FIP property and hence possesses a Nash equilibrium.*

Proof. Let Γ' be a finite subgame of Γ. Since $\triangleright^{\mathrm{Ind}}$ is acyclic in Γ, and hence in Γ' as well, *every* improvement path in Γ', if continued whenever possible, finds a Nash equilibrium at some stage. In terms of Monderer and Shapley (1996), Γ' has the finite improvement property (FIP); obviously, Γ' has the weak FIP. Therefore, Γ has the QwFIP property and hence the very weak FIP property by Proposition 3. □

5 Strategic complementarities

We reproduce standard definitions useful for monotone comparative statics.

Let X and S be *partially ordered sets* (*posets*) and $\mathcal{C}$ be a chain. We say that a function $u\colon X \times S \to \mathcal{C}$ satisfies the *single crossing* conditions (Milgrom and Shannon, 1994) iff, for all $x, y \in X$ and $s, s' \in S$, there holds

$$[y > x \ \&\ s' > s \ \&\ u(y,s) > u(x,s)] \Rightarrow u(y,s') > u(x,s'); \tag{1a}$$

$$[y < x \ \&\ s' < s \ \&\ u(y,s) > u(x,s)] \Rightarrow u(y,s') > u(x,s'). \tag{1b}$$

u satisfies the *weak single crossing* condition (Shannon, 1995) iff

$$[y > x \ \&\ s' > s \ \&\ u(y,s) > u(x,s)] \Rightarrow u(y,s') \ge u(x,s') \tag{2}$$

for all $x, y \in X$ and $s, s' \in S$. Either condition (1) implies (2).

Let X be a *lattice*. A function $u\colon X \to \mathcal{C}$ is *quasisupermodular* (Milgrom and Shannon, 1994; LiCalzi and Veinott, 1992) iff, whenever $y, x \in X$,

$$u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y); \tag{3a}$$

$$u(y) > u(y \vee x) \Rightarrow u(y \wedge x) > u(x). \tag{3b}$$

Kukushkin (2013b) partitioned conditions (3) into four independent conditions, two of which will be used here:

$$\forall x, y \in X \left[u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > \min\{u(x), u(y)\} \right]; \tag{4a}$$
$$\forall x, y \in X \left[u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > \min\{u(x), u(y)\} \right]. \tag{4b}$$

A function $u \colon X \to \mathcal{C}$ is *weakly quasisupermodular* (Shannon, 1995; LiCalzi and Veinott, 1992) iff

$$\forall x, y \in X \left[u(x) > u(y \wedge x) \Rightarrow u(y \vee x) \geq \min\{u(x), u(y)\} \right]; \tag{5a}$$
$$\forall x, y \in X \left[u(x) > u(y \vee x) \Rightarrow u(y \wedge x) \geq \min\{u(x), u(y)\} \right]. \tag{5b}$$

These implications are obvious: (3a) $\Rightarrow$ (4a) $\Rightarrow$ (5a); (3b) $\Rightarrow$ (4b) $\Rightarrow$ (5b). Meanwhile, (3a) does not imply (5b), and (3b) does not imply (5a).

Theorem 2. *Let Γ be a strategic game such that each strategy set X_i is simultaneously a compact topological space and a lattice. Let each utility function u_i satisfy the condition* (1a) *with $X := X_i$, $S := X_{-i}$, and $\mathcal{C} := \mathcal{C}_i$. Let every function $u_i(\cdot, x_{-i}) \colon X_i \to \mathcal{C}_i$ ($i \in N$, $x_{-i} \in X_{-i}$) satisfy the condition* (4a). *Let Γ have the local finite deviations property. Then Γ has the very weak FIP property and hence possesses a Nash equilibrium.*

Essentially, this theorem follows from Proposition 3 above and Theorem 1 of Kukushkin et al. (2005). Since the assumptions of the latter theorem were somewhat stronger than those made here, a complete proof is given in Appendix, Section B.

Theorem 2′. *Let Γ be a strategic game such that each strategy set X_i is simultaneously a compact topological space and a lattice. Let each utility function u_i satisfy the condition* (1b) *with $X := X_i$, $S := X_{-i}$, and $\mathcal{C} := \mathcal{C}_i$. Let every function $u_i(\cdot, x_{-i}) \colon X_i \to \mathcal{C}_i$ ($i \in N$, $x_{-i} \in X_{-i}$) satisfy the condition* (4b). *Let Γ have the local finite deviations property. Then Γ has the very weak FIP property and hence possesses a Nash equilibrium.*

The proof is dual to that of Theorem 2.

6 Aggregative games

We call a strategic game *aggregative* iff there are mappings $\sigma_i \colon X_{-i} \to \mathbb{R}$ ($i \in N$), *aggregation rules*, and $U_i \colon \sigma_i(X_{-i}) \times X_i \to \mathcal{C}_i$ ($i \in N$) such that

$$u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i)$$

for all $i \in N$ and $x_N \in X_N$. For each $i \in N$, we denote $S_i := \sigma_i(X_{-i}) \subseteq \mathbb{R}$. An aggregative game is *appropriately aggregative* iff each strategy set X_i is a poset, while there are mappings $g \colon X_N \to \mathbb{R}$, $F_i \colon S_i \times X_i \to \mathbb{R}$ and $v_i \colon X_{-i} \to \mathbb{R}$ ($i \in N$) satisfying the following conditions.

First, for all $i \in N$ and $x_N \in X_N$,

$$g(x_N) = F_i(\sigma_i(x_{-i}), x_i) + v_i(x_{-i}). \tag{6}$$

Second, each F_i has the *strictly increasing differences* property (Topkis, 1978):

$$\forall s_i, s_i' \in S_i \, \forall y_i, x_i \in X_i \left[[y_i > x_i \ \& \ s_i' > s_i] \Rightarrow F_i(s_i', y_i) - F_i(s_i', x_i) > F_i(s_i, y_i) - F_i(s_i, x_i) \right]. \qquad (7)$$

A straightforward example of appropriate aggregation is given by $\sigma_i(x_{-i}) := \sum_{j \neq i} x_j$ (with $X_i \subset \mathbb{R}$), in which case $g(x_N) := \sum_{i \neq j} x_i \cdot x_j / 2$, $F_i(s_i, x_i) := s_i x_i$, and $v_i(x_{-i}) := 0$ satisfy (6) and (7). Jensen (2010) provides a number of less straightforward examples.

Theorem 3. *Let Γ be an appropriately aggregative game such that each strategy set X_i is simultaneously a compact topological space and a lattice. Let each U_i satisfy the weak single crossing condition (2) with $X := X_i$, $S := S_i$, and $\mathcal{C} := \mathcal{C}_i$. Let, for each $i \in N$, there be $s_i^* \in S_i$ such that $U_i(s_i, \cdot)$ satisfies (4a) for all $x_i, y_i \in X_i$ and $s_i < s_i^*$, while satisfying (4b) for all $x_i, y_i \in X_i$ and $s_i > s_i^*$. Let Γ have the local finite deviations property. Then Γ has the very weak FIP property and hence possesses a Nash equilibrium.*

The proof, based on Proposition 3 and a combination of ideas from Jensen (2010) and Kukushkin (2016), is deferred to Appendix, Section C.

Theorem 3′. *Let Γ be an appropriately aggregative game such that each strategy set X_i is simultaneously a compact topological space and a lattice. Let each U_i satisfy the single crossing conditions (1) with $X := X_i$, $S := S_i$, and $\mathcal{C} := \mathcal{C}_i$. Let, for each $i \in N$, there be $s_i^* \in S_i$ such that $U_i(s_i, \cdot)$ satisfies (5a) for all $x_i, y_i \in X_i$ and $s_i < s_i^*$, while satisfying (5b) for all $x_i, y_i \in X_i$ and $s_i > s_i^*$. Let Γ have the local finite deviations property. Then Γ has the very weak FIP property and hence possesses a Nash equilibrium.*

The proof is virtually the same as that of Theorem 3; only the reference to Proposition 26 from Kukushkin (2013b) should be replaced with the reference to Proposition 28 from the same paper.

7 Concluding remarks

7.1. The description of the preferences of the players with "generalized" utility functions is equivalent to the description with complete binary relations as in Reny (2016). An even more general description would emerge if each $\mathcal{C}_i$ were just a poset. Theorem 1 would remain valid in this case with the same proof, cf. Kukushkin (2011). Whether Theorems 2 and 3 allow such a broad generalization is not clear at the moment; most likely, additional assumptions would be needed.

7.2. The compactness assumption in Proposition 1 cannot simply be dropped. If each $\mathcal{C}_i$ is just $\mathbb{R}$, it boils down to the condition that each u_i is bounded, both above and below. The fact that the proposition may become wrong without an upper bound on utilities may be demonstrated with a one-person game. As to the lower bound, two players are needed, but one of them may be a dummy.

Example 1. Let us consider a game where $N := \{1, 2\}$, $X_1 := [0, 1]$, $X_2 := \{0\}$, and the utility mapping is this:

$$u_N(x_N) := \begin{cases} (1 - x_1, -1/x_1), & \text{if } x_1 > 0, \\ (0, 0), & \text{if } x_1 = 0. \end{cases}$$

The game is better-reply secure since the graph G of the utility mapping u_N is closed and a payoff strictly above $u_1(x_N)$ is secured by any $y_1 \in]0, 1[$ if $x_1 = 0$, or by any $y_1 \in]0, x_1[$ if $x_1 > 0$. Thus, all assumptions of Proposition 1 are satisfied except that u_2 is not bounded below. On the other hand, no open neighborhood of $(0, 0)$ has the finite deviations property; moreover, there is no Nash equilibrium.

As suggested by Reny (1999) himself, the proposition can be made applicable to unbounded utilities via a re-interpretation of better-reply security. Namely, we could perceive u_N as a mapping $X_N \to \bar{\mathbb{R}}^N$, where $\bar{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, and require the inequality in the definition to hold for vectors in $\bar{G}(x_N)$ with infinite coordinates as well. (It should be noted that $\bar{\mathbb{R}}$ is compact in *its* order interval topology.) In Example 1, G will no longer be closed under this interpretation, $\bar{G}(0, 0) = \{(0, 0), (1, -\infty)\}$, and player 1 cannot secure any payoff above 1. In other words, another assumption will fail and there will be no surprise in the absence of an equilibrium.

One could suspect the compactness assumption to imply that the preferences can actually be described with a *real-valued* utility function. However, this is not the case: if $\mathcal{C}_i$ is $\mathbb{R} \times \{0, 1\}$ with the lexicographic order, then the closure of every bounded subset of $\mathcal{C}_i$ is compact, but its embedding into the real line may be impossible (Wakker, 1988, Lemma 3.1).

7.3. The restriction of the main requirement in the definition of the finite deviations property to $x_N \in Y \setminus \mathrm{E}(\Gamma)$ (rather than *all* $x_N \in Y$) was taken from Prokopovych (2013). As can easily be seen from Section A, Proposition 1 would remain valid without such a restriction. One could argue that the two versions of the definition only differ when $Y \cap \mathrm{E}(\Gamma) \neq \emptyset$, so it makes no sense to distinguish between them when the sole purpose is to establish the existence of an equilibrium. Since the very weak FIP we study in this paper is *more* than the mere existence, the fact that the weaker condition is sufficient might be worth noting.

7.4. Similarly to Kukushkin (2011), our Theorem 1 implies a new generalization of the old theorem of Bergstrom (1975) and Walker (1977): An acyclic binary relation $\rhd$ on a compact topological space X admits a maximizer if, whenever $y \rhd x$, there is an open neighborhood O of x and a finite set $\{z^1, \ldots, z^m\} \subset X$ such that for every $x' \in O$ there is k for which $z^k \rhd x'$. Funnily, this particular result seems to have never been published although there are quite a few of much more straightforward generalizations in the literature.

7.5. The key role in the proof of Theorem 3 is played by a construction essentially invented by Jensen (2010), who built on Huang (2002), Dubey et al. (2006), and Kukushkin (2005). Unfortunately, there were technical oversights in Jensen (2010): the proof needed stronger assumptions than were made explicitly (Jensen, 2012). In a personal communication, Jensen conjectured that his main theorem is nonetheless valid as stated. Our Theorem 3 makes a significant step towards the vindication of his position.

Appendix: Proofs

A Proof of Proposition 1

Since Γ is better-reply secure, for every $v_N \in \bar{G}(\bar{x}_N)$, there are $j(v_N) \in N$, $\alpha_{j(v_N)}(v_N) \in \mathcal{C}_{j(v_N)}$, $y_{j(v_N)} \in X_{j(v_N)}$, and $V_{-j(v_N)}(v_N) \subseteq X_{-j(v_N)}$ such that $V_{-j(v_N)}(v_N)$ is open, $\bar{x}_{-j(v_N)} \in V_{-j(v_N)}(v_N)$, $\alpha_{j(v_N)}(v_N) >$

$v_{j(v_N)}$, and, whenever $x_{-j(v_N)} \in V_{-j(v_N)}(v_N)$, there holds $u_{j(v_N)}(y_{j(v_N)}, x_{-j(v_N)}) \geq \alpha_{j(v_N)}(v_N)$. Denoting $W(v_N) := \{w_N \in \mathcal{C}_N \mid \alpha_{j(v_N)}(v_N) > w_{j(v_N)}\}$, we have $v_N \in W(v_N)$ and hence $\bar{G}(\bar{x}_N) \subseteq \bigcup_{v_N \in \bar{G}(\bar{x}_N)} W(v_N)$. Since every $W(v_N)$ is open while $\bar{G}(\bar{x}_N)$ is compact, there are $v_N^1, \ldots, v_N^m \in \bar{G}(\bar{x}_N)$ such that $\bar{G}(\bar{x}_N) \subseteq \bigcup_{h=1}^m W(v_N^h) =: \tilde{W}$.

Claim A.1. *There is an open neighborhood V of $\bar{x}_N$ such that $u_N(x_N) \in \tilde{W}$ whenever $x_N \in V$.*

Remark. In principle, this claim belongs to textbook material. Since our assumptions are broader than usual, a complete proof is given.

Proof of Claim A.1. We set $F := (\operatorname{cl} u_N(X_N)) \setminus \tilde{W} \subset \mathcal{C}_N$; F is compact. For every $w_N \in F$, we have $(\bar{x}_N, w_N) \notin \bar{G}$. Since $\bar{G}$ is closed, there is an open neighborhood $V'(w_N)$ of $(\bar{x}_N, w_N)$ in $X_N \times \mathcal{C}_N$ such that $V'(w_N) \cap \bar{G} = \emptyset$; without restricting generality, we have $V'(w_N) = V'_X(w_N) \times V'_C(w_N)$, where $V'_X(w_N)$ is open in X_N, while $V'_C(w_N)$ is open in $\mathcal{C}_N$. Since $\{\bar{x}_N\} \times F$ is compact, it is covered by a finite number of such neighborhoods: $V'(w_N^1)$, ..., $V'(w_N^{m'})$. We define $V := \bigcap_{h=1}^{m'} V'_X(w_N^h)$; V is open and $\bar{x}_N \in V$.

Now if $x_N \in V$ and $u_N(x_N) \notin \tilde{W}$, we would have $u_N(x_N) \in F$; therefore, $\langle \bar{x}_N, u_N(x_N) \rangle \in V'(w_N^h)$ for some h. Since $x_N \in V'_X(w_N^h)$, we have $\langle x_N, u_N(x_N) \rangle \in V'(w_N^h)$ as well. Therefore, $\langle x_N, u_N(x_N) \rangle \notin G$, which is impossible. □

Picking such an open neighborhood V, we define $O := V \cap \bigcap_{h=1}^m [X_{j(v_N^h)} \times V_{-j(v_N^h)}(v_N^h)]$. Again, O is open and $\bar{x}_N \in O$. Setting $H := \{1, \ldots, m\}$, $i(h) := j(v_N^h)$ and $y_{i(h)}^h := y_{j(v_N^h)}$ for all $h \in H$, we check that O satisfies the requirement.

Let $x_N \in O$; hence $u_N(x_N) \in \bigcup_{h \in H} W(v_N^h)$ by Claim A.1 and hence $\alpha_{-j(v_N^h)}(v_N^h) > u_{j(v_N^h)}(x_N^h)$ for some h. Since $x_{-j(v_N^h)} \in V_{-j(v_N^h)}(v_N^h)$, we have $u_{j(v_N^h)}(y_{j(v_N^h)}, x_{-j(v_N^h)}) \geq \alpha_{-j(v_N^h)}(v_N^h) > u_{j(v_N^h)}(x_N^h)$. Thus, $(y_{i(h)}^h, x_{-i(h)}) \rhd_{i(h)}^{\mathrm{Ind}} x_N$ indeed.

B Proof of Theorem 2

In light of Proposition 3, it is enough to show that Γ has the QwFIP property.

Let $\Gamma' \leq \Gamma$ be finite; for each $i \in N$, we define X''_i as the minimal sublattice of X_i containing X'_i. Then X''_i is still finite; hence we can argue similarly to the proof of Theorem 1 of Kukushkin et al. (2005). We define

$$X^{\uparrow} := \{x_N \in X''_N \mid \exists y_N \in X''_N\, [y_N > x_N \;\&\; y_N \rhd^{\mathrm{Ind}} x_N]\}; \quad X^{\downarrow} := X''_N \setminus X^{\uparrow};$$

$$y_N \succ x_N \rightleftharpoons \big[[y_N \in X^{\downarrow} \;\&\; x_N \in X^{\uparrow}] \text{ or } [x_N, y_N \in X^{\uparrow} \;\&\; y_N > x_N] \text{ or } [x_N, y_N \in X^{\downarrow} \;\&\; y_N < x_N] \big]. \quad (8)$$

Clearly, $\succ$ is irreflexive and transitive.

Claim B.1. *If $x_N \in X''_N \setminus \mathrm{E}(\Gamma'')$, then there exists $y_N \in X''_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ and $y_N \succ x_N$.*

Proof of Claim B.1. If $x_N \in X^{\uparrow}$, then we pick $y_N \in X''_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ and $y_N > x_N$. If $y_N \in X^{\downarrow}$, then $y_N \succ x_N$ by the first disjunctive term in (8). If $y_N \in X^{\uparrow}$, then $y_N \succ x_N$ by the second disjunctive term in (8).

Let $x_N \in X^{\downarrow}$. We pick $i \in N$ and $y_N \in X''_N$ such that $y_N \rhd^{\mathrm{Ind}}_i x_N$. Denoting $Y_i := \{z_i \in X''_i \mid z_i \leq x_i\}$, we pick $\bar{z}_i \in \mathrm{Argmax}_{z_i \in Y_i} u_i(z_i, x_{-i})$, which is possible because Y_i is finite. Since $x_N \in X^{\downarrow}$, $y_i > x_i$ is impossible. If $y_i < x_i$, then $u_i(\bar{z}_i, x_{-i}) \geq u_i(y_i, x_{-i})$; hence $u_i(\bar{z}_i, x_{-i}) > u_i(x_N)$ and hence $\bar{z}_i < x_i$. If y_i and x_i are incomparable in the order, then $y_i \vee x_i > x_i$ and $y_i \wedge x_i < x_i$. An assumption that $u_i(x_N) \geq u_i(y_i \wedge x_i, x_{-i})$ would imply $u_i(y_i, x_{-i}) > u_i(y_i \wedge x_i, x_{-i})$, and hence $u_i(y_i \vee x_i, x_{-i}) > u_i(x_N)$ by (4a), contradicting our assumption that $x_N \in X^{\downarrow}$. Therefore, $u_i(y_i \wedge x_i, x_{-i}) > u_i(x_N)$; hence $u_i(\bar{z}_i, x_{-i}) > u_i(x_N)$ and $\bar{z}_i < x_i$ again. Denoting $z_N := (\bar{z}_i, x_{-i})$, we see that $z_N \rhd^{\mathrm{Ind}} x_N$ and $z_N < x_N$. To show that $z_N \succ x_N$, we only have to show that $z_N \in X^{\downarrow}$.

Suppose the contrary: there are $j \in N$ and $y_j > z_j$ such that

$$u_j(y_j, z_{-j}) > u_j(z_N). \tag{9}$$

Let us consider two alternatives.

If $j = i$ (hence $z_{-j} = x_{-i}$), $y_i > x_i$ would contradict $x_N \in X^{\downarrow}$ while $y_i < x_i$ would contradict the choice of $\bar{z}_i$; therefore, we have to assume that y_i and x_i are incomparable, hence $y_i \vee x_i > x_i$. The choice of $\bar{z}_i$ implies $u_i(\bar{z}_i, x_{-i}) \geq u_i(y_i \wedge x_i, x_{-i})$ and hence, by (9) and (4a), $u_i(y_i \vee x_i, x_{-i}) > u_i(x_N)$, contradicting the assumption $x_N \in X^{\downarrow}$.

Thus, we are led to $j \neq i$; hence $y_j > z_j = x_j$ and $z_{-j} < x_{-j}$. Now (9) and (1a) imply $u_j(y_j, x_{-j}) > u_j(x_N)$, again contradicting the assumption $x_N \in X^{\downarrow}$. $\square$

Finally, having $x^0_N \in X''_N \setminus \mathrm{E}(\Gamma'')$, we start building an improvement path, applying Claim B.1 at each step, i.e., picking $x^{k+1}_N \in X''_N$ such that $x^{k+1}_N \rhd^{\mathrm{Ind}} x^k_N$ and $x^{k+1}_N \succ x^k_N$, as long as $x^k_N \notin \mathrm{E}(\Gamma'')$. Since $\succ$ is an order, we cannot return back. Since X''_N is finite, we reach $\mathrm{E}(\Gamma'')$ at some stage.

C Proof of Theorem 3

In light of Proposition 3, it is enough to show that Γ has the QwFIP property. Let $\Gamma' \leq \Gamma$ be finite. Exactly as in the case of Theorem 2, we define X''_i, for each $i \in N$, as the minimal sublattice of X_i containing X'_i. Then X''_i is still finite.

To establish that Γ'' has the weak FIP, we argue similarly to Jensen (2010) or Kukushkin (2016).

For each $i \in N$, we define the *best response correspondence*:

$$R_i(s_i) := \operatorname*{Argmax}_{x_i \in X''_i} U_i(s_i, x_i).$$

Since X''_i is finite, $R_i(s_i) \neq \emptyset$ for each $s_i \in S''_i$.

By Proposition 26 from Kukushkin (2013b), and conditions (4a) and (2), the correspondence R_i is *weakly ascending* in the sense of Veinott (1989):

$$[s'_i > s_i \ \&\ y_i \in R_i(s'_i) \ \&\ x_i \in R_i(s_i)] \Rightarrow [y_i \vee x_i \in R_i(s'_i) \ \text{ or } \ y_i \wedge x_i \in R_i(s_i)].$$

Therefore, by Theorem 3.2 of Veinott (1989), or, easier to find, Proposition 2.5 from Kukushkin (2013a), there exists an increasing selection r_i from R_i. Henceforth, we fix such a selection for each $i \in N$ and denote $X_i^0 := r_i(S_i')$. Clearly, $X_i^0 \subseteq X_i''$ is a chain.

Now, we introduce this *admissible best response improvement relation* $\rhd^{\mathrm{BR}}$ on X_N'' ($i \in N$, y_N, $x_N \in X_N''$):

$$y_N \rhd_i^{\mathrm{BR}} x_N \rightleftharpoons [y_N \rhd_i^{\mathrm{Ind}} x_N \;\&\; y_i = r_i(x_{-i})];$$
$$y_N \rhd^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N \, [y_N \rhd_i^{\mathrm{BR}} x_N].$$

Since $r_i(x_{-i})$ is defined for every $x_{-i} \in X_{-i}''$, every maximizer of $\rhd^{\mathrm{BR}}$ on X_N'' is a Nash equilibrium in Γ''. Since X_N'' is finite, it is sufficient to show that $\rhd^{\mathrm{BR}}$ is acyclic. We achieve this objective by producing an *order potential* of $\rhd^{\mathrm{BR}}$, i.e., an irreflexive and transitive binary relation $\succ$ on X_N'' such that

$$\forall x_N, y_N \in X_N'' \left[y_N \rhd^{\mathrm{BR}} x_N \Rightarrow y_N \succ x_N \right].$$

For each $i \in N$, we, henceforth, assume that $S_i' := \sigma_i(X_{-i}'') = \{s_i^0, s_i^1, \ldots, s_i^m\}$ with $s_i^k > s_i^h$ whenever $k > h$; for each $x_i \in X_i^0$, we define $\varkappa_i(x_i) := \min\{k \mid x_i = r_i(s_i^k)\}$ and

$$\Phi_i(x_i) := -F_i(s_i^{\varkappa_i(x_i)}, x_i) + \sum_{k < \varkappa_i(x_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))]. \tag{10}$$

For $x_i \in X_i'' \setminus X_i^0$, we define $\Phi_i(x_i)$ arbitrarily, e.g., $\Phi_i(x_i) := 0$. For every $x_N \in X_N''$, we define a set $N^0(x_N) := \{i \in N \mid x_i \in X_i^0\}$ and a function

$$H(x_N) := g(x_N) + \sum_{i \in N} \Phi_i(x_i). \tag{11}$$

Now, we are ready to define our potential, a binary relation on X_N'':

$$\begin{aligned} y_N \succ x_N \rightleftharpoons \big[N^0(y_N) \supset N^0(x_N) \;\text{ or }\; &[N^0(y_N) = N^0(x_N) \;\&\; H(y_N) > H(x_N)] \;\text{ or} \\ &\big(N^0(y_N) = N^0(x_N) \;\&\; H(y_N) = H(x_N) \;\& \\ &\quad \forall i \in N \, [y_i = x_i \;\text{ or }\; y_i \geq x_i] \;\&\; \exists i \in N \, [y_i > x_i]\big)\big]. \end{aligned} \tag{12}$$

Obviously, $\succ$ is irreflexive and transitive.

Claim C.1. *If $x_N, y_N \in X_N''$ and $y_N \rhd^{\mathrm{BR}} x_N$, then $y_N \succ x_N$.*

Proof of Claim C.1. Let $y_N \rhd_i^{\mathrm{BR}} x_N$ and $\sigma_i(x_{-i}) = s_i^{\bar{k}}$. We have $y_i = r_i(s_i^{\bar{k}}) \neq x_i$ by definition; hence $y_i \in X_i^0$ and $N^0(y_N) \supseteq N^0(x_N)$. If the inclusion is strict, we have $y_N \succ x_N$ by the first term in (12).

Let us assume $N^0(y_N) = N^0(x_N)$, i.e., $x_i \in X_i^0$. Taking into account (10), we can rewrite (11) as

$$H(x_N) = \sum_{k < \varkappa_i(x_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] + F_i(s_i^{\bar{k}}, x_i) - F_i(s_i^{\varkappa_i(x_i)}, x_i) + C(x_{-i}); \tag{13a}$$

$$H(y_N) = \sum_{k < \varkappa_i(y_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] + F_i(s_i^{\bar{k}}, y_i) - F_i(s_i^{\varkappa_i(y_i)}, y_i) + C(x_{-i}). \tag{13b}$$

Let us assume that $x_i > y_i$; then $\varkappa_i(y_i) \le \bar{k} < \varkappa_i(x_i)$. Subtracting (13a) from (13b), we obtain

$$H(y_N) - H(x_N) = [F_i(s_i^{\varkappa_i(x_i)}, x_i) - F_i(s_i^{\bar{k}}, x_i)] - \sum_{\bar{k} \le k < \varkappa_i(x_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))]$$
$$= \sum_{\bar{k} \le k < \varkappa_i(x_i)} \Big([F_i(s_i^{k+1}, x_i) - F_i(s_i^k, x_i)] - [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] \Big).$$

By (7), the difference is strictly positive. Therefore, $y_N \succ x_N$ by the second term in (12).

Now let us assume that $x_i < y_i$; then $\varkappa_i(x_i) < \varkappa_i(y_i) \le \bar{k}$. Subtracting (13a) from (13b), we obtain

$$H(y_N) - H(x_N) =$$
$$\sum_{\varkappa_i(x_i) \le k < \varkappa_i(y_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] + [F_i(s_i^{\bar{k}}, y_i)] - F_i(s_i^{\varkappa_i(y_i)}, y_i) - [F_i(s_i^{\bar{k}}, x_i) - F_i(s_i^{\varkappa_i(x_i)}, x_i)]$$
$$= \sum_{\varkappa_i(x_i) \le k < \varkappa_i(y_i)} \Big([F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] - [F_i(s_i^{k+1}, x_i) - F_i(s_i^k, x_i)] \Big) +$$
$$\Big([F_i(s_i^{\bar{k}}, y_i) - F_i(s_i^{\varkappa_i(y_i)}, y_i)] - [F_i(s_i^{\bar{k}}, x_i) - F_i(s_i^{\varkappa_i(y_i)}, x_i)] \Big).$$

By (7), the difference is non-negative; it can only be zero if $\varkappa_i(y_i) = \bar{k} = \varkappa_i(x_i) + 1$. Thus, $y_N \succ x_N$ by the second or the third term in (12). $\square$

To summarize, we established that the admissible best response improvement relation $\rhd^{\mathrm{BR}}$ is acyclic on X''_N. Starting from $x_N^0 \in X''_N$ an admissible best response improvement path in Γ'', we inevitably reach a Nash equilibrium at some stage. Therefore, Γ'' has the weak FIP.

Acknowledgments

Financial support from the Russian Science Foundation (project 16-11-10246) is acknowledged. I thank Christian Ewerhart, who roused my interest in the workability of Reny's approach outside its original context.

References

Bergstrom, T., 1975. Maximal elements of acyclic relations on compact sets. Journal of Economic Theory 10, 403–404.

Bich, P., 2009. Existence of pure Nash equilibria in discontinuous and non quasiconcave games. International Journal of Game Theory 38, 395–410.

Dubey, P., O. Haimanko, and A. Zapechelnyuk, 2006. Strategic complements and substitutes, and potential games. Games and Economic Behavior 54, 77–94.

Huang, Z., 2002. Fictitious play in games with a continuum of strategies. Ph.D. Thesis, State University of New York at Stony Brook, Department of Economics.

Jensen, M.K., 2010. Aggregative games and best-reply potentials. Economic Theory 43, 45–66.

Jensen, M.K., 2012. Aggregative games and best-reply potentials: Erratum. https://sites.google.com/site/mkaaejensen/research

Kukushkin, N.S., 1994. A fixed-point theorem for decreasing mappings. Economics Letters 46, 23–26.

Kukushkin, N.S., 2005. "Strategic supplements" in games with polylinear interactions. EconWPA Paper #0411008.

Kukushkin, N.S., 2011. Nash equilibrium in compact-continuous games with a potential. International Journal of Game Theory 40, 387–392.

Kukushkin, N.S., 2013a. Increasing selections from increasing multifunctions. Order 30, 541–555.

Kukushkin, N.S., 2013b. Monotone comparative statics: changes in preferences versus changes in the feasible set. Economic Theory 52, 1039–1060.

Kukushkin, N.S., 2016. Cournot tatonnement in aggregative games with monotone best responses. In: Contributions on Equilibrium Theory for Cournot and Related Games: Essays in Honour of Koji Okuguchi. von Mouche, P., and F. Quartieri, eds. Springer, 45–59.

Kukushkin, N.S., S. Takahashi, and T. Yamamori, 2005. Improvement dynamics in games with strategic complementarities. International Journal of Game Theory 33, 229–238.

LiCalzi, M., and A.F. Veinott, Jr., 1992. Subextremal functions and lattice programming. EconWPA paper #0509001.

McLennan, A., P.K. Monteiro, and R. Tourky, 2011. Games with discontinuous payoffs: A strengthening of Reny's existence theorem. Econometrica 79, 1643–1664.

Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

Monderer, D., and L.S. Shapley, 1996. Potential games. Games and Economic Behavior 14, 124–143.

Novshek, W., 1985. On the existence of Cournot equilibrium. Review of Economic Studies 52, 85–98.

Prokopovych, P., 2013. The single deviation property in games with discontinuous payoffs. Economic Theory 53, 383–402.

Reny, P.J., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67, 1029–1056.

Reny, P.J., 2016. Nash equilibrium in discontinuous games. Economic Theory 61, 553–569.

Shannon, C., 1995. Weak and strong monotone comparative statics. Economic Theory 5, 209–227.

Topkis, D.M., 1978. Minimizing a submodular function on a lattice. Operations Research 26, 305–321.

Veinott, A.F., Jr., 1989. Lattice Programming. Unpublished notes from lectures delivered at Johns Hopkins University.

Vives, X., 1990. Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305–321.

Wakker, P., 1988. Continuity of preference relations for separable topologies. International Economic Review 29, 105–110.

Walker, M., 1977. On the existence of maximal elements. Journal of Economic Theory 16, 470–474.