Can forbidden zones for the expectation explain noise influence in behavioral economics and decision sciences?

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The present article is devoted to discrete random variables that take a limited number of values in finite closed intervals. I prove that if non-zero lower bounds exist for the variances of the variables, then non-zero bounds or forbidden zones exist for their expectations near the boundaries of the intervals. This article is motivated by the need in rigorous theoretical support for the analysis of the influence of scattering and noise on data in behavioral economics and decision sciences.

1. Introduction

The construction of bounds for functions of random variables is considered in a number of works that use information about their moments.

Bounds for the probabilities and expectations of convex functions of discrete random variables with finite support are studied in [1].

Inequalities for the expectations of functions are studied in [2]. These inequalities are based on information of the moments of discrete random variables.

A class of lower bounds on the expectation of a convex function using the first two moments of the random variable with a bounded support is considered in [3].

Bounds on the exponential moments of $\min(y, X)$ and $X I\{X < y\}$ using the first two moments of the random variable X are considered in [4].

In the present short article, information about the variance of a random variable that takes on limited number of values in a finite closed interval is used to reveal and estimate bounds on its expectation. It is proven that if there is a non-zero lower bound on the variance of the variable, then non-zero bounds on its expectation exist near the boundaries of the interval.

The obtained bounds (or bounding inequalities) can be treated as non-zero forbidden zones for the expectation near the boundaries of the interval.

The simplest case of a discrete random variable with finite support is considered.

Keeping in mind the above bounds on functions of random variables [1-4], functions of the expectation of a random variable can be further investigated.

The present article is motivated mainly by the need for rigorous theoretical support in the analysis of the influence of scattering and noisiness of data in behavioral economics, decision sciences, utility and prospect theories.

The idea of this theorem has explained, at least partially, some problems of utility and prospect theories, including the underweighting of high and the overweighting of low probabilities, risk aversion, etc. (see, e.g., [5]).

The plenary report [6] was devoted to general questions of the description of noise.

Due to the convenience of abbreviations and consonant with the usage in previous works, here a bound will sometimes be referred to with the term "restriction," especially in mathematical expressions, using its first letter "*r*," for example "r_{Expect}."

2. Preliminaries

Let us consider a probability space *(Ω, Æ, P)* and a discrete random variable *X*, such that *Ω → R*. Let us suppose that the values of *X* are

$$\{x_k\}, \; k = 1, 2, ..., K, \; \text{ where } \; 2 \le K < \infty,$$

and

$$a \le x_k \le b, \; \text{ where } \; 0 < (b - a) < \infty .$$

The probability mass function of *X* is defined as

$$f_X(x) = P(X = x) \equiv P(\{\omega \in \Omega : X(\omega) = x\}) .$$

Let us consider the expectation of *X*

$$E(X) \equiv \sum_{k=1}^{K} x_k f_X(x_k) \equiv \mu ,$$

its variance

$$E(X - \mu)^2 = \sum_{k=1}^{K} (x_k - \mu)^2 f_X(x_k) \equiv \sigma^2$$

and possible interrelationships between them.

3. Non-zero bounds

3.1. Conditions of variance maximality

The maximal value of the variance is intuitively obtained for the probability mass function that is concentrated at the boundaries of the interval. This statement is nevertheless proven in the Appendix. Such a probability mass function can be represented as $f_X(a) = (b-\mu)/(b-a)$ and $f_X(b) = (\mu-a)/(b-a)$. The following inequality holds consequently for the variance of X

$$E(X-\mu)^2 \le (\mu-a)^2 \frac{b-\mu}{b-a} + (b-\mu)^2 \frac{\mu-a}{b-a} = (\mu-a)(b-\mu) \cdot \qquad (1)$$

3.2. Existence theorem

Theorem. Suppose a random variable X takes on values $\{x_k\}$, $k = 1, 2, \dots ,$ K, in an interval $[a, b]$, $0 < (b-a) < \infty$, and $2 \le K < \infty$. If there exists a non-zero lower bound σ^2_{Min} on the variance $E(X-\mu)^2$ of the variable, such that $E(X-\mu)^2 \ge \sigma^2_{Min} > 0$, then non-zero bounds $r_{Expect} > 0$ on its expectation $E(X)$ exist near the boundaries of the interval $[a, b]$, that is,

$$a < (a + r_{Expect}) \le E(X) \le (b - r_{Expect}) < b \,. \qquad (2).$$

Proof. It follows from (1) and the hypotheses of the theorem that

$$0 < \sigma^2_{Min} \le E(X-\mu)^2 \le (\mu-a)(b-\mu) \,.$$

For the boundary a this leads to the inequalities $\sigma^2_{Min} \le (\mu-a)(b-a)$ and

$$\mu \ge a + \frac{\sigma^2_{Min}}{b-a} \,. \qquad (3).$$

For the boundary b the consideration is similar and gives the inequality

$$\mu \le b - \frac{\sigma^2_{Min}}{b-a} \,. \qquad (4).$$

Denoting the bounds (restrictions r_{Expect}) on the expectation as

$$r_{Expect} \equiv \frac{\sigma^2_{Min}}{b-a} \,,$$

and using (3) and (4), we obtain the generalized inequalities

$$a + r_{Expect} \le \mu \le b - r_{Expect} \,.$$

Therefore, if the inequalities $0 < (b-a) < \infty$ and $\sigma^2_{Min} > 0$ hold, then the bounds $r_{Expect} > 0$ exist, such that the inequalities (2)

$$a < (a + r_{Expect}) \le \mu \le (b - r_{Expect}) < b$$

are satisfied which proves the theorem.

4. Applications

4.1. Applications in behavioral economics

The idea of the considered bounds was applied, e.g., in [5].

The work [5] was devoted to the well-known problems of utility and prospect theories. Such problems had been pointed out, e.g., in [7]. In [5] some examples of typical paradoxes were studied. Similar paradoxes may concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, etc.

The dispersion and noisiness of the initial data can lead to bounds (restrictions) on the expectations of these data. This should be taken into account when dealing with this kind of problems. The proposed bounds explained, at least partially, the analyzed examples of paradoxes.

4.2. Possible general applications

The plenary report [6] presented the idea of these new general bounds (restrictions) on the expectations of random variables in the presence of a non-zero minimal variance. Possible contributions to engineering and the economics, involving the dispersion of the data were considered.

Possible general consequences of these bounds can include:

1) A quantitative reduction of the available space of the parameters.

2) Qualitative changes in the connectivity of the space of the parameters.

3) Discontinuities in functions under the presence of noise.

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Appendix. Proof of variance maximality conditions

Let us search for the probability mass function $f_X(x)$ such that the variance of X attains its maximal possible value under the condition that it has a given expectation μ.

Let us consider two arbitrary possible realizations $x_a < x_b$ of the random variable X and the corresponding probabilities $f_X(x_a)$ and $f_X(x_b)$.

For the points x_a and x_b, one can define the point

$$\mu_2 \equiv \frac{x_a f_X(x_a) + x_b f_X(x_b)}{f_X(x_a) + f_X(x_b)},$$

as the "two-point expectation" of $f_X(x_a)$ and $f_X(x_b)$, and the expression

$$E_2(X-\mu_2)^2 \equiv (x_a-\mu_2)^2 f_X(x_a) + (x_b-\mu_2)^2 f_X(x_b),$$

as the "two-point variance" of $f_X(x_a)$ and $f_X(x_b)$.

One can denote the sum of the probabilities $f_X(x_a)$ and $f_X(x_b)$ by w_2. The expression for the "two-point variance" $E_2(X\text{-}\mu_2)^2$ can be easily transformed to

$$E_2(X-\mu_2)^2 = (\mu_2-x_a)^2 \frac{x_b-\mu_2}{x_b-x_a} w_2 + (x_b-\mu_2)^2 \frac{\mu_2-x_a}{x_b-x_a} w_2 .$$

Let us take the derivative of the "two-point variance" $E_2(X\text{-}\mu)^2$ with respect to x_a under the conditions of constant w_2 and μ_2

$$\begin{aligned}\frac{\partial(E_2(X-\mu_2)^2)}{\partial x_a} &= \{[-2(x_b-x_a)+(\mu_2-x_a)](\mu_2-x_a)+ \\ &+[-(x_b-x_a)+(\mu_2-x_a)](x_b-\mu_2)\}\frac{x_b-\mu_2}{(x_b-x_a)^2}w_2 = \\ &=\{[(\mu_2-x_a)-2(x_b-x_a)](\mu_2-x_a)-(x_b-\mu_2)^2\}\frac{x_b-\mu_2}{(x_b-x_a)^2}w_2\end{aligned}.$$

If the inequality $x_a < \mu_2 < x_b$ is true, then the inequalities $(\mu_2\text{-}x_a) < (x_b\text{-}x_a)$,

$$(\mu_2-x_a)-2(x_b-x_A)<0$$

and, for this derivative, $\partial(E_2(X\text{-}\mu_2)^2)/\partial x_a < 0$ are true also. So, under the condition $a \le x_a < \mu_2 < x_b \le b$, the "two-point variance" reaches its maximum at $x_a = a$.

Note that, if $x_a = \mu_2 = x_b$, then $E_2(X\text{-}\mu_2)^2 = 0$.

Analogously, one can easily prove that the "two-point variance" $E_2(X\text{-}\mu_2)^2$ reaches its maximum at $x_b = b$.

So, the "two-point variance" reaches its maximum at $x_a = a$ and $x_b = b$.

Under the hypothesis that $K \ge 2$, every point x_a, such that $a \le x_a < \mu$, has the corresponding point x_b, such that $\mu < x_b \le b$ (note, the point x_b may be the same for more than one x_a), and vice versa.

The conditions of given w_2 and μ_2 allow satisfying the condition of having the given expectation μ: For arbitrary pairs of points $x_{a.1}$ and $x_{b.1}$ varied to $x_{a.2}$

and $x_{b.2}$, such that $w_{2.1} = w_{2.2} = w_2$ and $\mu_{2.1} = \mu_{2.2} = \mu_2$, one can write indeed

$$(x_{a.1} - \mu_2 + \mu - \mu) f_X(x_{a.1}) + (x_{b.1} - \mu_2 + \mu - \mu) f_X(x_{b.1}) =$$
$$= (x_{a.2} - \mu_2 + \mu - \mu) f_X(x_{a.2}) + (x_{b.2} - \mu_2 + \mu - \mu) f_X(x_{b.2})$$

and easily draw

$$(x_{a.1} - \mu) f_X(x_{a.1}) + (x_{b.1} - \mu) f_X(x_{b.1}) =$$
$$= (x_{a.2} - \mu) f_X(x_{a.2}) + (x_{b.2} - \mu) f_X(x_{b.2}) .$$

That is, the expectation $\mu = E(X)$ remains the same value for arbitrary pairs of varied points, under the above conditions of the constant w_2 and μ_2.

Therefore, under the condition of having the given expectation μ, the variance $E(X\text{-}\mu)^2$ attains its maximum for the $f_X(x)$, that is concentrated at the boundaries of the interval.

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