The Paradox of Thrift in an Inegalitarian Neoclassical Economy

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Abstract

[Schilcht 1975] and [Bourguignon 1981] studied the case of a convex saving function in the [Stiglitz 1969] model. They have shown that if one of the two proportions of the rich or the poor is below a certain threshold, there is a two-class equilibrium. However, they have only proved the existence of this threshold. We give here a system of equations to calculate this threshold which we interpret as the maximum proportion of rich for having a stable two-class configuration. If the proportion of rich exceeds this threshold, the economy enters a phase of decline although the golden-rule capital has not yet been reached. This decline is due to a specific articulation between the rate of decrease in the productivity of capital and the rate of increase in the depreciation of capital. The mechanism of this decline recalls the description given in [Keynes 1936], of the decline which happens when there is too much savings in an inegalitarian context. This is an example of what is known as the "paradox of thrift". It is remarkable that this paradox takes place in a neoclassical setting that does not include key Keynesian elements such as saturation of demand, monetization of savings, short-term effects, expectation problems, involuntary unemployment and rigidities. Numerical simulations are given to illustrate and analyze the mechanisms involved.

Keywords : Paradox of Thrift, Inequality, Saving, Growth.

1 Introduction

[Schilcht 1975] and [Bourguignon 1981] studied the case of a convex relationship between savings and income in the [Stiglitz 1969] model. The purpose of the Stiglitz model was to show the influence of income and wealth distribution on economic growth and on the convergence of social classes. Although there is no evidence of the convex or non-convex nature of the relationship between savings and income at the aggregate level, at the individual and static level the convexity hypothesis is the most likely [Dynan-Skinner-Zeldes 2004, Boushey-Hersh 2012]. Therefore, the present study is based on this hypothesis of convexity. [Schilcht 1975] has shown that if this hypothesis is adopted instead of the concavity or linearity of the relationship between individual savings and income, and if the proportion of one of the social classes is less than a certain threshold, the convergence of social classes no longer takes place and

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the system evolves towards an inegalitarian equilibrium with two social classes. Therefore, the spontaneous and generally observable trend towards a rich / poor social structure rather than an egalitarian structure is further confirmation of the convexity hypothesis. [Bourguignon 1981] shows that under this hypothesis, inegalitarian equilibria Pareto-dominate the egalitarian equilibrium.

In this paper we give a system of equations which allows calculating this threshold and we interpret it as the maximum proportion of rich to have a stable two-class configuration. If the proportion of rich people exceeds this threshold, the economy enters a phase of decline.

The purpose of this paper is also to examine in detail this decline in the light of the description in [Keynes 1936] of the economic decline caused by an excess of savings in a context of inequality.

We begin in section 2 by presenting the characteristics of the model and the main results obtained by [Schilcht 1975] and [Bourguignon 1981]. We mostly keep the notations and method of [Bourguignon 1981].

In Section 3, we give the equations for calculating the maximal sustainable proportion of rich (MSPOR). We calculate the MSPOR from numerical values proposed for the rate of capital depreciation and for production and saving functions. The calculation is carried out for different values of the propensity to save. These numerical values are also used for the following sections to illustrate the findings.

In Section 4, we analyze the dynamics of the decline. Given a certain resemblance to the description in [Keynes 1936], we refer to it as the "Keynesian decline".

We then discuss the following questions:

- How does the equilibrium of the economy behave according to the distribution of wealth? (Section 5) This section shows that a tiny proportion of rich people makes it possible to push a locked economy into insufficient savings and egalitarian poverty towards a level close to the golden-rule. On the other hand, it also shows that the increase in this proportion is harmful.

- How does equilibrium behave according to the propensity to save, for a given distribution of wealth? (Section 6). This section highlights the phenomenon of "paradox of thrift" although the model does not include strictly Keynesian elements, such as saturation of demand, monetization of savings, short-term effects, expectation problems, unemployment and rigidities.

Section 7 concludes and presents possible directions for further study.

2 Notation and main features of the model

We mainly use the assumptions, notations, method and results of [Bourguignon 1981].

The economy is represented by a per capita production function $f(k)$ where k is the average capital per capita. f is increasing, concave and twice differentiable. Individual savings are assumed to depend on income according to the function $S(y)$ where y is the income of the individual concerned. S is convex, increasing, differentiable and checks $\lim_{y\to\infty} S'(y) = 1$.

The capital undergoes depreciation at a rate δ per unit of time and capital. This depreciation plays the same role as population growth in [Bourguignon 1981]. We have thought that it would be more appropriate, in modern economic conditions, to speak of depreciation of capital rather than demography. But the interpretation of δ as a population growth rate remains valid.

We assume that the economy has a unique stable egalitarian equilibrium k_0. Mathematically, this condition is equivalent to saying that k_0 is the unique solution of the equation $S(f(k)) = \delta k$ and that $S'(f(k_0)).f'(k_0) < \delta$.

We denote by k^* the per capita capital of the golden-rule defined by

$$f'(k^*) = \delta$$

The society is composed of 2 classes: the poor, in proportion a_1 and the rich in proportion $a_2 = 1 - a_1$. In a theoretical perspective, this assumption is not restrictive because the convexity of saving implies that the equilibrium has at most two classes [Bourguignon 1981]. In the spirit of the present study, the concept of "poor class" includes the middle class. Consequently, the poor class is the majority. So, we have $a_2 < a_1$. We will assume this for all the following. The capital stock per capita is c_1 for the poor and c_2 for the rich. The average per capita capital therefore satisfies $k = a_1c_1 + a_2c_2$.

As stated by the neoclassical theory of distribution, capital is paid for according to its marginal productivity. The remuneration of per capita capital is therefore: $k.f'(k)$. By deduction, the per capita wage is $f(k) - kf'(k)$. All individuals receive the same payment in exchange for their contributions to work, i.e. $f(k) - kf'(k)$. For capital, individuals are remunerated according to the shares of capital they hold. Thus, an individual of class i (with $i = 1$ or 2) receives $c_i.f'(k)$ in exchange for his contribution to capital. Moreover, he bears the share of the depreciation of the capital he owns: $\delta.c_i$.

The equation of capital evolution for class i is therefore

$$\dot{c_i} = S\left[f(k) - kf'(k) + c_i.f'(k)\right] - \delta.c_i$$

The equilibrium is thus characterized by the following 3 equations for the 3 unknowns c_1, c_2 and k:

$$\begin{aligned} S\left[f(k) + (c_1 - k).f'(k)\right] - \delta.c_1 &= 0 \\ S\left[f(k) + (c_2 - k).f'(k)\right] - \delta.c_2 &= 0 \\ k &= a_1c_1 + a_2c_2 \end{aligned} \tag{1}$$

Denote T the inverse function of S. We have

$$T' > 1, T'' < 0$$

and

$$\lim_{x\to\infty} T'(x) = 1$$

Let (E) be the curve in the space (c, k) defined by the equation :

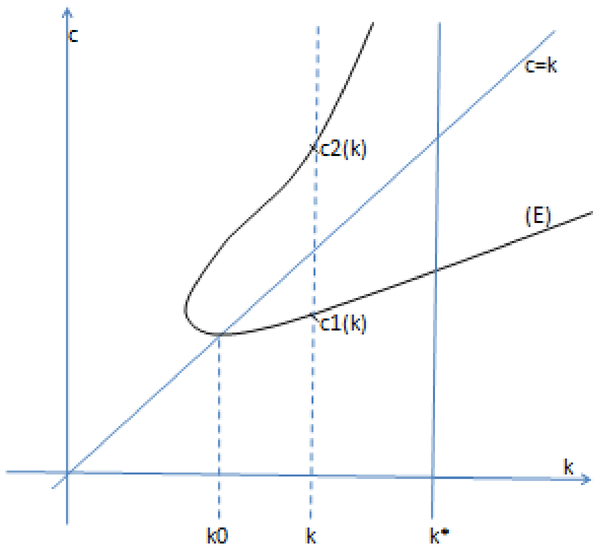
$$S\left[f(k) + (c-k).f'(k)\right] = \delta.c$$

or, equivalently:

$$f(k) + (c-k).f'(k) = T(\delta.c)$$

The curve (E) intersects the line $(c = k)$ at the points satisfying $f(k) = T(\delta.k)$. By assumption, this equation is verified only in k_0. Therefore the curve (E) intersects the line $(c = k)$ only in k_0.

[Bourguignon 1981] shows that a necessary condition for an equilibrium with two social classes is $k_0 < k^*$. In this case and for $k \in [k_0, k^*[$ he establishes that the equation $f(k) + (c-k).f'(k) = T(\delta.c)$ admits two solutions $c_1(k)$ and $c_2(k)$ such that $c_1(k) \leqslant k$ and $c_2(k) > k$. These two solutions are candidates for per capita capital values of the two social classes at equilibrium.



All details and justifications concerning the elaboration of the curve (E), the phase plan and the dynamics of the system can be found in [Bourguignon 1981]. We have reproduced here the notations of [Bourguignon 1981] in order to facilitate the consultation of this reference at the same time as the present paper.

We assume for all the following that the condition $k_0 < k^*$ is checked because without it all social classes would necessarily converge. Indeed, in continuity with the work of [Schilcht 1975] and [Bourguignon 1981], our concern is to study the consequences of a persistent inequality, a pattern that seems to be more realistic.

If we consider the production parameters as given (i.e. the production function and the depreciation coefficient) then the position of k_0 with respect to k^* depends on the saving behavior, that is, on the function S. The intuitive economic interpretation of the condition $k_0 < k^*$ is that the poor class, if it where alone, would not have the sufficient saving propensity to reach the golden-rule [2].

Defining the function $A(k)$ by the equation $(1-A(k)).c_1(k)+A(k).c_2(k) = k$, [Bourguignon 1981] shows that A is positive and continuous over $]k_0, k^*[$, that $A(k_0) = 0$ and $\lim\limits_{k\to k^*} A(k) = 0$. It follows that $A(k)$ admits a maximum on $]k_0, k^*[$ denoted $\overline{A}$, and that under the condition $k_0 < k^*$, for a stable inegalitarian equilibrium to exist, we must have $\inf(a_1, a_2) < \overline{A}$. This condition is also sufficient[3] and the inegalitarian equilibrium Pareto-dominates the egalitarian equilibrium.

Since we have assumed $a_2 < a_1$, the necessary and sufficient condition becomes $a_2 < \overline{A}$. Let us observe that the social class which was initially poor will never be able to surpass the rich class. Indeed, assuming that the system inverts the situations along the way, then, by continuity of the state variables c_1 and c_2, it would be necessary that at a certain date these two variables become equal. Equations (1) show that these two variables would then always remain equal from this date on.

We deduce that a_2 constitutes the proportion of the rich class at the beginning and at the end. One can therefore reformulate the necessary and sufficient condition for the existence of a stable inegalitarian equilibrium by saying that the proportion of rich must be less than $\overline{A}$.

3 The maximal sustainable proportion of rich

If the proportion of rich exceeds $\overline{A}$, [Bourguignon 1981] shows that there can be only an egalitarian equilibrium Pareto-dominated by the inegalitarian equilibria achievable with proportions of rich less than $\overline{A}$. As soon as the proportion of rich exceeds $\overline{A}$, we will see that the economy ends up being trapped in a decline. For this reason we refer to $\overline{A}$ as the maximal sustainable proportion of rich (MSPOR).

In this section we establish a system of equations for calculating $\overline{A}$. Then, as example, different values of $\overline{A}$ corresponding to different values of certain parameters are calculated.

[2] This intuitive interpretation of $k_0 < k^*$ entails that k_0 increases with the saving propensity, what is checked in all the following. The precise definition of the saving propensity is given in next section.

[3] In fact, Bourguignon asserts that $\inf(a_1, a_2) \leqslant \overline{A}$ is a necessary and sufficient condition, but if $\inf(a_1, a_2) = \overline{A}$, the stability is lost.

To calculate $\overline{A}$, we start from the system (1) replacing a_2 by $A(k)$. From now on, it is assumed that the system (1) is smooth enough for the functions $c_1(k), c_2(k)$ and $A(k)$ to be differentiable. Then we derive the 3 equations with respect to k and we write that $\frac{dA}{dk} = 0$.

We have

$$k = (1 - A(k)).c_1 + A(k).c_2$$

Deriving with respect to k, we get:

$$1 = A(k).\frac{d(c_2 - c_1)}{dk} + (c_2 - c_1)\frac{dA}{dk} + \frac{dc_1}{dk}$$

We write that $\frac{dA}{dk} = 0$ at $\overline{A}$, which gives:

$$\overline{A} = \frac{1 - \frac{dc_1}{dk}}{\frac{dc2}{dk} - \frac{dc_1}{dk}} \tag{2}$$

Furthermore, the derivatives of c_1 and c_2 with respect to k are obtained by deriving the first two equations of the system (1):

$$\frac{dc_1}{dk} = \frac{f"(k).(c_1 - k)}{\delta T'(\delta c_1) - f'(k)} \tag{3}$$

$$\frac{dc_2}{dk} = \frac{f"(k).(c_2 - k)}{\delta T'(\delta c_2) - f'(k)} \tag{4}$$

Lastly, the third equation of the system (1) provides:

$$\overline{A} = \frac{k - c_1}{c_2 - c_1} \tag{5}$$

We obtain equations (2) to (5) for the unknowns: $\frac{dc_1}{dk}, \frac{dc_2}{dk}, k$ and $\overline{A}$. By adding the first two equations of the system (1), we obtain 6 equations for the 6 unknowns $c_1, c_2, \frac{dc_1}{dk}, \frac{dc_2}{dk}, k$ and $\overline{A}$.

It is noteworthy that the value of $\overline{A}$ depends only on the production function, the rate of depreciation and the saving function, and not on the initial state of the economy (i.e. initial capital and wealth distribution).

Since there is no explicit formula for $\overline{A}$, we have thought useful to take numerical values for these 3 data (production function, depreciation rate and saving function) to illustrate our point and get an idea of the order of magnitude of $\overline{A}$ for these numerical values. It is not argued that the following calculations express the actual situation of a particular country[4].

The production function is chosen so that the marginal productivity of capital can decrease rapidly. The choice is a Cobb-Douglas with a share of the capital income equal to 0.3. The parameters of the production function have been adjusted so that the capital coefficient is 2.5 for an average per capita

[4] The model is still at the rudimentary stage to lend itself to empirical work.

income normalized to 1. Consequently, the production function per capita is $f(k) = \frac{3}{4}k^{0.3}$.

An analytic form has been adjusted for the individual saving function to ensure that it is increasing, convex and that the limit of the marginal propensity to save equal to 1:

$$S(y) = b + \frac{1}{2}(1+c)(y-a) + \frac{1-c}{1+c}\sqrt{c' + \left[\frac{1}{2}(1+c)(y-a)\right]^2}$$

This form checks the requested conditions. The coefficients a, b, c and c' are adjusted to have the following values for individual savings rates at different levels of income:

income	0.1	1	1.5	2
savings rate	7%	15%	20%	30%

By minimizing the sum of the absolute values of the deviations, the adjusted values for a, b, c and c' are:

$$\begin{aligned} a &= 1.7105249 \\ b &= 0.0255809 \\ c &= 0.0677230 \\ c' &= 0.1889504 \end{aligned}$$

The term "social propensity to save" is used hereafter to indicate the general state of mind of society about the willingness to save. If function S represents the saving behavior, the change in the level of the social propensity to save can be obtained by the form:

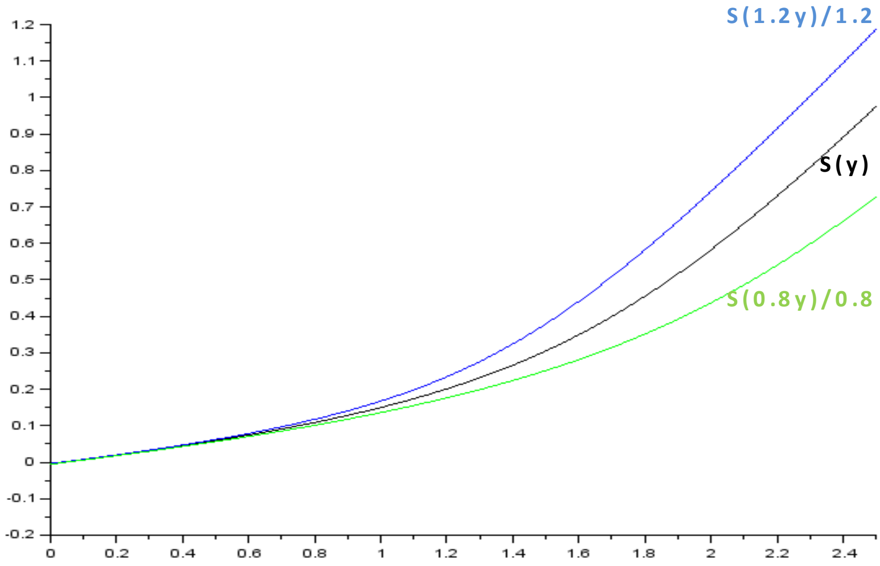
$$S_\beta(y) = \frac{1}{\beta}S(\beta y)$$

The variation of the coefficient β thus represents the variation of the overall willingness to save of society (see the following graph). If β increases, the willingness to save increases. β is referred to as the "social propensity to save". It is obvious, however, that the variation of the coefficient β can not in itself represent all the possibilities of modifying the profile of the willingness to save. For example, one can think of an increase in the willingness to save among the poor at the same time as a decrease among the rich. Such a change is not captured by the parameter β and is not considered in the present study.

If $\beta > 1$ the propensity to save increases for all incomes. It decreases if $\beta < 1$:

income	0.1	1	1.5	2
savings rate with $\beta = 1.2$	7.8%	16.7%	25.3%	37.1%
savings rate with $\beta = 0.8$	5.8%	13.6%	16.7%	21.8%

We obtain the following curves that give the individual savings rate as a function of income for $\beta = 0.8$, $\beta = 1$ and $\beta = 1.2$:



Lastly, the annual capital depreciation rate is set at 3.7%.

With the various parameters specified above, the following results for $\overline{A}$ as a function of β are obtained by computer:

β	1	1.2	1.1	0.9	0.8
$\overline{A}$	5.44%	1.33%	4.45%	5.35%	4.85%

We see that the MSPOR $\overline{A}$ decreases quite sharply if the social propensity to save increases from the reference situation $\beta = 1$.

For each value of β and with a proportion of rich equal to $\overline{A}$, values of per capita and per class capital and output at inegalitarian equilibrium are given as well as per capita capital and output at egalitarian equilibrium:

β	1	1.2	1.1	0.9	0.8
average per capita capital	8.89	11.75	9.89	8.29	7.86
average per capita income	1.44	1.57	1.49	1.41	1.39
per capita capital of the poor	6.51	10.73	7.98	5.62	5.01
per capita capital of the rich	50.2	87.5	50.86	55.53	63.72
per capita income of the poor	1.33	1.53	1.41	1.28	1.24
per capita income of the rich	3.46	4.61	3.35	3.83	4.36
per capita capital at egalitarian equilibrium	6.25	10.66	7.77	5.35	4.75
per capita income at egalitarian equilibrium	1.30	1.53	1.39	1.24	1.2
per capita capital at the golden-rule	13.18	13.18	13.18	13.18	13.18

We see that the best situation for both the poor and the rich is the situation $\beta = 1.2$, where the social propensity to save is high and the proportion of wealthy low. The most damaging situation for the poor is the situation $\beta = 0.8$ where the social propensity to save is low and the proportion of rich is quite high.

We now give the savings rates $S_\beta(y)/y$ at equilibrium by social class and for society as a whole, for each value of β.

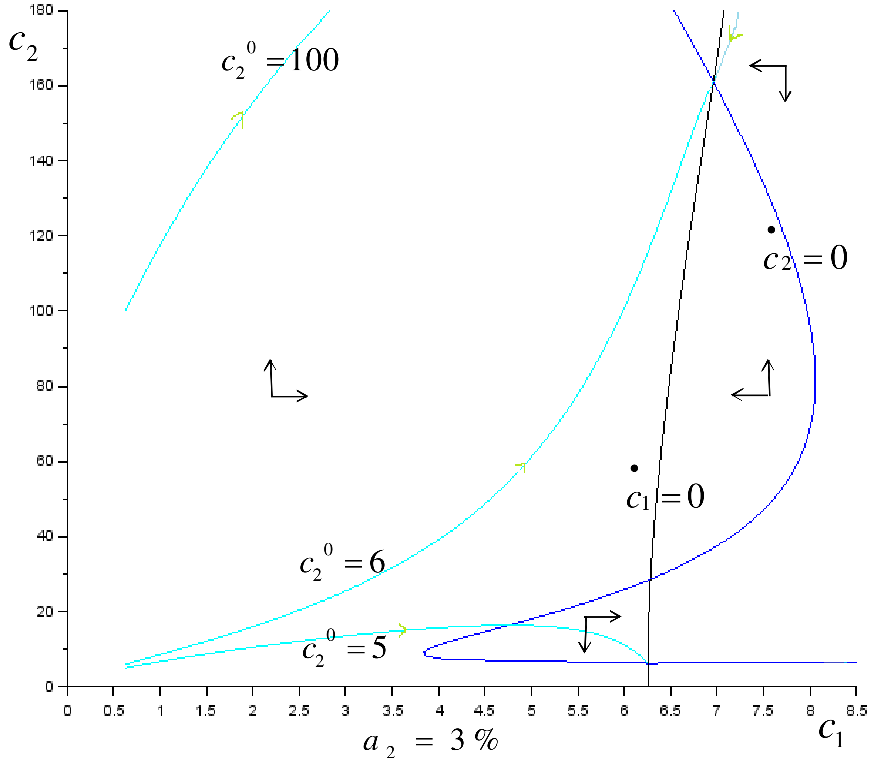
β	1	1.2	1.1	0.9	0.8
savings rate of the poor	18.1%	26%	21%	16.3%	15%
savings rate of the rich	53.7%	70.2%	56.2%	53.6%	54%
aggregate savins rate	22.7%	27.7%	24.5%	21.7%	20.9%

We see that, apart from the case $\beta = 1.2$, the aggregate savings rates are relatively close. However, the social propensities to save, individual savings rates and equilibrium incomes differ significantly. In fact the aggregate savings rate is a parameter which, considered alone, does not reflect the saving behavior. Other characteristics are important such as the level of average income, the distribution of wealth and income, or the position in the accumulation process (more or less close to equilibrium). For example, the aggregate savings rate may increase due to a higher concentration of income while the average income falls. This may explain the inconclusive results of the studies on the relationship between aggregate savings rates and income [Dynan-Skinner-Zeldes 2004]. But it should not be concluded that at the individual level, the savings rate does not increase as income increases.

4 The Keynesian decline

We are interested here in what happens when the proportion of rich exceeds $\overline{A}$. After a period of growth, the economy declines towards the egalitarian configuration which happens to be Pareto-dominated by inegalitarian equilibria, as showed by [Bourguignon 1981]. We try to see the mechanisms of this decline through a numerical example.

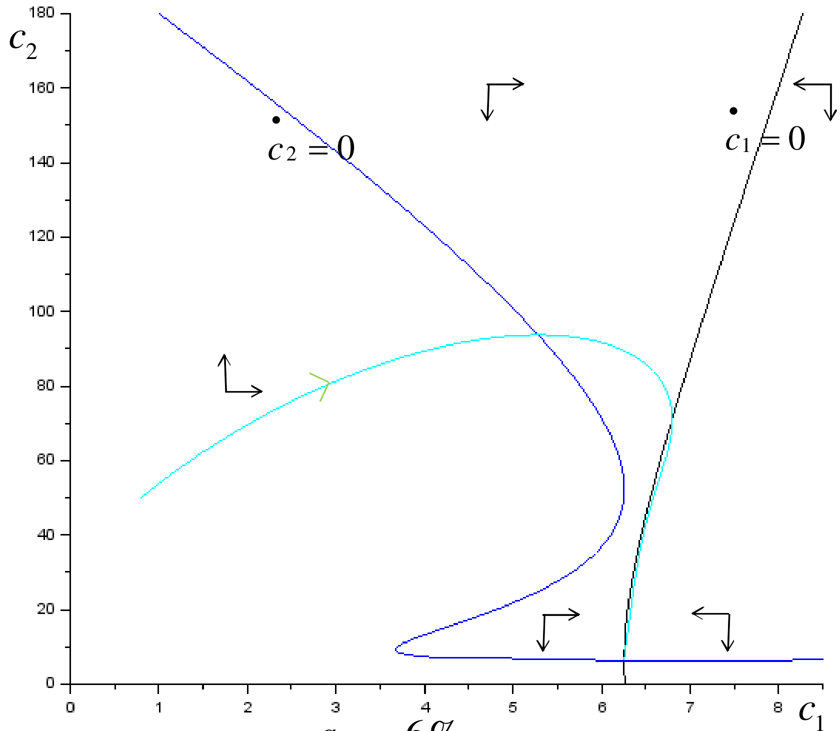
The parameters of the section 3 are used again: production function, saving function with $\beta = 1$ and depreciation rate. The following figure shows the phase plan if we take a proportion of rich of 3%, less than the MSPOR which is 5.44% for $\beta = 1$. The initial per capita capital of the rich class is given the values $c_2^0 = 5$ and 6 then 100, and the initial per capita capital of the poor class the value $c_1^0 = 0.6$. The following trajectories (in green) are obtained:



We observe that if $c_2^0 = 6$, the economy is freed from from the path to the poor egalitarian equilibrium and grows towards the rich inegalitarian equilibrium. Whereas if one begins with $c_2^0 = 5$, the income of the rich class is not sufficient to allow a saving capable to release the economy from the path of egalitarian poverty. This conclusion is not surprising. It is consistent with the intuition that capital weakness can trap the economy into poverty.

It is less immediate to admit that an excess of capital can lead to trapping the economy in poverty. Yet, if we take a proportion of rich above the MSPOR, this is what we observe. This is the case that is interesting to analyze.

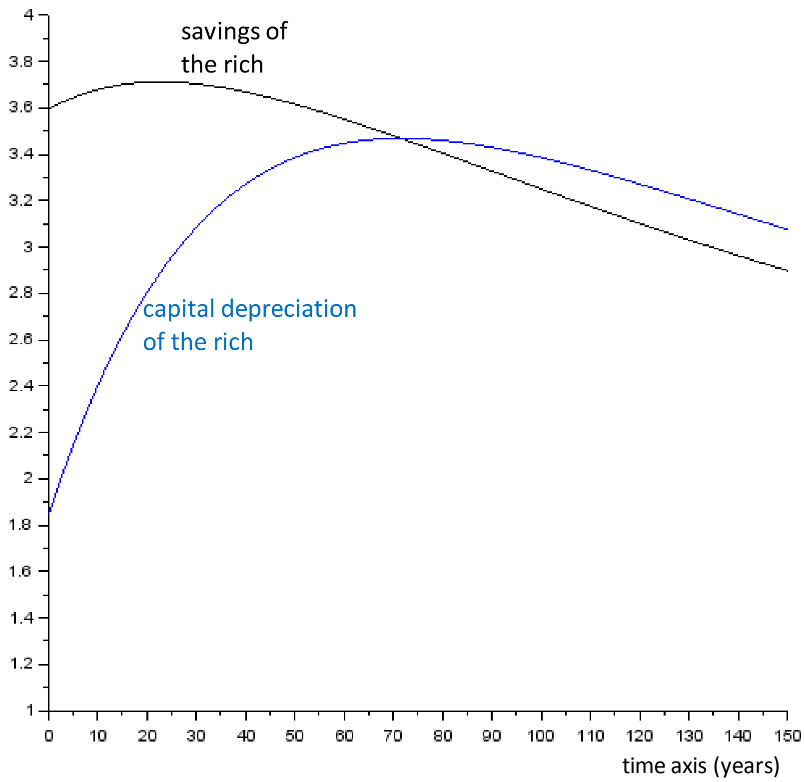
We take $a_2 = 6\%$. The curves $\left\{\dot{c_1} = 0\right\}$ and $\left\{\dot{c_2} = 0\right\}$ intersect only in the poor egalitarian equilibrium. The following trajectory is obtained for $c_2^0 = 50$ and $c_1^0 = 0,8$:



$a_2 = 6\%$

In this setting, the rich begin with a per capita capital of 50. They then climb to more than 90 to finally plummet to 6.25 which is the capital per capita of the poor egalitarian equilibrium. The poor also experience a drop at the end of the trajectory from 6.90 to 6.25. But this decline is less marked and the overall balance is positive for them: from 0.8 to 6.25.

To understand the reason for this decline, we are interested in what governs the capital dynamics for the rich, that is, their savings on the one hand and the depreciation of their capital on the other.



At the start, both classes take advantage of the existence of inequality. Indeed, the poor benefit from a good level of production made possible by the capital of the rich, whereas the rich profit from a good productivity of their capital thanks to the labor of the poor, or in other words, thanks to a still modest macroeconomic capital per capita ratio. The economy is growing considerably.

This strong growth has the effect of an increase in the capital stock and a rapid decline in capital marginal productivity. This decline doubly affects the income of the rich in comparison with the case of an equal distribution of wealth. Indeed, it curbs the increase in production, as is also the case in an egalitarian society where capital stock is growing. But in addition to this, it diminishes the income share of the wealthy acquired through the existence of inequalities.

In the above graph, the income and savings of the rich begin to decline after about 20 years. However, their savings remain abundant. Their capital therefore continues to rise and it begins to fall only after about 50 years of the date of the decline in income. This discrepancy is the cause of an excessive accumulation which leads to a situation where it is no longer possible to cover the depreciation of capital by saving. The decline then begins and it is no longer recoverable. In fact, this dynamic depends on the comparison between the decline in the productivity of capital and the increase in the depreciation of capital. It should be noted that at the macroeconomic level, average per capita capital does not reach the golden-rule stage beyond which capital productivity falls below depreciation rate. Thus, inequality makes the economic growth stop before reaching the golden-rule stage. But it will be seen below (section 5) that inequality can also make it possible to approach the golden-rule by compensating the weakness of the savings of the poor class.

Thus the initial abundance of wealth is the very cause of subsequent decline. A smaller proportion of rich in the beginning could have delayed capital growth and marginal productivity decline so that the economy stabilizes without tumbling into poverty, as the case $a_2 = 3\%$ shows.

The mechanism of this decline reminds one of the description in [Keynes 1936], of the decline that occurs when there is too much unevenly distributed wealth. That's what he calls "the paradox of poverty in the midst of plenty, where excessive wealth and saving of the rich can lead to a decline in both aggregate wealth and savings" [Keynes 1936, chapter 3, section II]. In this regard, he asserts that:

> "... the richer the community, the wider will tend to be the gap between its actual and its potential production; and therefore the more obvious and outrageous the defects of the economic system. For a poor community will be prone to consume by far the greater part of its output, so that a very modest measure of investment will be sufficient to provide full employment; whereas a wealthy community will have to discover much ampler opportunities for investment if the saving propensities of its wealthier members are to be compatible with the employment of its poorer members. If in a potentially wealthy community the inducement to invest is weak, then, in spite of its potential wealth, the working of the principle of effective demand will compel it to reduce its actual output, until, in spite of its potential wealth, it has become so poor that its surplus over its consumption is sufficiently diminished to correspond to the weakness of the inducement to invest."

The decline in investment opportunities in this paragraph of Keynes corresponds in the present model to declining productivity as capital accumulation progresses. However, there is no question of capital depreciation in this paragraph of Keynes, but of underemployment.

Other elements generally present in Keynesian economics, such as demand-driven economy, monetization of savings, short-term effects, expectation problems and rigidities, are not included in the present model of neoclassical essence. It is remarkable that, despite this, the decline does occur anyway.

5 The proportion of rich and the aggregate savings rate

A number of economists share the view that greater inequality, by shifting income toward more saving agents, increases the aggregate savings rate, thus accelerating capital accumulation and growth. This idea can be found, for example, in [Barro 2000].

On the contrary, more recent opinions reconnect with the vision expressed in Keynes's quote (section 4) and attribute a less positive role to inequalities with respect to their impact on the economy and consequently saving [Stiglitz 2011, Ostry-Berg-Tsangarides 2014].

It should be noted that what is generally referred to as "inequality" is meant to describe a situation with a large income gap between rich and poor. This concept of inequality is not only dependent on the proportion of rich. It can evolve even in the opposite direction to the proportion of rich if one keeps personal incomes constant and if one measures inequality by the Gini index. However, this section only examines the relationship between the proportion of rich and the aggregate savings rate, what is nevertheless a topical issue as the number of billionaires has doubled since 2008 financial crisis [Oxfam report "Even It Up" 2014].

Within the present framework, we show that if we start from an egalitarian situation and introduce a tiny proportion of rich people, the aggregate savings rate at equilibrium improves significantly. But if we start from a situation where there are already some rich people, the addition of new rich people deteriorates the income and the aggregate savings rate at equilibrium.

[Bourguignon 1981] shows that, for a given proportion of rich a_2 satisfying $0 < a_2 < \overline{A}$, the possible equilibria are pairs (k_1, k_2) each consisting of an unstable equilibrium k_1 and a stable equilibrium k_2 with $k_2 > k_1$. We deduce that the equilibrium determined by $\widetilde{k}(a_2) = \sup\{k/A(k) = a_2\}$ is a stable equilibrium. As stated in [Bourguignon 1981], the equilibrium $\widetilde{k}$ Pareto-dominates all the other equilibria where the proportion of rich is a_2.

Let us show that the equilibrium capital $\widetilde{k}(a_2)$ and the aggregate savings rate are decreasing as functions of a_2 as long as $a_2 > 0$:

At equilibrium, aggregate savings are necessarily equal to the depreciation of the total capital:

$$S = \delta\widetilde{k}$$

The aggregate savings rate as a function of $\widetilde{k}$ is therefore $s(\widetilde{k}) = \frac{\delta\widetilde{k}}{f(\widetilde{k})}$. It is easily checked that $s(\widetilde{k})$ is an increasing function of $\widetilde{k}$ because f is concave and positive. We now prove that $\widetilde{k}(a_2)$ is a decreasing function of a_2, which will establish the decrease of the aggregate savings rate s as a function of a_2.

Suppose not. There would be two real numbers a, b in $]0, 1[$ such that $a < b$, which would check $\widetilde{k}(a) < \widetilde{k}(b)$ or $\widetilde{k}(a) = \widetilde{k}(b)$. Suppose $\widetilde{k}(a) < \widetilde{k}(b)$. Define the function $\varphi(x) = A(x) - a$ on the interval $\left[\widetilde{k}(b), k^*\right]$. The function $A(.)$ is assumed to be continuous on $[k_0, k^*]$ (by setting $A(k^*) = 0$ - for the definition and properties of A, see sections 2 and 3).

We have $\varphi(\widetilde{k}(b)) = A(\widetilde{k}(b)) - a = b - a > 0$ and $\varphi(k^*) = A(k^*) - a = -a < 0$. The function φ being continuous, there would exist y in $\left[\widetilde{k}(b), k^*\right]$ such that $\varphi(y) = 0$. We would have $A(y) - a = 0$, with $y \geqslant \widetilde{k}(b) > \widetilde{k}(a)$. This contradicts the definition of $\widetilde{k}(a) = \sup\{k/A(k) = a\}$. We thus have $\widetilde{k}(a) \geqslant \widetilde{k}(b)$. In fact, we have $\widetilde{k}(a) > \widetilde{k}(b)$. Indeed, since A is continuous, we have $A(\widetilde{k}(a)) = a$. If we suppose $\widetilde{k}(a) = \widetilde{k}(b)$, then $A(\widetilde{k}(a)) = A(\widetilde{k}(b))$, which implies $a = b$ and contradicts $a < b$. QED

We now show that the limit of $\widetilde{k}(a_2)$ when $a_2 \to 0$ is k^*:

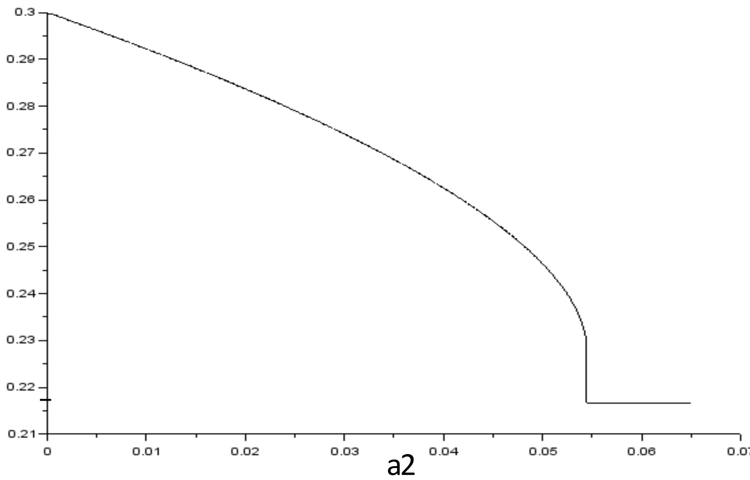
Since the function $\widetilde{k}(.)$ is decreasing, it has a limit when $a_2 \to 0$. We show that this limit is k^*. Consider an increasing sequence k_n which tends to k^* from the left. Denote $a_n = A(k_n)$. The function A being positive on $]k_0, k^*[$ and continuous on $[k_0, k^*]$, the sequence a_n is positive and tends to 0. Moreover $\widetilde{k}(a_n) \geqslant k_n$. So $\widetilde{k}(a_n) \to k^*$, what shows that the limit of $\widetilde{k}$ when a_2 tend to 0 from the right is k^*. QED

Therefore, when the proportion of rich decreases, the average per capita capital tends to the golden-rule level, where the average net income is maximum. The limit of the aggregate savings rate is the golden-rule's savings rate $s(k^*) = \frac{\delta k^*}{f(k^*)}$. In the Cobb-Douglas case, this rate is equal to the share of capital, which is 0.3 in our numerical simulation.

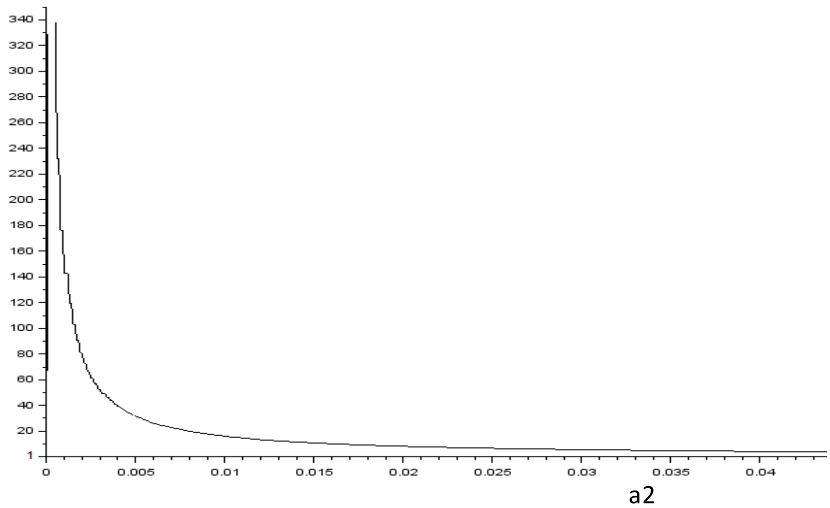
It is remarkable that this result does not depend on the saving function, provided it is increasing and convex. It should be noted that in our inegalitarian economy, this rate does not correspond to the individual savings rates. The poor save less and the rich save much more. But it happens that capital is distributed mechanically during the growth process so that the equilibrium approaches spontaneously the golden-rule.

However, while it is true that the decline in the proportion of rich increases the aggregate savings rate and brings the economy closer to the golden-rule, the total suppression of the rich reduces this rate and drops the economy in a situation Pareto-dominated by all the inegalitarian situations.

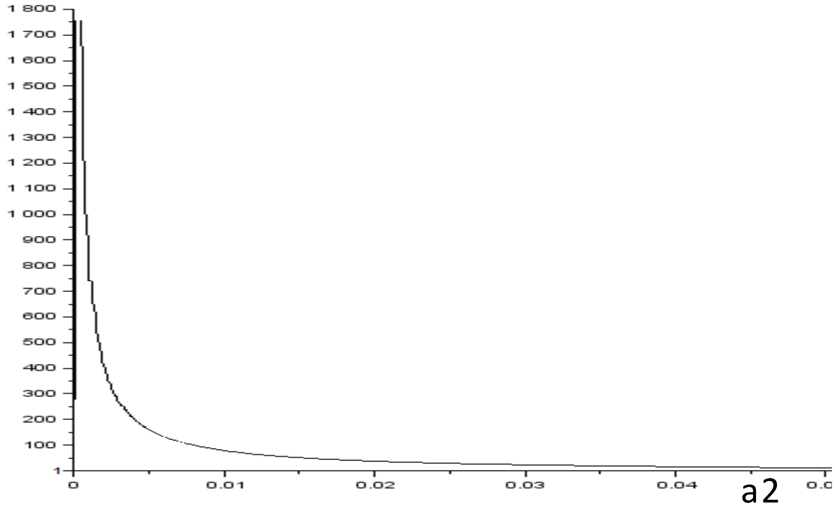
In the case $\beta = 1$, the following graphs represent the aggregate savings rate, the ratios of income and capital between rich and poor as functions of the proportion of rich a_2:



aggregate savings rate at equilibrium as a function of a_2



ratio of income between rich and poor at equilibrium as a function of a_2



ratio of capital between rich and poor at equilibrium as a function of a_2

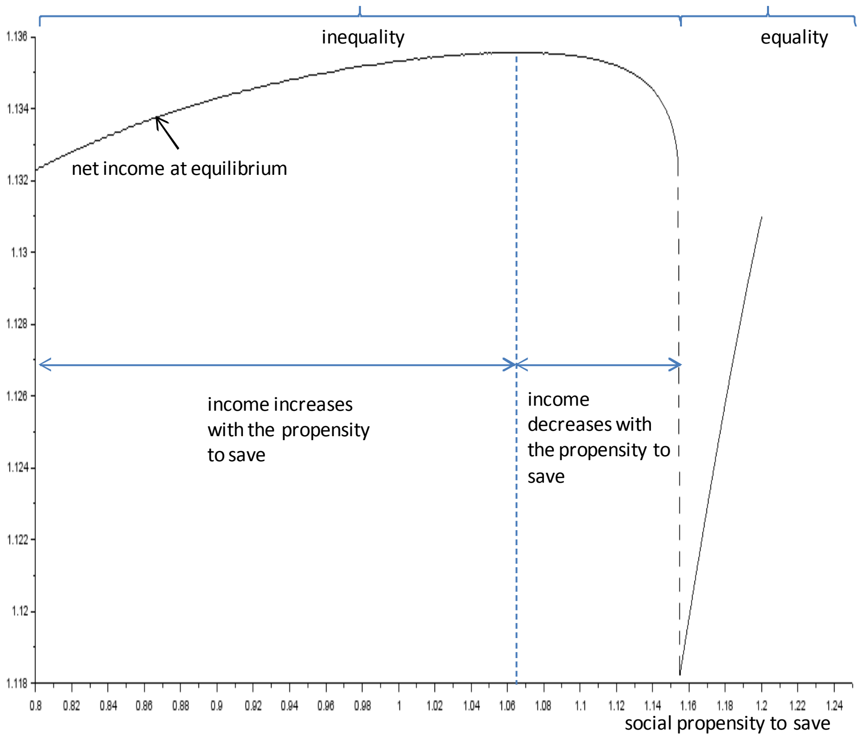
The relationship between the aggregate savings rate at equilibrium and the proportion of rich is thus contrary to the immediate impression that wealth increases savings. This finding supports the idea, suggested by Keynes's quote (section 4), that excessive wealth creates poverty. However, it does not advocate egalitarianism since it also shows that a sufficiently small proportion of rich makes it possible to approach the level of savings of the golden-rule and to rescue the economy from egalitarian poverty.

6 The paradox of thrift

What has been called "Keynesian decline" presupposes that above a certain point, saving plays a counterproductive role. This phenomenon is known as the "paradox of thrift". According to [Keynes 1936, Chapter 23, Section VII], the existence of this paradox has been the subject of controversy between economists. Indeed, it is not easy to admit that abundance can create scarcity when one is accustomed to reasoning in terms of supply-demand balance.

[Keynes 1936, Chapter 23, Section VII] presents the fable of the bees of Mandeville where he sought to explain the counterproductive effect of an excess of savings, as well as the hostile reactions of some English authors of the 18th century. Still according to [Keynes 1936], controversy continued in the 19th century between Ricardo and Malthus in the form of a debate on the possibility of a situation of overproduction, which amounts to a debate on the paradox of thrift. Indeed, if there is overproduction, there is under-consumption therefore over-saving. After the First World War, Hayek and Schumpeter stood up against the paradox contrary to Keynes [Hayek 1931, Earley 1994]. Today, opinions still seem to be divided. On the side of the paradox, we find for example [Krugman 2009] and the septic side we find [Barro 2000].

In this section we examine numerically the relationship between the social propensity to save and the average income at equilibrium, the propensity of rich being fixed. The social propensity to save β varies from 0.8 to 1.2. The proportion of rich is $a_2 = 3\%$. We obtain the following graph for the equilibrium net income $y = f(k) - \delta k$:



We observe that there is an optimal value for the social propensity to save $\beta^* = 1,064$ (with a maximum error of 10^{-3}). The net equilibrium income for $\beta = \beta^*$ is then $y = 1,1356$. This income is slightly less than the net income of the golden-rule $y^* = 1,1380$. But it is much higher than the egalitarian net income for $\beta = \beta^*$, which is $y_0(\beta^*) = 1,0880$.

Beyond $\beta^*$and before reaching $\beta = 1,155$, net income declines although the economy remains in the inegalitarian and rich part. At $\beta = 1,155$, the economy crashes sharply in the egalitarian and poor area. It looks like the Marxist transition from capitalism to socialism, but without the class struggle! To recover once in the egalitarian structure, it would require a social propensity to save of about $\beta = 1,26$, what means an increase in aggregate savings rate from 23% to 30%. Also observe that the decline in the inegalitarian situation begins at $k(\beta^*) = 11,652$, whereas in the egalitarian situation it begins only when the economy is overaccumulated, i.e. when capital exceeds the golden-rule level $k^* = 13,182$.

To sum up, if the social propensity to save is not very high (less than β^*), the introduction of a proportion of rich by 3% makes it possible to significantly exceed the egalitarian net income and to approach the net income of the golden-rule. But this gain may quickly vanish if the social propensity to save increases.

7 Conclusion

This study highlighted one aspect of the consequences of inequality on the macroeconomic relationship between savings and income in a basic neoclassical model. Inequality is at the same time useful and harmful. It is useful because it makes it possible to achieve an aggregate income out of reach if the savings of the majority class is insufficient. It is harmful in the sense that it renders the economic equilibrium that it has achieved fragile. Indeed, the economy risks a great decline if the size of the rich class or the social propensity to save exceeds certain thresholds. This decline is due to a specific articulation between the rate of decline in the productivity of capital and the rate of increase in the depreciation of capital. The dynamics of such a decline reminds one of Keynes's description of the consequences of excess savings in a context of inequality. It is noteworthy that this decline takes place in a neoclassical model that does not include key Keynesian elements such as saturation of demand, monetization of savings, short-term effects, expectation problems, involuntary unemployment and rigidities. It is remarkable that, despite this, the decline does occur anyway.

The following directions should be further explored: taking into account taxation, technical progress, imperfect competition and rent seeking behavior.

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