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Abstract. In this paper, we consider a standard model of many-to-one matching markets. First, we study the relation between *strategy-proofness* and *unanimity* under a certain requirement and we prove these two properties become equivalent. Second, we illustrate that this result has an immediate impact on the relation between strategy-proofness and *Maskin monotonicity*. Finally, we determine a close connexion between strategy-proofness and implementation literature. We provide under certain minimal requirements the foundation for reasoning the equivalence among dominant strategy implementation, standard Nash implementation, and partially honest Nash implementation.

Keywords: Many-to-one matching markets; strategy-proofness; unanimity; Maskin monotonicity, implementation.

JEL classification: C72; D71

1 Introduction

The matching problems are a branch of public decision problems that have been extensively analyzed in the literature. They are two principle classes for these problems: *one-side matching markets* and *two-side matching markets*. The first class studies the assignment of a set of objects with finite capacities to some agents. As examples, we cite the matching of office spaces to faculty members, the matching of student to dormitories

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etc. The second class, that we are focusing on in this paper, is the one of the most "popular" classes in matching theory; it explores two sides where each side is assumed to have strict preferences over the opposite side. Gale and Shapley (1962) might be the first to study these classes of matching theory. In these environments, *stability* is considered as a key property to ensure a such assignment. It is checked if the two properties of *individual rationality* and *no-blocking* are held. Individual rationality requires that each agent finds her or his match acceptable. The property of no-blocking means that no agent can improve by breaking up with their current match in order to match up with each other instead. Gale and Shapley (1962) proved the existence of stable matchings by providing an algorithm, called the *Deferred Acceptance Algorithm*, that has been a central role to design matching programs across several institutions and markets. We distinguish among three categories: *one-to-one*, *many-to-one*, and *many-to-many matching problems.* The first category studies the assignment of each agent on one side on the market to at most one agent on the other side. The best known example for this class is the problem of marriages where there is a certain community with a set of men and a set of women, where men only desire women and women only desire men that we would to match. Each man ranks a set of women in accordance with his preferences for a mariage and *vis versa.* For this class of matching, Gale and Shapley (1962) showed that there always is a stable set of marriages. The second category establishes different examples of matching problems in real world applications: on one side, colleges admit many student, firms hire many workers, hospitals employ many interns, and on the other side, student attend one college, workers work for one firm, and interns work for one hospital. The very known example is the college admission problem that is introduced by Gale and Shapley (1962). Each student ranks a set of college in accordance with his preferences for a matching and each college ranks a set of students with a quota. They proved that a stable college admissions matching exists. The third category analyses many situations where the very known example is given by Roth and Sotomayor (1990) who examine the assignment problem in the market for medical interns in the U.K. .

A *solution* is a logical process to select a set of matching outcomes. By solutions we generally indicate *matching rules* which differ according some desirable properties. As examples of matching rules: the stable rule, the Pareto rule, and the individually rational rule.

Although the basic models of two side matching problems are based on the notion of *stability*, many works in the matching literature are concerned with the question of the strategic behavior of agents. Generally, in most cases of public decision problems, agents have private information about their own preferences and they do not provide them exactly. So that agents reveal truthfully their preferences, a central planner should find mechanisms which give agents the incentive to reveal truthfully their private information independently of the other agents' behavior; the one of the most frequently mechanism used in the recent literature in social choice theory is known as *strategy-proofness.* This property is widely studied in the previous literature as long as it obligates the institutions to reveal truthfully their preferences. It requires that no agent can manipulate the outcomes of social choice rule in his favor by misrepresenting his preferences. It constitutes a key property for the fundamental result in social choice theory, which is known as the Gibbard (1973)-Satterthwaite (1975) Theorem. This result shows that if there exist at least three alternatives and individual preferences are not restricted, then every strategy-proof social choice rule is dictatorial.

In relation with matching problems, many authors studied the relation between strategy-proofness and efficiency. In one-to-one matching markets, Roth (1982) provided an impossibility result by showing that there is no stable and strategy-proof mechanism. Alcade and Barberà (1994) proved that there is no solution that is Pareto efficient, individually rational and strategy-proof. In many-to-one matching problems, Sönmez (1996a) axiomatized matching rule and he proved that there is a rule that is Pareto efficient, individually rational, and strategy-proof, which selects the unique stable matching, if and only if the capacities are unlimited.

In this paper, we deal with many-to-one matching problems. As solutions, we focus in particular on all sub-solutions of stable rule. Differently to the previous studies that often relate the axiomatization of strategy-proofness with Pareto efficiency and individual rationality[1], in this study we introduce the property of unanimity as a principle requirement to masure efficiency. This intuitive property is very attractive and checked by many solutions in the literature. We first provide a close connexion between strategy-proofness and unanimity. Second, we use a part of the first connexion to determine the relation between strategy-proofness and Maskin monotonicity. Alike connexion is given recently but in a different topic by Diss et *al.* (2016) who explored the relation between strategy-proofness, unanimity and Maskin monotonicity in a domain of single-peaked preferences with private values. Their results provided a connexion with implementation literature where these properties are central.

In our context and in relation with implementation theory, we show that there is a close connexion with strategy-proofness and the implementability of the sub- solution of the stable correspondence in dominant strategies equilibria and in Nash equilibria. Before exposing our contribution in this direction, we start by giving a brief review of implementation theory in matching problems. In these domains, many authors have tried to apply a certain existing theoretical results to implement the stable SCCs in various models. In the class of one-to-one matching models that include both marriage problems (Gale and Shapley,1962) and housing markets (Shapley and Scarf, 1974), Sönmez (1996b) used the results of Danilov (1992) and Yamato (1992) that are based on a strong version of Maskin monotonicity. He proved that the solution of nonempty core is Nash implementable with at least three agents. In one-to-one and many-to-one matching problems, Kara and Sönmez (1996,1997) applied the results of Maskin (1977,1999), Danilov (1992), and Yamato (1992). They examined the implementability of certain examples of the stable matching SCCs. Haake and Klaus (2009) considered a general class of many-to-one matching rules in studying a class of matching with contracts markets (Hatfield and Milgrom, 2005). To extend the result of Tadenuma and Toda (1998) from pure one-to-one matching rules to general many-to-one matching rules, Korpela (2013) referred to a general variant of Maskin monotonicity developed in Korpela (2010) and provided a full characterization for Nash implementation. In this topic, Doghmi and Ziad (2015) have recently provided a full characterization based only on the property of Maskin monotonicity.

Although several authors are interested to Nash implementability of stable rules in matching problems, they are few works that examined the implementability of these rules

[1]See for example Schummer (1997) and Hashimoto (2008) in the domain of exchange economies.

in dominant strategy equilibria. For instance, Kumano and Watabe (2012) studied the implementability of the deferred acceptance algorithm, but on *one-side matching markets*, where priorities objects cannot be private information and in using a direct mechanism. They provided a full characterization based on strategy-proofness. In this paper we study *two-sided matching markets* in which both preferences of workers and firms are private information and prove that it is possible to implement the stable correspondence in both dominant strategy equilibria and Nash equilibria by indirect mechanism. Hence, we bridge the gap between Nash implementation and dominant strategy implementation. We introduce a new condition and we prove that strategy-proofness is not only a necessary property in general environments as known in the literature, but also becomes sufficient for a sub-solution of the stable matching correspondence to be implementable with at least three agents and at least three matchings when the requirement of citizen sovereignty is fulfilled. In standard Nash implementation, strategy-proofness provides a full characterization without any restriction on the number of matchings. However, in partially honest Nash implementation, all sub-solutions of the stable matching correspondences are implementable independently to strategy-proofness. From these results we obtain a certain equivalence among these theories.

The rest of this paper is organized as follows: In Section 2, we introduce notations and definitions. In Section 3, we state and prove our main result. In sections 4, we give some implications. In Section 5, we study the relation between strategy-proofness and implementation literature. In Section 6, we conclude.

2 Model and definitions

Let $\mathcal{F} = \{f_1, ..., f_n\}$ be the set of Firms. Let $\mathcal{W} = \{w_1, ..., w_m\}$ be the set of workers. Let $q = (q_{f_1}, ..., q_{f_n})$ be a vector of positive natural numbers, where q_{f_i} is the *capacity* of firm $f_i \in \mathcal{F}$. Let $R = (R_i)_{i \in \mathcal{F} \cup \mathcal{W}}$ be a list of preferences of firms and workers. The asymmetrical and symmetrical parts of R_i are denoted respectively by P_i and $\sim_i$.

In this model, we assume that the indifferences are not allowed as in Korpela's model (2013). We consider the case where $\mathcal{F}$, $\mathcal{W}$ and q are fixed. The four-tuple $(\mathcal{F}, \mathcal{W}, q, R)$ constitute a *many-to-one matching labor market*.

The preference relation R_w of worker $w \in \mathcal{W}$ is a linear order on the set $\mathcal{F} \cup \{\emptyset\}$, where $\emptyset$ is interpreted as w not being matched with any $f \in \mathcal{F}$. Let $\mathscr{L}_w$ be the class of all such preference relations for worker $w \in \mathcal{W}$. The preference relation R_f of firm $f \in \mathcal{F}$ is a linear order on the set $\mathcal{W}_f = \{G \mid G \subseteq \mathcal{W} \text{ and } \mid G \mid \leq q_f\}$, where $\emptyset$ is also interpreted as w not being matched with any $f \in \mathcal{F}$. As Roth (1985); Kara and Sönmez (1996), we assume that R_f is **responsive** for all $f \in \mathcal{F}$. That is, for all $G \subseteq \mathcal{W}$ such that $\mid G \mid < q_f$, the following two properties are checked:
1) For all $w, w' \in \mathcal{W} \setminus G, G \cup \{w\} P_f G \cup \{w'\}$ if, and only if $w P_f w'$. [2]
2) For all $w \in \mathcal{W} \setminus G, G \cup \{w\} P_f G$ if, and only if $w P_f \emptyset$.

Let $\mathscr{L}_f$ be the class of all such preferences for firm $f \in \mathcal{F}$. Let $\mathscr{L} = \mathscr{L}_f^n \times \mathscr{L}_w^m$. We define the choice H of a firm f from a group of workers $G \subseteq \mathcal{W}$ under the preference R_f as follows: $H_f(G; R_f) = \{G' \subseteq G : \mid G' \mid \leq q_f, G' R_f G'' \text{ for all } G'' \subseteq G \text{ such that } \mid G'' \mid \leq q_f\}$.

[2] We write w instead of {w}.

A many-to-one matching μ is a mapping from the set $\mathcal{F} \cup \mathcal{W}$ into $2^{\mathcal{F} \cup \mathcal{W}}$ such that:
For all $w \in \mathcal{W}, \mid \mu(w) \mid \leq 1$ and $\mu(w) \subseteq \mathcal{F}$;
For all $f \in \mathcal{F}, \mid \mu(f) \mid \leq q_f$ and $\mu(f) \subseteq \mathcal{W}$;
For all $(f, w) \in \mathcal{F} \times \mathcal{W}, \mu(w) = f$ if and only if $w \in \mu(f)$.

For any $i \in \mathcal{F} \cup \mathcal{W}$, we call $\mu(i)$ the assignment of i under μ. We denote the set of all possible matchings by $\mathcal{M}$. We extend the preference relation R_f of a firm $f \in \mathcal{F}$ from the class of subsets $2^{\mathcal{W} \cup \{\emptyset\}}$ to the set of matching $\mathcal{M}$ as follows: f prefers the matching μ to the matching μ' if and only if it prefers its assignment under μ to its assignment under μ'. Similarly, we extend the preference relation for a worker and we use R_f and R_w to denote this extension.

A matching μ is *blocked by a worker* $w \in \mathcal{W}$ under a preference profile R if $\emptyset R_w \mu(w)$. A matching μ is *blocked by a firm* $f \in \mathcal{F}$ under a preference profile R if $\mu(f) \neq H_f(\mu(f); R_f)$. Under responsive preferences, this statement is equivalent to the following: A matching μ is blocked by a firm $f \in \mathcal{F}$ under a preference profile R if there is a worker $w \in \mu(f)$ such that $\emptyset R_f w$.

A matching μ is *individually rational* under a preference profile R if it is not blocked by a worker or a firm under R; i.e., if $\mu(w) P_w \emptyset$ for all $w \in \mathcal{W}$ and $\mu(f) = H_f(\mu(f); R_f)$ for all $f \in \mathcal{F}$.

A matching μ is *blocked by a firm-worker pair* $(f, w) \in \mathcal{F} \times \mathcal{W}$ under a preference profile R if (i) $f P_w \mu(w)$, and (ii) $w P_f w'$ for some $w' \in \mu(f)$, or $w P_f \emptyset$ if $\mid \mu(f) \mid < q_f$.

A matching μ is *stable* under a preference profile R if it is both individually rational and not blocked by any firm-worker pair.

We denote the set of all stable matchings under R by $\mathcal{S}(R)$. The stable many-to-one matching rule is a correspondence $\psi : \mathscr{L} \to \mathcal{M}$ such that $\psi(R) = \mathcal{S}(R)$ for all $R \in \mathscr{L}$. A sub-solution of the stable many-to-one matching rule is any correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ such that $\varphi(R) \subseteq \mathcal{S}(R)$ for all $R \in \mathscr{L}$. Gale and Shapley (1962) showed that a many-to-one matching problem has at least one stable matching for any admissible preference profile.

Now, we formally define the notion of strategy-proofness, which is the key property in this paper. It requires that no agent can benefit from misrepresenting his true preferences.

Definition 1 *(Strategy-proofness)*
A sub-solution to the stable matching correspondence φ satisfies the strategy-proofness property if for all $R \in \Re$, all $i \in \mathcal{F} \cup \mathcal{W}$, and all $R'_i \in \Re_i$, $\varphi(R) P_i \varphi(R'_i, R_{-i})$.

The second fundamental property is unanimity. It means that if for all preference profile, a matching is top-ranked for all agents, then this matching must be chosen. Formally, it is defined as follows.

Definition 2 (Unanimity) *The unanimity property states that if for all profile $R \in \Re$, for all $i \in \mathcal{F} \cup \mathcal{W}$ and for all matching $\nu \in \mathcal{M}$ for which $L(\nu, R_i) = \mathcal{M}$, then $\nu \in \varphi(R)$.*

According to Doghmi and Ziad (2015), the property of unanimity is checked by all sub-solution to the stable matching correspondence as shows the following proposition.

Proposition 1 (Doghmi and Ziad 2015) *Any sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ satisfies unanimity.*

Next, we use the following condition.

Condition 1. Let φ be a sub-solution to the stable matching correspondence. Let $R \in \mathscr{L}$, $\mu, \nu \in \mathcal{M}$, $i \in \mathcal{F} \cup \mathcal{W}$, and $\nu \in \varphi(R)$, if $\nu \in LS(\mu, R_i)$, there exists $\overline{R} \in \mathscr{L}$ such that $\mu \in \varphi(\overline{R})$, and $L(\nu, \overline{R}_j) = \mathcal{M}$ for all $j \in \mathcal{F} \cup \mathcal{W}$.

This condition means that if in a first profile R, a socially chosen matching ν is strictly dominated by a matching μ for an agent i , then there exists a new profile $\overline{R}$, in which the matching μ is chosen by the society, and ν improves her ranking and becomes top-ranked for all agents. It has been recently introduced by Diss and *al.* (2016) to study the relation between strategy proofness and unanimity in fair allocation problems with single-peaked preferences. It is checked by all solutions satisfying the requirements of *citizen sovereignty*[3] and *external stability* [4] in the problems of fair allocation.

In our context of many-to-one matching markets, we refer to the result of Roth (1982)[5] and we provide the following example: let $\varphi : \mathscr{L} \rightarrow \mathcal{M}$ be a sub-solution to the stable matching correspondence which associates the set of weakly Pareto efficient outcomes P_w in $\mathcal{M}$ with a preference profile R defined as follows: $\varphi^{P_w}(R) = \{\mu \in \mathcal{M} : \nexists \mu' \in \mathcal{M}$ such that $\mu' P_i \mu$ for all $i \in \mathcal{F} \cup \mathcal{W}\}$.

We show that φ^{P_w} satisfies Condition 1. Assume not, i.e., for $\varphi^{P_w}(R) \subseteq \mathcal{S}(R)$ for all $R \in \mathscr{L}$, $\mu, \nu \in \mathcal{M}$, $i \in \mathcal{F} \cup \mathcal{W}$, and $\nu \in \varphi^{P_w}(R)$, we have $\nu \in LS(\mu, R_i)$ (1), but for all $\overline{R} \in \mathscr{L}$ we have either (a) $\mu \notin \varphi^{P_w}(\overline{R})$, or (b) $L(\nu, \overline{R}_j) \neq \mathcal{M}$ for some $j \in \mathcal{F} \cup \mathcal{W}$. For statement (a), $\mu \notin \varphi^{P_w}(\overline{R})$ for all $\overline{R} \in \mathscr{L}$ means that there exists $\mu' \in \mathcal{M}$ such that $\mu' \overline{P}_i \mu$ for all $i \in \mathcal{F} \cup \mathcal{W}$. Since statement (a) is verified for all $\overline{R} \in \mathscr{L}$, then we have for R, $\mu' P_i \mu$ for all $i \in \mathcal{F} \cup \mathcal{W}$. Hence, from (1) we obtain $\mu' P_i \nu$ for some $i \in \mathcal{F} \cup \mathcal{W}$. If $\mu' = \emptyset$, then ν can not be individually rational, and so it can not be stable, a contradiction. In all other cases, there is a pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ such that $w \in \mu'(f)$ and $f \in \mu'(w)$, and we have: (i) for some $f \in \mu'(w)$, $f P_w \nu(w)$ and (ii) for some $w \in \mu'(f)$, $w P_f w'$ for some $w' \in \nu(f)$ or $w P_f \emptyset$ if $\mid \nu(f) \mid < q_f$. Therefore, ν is blocked by a firm-worker pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ under R, and so it is not stable, a contradiction. For statement (b), we have for all $\overline{R} \in \mathscr{L}$, $L(\nu, \overline{R}_j) \neq \mathcal{M}$ for some $j \in \mathcal{F} \cup \mathcal{W}$. This means that there exists $\mu'' \in \mathcal{M}$ such that $\mu'' P_i \nu$ for some $j \in \mathcal{F} \cup \mathcal{W}$, and hence we follow the same reasoning of statement (a).

In the following proposition, we prove that if Condition 1 holds, the property of strategy-proofness is satisfied.

Proposition 2 *If Condition* 1 *holds, any sub-solution to the stable matching correspondence* $\varphi : \mathscr{L} \rightarrow \mathcal{M}$ *satisfies strategy-proofness.*

Proof. Assume not. Therefore, there exists R and R'_i such that $\varphi(R'_i, R_{-i}) P_i \varphi(R)$. Let $\nu \in \varphi(R)$ and $\mu \in \varphi(R'_i, R_{-i})$. Hence, $\nu \in LS(\mu, R_i)$ (1). By Condition 1, there exists $\overline{R} \in \mathscr{L}$ such that $\mu \in \varphi(\overline{R})$, and $L(\nu, \overline{R}_j) = \mathcal{M}$ for all $j \in \mathcal{F} \cup \mathcal{W}$ (2). From

[3] A sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \rightarrow \mathcal{M}$ satisfies the property of citizen sovereignty if for each $\mu \in \mathcal{M}$, there is a profile $R \in \mathscr{L}$ such that $\mu \in \varphi(R)$.

[4] An SCF $f(R) \subseteq X$ is externally stable under R if every allocation in $X \setminus f(R)$ is dominated by $f(R)$.

[5] Roth (1982) proved that if a matching is an agent-optimal stable, then it is weakly Pareto optimal.

Proposition 1, φ satisfies unanimity, and hence $\nu \in \varphi(\overline{R})$. Therefore, $\varphi(\overline{R}) = \{\mu, \nu\}$. From (2), stability, and strictness we have $\mu = \nu$,[6] which contradicts (1). Q.E.D

From Propositions 1 and 2 we give the following first main result in this paper.

Theorem 1 *If Condition* 1 *holds, a sub-solution to the stable matching correspondence* $\varphi : \mathscr{L} \to \mathcal{M}$ *satisfies unanimity if and only if* φ *is strategy-proof.*

The second property that has a close connexion with strategy-proofness is Maskin monotonicity. It means that if a matching μ is socially chosen in a profile R and if the matchings ranked below μ for all agents remain ranked below it (in a large sense) in a new profile R', then the matching μ must be socially chosen in R'.

Definition 3 *(Maskin monotonicity)*
A sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ *satisfies Maskin monotonicity if for all* $R, R' \in \mathscr{L}$*, for any* $\mu \in \varphi(R)$*, if for any* $i \in \mathcal{F} \cup \mathcal{W}$*,* $L(\mu, R_i) \subseteq L(\mu, R'_i)$*, then* $\mu \in \varphi(R')$*.*

If strategy-proofness is known as the key necessary condition in the literature of dominant strategy implementation, Maskin monotonicity plays a fundamental role for Nash implementation. According to Maskin (1999), any Nash implementable social choice rules must satisfies the property of monotonicity. Studying the link between the two properties allow us to inspect the relation between dominant strategy implementation and Nash implementation. We return to this relationship for more details in Section 3.

From Proposition 2, it is obvious that Maskin monotonicity implies strategy-proofness. Hence, to prove the converse implication, we once again use Condition 1. We show that, under this property, strategy-proofness implies Maskin monotonicity.[7]

Proposition 3 *If Condition* 1 *holds, any sub-solution to the stable matching correspondence* $\varphi : \mathscr{L} \to \mathcal{M}$ *satisfies strategy-proofness is Maskin monotonic.*

Proof. Assume that a sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ satisfies strategy-proofness, but not Maskin monotonicity. Then, for any $R, R' \in \mathscr{L}$, any $\mu \in \varphi(R)$, and any $i \in \mathcal{F} \cup \mathcal{W}$, $L(\mu, R_i) \subseteq L(\mu, R'_i)$ (1), but $\mu \notin \varphi(R')$ (2). Since φ satisfies strategy-proofness, then for all $R \in \mathscr{L}$, all $i \in \mathcal{F} \cup \mathcal{W}$, and all $R'_i \in \mathscr{L}_i$, $\varphi(R) P_i \varphi(R'_i, R_{-i})$. Let $\nu \in \varphi(R'_i, R_{-i})$, hence $\mu R_i \nu$ (3). From Proposition 1, φ is unanimous. Hence, by (2), $\mu \notin \varphi(R')$ implies that there exist $i \in \mathcal{F} \cup \mathcal{W}$ and $\omega \in \mathcal{M}$ such that $\omega P'_i \mu$ and by (1) we

[6]From (2), stability, and strictness we necessarily have $\mu = \nu$ on $\mathcal{F} \cup \mathcal{W}$, i.e., $\mu(j) = \nu(j)$ for all $j \in (\mathcal{F} \cup \mathcal{W})$. Assume not, i.e., there exists $j \in (\mathcal{F} \cup \mathcal{W})$ such that $\mu(j) \neq \nu(j)$. As $L(\nu, \overline{R}_j) = \mathcal{M}$ and the individual preferences are strict, we have $\nu(j)\overline{P}_j\mu(j)$. If $\nu(j) = \emptyset$, then μ can not be individually rational, and so it can not be stable. In all other cases, there is a pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ such that $w \in \nu(f)$ and $f \in \nu(w)$, and we have: (i) for some $f \in \nu(w)$, $f\overline{P}_w\mu(w)$ and (ii) for some $w \in \nu(f)$, $w\overline{P}_f w'$ for some $w' \in \mu(f)$ or $w\overline{P}_f\emptyset$ if $\mid \mu(f) \mid < q_f$. Therefore, μ is blocked by a firm-worker pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ under $\overline{R}$, and so it is not stable, a contradiction.

[7]For more details on the literature that examined the relation between strategy-proofness and Maskin monotonicity in other domains, see Muller and Satterthwaite (1977), Bochet and Storcken (2010), and Klaus and Bochet (2013).

have $\omega P_i \mu$, and hence by (3), $\omega P_i \nu$ (4). By Condition 1, there exists $\overline{R} \in \mathscr{L}$ such that $w \in \varphi(\overline{R})$, and $L(\nu, \overline{R}_j) = \mathcal{M}$ for all $j \in \mathcal{F} \cup \mathcal{W}$ (5). By unanimity, $\nu \in \varphi(\overline{R})$, and hence $\varphi(\overline{R}) = \{w, \nu\}$. From (5), stability, and strictness we have $w = \nu$,[8] which contradicts (4). Q.E.D

From Propositions 2 and 3 we complete the proof of the second main result in this paper.

Theorem 2 *If Condition* 1 *holds, a sub-solution to the stable matching correspondence* $\varphi : \mathscr{L} \to \mathcal{M}$ *satisfies Maskin monotonicity if and only if* φ *is strategy-proof.*

3 Connexion with implementation literature

A sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ represents the welfare for a society in providing desired outcomes for a social designer. To implement this sub-solution, the social designer does not know exact preferences that are private information of the agents on the assignments. Hence, she/he organizes a non-cooperative game (game form) among agents. This procedure is a pair $\Gamma = (S, g)$ with $S = S_1 \times ... \times S_n$ and $g : S \to X$. For each agent $i \in \mathcal{F} \cup \mathcal{W}$, S_i is agent i's strategy space, and g is the outcome function that associates an outcome with each profile of strategies. A game form (mechanism) $\Gamma = (S, g)$ implements $\varphi : \mathscr{L} \to \mathcal{M}$ in a set of solution concepts SC if for all $R \in \mathscr{L}$, $\varphi(R) = g(SC(S, g, R))$. We say that φ is implementable in a solution concept if there is a mechanism which implements it.

3.1 Strategy-proofness and dominant strategies implementation

In general environment, strategy-proofness is known as a necessary condition for dominant strategies implementation. It becomes necessary and sufficient together with an additional property, termed *weak-non-bossiness*, introduced recently by Saijo and al. (2007). This full characterization is based on a restricting class of mechanisms and on single-valued rules. In this subsection, we use an indirect mechanism rather than direct mechanism to characterize multi-valued-rules. We prove that strategy-proofness becomes also sufficient under Condition 1. As we illustrate in the next, this result is based on the connexion between strategy-proofness and Maskin monotonicity, proved in the above, and on the finding of Yao and Yi (2007).

Let dominant strategy equilibrium be a solution concept of the game (Γ, R). The set of dominant strategy equilibria at state R is denoted by $DSE(S, g, R)$ and the set of strategy dominant equilibria outcomes is $g(DSE(S, g, R))$. A mechanism $\Gamma = (S, g)$ implements $\varphi : \mathscr{L} \to \mathcal{M}$ in dominant strategy equilibria if for all $R \in \mathscr{L}$, $\varphi(R) = g(DSE(S, g, R))$.

Next, we define the strict version of Maskin monotonicity developed by Yao and Yi (2007) that we call Y-monotonicity. This condition is formally defined as follows:

Definition 4 *(Y-monotonicity)*
A sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ *satisfies* Y-

[8] To prove this, we follow the same statement in Proposition 2.

monotonicity if for all $R, R' \in \mathscr{L}$, for any $\mu \in \varphi(R)$, if for any $i \in \mathcal{F} \cup \mathcal{W}$, $L(\mu, R_i) \setminus \{\mu\} \subseteq LS(\mu, R'_i)$, then $\varphi(R') = \{\mu\}$.

In other terms, Y-monotonicity means that if a matching μ is socially chosen in a profile R and if the matchings ranked below μ for all agents, in excluding μ, become strictly ranked below it in a new profile R', then the matching μ must be alone socially chosen in R'.

Now, we present a strong variant of the requirement of *citizen sovereignty* provided in Yao and Yi (2007).

Definition 5 *(Citizen sovereignty)*
A sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ satisfies the property of citizen sovereignty if for each $\mu \in \mathcal{M}$, there is a profile $R \in \mathscr{L}$ such that $\varphi(R) = \{\mu\}$.

This property requires that every possible ranking of matchings can be achieved from an individual preference outcome.

Corollary 1 (Yao and Yi (2007)) *Let $\mid \mathcal{M} \mid \geq 3$. If the property of citizen sovereignty holds, then a sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ is implementable in dominant strategies if and only if it satisfies Maskin monotonicity and Y-monotonicity.*

Proposition 4 *In many-to-one matching markets, Y-monotonicity becomes equivalent to Maskin monotonicity.*

Proof. Let $R, R' \in \mathscr{L}$, $\mu \in \mathcal{M}$, and $\mu \in \varphi(R) \subseteq \mathcal{S}(R)$. *i)* Y-monotonicity implies Maskin monotonicity; this first implication is immediate from the inclusions $L(\mu, R_i) \setminus \{\mu\} \subseteq L(\mu, R_i)$ and $LS(\mu, R'_i) \subseteq L(\mu, R'_i)$. *ii)* Maskin monotonicity implies Y-monotonicity; in this case, assume that $L(\mu, R_i) \setminus \{\mu\} \subseteq LS(\mu, R'_i)$. From this, we have $L(\mu, R_i) = L(\mu, R_i) \setminus \{\mu\} \cup \{\mu\} \subseteq LS(\mu, R'_i) \cup \{\mu\}$. Since $LS(\mu, R'_i) \cup \{\mu\} \subseteq L(\mu, R'_i)$, we obtain $L(\mu, R_i) \subseteq L(\mu, R'_i)$ for all $i \in \mathcal{F} \cup \mathcal{W}$ (*). By Maskin monotonicity, $\mu \in \varphi(R')$. Now we show that $\varphi(R') = \{\mu\}$. Assume that there is $\mu' \in \varphi(R')$, we want to show that $\mu(i) = \mu'(i)$ for all $i \in \mathcal{F} \cup \mathcal{W}$. Assume not, i.e., there is $j \in \mathcal{F} \cup \mathcal{W}$ such that $\mu(j) \neq \mu'(j)$. By strictness of preferences we have either (1) $\mu(j) P'_j \mu'(j)$ or (2) $\mu'(j) P'_j \mu(j)$. For case (1), if $\mu(j) = \emptyset$, then μ' can not be individually rational, and so it can not be stable. In all other cases, there is a pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ such that $w \in \mu(f)$ and $f \in \mu(w)$, and we have: (i) for some $f \in \mu(w)$, $f P'_w \mu'(w)$ and (ii) for some $w \in \mu(f)$, $w P'_f w'$ for some $w' \in \mu'(f)$ or $w P'_f \emptyset$ if $\mid \mu'(f) \mid < q_f$. Therefore, μ' is blocked by a firm-worker pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ under R', and so it is not stable, a contradiction. For case (2), we have from (*), $\mu'(j) P_j \mu(j)$. If $\mu'(j) = \emptyset$, then μ can not be individually rational, and so it can not be stable, a contradiction. In all other cases, there is a pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ such that $w \in \mu'(f)$ and $f \in \mu'(w)$, and we have: (i) for some $f \in \mu'(w)$, $f P_w \mu(w)$ and (ii) for some $w \in \mu'(f)$, $w P_f w'$ for some $w' \in \mu(f)$ or $w P_f \emptyset$ if $\mid \mu(f) \mid < q_f$. Therefore, μ is blocked by a firm-worker pair $(f, w) \in \mathcal{F} \times \mathcal{W}$ under R, and so it is not stable, a contradiction. Q.E.D

From Corollary 1 of Yao and Yi (2007) and Proposition 5 in this paper we complete the proof of the the following theorem.

Theorem 3 *Let $\mid \mathcal{M} \mid \geq 3$. If the property of citizen sovereignty holds, then a sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ is implementable in dominant strategies if and only if it satisfies Maskin monotonicity.*

From Theorems 2 and 3 we give the following result.

Corollary 2 *Let $n \geq 3$ and $\mid \mathcal{M} \mid \geq 3$. If Condition 1 and the requirement of citizen sovereignty hold, then a sub-solution to the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ is implementable in dominant strategies if and only if it satisfies strategy-proofness.*

3.2 Strategy-proofness and Nash implementation with standards agents

In this subsection we provide a full characterization to implement a sub-solution of the stable matching correspondence in Nash equilibria. Differently to the previous literature, we use the property of strategy-proofness as an alternative requirement. We shows that strategy-proofness is not only a necessary and sufficient condition for dominant strategy implementation, but it provide a full characterization for Nash implementation in this environment. This shows that both theories are very near each other in this domain.

Let $R \in \mathscr{L}$ a profile of preferences, and let Nash equilibrium be a solution concept of the game (Γ, R). The set of Nash equilibria at state R is denoted by $NE(S, g, R)$ and the set of Nash equilibria outcomes is $g(NE(S, g, R))$. A mechanism $\Gamma = (S, g)$ implements a sub-solution of the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ in Nash equilibria if for all $R \in \mathscr{L}$, $\varphi(R) = g(NE(S, g, R))$.

To illustrate the connection between Nash implementability and strategy-proofness of a sub-solution of the stable matching correspondence, we appeal to the result of Doghmi and Ziad (2015).

Corollary 3 (Doghmi and Ziad (2015)) *Let $n \geq 3$. A sub-solution of the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ is Nash implementable if and only if it satisfies Maskin monotonicity.*

From Theorem 2 and Corollary 4 we give the connexion between strategy-proofness and Nash implementation with standards agents.

Corollary 4 *Let $n \geq 3$. If Condition 1 and the requirement of citizen sovereignty hold, a sub-solution of the stable matching correspondence $\varphi : \mathscr{L} \to \mathcal{M}$ is Nash implementable if and only if it satisfies strategy-proofness.*

3.3 Strategy-proofness and Nash implementation with partially honest agents

Here, we present an environment of partial honest agents. We consider the same well-known model as the one considered in Dutta and Sen (2012), and Doghmi and Ziad (2013), Kartik et al. (2014), Korpela (2014), Saporiti (2014), Doghmi (2015), Ortner (2015), and Hagiwara et al.(2016) among others. More precisely, we assume that there are some players who have a “small” intrinsic preference for honesty and each honest individual expresses her preferences in a *lexicographic* way. For a domain of single-peaked preferences $\mathscr{L}$, let C_i be the other components of the strategy space (which depends on individual preferences, social states, ...). The set $S_i = \mathscr{L} \times C_i$ represents the strategy profiles for a player $i \in \mathcal{F} \cup \mathcal{W}$ and $S = S_1 \times ... \times S_n$ is a set of strategy profiles. The elements of S are denoted by $s = (s_1, ..., s_n)$. A *domain* is a set $\mathscr{D} \subset \mathscr{L}$. For each $i \in \mathcal{F} \cup \mathcal{W}$, and $R \in \mathscr{D}$, let $\tau_i(R) = \{R\} \times C_i$ be the set of truthful messages of agent i. We denote by $s_i \in \tau_i(R)$ a truthful strategy as player i is reporting the true preference profile. We extend a player's ordering over the set X to an ordering over strategy space S. This is because, the players' preference between being honest and dishonest depends on strategies that the others played and the outcomes which they obtain. Let $\succeq_i^R$ be the preference of player i over S in preference profile R. The asymmetrical and symmetrical parts of $\succeq_i^R$ are denoted respectively by $\succ_i^R$ and $\sim_i^R$. Let Γ be a mechanism (game form) represented by the pair (S, g), where $S_i = \mathscr{D} \times C_i$ and $g : S \to A$ is a payoff function.

Definition 6 *A player i is partially honest if for all preference profile $R \in \mathscr{D}$ and $(s_i, s_{-i}), (s'_i, s_{-i}) \in S$,*
(i) When $g(s_i, s_{-i}) R_i g(s'_i, s_{-i})$ and $s_i \in \tau_i(R)$, $s'_i \notin \tau_i(R)$, then $(s_i, s_{-i}) \succ_i^R (s'_i, s_{-i})$.
(ii) In all other cases, $(s_i, s_{-i}) \succeq_i^R (s'_i, s_{-i})$ iff $g(s_i, s_{-i}) R_i g(s'_i, s_{-i})$.

Let $NE(g, \succeq^R, S)$ be the set of Nash equilibria of the game $(\Gamma, \succeq^R)$. A mechanism $\Gamma = (S, g)$ implements a sub-solution to the stable matching correspondence $\varphi : \mathscr{D} \to \mathcal{M}$ in Nash equilibria if for all $R \in \mathscr{D}$, $\varphi(R) = g(NE(g, \succeq^R, S))$. We say that φ is partially honest implementable in Nash equilibria if there is a mechanism which implements it in these equilibria. In this framework, we recall the Assumption A of Dutta and Sen (2012).

Assumption 1 *There exists at least one partially honest individual and this fact is known to the planner. However, the identity of this individual is not known to her.*

Next, we define a weak variant of no-veto power.

Definition 7 *(Strict-weak no-veto power)*
A sub-solution to the stable matching correspondence $\varphi : \mathscr{D} \to \mathcal{M}$ satisfies strict-weak no-veto power if for each $i \in \mathcal{F} \cup \mathcal{W}$, each $R \in \mathscr{D}$, each $\mu \in \varphi(R)$, and each $\nu \in \mathcal{M}$, if for $R' \in \mathscr{D}$, $\nu \in LS(\mu, R_i) \subseteq L(\nu, R'_i)$ and $L(\mathcal{M}, R'_j) = \mathcal{M}$ for all $j \in (\mathcal{F} \cup \mathcal{W}) \backslash \{i\}$, then $\nu \in \varphi(R')$.

Doghmi and Ziad (2012, 2013) showed that the properties of strict-weak no-veto power and unanimity are sufficient for a social choice correspondence to be partially honest Nash implementable.

Theorem 4 (Doghmi and Ziad (2012, 2013)) *Let $n \geq 3$ and suppose Assumption A holds. If a sub-solution to the stable matching correspondence $\varphi : \mathscr{D} \to \mathcal{M}$ satisfies strict-weak no-veto power and unanimity, then φ can be implemented in Nash equilibria.*

According to Doghmi and Ziad (2015), the property of strict-weak no-veto power is automatically checked for all sub-solution to the stable matching correspondence.

Proposition 5 (Doghmi and Ziad 2015) *Any sub-solution to the stable matching correspondence $\varphi : \mathscr{D} \to \mathcal{M}$ satisfies strict-weak no-veto power.*

It follows from Theorem 3 and Propositions 1 and 6 that any sub-solution to the stable matching correspondence can be implemented Nash equilibria in an environment of partial honesty.

Corollary 5 *Let $n \geq 3$ and suppose Assumption A holds. Any sub-solution to the stable matching correspondence $\varphi : \mathscr{D} \to \mathcal{M}$ is partially honest Nash implementable.*

We remark that the partial honest Nash implementability of a sub-solution to the stable matching correspondence is checked independently to the property of strategy-proofness.

3.4 Equivalence results

Condition 1 and the property of citizen sovereignty allow us to capture the equivalence between dominant strategy implementation, standard Nash implementation, and partially honest Nash implementation. We deduce this from the corollaries 4, 6, and 7.

Corollary 6 *Let $n \geq 3$ and $\mid \mathcal{M} \mid \geq 3$. If Condition 1 and the requirement of citizen sovereignty hold, then dominant strategy implementation, standard Nash implementation, and partially honest Nash implementation are equivalent.*

This equivalence result can be related to the recent development in implementation literature concerning *secure implementation with standard agents* provided in Saijo and al. (2007) and Mizukami and Wakayama (2016), and the one with *partially honest agents* developed in Saporiti (2014). Since secure implementation is a double implementation in Nah equilibria and in dominant strategy equilibria, Corollary 7 extends the equivalence of implementability to this concept of secure implementability.

4 Conclusion

In this paper we have inspected the relation between strategy-proofness and unanimity. We have first, introduced a new condition and we have proved that these properties become equivalent if this new requirement is met. We have second, showed that this result help us us to detect the link between strategy-proofness and Maskin monotonicity. These results allowed us to determine a certain connexion between strategy-proofness and implementation literature. In particular, we have proved that dominant strategy implementation, standard Nash implementation, and partially honest Nash implementation are equivalent under our new requirement.

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