Rating Transition Probability Models and CCAR Stress Testing: Methodologies and implementations

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September 2016

RATING TRANSITION PROBABILITY MODELS AND CCAR STRESS TESTING*
- Methodologies and implementations

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Abstract

Rating transition probability models, under the asymptotic single risk factor model framework, are widely used in the industry for stress testing and multi-period scenario loss projection. For a risk-rated portfolio, it is commonly believed that borrowers with higher risk ratings are more sensitive and vulnerable to adverse shocks. This means the asset correlation is required be differentiated between ratings and fully reflected in all respects of model fitting. In this paper, we introduce a risk component, called credit index, representing the part of systematic risk for the portfolio explained by a list of macroeconomic variables. We show that the transition probability, conditional to a list of macroeconomic variables, can be formulated analytically by using the credit index and the rating level sensitivity with respect to this credit index. Approaches for parameter estimation based on maximum likelihood for observing historical rating transition frequency, in presence of rating level asset correlation, are proposed. The proposed models and approaches are validated on a commercial portfolio, where we estimate the parameters for the conditional transition probability models, and project the loss for baseline, adverse and severely adverse supervisory scenarios provided by the Federal Reserve for the period 2016Q1-2018Q1. The paper explicitly demonstrates how Miu and Ozdemir's original methodology ([5]) on transition probability models can be structured and implemented with rating specific asset correlation. It extends Yang and Du's earlier work on this subject ([9]).We believe that the models and approaches proposed in this paper provide an effective tool to the practitioners for the use of transition probability models.

Keywords: CCAR stress testing, multi-period scenario, loss projection, credit index, risk sensitivity, asset correlation, transition frequency, transition probability, through-the-cycle, maximum likelihood

1. Introduction

The largest bank holding companies with assets above $10 billion operating in the United States are subject to the Comprehensive Capital Analysis and Review (CCAR, [2]) annual exercise by the Federal Reserve to assess whether they have sufficient capital to continue operations throughout times of economic and financial stress, and whether they have robust, forward-looking capital-planning processes that account for their unique risks. The CCAR stress testing includes the assessment of loss on baseline, adverse, and severely adverse scenarios on a quarterly basis provided by the Federal Reserve covering a period of nine quarters.

Under the AIRB (Advanced Internal Rating-Based) framework for a bank, a dynamic rating transition probability model provides a tool for multi-period scenario loss assessment: Given the risk rating distribution for a risk-rated portfolio at the beginning of a horizon, the risk rating distribution at the end of the horizon can be derived by using the conditional transition probabilities given by a scenario. This calculation is re-iterated through a period of time and loss projection for a multi-period scenario is thus obtained, given the EAD and LGD components.

Let $\{R_i \mid 1 \le i \le k\}$ denote a rating system with k ratings, where a lower index i indicates the lower default risk. Thus R_1 is the best quality rating and R_k is the worst rating, i.e., the default rating. It is assumed that, under the asymptotic single risk factor (ASRF) model framework, the risk for an entity with a non-default rating R_i is governed by a latent random variable z_i, called the firm's normalized asset value, which splits into two parts as ([1], [3], [4], [5], [6], [9]):

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$$z_i = s\sqrt{\rho_i} + \varepsilon_i\sqrt{1-\rho_i}\ ,\ 0<\rho_i<1,\ \ s \sim N(0,1),\ \ \varepsilon_i \sim N(0,1) \tag{1.1}$$

where s denotes the systematic risk (common to all non-default ratings) and ε_i is the idiosyncratic risk independent of s. The quantity ρ_i is called the asset correlation for rating R_i. It is assumed that there exist threshold values $\{b_{ij}\}$ such that a firm's rating migrates from R_i to R_j or worse (called downgrade) in horizon when z_i falls below the threshold value $b_{i(k-j+1)}$.

The idiosyncratic risk can be factored in a transition probability model by the threshold values $\{b_{ij}\}$. In contrast, modeling for systematic risk, in presence of rating level asset correlation, is challenging.

When the risk for a portfolio is believed to be homogenous, uniform asset correlation can be assumed, and fitting for conditional transition probability models is relatively simple. However, for a risk-rated portfolio, it is commonly believed that borrowers with higher risk ratings are more sensitive and vulnerable to adverse shocks, and risk behaviours differ from rating to rating. This means the asset correlation is required be differentiated between ratings and fully recognized in all respects of model fitting. This paper explicitly demonstrates how Miu and Ozdemir's original methodology ([5]) on rating transition models can be structured and implemented with rating specific asset correlation.

Rating level asset correlation is recognized for transition probability models by Yang and Du ([9]). However, the macroeconomic variable coefficients (i.e., the coefficients in (a) below) are fitted via a regression ([9], expression (3.3)) without fully reflecting the rating level asset correlation. New approaches for parameter fitting are proposed in this paper with rating level asset correlation being fully recognized.

A credit index, as introduced in the next section and as in its simplest form, is a linear combination of a list of given macroeconomic variables, normalized to have zero mean and one standard deviation, under some appropriate assumption. As shown in Theorem 2.3 in the next section, the conditional transition probabilities, given the list of macroeconomic variables, can be formulated analytically by the following three types of parameters:

(a) The coefficients of macroeconomic variables for the credit index
(b) The risk sensitivity for a rating with respect to the credit index
(c) The threshold values $\{b_{ij}\}$

Threshold values in (c) can be estimated separately (Section 2.3). For parameters in (a) and (b), we will propose the estimation approaches by maximizing the log-likelihood for observing the historical rating transition frequency, with rating level sensitivity fully incorporated.

The advantages of the proposed transition probability models and the parameter estimation approaches include the following:

1. Rating transition probability models are structured by a credit index (representing the part of systematic risk for the portfolio explained by a list of given macroeconomic variables) and the risk sensitivity with respect to this credit index for each rating.
2. The proposed parameter estimation approaches are based on maximum likelihood for observing historical rating transition frequency. The rating level asset correlation is fully recognized in all aspects of model fitting.
3. Transition probability models fitted in this way are robust, not only at the portfolio level, but at the rating level as well.

The paper is organized as follows: In section 2, we introduce the concept of credit index, and show the analytical formulation of the conditional transition probability models given a list of macroeconomic variables. In section 3, we propose the parameter estimation approaches based on maximum likelihood for observing rating transition frequency. The proposed models and parameter estimation approaches are validated in section 4, where we fit a transition probability models and project the loss for a commercial portfolio based on the supervisory scenarios provided by the Federal Reserve for the period 2016Q1-2018Q1.

2. Rating Transition Probability Models

In this section, we introduce the concept of credit index, a component representing the part of systematic risk explained by a given list of macroeconomic variables. We then show the analytical formulation of the condition rating transition probabilities given a list of macroeconomic variables using this credit index.

2.1 Rating transition probabilities given the single latent systematic risk factor s

Let $p_{ij}(s)$ denote the transition probability, given the single latent risk factor s, for a firm with a non-default risk rating R_i at the beginning of a horizon and migrating to R_j at the end of the horizon.

Proposition 2.1. The following equations hold for the transition probability $p_{ij}(s)$:

$$\begin{aligned} p_{ij}(s) &= \Phi[(b_{i(k-j+1)} - s\sqrt{\rho_i})/\sqrt{1-\rho_i}\,] - \Phi[(b_{i(k-j)} - s\sqrt{\rho_i})/\sqrt{1-\rho_i}\,] \\ &= \Phi(\tilde{b}_{i(k-j+1)} - r_i s) - \Phi(\tilde{b}_{i(k-j)} - r_i s) \end{aligned} \tag{2.1}$$

where

$$r_i = \sqrt{\rho_i}/\sqrt{1-\rho_i} \tag{2.2}$$

$$\tilde{b}_{i\,h} = b_{i\,h}/\sqrt{1-\rho_i} = b_{i\,h}\sqrt{1+r_i^2} \tag{2.3}$$

Proof. Expressions (2.1) and (2.2) follow from (1.1) and the definition of threshold values $\{b_{ij}\}$. For (2.3), we have by (2.2):

$$\begin{aligned} &\sqrt{1+r_i^2} = 1/\sqrt{1-\rho_i} \\ &\Rightarrow \tilde{b}_{i\,h} = b_{i\,h}/\sqrt{1-\rho_i} = b_{i\,h}\sqrt{1+r_i^2} \end{aligned}$$

□

We call r_i the risk sensitivity for rating R_i with respect to the systematic risk factor s.

By (2.1)-(2.3), the default probability $p_{ik}(s)$ and downgrade probability $p_{i(i+1)}(s)$ are given by:

$$p_{ik}(s) = \Phi(b_{i1}\sqrt{1+r_i^2} - r_i s)\,,\;\; p_{i(i+1)}(s) = \Phi(b_{i(k-i)}\sqrt{1+r_i^2} - r_i s)$$

Given a non-default rating R_i, the risk sensitivity r_i can be estimated by maximizing the log-likelihood for observing the default or downgrade frequency, using for example, SAS PROC NLMIXED ([9]).

2.2. The credit index and transition probabilities given macroeconomic variables

Given a list of macroeconomic variables $x = (x_1, x_2, ..., x_m)$, let u_i be the mean value of x_i. Consider a linear combination of the form:

$$c(x) = a_1 x_1 + a_2 x_2 + ... + a_m x_m,$$

Let $\tilde{x}_i = (x_i - u_i)$. Normalize $c(x)$ by setting

$$ci(x) = [c(x) - u]/v = (a_1\tilde{x}_1 + a_2\tilde{x}_2 + ... + a_m\tilde{x}_m)/v$$

where u and v denote respectively the mean and standard deviation of $c(x)$. We assume that, in presence of the given macroeconomic variables, the systematic risk factor s splits into two parts as in (2.4) below:

$$\begin{aligned} s &= -\lambda ci(x) - e\sqrt{1-\lambda^2}, \quad e \sim N(0,1), \quad 0 < \lambda < 1 \\ &= -[\lambda(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m) + \sigma e] \end{aligned} \tag{2.4}$$

where

$$\tilde{a}_i = a_i / v, \quad \sigma = \sqrt{1-\lambda^2}$$

By (2.1)-(2.4), we have the following expressions for transition probability $p_{ij}(s)$:

$$p_{ij}(s) = \Phi[\tilde{b}_{i(k-j+1)} + r_i(\lambda ci(x) + \sigma e)] - \Phi[\tilde{b}_{i(k-j)} + r_i(\lambda ci(x) + \sigma e)] \tag{2.5}$$

$$\begin{aligned} &= \Phi[\tilde{b}_{i(k-j+1)} + r_i\lambda(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m) + r_i\sigma e] \\ &\quad - \Phi[\tilde{b}_{i(k-j)} + r_i\lambda(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m) + r_i\sigma e] \end{aligned} \tag{2.6}$$

Let $E_s[\Phi(a_0 + a_1 s)]$ denote the expected value of $\Phi(a_0 + a_1 s)$ with respect to s. In the subsequent discussions, we need the following lemma:

Lemma 2.2. ([8]) $E_s[\Phi(a_0 + a_1 s)] = \Phi(a_0 / \sqrt{1 + a_1^2})$, where $s \sim N(0,1)$

Let $p_{ij}(x) = E[p_{ij}(s) \mid x]$ be the expected value of transition probability $p_{ij}(s)$ given macroeconomic variables $x = (x_1, x_2, ..., x_m)$.

Theorem 2.3. Assume that e in (2.4) is independent of $x_1, x_2, ..., x_m$. Then the following equations hold for transition probability $p_{ij}(x)$:

$$p_{ij}(x) = \Phi(\bar{b}_{i(k-j+1)} + \tilde{r}_i ci(x)) - \Phi(\bar{b}_{i(k-j)} + \tilde{r}_i ci(x)) \tag{2.7}$$

$$\begin{aligned} &= \Phi[\bar{b}_{i(k-j+1)} + \tilde{r}_i(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m)] \\ &\quad - \Phi[\bar{b}_{i(k-j)} + \tilde{r}_i(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m)] \end{aligned} \tag{2.8}$$

where

$$\tilde{r}_i = r_i\lambda / \sqrt{1 + r_i^2\sigma^2} = r_i\lambda / \sqrt{1 + r_i^2(1-\lambda^2)} \tag{2.9}$$

$$\bar{b}_{ih} = \tilde{b}_{ih} / \sqrt{1 + r_i^2\sigma^2} = b_{ih}\sqrt{1 + \tilde{r}_i^2} \tag{2.10}$$

Proof. Expressions (2.7) and (2.8) follow respectively from (2.5) and (2.6) by Lemma 2.2. For (2.9) and (2.10), we have by Lemma 2.2 and (2.3):

$$\begin{aligned}\overline{b}_{i\,h} &= b_{i\,h}\sqrt{1+r_i^2}\,/\sqrt{1+r_i^2\sigma^2} \\ &= b_{i\,h}\sqrt{1+r_i^2}\,/\sqrt{1+r_i^2(1-\lambda^2)}\end{aligned}$$

By definition of $\tilde{r}_i$, we have

$$\begin{aligned}&\tilde{r}_i = r_i\lambda/\sqrt{1+r_i^2\sigma^2} = r_i\lambda/\sqrt{1+r_i^2(1-\lambda^2)} \\ &\Rightarrow \sqrt{1+\tilde{r}_i^2} = \sqrt{1+r_i^2}\,/\sqrt{1+r_i^2(1-\lambda^2)} \\ &\Rightarrow \overline{b}_{i\,h} = b_{i\,h}\sqrt{1+\tilde{r}_i^2}\end{aligned}$$

□

The linear combination $(\tilde{a}_1\tilde{x}_1+\tilde{a}_2\tilde{x}_2+\ldots+\tilde{a}_m\tilde{x}_m)$ in (2.8) is constrained by:

$$v(\tilde{a}_1\tilde{x}_1+\tilde{a}_2\tilde{x}_2+\ldots+\tilde{a}_m\tilde{x}_m)=1 \qquad (2.11)$$

where $v($ x) denotes the standard deviation of the random variable x.

The default and downgrade probabilities have a simpler form and are given respectively by:

$$p_{ik}(x)=\Phi[(b_{i\,1}\sqrt{1+\tilde{r}_i^2}+\tilde{r}_i(\tilde{a}_1\tilde{x}_1+\tilde{a}_2\tilde{x}_2+\ldots+\tilde{a}_m\tilde{x}_m)] \qquad (2.12)$$

$$p_{i(i+1)}(x)=\Phi[b_{i\,k-i}\sqrt{1+\tilde{r}_i^2}+\tilde{r}_i(\tilde{a}_1\tilde{x}_1+\tilde{a}_2\tilde{x}_2+\ldots+\tilde{a}_m\tilde{x}_m)] \qquad (2.13)$$

Given the asset correlations $\{\rho_i\}$ in (2.1) and thus $\{r_i\}$, the risk sensitivity $\tilde{r}_i$ with respect to $ci(x)$ is driven by the parameter λ as in (2.9).

We define the credit index for a portfolio to be the $ci(x)$ where the following conditions are satisfied:

(a) The residual e in (2.4) is independent of $x_1, x_2, \ldots, x_m$.

(b) $ci(x)$ is obtained from a normalization of a linear combination $a_1\tilde{x}_1+a_2\tilde{x}_2+\ldots+a_m\tilde{x}_m$ with which the model $\{p_i(x)\}$ best predicting (through maximum likelihood as stated more specifically in section 3.2) the default probability of the portfolio, where

$$p_i(x)=\Phi[c_i+\tilde{r}_i(a_1\tilde{x}_1+a_2\tilde{x}_2+\ldots+a_m\tilde{x}_m)] \qquad (2.14)$$

is a model predicting the default probability for the rating R_i. The quantity $\tilde{r}_i$ in (2.14) is driven by (2.9). No constraint is imposed for parameters $\{c_i\}$ and $\{a_1, a_2, \ldots, a_m\}$.

Condition (b) can be adapted to targeting the downgrade risk (rather than default risk) when default rate for the portfolio is low.

Similarly to the quantities r_i and ρ_i, defined under model (2.1) with respect to the single latent risk factor s, we have the risk sensitivity $\tilde{r}_i$ for rating R_i with respect to the credit index $ci(x)$, and $\tilde{\rho}_i$, which is defined as:

$$\tilde{\rho}_i = \rho_i \lambda^2$$

By (2.1) and (2.4), we can think $\tilde{\rho}_i$ as the part of asset correlation ρ_i explained by the credit index $ci(x)$.

Proposition 2.4. For quantities $\tilde{r}_i$ and $\tilde{\rho}_i$, the following statements hold:

(a) Similarly to expression (2.2) for r_i and ρ_i,
$$\tilde{r}_i = \sqrt{\tilde{\rho}_i} / \sqrt{1-\tilde{\rho}_i}$$
(b) $\tilde{r}_i < r_i$ and $\tilde{\rho}_i < \rho_i$

Proof. Statement (b) follows from the facts $0 < \lambda < 1$ and $\sigma^2 = 1 - \lambda^2$. For (a), recall:

$$\begin{aligned} &\tilde{r}_i = r_i \lambda / \sqrt{1 + r_i^2 \sigma^2} \\ &\Rightarrow 1 + \tilde{r}_i^2 = 1 + r_i^2 \lambda^2 / (1 + r_i^2 \sigma^2) \\ &\qquad\qquad = (1 + r_i^2)/(1 + r_i^2 \sigma^2) \\ &\Rightarrow \tilde{r}_i^2 / (1 + \tilde{r}_i^2) = r_i^2 \lambda^2 / (1 + r_i^2) \end{aligned} \tag{2.15}$$

By (2.2), we have:

$$\begin{aligned} &r_i = \sqrt{\rho_i} / \sqrt{1-\rho_i} \\ &\Rightarrow \rho_i = r_i^2 / (1 + r_i^2) \end{aligned} \tag{2.16}$$

By (2.15) and (2.16), we have

$$\begin{aligned} &\tilde{r}_i^2 / (1 + \tilde{r}_i^2) = \rho_i \lambda^2 = \tilde{\rho}_i \\ &\Rightarrow \tilde{r}_i = \sqrt{\tilde{\rho}_i} / \sqrt{1-\tilde{\rho}_i} \end{aligned}$$

□

Consequently, by (2.7), for the determination of the transition probabilities $\{p_{ij}(x)\}$, the following parameters (total $(k+1)(k-1)+m$) are required:

(a) Parameters $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ for macroeconomic variables in credit index $ci(x)$
(b) Rating level risk sensitivities $\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_{k-1}$
(c) Threshold values $\{b_{ij}\}, 1 \le i \le k-1, 1 \le j \le k$

Remark 2.5. The threshold values $\{b_{ij}\}$ can be estimated separately, as shown in the next section. Therefore, the key to the transition probabilities $\{p_{ij}(x)\}$ is the determination of parameters: $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ and $\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_{k-1}$.

2.3. Determination of the threshold values $\{b_{ij}\}$

Let $p_{ij} = E[p_{ij}(s)]$ be the through-the-cycle (TTC) transition probability. By Lemma 2.2 and (2.1)-(2.3), we have:

$$\begin{aligned} p_{ij} &= E[\Phi(b_{i(k-j+1)}\sqrt{1+r_i^2} - r_i s)] - E[\Phi(b_{i(k-j)}\sqrt{1+r_i^2} - r_i s)] \\ &= \Phi(b_{i(k-j+1)}) - \Phi(b_{i(k-j)}) \end{aligned}$$

This means, $\{b_{ij}\}$ can be found by using the TTC transition probabilities $\{p_{ij}\}$.

We now describe how the TTC transition probabilities $\{p_{ij}\}$ can be determined by using the maximum likelihood approach.

Given i, the log-likelihood for observing the rating transition frequencies $\{n_{ij}\}$ (through-the-cycle frequencies are used here) is (up to a constant given by an appropriate multinomial coefficient number):

$$LL = n_{i1}\log\ p_{i1} + n_{i2}\log p_{i2} + ... + n_{ik}\log p_{ik} \tag{2.17}$$

Using the relation $p_{ik} = 1 - p_{i1} - p_{i2} - ... - p_{i\,k-1}$ and setting to zero the derivative of (2.17) with respect to p_{ij}, $1 \le j \le k-1$, we have

$$\begin{aligned} & n_{ij} / p_{ij} - n_{ik} /(1 - p_{i1} - p_{i2} - ... - p_{i\,k-1}) = 0 \\ & \Rightarrow n_{ij} / p_{ij} = n_{ik} / p_{ik} \end{aligned}$$

Because this holds for each j $(1 \le j < k)$ for the fixed k, the vector $(p_{i1}, p_{i2}, ..., p_{ik})$ is in proportion with $(n_{i1}, n_{i2}, ..., n_{ik})$. Therefore, the maximum likelihood estimate for p_{ij} is given by:

$$p_{ij} = n_{ij} /(n_{i1} + n_{i2} + ... + n_{ik}) = n_{ij} / n_i \tag{2.18}$$

where

$$n_i = n_{i1} + n_{i2} + ... + n_{ik}$$

In general, we expect that a rating migrates more likely to a closer non-default rating at the end of the horizon than a faraway non-default rating; and higher risk rating carries higher default probability. Therefore monotonicity constraints as below are usually imposed:

$$\begin{aligned} & p_{i\,i+1} \ge p_{i\,i+2} \ge ... \ge p_{ik-1} \\ & p_{i\,1} \le p_{i\,2} \le ... \le p_{i\,i-1} \\ & p_{1\,k} \le p_{2\,k} \le ... \le p_{k-1\,k} \end{aligned}$$

3. The Proposed Parameter Estimation Approaches

In this section, we propose the parameter estimation approaches based on maximum likelihood for observing historical rating transition frequency.

3.1 Log-likelihood functions for observing rating transition frequency

Given a non-default rating R_i at the beginning of a horizon, we consider the following three rating transition frequencies:

(a) n_{ij} - The frequency transiting to R_j at the end of the horizon

(b) d_i - The default frequency at the end of the horizon

(c) dd_i - The frequency downgraded to a worse or default rating at the end of the horizon.

With the multinomial probability distribution, we have the corresponding log-likelihood functions (for all ratings for a single horizon) as below (up to a constant independent of the parameters in $\{p_{ij}(x)\}$):

$$LL = \sum_{j=1}^{k} n_{1j} \log(p_{1j}(x)) + \sum_{j=1}^{k} n_{2j} \log(p_{2j}(x)) + ... + \sum_{j=1}^{k} n_{(k-1)j} \log(p_{(k-1)j}(x)) \quad (3.1)$$

$$LL = \sum_{i=1}^{k-1} [(n_i - d_i)\log(1 - p_{ik}(x)) + d_i \log(p_{ik}(x))] \quad (3.2)$$

$$LL = \sum_{i=1}^{k-1} [(n_i - dd_i)\log(1 - p_{i(i+1)}(x)) + dd_i \log(p_{i(i+1)}(x))] \quad (3.3)$$

where $n_i = n_{i1} + n_{i2} + ... + n_{ik}$.

For parameter fitting for the credit index, we use only (3.2) (or (3.3) when default rate is low for the portfolio), with $p_{ik}(x)$ given by $p_i(x)$ in (2.14). For risk sensitivity fitting of $\{\tilde{r}_i\}$, we use (3.2) or (3.3) with $p_{ik}(x)$ or $p_{ik}(x)$ given by (2.7) as below:

$$p_{ik}(x) = \Phi[(b_{i\,1}\sqrt{1+\tilde{r}_i^2} + \tilde{r}_i ci(x)] \quad (3.4)$$

$$p_{i(i+1)}(x) = \Phi[b_{i\,k-i}\sqrt{1+\tilde{r}_i^2} + \tilde{r}_i ci(x)] \quad (3.5)$$

The total log-likelihood for a time series sample is the sum of all the horizon log-likelihoods over all horizons.

A function is log concave if its logarithm is concave. If a function is concave, a local maximum is actually a global maximum, and the function is unimodal. This property is important for maximum likelihood search.

Proposition 3.1. The log likelihood functions (3.2) and (3.3) are concave as a function of $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$, and (3.4) or (3.5) is log concave as a function of $\tilde{r}_i$. This concavity holds when the standard normal cumulative distribution Φ is replaced by any probability cumulative distribution which is log concave (e.g., the cumulative distribution for logistic distribution).

Proof. It is well-known that the standard normal cumulative distribution is log concave, and the sum of concave functions is again concave. It is also known that, if $f(x)$ is log concave, then so is $f(Az+b)$, where $Az + b : R^m \rightarrow R^1$ is any affine transformation from the *m*-dimensional Euclidean space to the 1-

dimensional Euclidean space. This means both the cumulative distribution $\Phi(x)$ and $F(x)=\Phi(-x)$ are log concave, and (3.2) or (3.3) is concave as a function of $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$.

For the concavity of (3.4) or (3.5) as a function of $\tilde{r}_i$, it suffices to show that the 2nd derivative of the function

$$L(r) = \log[\,\Phi(b\sqrt{1+r^2}+ra)] \tag{3.6}$$

is non-positive for any constants a and b. The 2nd derivative $d^2[L(r)]/dr^2$ is given by:

$$\begin{aligned}&(br/\sqrt{1+r^2}+a)^2\{-[\phi(b\sqrt{1+r^2}+ra)]^2/[\Phi(b\sqrt{1+r^2}+ra)]^2+\phi'(b\sqrt{1+r^2}+ra)/\Phi(b\sqrt{1+r^2}+ra)\}\\&\qquad+\phi(b\sqrt{1+r^2}+ra)(b)(1+r^2)^{-3/2}/\Phi(b\sqrt{1+r^2}+ra)\\&=I+II\end{aligned} \tag{3.7}$$

where ϕ and ϕ' denote the 1st and 2nd derivatives of Φ. Because the factor in the 1st term of (3.7) below

$$\{-[\phi(b\sqrt{1+r^2}+ra)]^2/[\Phi(b\sqrt{1+r^2}+ra)]^2+\phi'(b\sqrt{1+r^2}+ra)/\Phi(b\sqrt{1+r^2}+ra)\}$$

corresponds to a 2nd derivative of $\log\Phi(x)$, it is non-positive. Thus the 1st term in (3.7) is non-positive. The 2nd term in (3.7) is non-positive if $b \le 0$. For the case $b > 0$, we can change b back to the negative case using the function $F(x)=\Phi(-x)$ and repeat the same discussion to have non-positivity of the 2nd derivative of (3.6).

□

3.2 Parameter estimation by maximum likelihood approaches

In this section, we assume that the threshold values $\{b_{ij}\}$ are known and so are $\{r_i\}$, where r_i is the risk sensitivity given by (2.2) for a non-default rating R_i with respect to the latent systematic risk factor s. This is because both $\{b_{ij}\}$ and $\{r_i\}$ are defined before observing any macroeconomic condition $x=(x_1, x_2, ..., x_m)$ (see section 2.1 for the estimation of $\{r_i\}$, and section 2.3 for $\{b_{ij}\}$).

As noted in remark 2.5, the key to the rating transition probabilities $\{p_{ij}(x)\}$ is the determination of the coefficients $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ for the credit index, and rating level risk sensitivities $\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_{k-1}$. Recall that the credit index enters the model via (2.4) and is defined by parameters: $\lambda, \tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$. By Proposition 2.3, the following relation is satisfied for $\tilde{r}_i$:

$$\tilde{r}_i = r_i\lambda/\sqrt{1+r_i^2(1-\lambda^2)} \tag{3.7}$$

Given $\{b_{ij}\}$ and $\{r_i\}$, recall that the coefficients $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ for the credit index are derived from a normalization of a linear combination $a_1\tilde{x}_1 + a_2\tilde{x}_2 + ... + a_m\tilde{x}_m$, with which the model $\{p_i(x)\}$ best predicting the default probability of the portfolio, where $p_i(x)$ is by (2.14) as:

$$p_i(x) = \Phi[c_i + \tilde{r}_i(a_1\tilde{x}_1 + a_2\tilde{x}_2 + ... + a_m\tilde{x}_m)] \quad (3.8)$$

This can be implemented by using the log likelihood function (3.2) with $p_{ik}(x)$ being replaced by $p_i(x)$ above. Maximize the corresponding total log likelihood for parameters $\lambda,\ \tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$.

When $\lambda,\ \tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ are known, $\{\tilde{r}_i\}$ can be determined in theory by (3.7). However, for a better and more robust model, we propose to perform additional recalibration for each $\tilde{r}_i$ at rating level by maximum likelihood using the total log likelihood via (3.4) or (3.5) across time for that rating. The final rating transition model is given by (2.7).

We thus propose the following two-step approach:

Step 1. Estimate $\tilde{a}_1,\ \tilde{a}_2,\ ...,\ \tilde{a}_m$ for the credit index

Maximize the total log likelihood by (3.2) and (3.7) as a function of $\lambda,\ \tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$. To ensure these estimates are the global maximum estimates, a series of additional searches are performed: Let $\lambda \in (0,1]$ vary through the set of values $\{i/N\,|1 \le i \le N\}$ for large integer N. For each value of λ, calculate $\{\tilde{r}_i\}$ using (3.7). Find the maximum likelihood estimates for $a_1,\ a_2,\ ...,\ a_m$ using the total log likelihood by (3.2) and (3.8) for all time for all ratings. By the concavity of (3.2) as a function of $a_1, a_2, ..., a_m$, any of these local maximum likelihood estimates $a_1,\ a_2,\ ...,\ a_m$ are the global maximum likelihood estimates for a given λ. Repeat this process for enough many times and compare with the initial results to obtain the global maximum likelihood estimate for $\lambda, a_1, a_2, ..., a_m$. Normalize the linear combination $a_1\tilde{x}_1 + a_2\tilde{x}_2 + ... + a_m\tilde{x}_m$ to obtain the estimate for $\tilde{a}_1,\ \tilde{a}_2,\ ...,\ \tilde{a}_m$.

Step 2. Estimate $\tilde{r}_i$ for each non-default rating R_i separately

Calculate credit index $ci(x)$ as

$$ci(x) = (\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m)/v$$

where v is the standard deviation of $\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m$. We then recalibrate and estimate $\tilde{r}_i$ by maximizing the total log-likelihood by (3.2) or (3.3) across time for rating R_i, with $p_{ik}(x)$ and $p_{i(i+1)}(x)$ given by (2.7) as:

$$p_{ik}(x) = \Phi[(b_{i\,1}\sqrt{1+\tilde{r}_i^2} + \tilde{r}_i ci(x)]$$

$$p_{i(i+1)}(x) = \Phi[b_{i\ k-i}\sqrt{1+\tilde{r}_i^2} + \tilde{r}_i ci(x)]$$

We implemented the above two-step optimization process by using SAS PROC NLMIXED procedure.

4. An Empirical Example: CCAR Stress Testing for a Commercial Portfolio

In this section, we fit the rating transition model for a commercial portfolio, and assess the nine quarter losses for the portfolio for the supervisory scenarios provided by the Federal Reserve for the period 2016Q1-2018Q1.

The data is created synthetically from the historical quarterly rating transition frequency for a US commercial portfolio (The sample default rate does not reflect the original portfolio true default rate). There are 7 ratings for the portfolio, with rating R_1 as the best quality rating and R_7 as the default rating.

We match the sample to the macroeconomic data (sourced from the Federal Reserve) by calendar quarter. We are focused on the following nine macroeconomic variables:

Table 1. Macroeconomic variables

#	Variable	Description
1	GDP_GQOQ_COM	Growth Rate of US Gross Domestic Product (quarter over quarter annualized by compounding)
2	LURC_DQOQ	Increase of US Civilian Unemployment Rate (quarter over quarter annualized)
3	PCREPI_GQOQ_COM	Growth Rate of US Commercial Real Estate Price (quarter over quarter annualized by compounding)
4	PPSDJT_GQOQ_COM	Growth Rate of Dow Jones Total Stock Market Index (quarter over quarter annualized by compounding)
5	RCBBB_DQOQ	Increase of US BBB 10-Year Corporate Yield (quarter over quarter annualized)
6	RCBBB_RT10Y	US 10-year BBB Corporate Credit Spread
7	RTB_DQOQ	Increase of US 3-Month Treasury Bill: Secondary Market Rate (quarter over quarter annualized)
8	RT10Y_DQOQ	Increase of US Constant Maturity Treasury Yield, 10 Yrs (quarter over quarter annualized)
9	VIX_FED	US Implied Volatility (Maximum of daily values per quarter)

Selection of variables is subject to a governance review process. Each variable should pass the unit root tests. Here we consider four lag variables for each macroeconomic variable: lag 0 (current), lag 1 (lag 1 quarter), lag 2 (lag two quarters), lag 3 (lag three quarters). Each lag variable is named by prefixing to the original name by a label "L" together with its lag number.

In the remainder of this section, we are focused on model fitting and scenario loss projection as described by (a)-(d) below:

(a) Variable selection

Let m denote the number of variables in a model. Due to the limited number of data points in the time series sample, we consider only models with $m \le 4$. A preliminary model selection process is performed via SAS logistic regression with model selection option being set to "Score", targeting portfolio default frequency over the sample. The top best 1000 models (in the form of variable combination, no coefficients provided by SAS with this model selection option) for each value of m are selected for subsequent evaluations.

(b) Transition probability model fitting

For each list of macroeconomic variables $x_1, x_2, ..., x_m$ from step (a), follow the steps proposed in section 3.2 to fit for coefficients $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ and sensitivities $\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_{k-1}$.

The table below shows the top 10 transition probability models based on the accuracy in predicting the portfolio default rate in the downturn period 2008Q1-2009Q4, where the last 6 columns in the table denote the risk sensitivity for each non-default rating with respect to the corresponding credit index. The column MAD denotes the average deviation (average of absolute values for the prediction error) between the realized and predicted portfolio default rate over the period 2008Q1-2009Q4

Table 2. Top 10 models

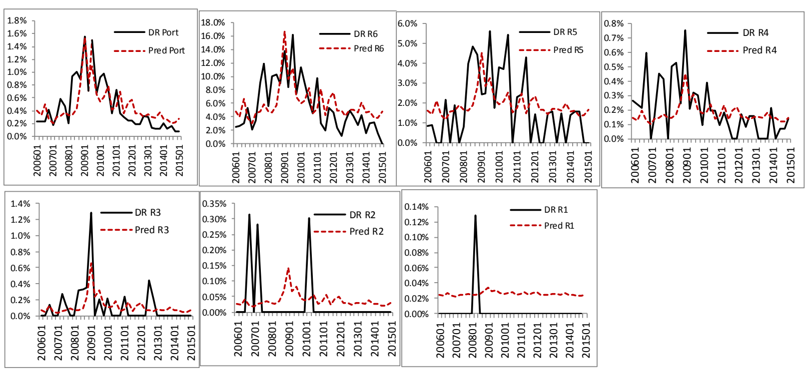
Model #	CI Model Variable V1	V2	V3	MAD	RSQ	CI Model Parameter Parm1	Parm2	Parm3	Rating Level Risk Sensitivity to CI 1	2	3	4	5	6
1	L1_VIX_FED	L1_RT10Y_DQOQ		0.0019	0.72	10.40	50.26		0.02	0.13	0.20	0.10	0.12	0.19
2	L1_PCREPI_GQOQ_COM	L1_RT10Y_DQOQ	L0_LURC_DQOQ	0.0019	0.75	-1.76	17.18	64.75	0.23	0.20	0.24	0.17	0.13	0.17
3	L0_GDP_GQOQ_COM	L1_PCREPI_GQOQ_COM		0.0020	0.75	-28.70	-4.69		0.23	0.16	0.25	0.14	0.14	0.18
4	L1_GDP_GQOQ_COM	L3_PCREPI_GQOQ_COM		0.0020	0.73	-34.86	-4.42		0.16	0.16	0.21	0.12	0.12	0.17
5	L1_RT10Y_DQOQ	L0_LURC_DQOQ		0.0020	0.71	23.89	77.82		0.13	0.20	0.24	0.17	0.11	0.16
6	L1_VIX_FED	L0_PPSDJT_GQOQ_COM	L2_LURC_DQOQ	0.0021	0.84	2.58	-1.47	60.12	0.18	0.20	0.21	0.16	0.15	0.21
7	L1_PCREPI_GQOQ_COM	L0_LURC_DQOQ		0.0021	0.72	-2.10	57.77		0.35	0.20	0.24	0.17	0.13	0.17
8	L3_PCREPI_GQOQ_COM	L0_LURC_DQOQ	L2_LURC_DQOQ	0.0021	0.82	-1.74	41.16	31.51	0.09	0.20	0.22	0.16	0.14	0.20
9	L1_RCBBB_RT10Y	L0_PPSDJT_GQOQ_COM	L2_LURC_DQOQ	0.0021	0.83	27.15	-1.69	59.64	0.19	0.20	0.22	0.16	0.14	0.21
10	L1_GDP_GQOQ_COM	L0_PPSDJT_GQOQ_COM	L2_LURC_DQOQ	0.0021	0.83	-10.24	-1.51	64.67	0.20	0.21	0.22	0.16	0.13	0.21

(c) Transition probability model performance

The charts below show the back-test results for the top transition model (#1 in table 2) between 2006Q1 and 2015Q1. In general, we expect the predicted default rate is higher in the downturn period 2008Q1-2010Q1.

The 1st and 2nd charts at top row show the predicted and realized default rates at the portfolio level and for rating R_6, respectively. It turns out that this model is able to pick up the default rate at the portfolio level and for ratings R_3, R_4, R_5 and R_6 as well. For rating R_1, the best quality rating, the predicted default rate is flat as expected, due to its low realized default rate (close to zero except for one quarter) for this rating.

Figure 1. Predicted vs. realized portfolio default rate for the top model



(d) Scenario loss projection

Let $s_1(t_0), s_2(t_0), ..., s_7(t_0)$ denote the percentage distribution for ratings $R_1, R_2, ..., R_7$ for the portfolio at the beginning t_0 of a horizon. Let $\{p_{ij}(x)\}$ be transition probabilities for this horizon given the list of macroeconomic variables $x = (x_1, x_2, ..., x_m)$. To facilitate the subsequent calculations, we add another row $\{p_{7j}\}$ to the transition matrix: $p_{7j}(x) = 0$ for $1 \le j \le 6$, and $p_{77}(x) = 1$. Then the rating distribution for the portfolio at the end of the horizon t_1 is given by:

$$s_i(t_1) = s_1(t_0)p_{1i}(x) + s_2(t_0)p_{2i}(x) + ... + s_7(t_0)p_{7i}(x)$$

Let $l_i(t_1)$ and $e_i(t_1)$ denote respectively the LGD and EAD factors for a default facility at the horizon end t_1 for risk rating R_i. Then the marginal portfolio default rate and marginal loss due to the period $(t_0, t_1]$ are given respectively by p_{t_1} and $L(t_1)$ below:

$$p_{t_1} = s_1(t_0)p_{17}(x) + s_2(t_0)p_{27}(x) + ... + s_6(t_0)p_{67}(x)$$
$$L(t_1) = s_1(t_0)p_{17}(x)f_1(t_1) + s_2(t_0)p_{27}(x)f_2(t_1) + ... + s_6(t_0)p_{67}(x)f_6(t_1)$$

where $f_i(t_1)$ is the sum of products $[l_i(t_1)][e_i(t_1)]$ over all facilities for the borrower. Using the top transition model selected from table 2, we calculate in the next table the portfolio default rate and loss for each quarter for baseline, adverse, and severely adverse scenarios, provided by the Fed for a period of nine quarters from 2016Q1 to 2018Q1. Here the cumulative portfolio default rate c_t at time t is calculated from the marginal default rate p_t by using the formula:

$$c_t = c_{t-1} + (1 - c_{t-1})p_t$$

The loss is presented as a percentage of the portfolio total exposure at the beginning of the period. The results show that, the model projects a loss of 3.42% for the baseline scenario, and 4.24% for the adverse scenario, and 6.11% for the severely adverse scenario:

Table 3. Loss projection on Fed's scenarios 2016Q1-2018Q1

Year / Quar	Marginal Port Default Rate			Cumulative Port Default Rate			Marginal Loss Projection			Cumulative Loss Projection		
	Base	Adv	Sev Adv	Base	Adv	Sev Adv	Base	Adv	Sev Adv	Base	Adv	Sev Adv
201601	0.40%	0.39%	0.38%	0.40%	0.39%	0.38%	0.0026	0.0026	0.0025	0.0026	0.0026	0.0025
201602	0.32%	0.27%	0.36%	0.72%	0.66%	0.73%	0.0032	0.0027	0.0036	0.0058	0.0053	0.0061
201603	0.38%	0.49%	0.85%	1.10%	1.15%	1.58%	0.0039	0.0050	0.0092	0.0097	0.0103	0.0153
201604	0.40%	0.58%	1.04%	1.50%	1.72%	2.60%	0.0040	0.0060	0.0112	0.0137	0.0162	0.0263
201701	0.44%	0.62%	1.05%	1.93%	2.33%	3.63%	0.0044	0.0064	0.0112	0.0180	0.0225	0.0372
201702	0.43%	0.55%	0.77%	2.34%	2.87%	4.37%	0.0043	0.0056	0.0080	0.0222	0.0280	0.0449
201703	0.42%	0.51%	0.64%	2.76%	3.36%	4.98%	0.0042	0.0052	0.0066	0.0263	0.0330	0.0512
201704	0.42%	0.52%	0.56%	3.16%	3.87%	5.51%	0.0042	0.0053	0.0058	0.0304	0.0382	0.0566
201801	0.39%	0.44%	0.46%	3.55%	4.29%	5.95%	0.0040	0.0044	0.0047	**0.0342**	**0.0424**	**0.0611**

Conclusions. Rating transition probability models are widely used in industry for multi-period scenario loss projection. This paper explicitly demonstrates how Miu and Ozdemir's original methodology can be structured and implemented with rating specific asset correlation. The models proposed in this paper are structured by using a risk component, called credit index, representing the part of systematic risk for the portfolio explained by a list of macroeconomic variables. Rating transition probabilities are formulated analytically by using the rating level sensitivity with respect to this credit index. The proposed parameter estimation approaches are based on maximum likelihood for observing the historical rating transition frequency. Rating level asset correlation is fully recognized in all respects of model fitting. The resulting models by these approaches are in general robust, not only at the portfolio level, but also at the rating level as well. These approaches can be implemented easily using, for example, SAS PROC NLMIXED ([9]) by modellers. We believe that the models and approaches proposed in this paper provide an effective tool to the practitioners for the use of migration matrix methodology for CCAR stress testing and loss projection.

Acknowledgements: The authors thank Carlos Lopez for proposing the use of credit index, his consistent inputs, insights, and supports for this research. Special thanks go to Clovis Sukam for his critical reading for this manuscript, and Nikolay Hovhannisyan, Wallace Law, and Biao Wu for many valuable discussions, insights, and comments.

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