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Abstract

The determination of the steady state of DSGE models remains the obscure stage of the resolution methods of this modeling. Researchers seek to solve a nonlinear system rather than understand the theory behind these models. In order to fill this gap and help in learning this methodology, this paper develops a procedure for finding the steady- state by using concepts that rest on Walras' Law.

1 Introduction

In 2014 fifty-four topics dealing with doubts about the steady state of DSGE models were opened on the Dynare discussion forum. In general, the vast majority of researchers that use this methodology to determine the steady state, just think of solving a system of nonlinear equations[1] and forget the underlying economic theory to these models.

Courses and textbooks on this macroeconomic approach neglect the economic intuition involved in the steady state. On the one hand, some steps from the resolution of these models are strongly consolidated in the literature[2]. On the other hand, almost nothing is known about how to determine the steady state of these models.

To fill this gap in the literature, this paper aims to demonstrate, in a theoretical and practical ways, that the procedure of definition of the steady state of a DSGE model relies on the foundations of Walras' Law. And this procedure enables the resolution of the steady state to be clearer and simpler than just solving a system of nonlinear equations.

Debreu (1952) proves the existence of equilibrium in a system formed by economic agents. Two years later, Arrow and Debreu (1954) intend to prove the existence of an equilibrium in an integrated model of production and consumption. In general equilibrium, prices are interpreted as elements that coordinate the plans of purchase and sales of all agents in the economy. Walras (1874) provides only a superficial treatment of the problem of uniqueness with regard to equilibrium, while Wald (1951) presents a systematic approach to this issue suggesting paths of proof of analytical nature.

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[1]An example is the procedure adopted by Bonaldi et al (2011).

[2]The problems of maximization of households and firms are solved using dynamic programming and Lagrange method, and the model is log-linearized using differentiation or the tools developed by Ulhig (1999).

The study of the theory of competitive equilibrium stability based on the economic structure of Arrow and Debreu initiated with the articles of Arrow and Hurwicks (1958) and Arrow, Block and Hurwics (1959). In the first article, the authors prioritize the problem of dynamic stability of a perfectly competitive market price that has an adjustment mechanism proportional to the demand excess. Having established the basic concepts, Arrow and Hurwicks (1958) describe the adjustment processes of price that are used as a means for studying the problem of stability in multiple competitive market: instantaneous adjustment process and lagged adjustment process.

In the second article, Arrow, Block and Hurwics (1959) extend the results of Arrow and Hurwicks (1958) by providing in principle a proof of global stability while considering all goods as gross substitutes. This result proves to be valid for processes whereby the price adjustment mechanism is a preserver of the continuous signal, but not in those cases in which the price adjustment mechanism show proportionality in relation to the excess demand function. Next, they work with systems in which one of the commodities plays the role of numeraire (normalized system) and with systems in which all goods are treated symmetrically (not normalized system).

Arrow, Block and Hurwics (1959) describe static and dynamic models used as the basis for construction of the main results. They also define the non-normalized price of k-th good as always non-negative. Once the price is known, they establish the function of excess demand of the i-th individual for the k-th good so that the budget constraint of the ith individual can be set up. Therefore, in order to generalize the results, they aggregate the individual excess demand functions, which naturaaly leads to the aggregate excess demand functions. In addition, they point out that the excess demand function complies with the property of being homogeneous relative to the price so as to avoid the problem of monetary illusion. Finally, they use this property for both systems: normalized and non normalized.

2 RBC model

2.1 Households

The economy of this model consists of a unitary set of households indexed by $j = 1, 2, \ldots, N$, whose problem is to maximize a certain intertemporal welfare function. Toward this end, an additively separable utility function in consumption (C) and in labor (L) is employed. Population growth is ignored. The structure of the labor market is of perfect competition. Thus, representative household optimizes the following welfare function:

$$\max_{C_{j,t}, L_{j,t}, K_{j,t+1}} E_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right) \tag{1}$$

where E_t is the operator of expectations, β is the discount factor, C stand for consumer goods, L is the amount of hours worked, σ is the coefficient of relative risk aversion and φ is the marginal disutility with respect to labor supply.

Households maximize their welfare function subject to its intertemporal budget constraint that indicates which resources are available and how they are allocated.

$$P_t C_{j,t} + P_t I_{j,t} = W_t L_{j,t} + R_t K_t j, t + \Pi_{j,t} \tag{2}$$

where P is the general price level, I is the level of investments, W is the level of wages, K

is the capital stock, R is the return on capital and Π is profit firms (dividends).

There is still the need for an additional equation that characterizes the capital accumulation process over time:

$$K_{j,t+1} = (1-\delta)K_{j,t} + I_{j,t} \tag{3}$$

where δ is the depreciation rate of capital.

Solving the previous problem, we arrive at the following first order conditions:

$$C_{j,t}^{\sigma} L_{j,t}^{\varphi} = \frac{W_t}{P_t} \tag{4}$$

and,

$$C_{j,t}^{-\sigma} = \beta E_t \left\{ C_{j,t+1}^{-\sigma} \left[(1-\delta) + \frac{R_{t+1}}{P_{t+1}} \right] \right\} \tag{5}$$

2.2 Firms

A representative firm is the agent that is dedicated to producing goods and services that will be consumed or saved (and later transformed into capital) by households. There are firms indexed by $j = 1, 2, \ldots, M$ that maximize profit following a perfect competition structure following structure through a Cobb-Douglas production function.

$$Y_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} \tag{6}$$

where A is the level of productivity, Y is the product, α is the share of capital in production, while $(1-\alpha)$ is the share of labor.

The problem of the firm is solved by maximizing the profit function (Π), choosing the quantities of each input (L, K):

$$\max_{L_{j,t}, K_{j,t}} P_t Y_{j,t} - W_t L_{j,t} - R_t K_{j,t} \tag{7}$$

Solving the previous maximization problem yields the following first order conditions:

$$L_{j,t} = (1-\alpha) \frac{Y_{j,t}}{\frac{W_t}{P_t}} \tag{8}$$

$$K_{j,t} = \alpha \frac{Y_{j,t}}{\frac{R_t}{P_t}} \tag{9}$$

The shock to the productivity level is assumed to follow a first order autoregressive process, such that:

$$\log A_t = (1-\rho_A) \log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t \tag{10}$$

where A_{ss} is the value of productivity level at steady state, ρ_A is the autoregressive parameter that its absolute value should be less than one, $|\rho_A| < 1$, to ensure the stationarity of the process , and $\epsilon_t \sim N(0, \sigma_A)$.

Because this model is a RBC, the general level of prices should equal marginal cost:

$$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha} \tag{11}$$

2.3 Equilibrium condition

Having defined households' and firms' problems, the next step is to establish an equilibrium condition for the goods market and the underlying general equilibrium theory to this model.

$$Y_t = C_t + I_t \tag{12}$$

Definition 2.1 (Walrasian equilibrium)**.** A Walrasian equilibrium for a given economy is a price vector **p**, a consumer basket y_j and a basket of inputs w_j such that

1. given prices **p**, y_j and w_j solve the problems of households and firms;
2. market clearing: $\sum_{j=1}^{N} \mathbf{p}y_j \leq \sum_{j=1}^{N} \mathbf{p}w_j$.

where $y_j = \{Y_1, Y_2, \ldots, Y_N\}$, $w_j = \{L_1, L_2, \ldots, L_N, K_1, K_2, \ldots, K_N\}$ e $\mathbf{p} = \{W_1, W_2, \ldots, W_N, R_1, R_2, \ldots, R_N, P_1, P_2, \ldots, P_M\}$.

2.4 Steady state

After setting the equilibrium of the economy, it is necessary to obtain the steady state values. An endogenous variable x_t is said to be at a steady state, in any t, if $E_t x_{t+1} = x_t = x_{t-1} = x_{ss}$.

Some endogenous variables have their values at steady state previously determined (exogenously determined). The steady-state value of the level of productivity, which is the source of shocks in the standard RBC models – steady state $E(\epsilon_t) = 0$ – by equation (10), cannot be known. In general, the literature sets $A_{ss} = 1$. The next step is to remove the time subscript of variable indicators, leaving the structural model as follows:

Predetermined steady state level

$$A = 1 \tag{13}$$

Households group

$$I_{ss} = \delta K_{ss} \tag{14}$$

$$C_{ss}^{\sigma} L_{ss}^{\varphi} = \frac{W_{ss}}{P_{ss}} \tag{15}$$

$$\frac{R_{ss}}{P_{ss}} = \frac{1}{\beta} - (1-\delta) \tag{16}$$

Firms group

$$Y_{ss} = K_{ss}^{\alpha} L_{ss}^{1-\alpha} \tag{17}$$

$$L_{ss} = (1-\alpha)\frac{Y_{ss}}{\frac{W_{ss}}{P_{ss}}} \tag{18}$$

$$K_{ss} = \alpha\frac{Y_{ss}}{\frac{R_{ss}}{P_{ss}}} \tag{19}$$

$$P_{ss} = \left(\frac{W_{ss}}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_{ss}}{\alpha}\right)^{\alpha} \tag{20}$$

Equilibrium conditions group

$$Y_{ss} = C_{ss} + I_{ss} \tag{21}$$

The system of equations (14)-(21) will be used to determine the value of the eight endogenous variables at steady state (Y_{ss}, C_{ss}, I_{ss}, K_{ss}, L_{ss}, W_{ss}, R_{ss} and P_{ss}). It is worth stressing that the first values being determined will be the prices (W_{ss}, R_{ss} and P_{ss}), and that the foundations of Walras' Law will be resorted to at that point.

Proposition 2.1 (Walras' law)**.** For any price vector **p**, there exists a $\mathbf{pz}(\mathbf{p}) \equiv 0$; i.e., the excess demand value is identically zero.

Proof. The definition of the excess demand is just written and multiplied by **p**:

$$\mathbf{pz}(\mathbf{p}) = \mathbf{p}\left[\sum_{i=1}^{n}\mathbf{y}_i(\mathbf{p},\mathbf{p}\,\mathbf{w}_i) - \sum_{i=1}^{n}\mathbf{w}_i\right] = \sum_{i=1}^{n}\left[\mathbf{p}\,\mathbf{y}_i(\mathbf{p},\mathbf{p}\,\mathbf{w}_i) - \mathbf{p}\,\mathbf{w}_i\right] = 0$$

since $\mathbf{y}_i(\mathbf{p},\mathbf{p}\,\mathbf{w}_i)$ satisfies the budget constraint $\mathbf{p}\,\mathbf{y}_i = \mathbf{p}\,\mathbf{w}_i$ for each individual i=1,. . . ,N. □

According to the Walras' law, if each individual satisfies their budget constraint, the value of her excess demand is zero, and the value of the sum of excess demands must also be zero – aggregate excess demand is also equal to zero.

Since the function of aggregate excess demand is homogeneous of degree zero, one can normalize prices and express the demands in terms of the relative price: $p_i = \frac{\widehat{p}_i}{\sum_{j=1}^{k}\widehat{p}_j}$. This has the consequence that the sum of the normalized prices p_i must always be equal to 1. Then, one can draw attention to the price vector belonging to Unit simplex dimension k-1: $S^{k-1} = \left\{\mathbf{p} \in R_{+}^{k} : \sum_{i=1}^{k} p_i = 1\right\}$ (Varian, 1992). In short, by invoking the Walras' Law, one can normalize the general price level of this economy, $P_{ss} = 1$.

To find R_{ss}, the equation (16) is used,

$$R_{ss} = P_{ss}\left[\left(\frac{1}{\beta}\right) - (1-\delta)\right] \tag{22}$$

Note that equation (22) has R_{ss} as function of the general price level (determined) and parameters, so its value was also determined. The steady state level of wages (W_{ss}) remains to be found. Thus, from the equation (20),

$$W_{ss} = (1-\alpha) P_{ss}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R_{ss}} \right)^{\frac{\alpha}{1-\alpha}} \tag{23}$$

After knowing the prices of the economy $\mathbf{p} = \{W_{ss}, R_{ss}, P_{ss}\}$, the market clearing condition is the next step to be determined.

Proposition 2.2 (Market clearing)**.** Given k markets, if demand equalizes supply in k-1 markets and $p_k > 0$, then the former must equal the latter in the k-th market.

Proof. Otherwise, the Walras' Law is violated. □

Knowing that the Walras' Law implies the existence of k-1 independent equations in equilibrium with k goods, so if demand equals supply in k-1 markets, there will also be equality between supply and demand in k-th market. Therefore, to find the equilibrium condition, the equilibrium conditions for the input markets must hold. For this purpose, it is necessary to find where supplies (provided by the households) and demands (provided by firms) for production inputs (labor and capital) meet (Figure 1).

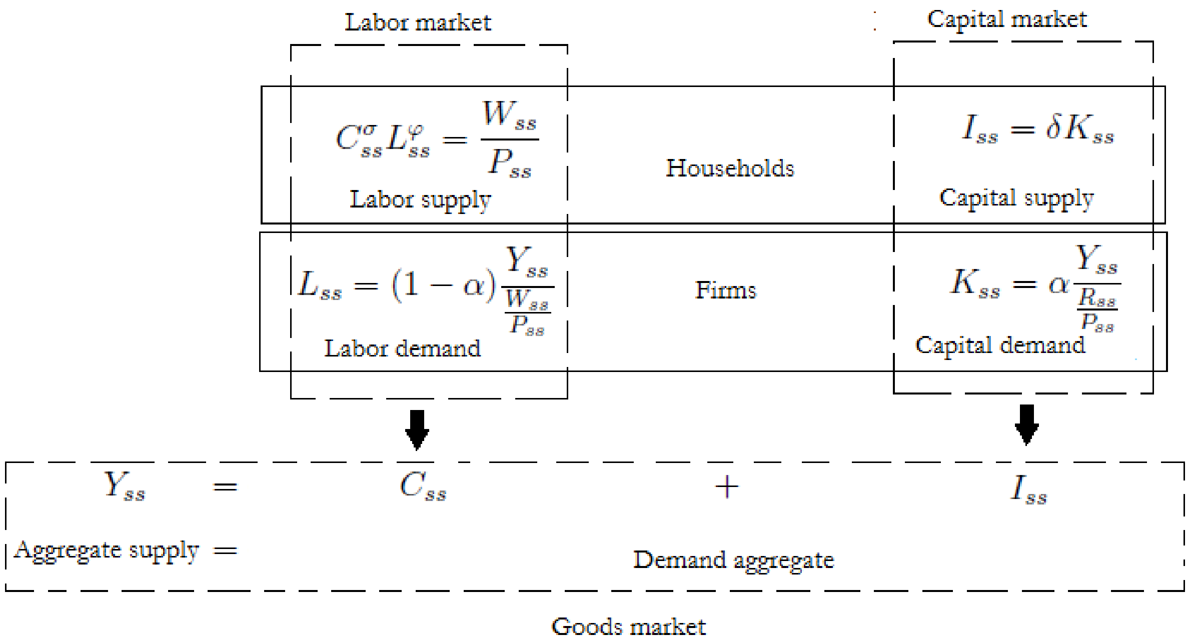

Figure 1: Market clearing structure. The dashed lines represent the labor, capital and goods markets.

Thus, consumption is obtained in the labor market, when the supply of labor (15) and the demand for labor (18) are equal:

$$C_{ss} = \left(\frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}} \right) (1-\alpha)^{\frac{1}{\sigma}} \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)} \right]^{\frac{\alpha(1+\varphi)}{\sigma(1-\alpha)}} \tag{24}$$

And the investment is obtained in the capital market, when the supply of capital (14) and the demand for capital (19) are equal:

$$I_{ss} = \left[\frac{\delta\alpha}{\frac{1}{\beta} - (1-\delta)}\right] Y_{ss} \tag{25}$$

Finally, by substituting equations (24) and (25) into equation (21), the equilibrium condition as regards the goods market must hold:

$$Y_{ss} = \left\{(1-\alpha)^{\frac{1}{\sigma}} \left[\frac{\frac{1}{\beta} - (1-\delta)}{\frac{1}{\beta} - 1 + (1-\alpha)\delta}\right] \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{\alpha(1+\varphi)}{\sigma(1-\alpha)}}\right\}^{\frac{\sigma}{\sigma+\varphi}} \tag{26}$$

Having determined the steady-state aggregate output level, finding the stationary states of other variables proves straightforward. From equations (24) and (25), the values C_{ss} and I_{ss} result. And from equations (18) and (19), we are left with L_{ss} and K_{ss}.

3 Conclusions

This article aimed to demonstrate that the procedure for setting the steady state of a DSGE model relies on Walras' Law. Therefore, the procedures adopted demonstrate that understanding the principles of general equilibrium that underlie DSGE models proves crucial to making the determination of the steady state a simple and easy procedure. The ideas presented in this article are not limited to just these two agents (households and firms). In including the government and the foreign sector, each of these sectors should be thought of as if a single market existed in such a way that the aggregate demand components – government spending and exports – would be provided by these two sectors, respectively.

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