Point-in-time PD term structure models for multi-period scenario loss projection: Methodologies and implementations for IFRS 9 ECL and CCAR stress testing

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POINT-IN-TIME PD TERM STRUCTURE MODELS FOR MULTI-PERIOD SCENARIO LOSS PROJECTION*

- Methodologies and implementations for IFRS 9 ECL and CCAR stress testing

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Abstract

Rating transition models ([8], [13]) have been widely used for multi-period scenario loss projection for CCAR stress testing and IFRS 9 expected credit loss estimation. Though the cumulative probability of default (PD) for a rating can be derived by repeatedly applying the migration matrix at each single forward scenario sequentially, divergence between the predicted and realized cumulative default rates can be significant, particularly when the predicting horizon extends to longer periods ([4]). In this paper, we propose approaches to modeling the forward PDs directly. The proposed models are structured via a credit index, representing the systematic risk for the portfolio explained by a list of macroeconomic variables, together with the risk sensitivity with respect to the credit index, for each rating and each forward term. An algorithm for parameter estimation is proposed based on maximum likelihood of observing the default frequency for each non-default rating and each forward term. The proposed models and approaches are validated on a corporate portfolio, where a forward PD model and a point-in-time rating transition model are fitted. It is observed that both models demonstrate strong strengths in predicting portfolio quarterly default rate (i.e. in one-term horizon), but the term model outperforms in general the transition model as the predicting horizon extends to longer periods (e.g., 1-year or 2-year horizons), due to the fact that the term model is calibrated over a longer horizon. We believe that the proposed models will provide practitioners a new and robust tool for modeling directly the PD term structure for multi-period scenario loss projection, for CCAR stress testing and IFRS 9 expected credit loss (ECL) estimation.

Keywords: CCAR stress testing, impairment loan, IFRS 9 expected credit loss, PD term structure, forward PD, marginal PD, credit index, risk sensitivity, maximum likelihood

1. Introduction

Let $p_k(t_k)$ denote the forward probability of default (PD) for a loan in the k^{th} period $(t_{k-1}, t_k]$ after the initial observation time t_0, i.e., the conditional probability of default for the loan in the period given that the loan does not default before the period. Then the marginal PD for the loan in the k^{th} period is given by:

$$(1 - c_{k-1}(t_{k-1}))\, p_k(t_k)$$

where $c_{k-1}(t_{k-1})$ denotes the cumulative PD for the period $(t_0, t_{k-1}]$, and $(1 - c_{k-1}(t_{k-1}))$ is the survival probability for the loan for the period $(t_0, t_{k-1}]$.

Let $l_i(t_i)$ and $e_i(t_i)$ denote respectively the point-in-time LGD and EAD factors for the i^{th} period after the initial observation time t_0. Let $f_i(t_i) = [l_i(t_i)][e_i(t_i)]$. Given the point-in-time PD term structure, the expected credit loss for a loan in a period from the initial observation time t_0 up to the k^{th} period can be estimated, assuming the point-in-time EAD and LGD term structures, by:

$$Loss = p_1(t_1) f_1(t_1) + (1 - c_1(t_1))\, p_2(t_2) f_2(t_2) + \ldots + (1 - c_{k-1}(t_{k-1}))\, p_k(t_k) f_k(t_k) \qquad (1.1)$$

Rating transition models ([4], [8], [12], [13]) have been widely used for multi-period scenario loss projection for CCAR ([5]) stress testing and IFRS 9 ([1], [2], [3]) expected credit loss estimation. Though cumulative PDs for a rating can be derived by repeatedly applying the migration matrix at each single forward scenario sequentially, divergence between the predicted and realized cumulative default rates can

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be significant, when the number of iterations increases ([4]). Forward looking point-in-time PD term structure comes into play as an option.

A credit index, as introduced in [13] and summarized in the next section, is a linear combination of a list of given macroeconomic variables that best predict the default risk of the portfolio under some appropriate assumptions. The linear combination is normalized to have zero mean and one standard deviation. As shown in Theorem 2.2 in the next section, forward PDs for a non-default risk rating R_i and a forward term can be structured via the credit index by using the following three types of parameters:

(a) The coefficients of macroeconomic variables for the credit index, which are common for all non-default ratings and forward terms, at the portfolio level
(b) The risk sensitivity with respect to the credit index for each rating and each forward term
(c) The threshold value for each rating and each forward term

Threshold values in (c) can be estimated separately (Lemma 2.1 (b)). For parameters in (a) and (b), we will propose estimation approaches based on maximum likelihood for observing the default frequency for each rating and each forward term.

The advantages for the proposed forward PD model for PD term structure include the following:

1. Analytical formulations for forward PDs can be derived under the Merton model framework
2. The model is structured via a credit index, representing the part of systematic risk for the portfolio explained by a list of given macroeconomic variables, together with the risk sensitivity with respect to the credit index, for each rating and each forward term. This means, given the credit index, the model for a rating and a forward term is determined by the sensitivity and the threshold value (for the intercept).
3. Parameters estimation is based on maximum likelihood for observing historical forward term default frequency, which can be implemented by using, for example, the SAS procedure PROC NLMIXED ([10]).

The paper is organized as follows: In section 2, we define the credit index for a portfolio, and derive the forward PD model under the Merton model framework. In section 3, we show how a PD term structure can be derived based on forward PDs and how loss can be evaluated over a multi-period scenario using the PD term structure. In section 4, we determine the log-likelihood function for observing the term default frequency. In section 5, we propose an algorithm for fitting the forward PD model. The proposed model and parameter estimation approaches are validated in section 6, where we fit a forward PD model and a point-in-time rating transition model for a corporate portfolio. Back-test and out-of-sample test results are provided.

2. Proposed Models for Forward Probability of Default

Given a borrower with a non-default risk rating R_i at the initial time t_0, assume the borrower did not default in the period $[t_0, t_{k-1}]$. We assume that the default risk for the borrower in the period $(t_{k-1}, t_k]$ is governed by a latent random variable $z_{ik}(t)$, called the firm's normalized asset value, which splits into two parts under the Merton model framework as ([6], [7], [8], [9], [12], [13]):

$$z_{ik}(t) = s(t)\sqrt{\rho_{ik}} + \varepsilon_{ik}(t)\sqrt{1-\rho_{ik}}\,,\ 0 < \rho_{ik} < 1,\ s(t) \sim N(0,1),\ \varepsilon_{ik}(t) \sim N(0,1) \quad (2.1)$$

where $s(t)$ denotes the systematic risk (common to all non-default ratings and all terms) at time t and $\varepsilon_{ik}(t)$ is the idiosyncratic risk independent of $s(t)$. The quantity ρ_{ik} is called the asset correlation given the initial risk rating R_i and forward term number k. It is assumed that there exist threshold values $\{b_{ik}\}$ such that the borrower will default in the k^{th} period $(t_{k-1}, t_k]$ if the normalized asset value $z_{ik}(t)$ falls below the threshold value b_{ik}. We call b_{ik} the default point for the k^{th} forward term for a borrower whose initial risk rating is R_i at time t_0.

For simplicity, we supress the time label t from $z_{ik}(t)$, $s(t)$, $\varepsilon_{ik}(t)$, and write them as z_{ik}, s, ε_{ik} respectively causing no confusions.

2.1. Forward probability of default

For a borrower with a non-default initial risk rating R_i at the initial time t_0, the k^{th} forward PD is the conditional probability that the borrower defaults in the k^{th} period $(t_{k-1}, t_k]$ given that the borrower does not default in the period $[t_0, t_{k-1}]$. For a given sample, the forward PD can be estimated by

$$d_{ik}(t_k)/n_{ik}(t_k) \tag{2.2}$$

where $n_{ik}(t_k)$ denotes the number of borrowers who survived the period $[t_0, t_{k-1}]$ with an initial risk rating R_i at the initial time t_0, and $d_{ik}(t_k)$ is the number of borrowers, within those $n_{ik}(t_k)$ borrowers, who defaulted in the period $(t_{k-1}, t_k]$.

Let $p_{ik}(s)$ denote the k^{th} forward PD given the systematic risk s in the k^{th} period. Under model (2.1), we have

$$\begin{aligned} p_{ik}(s) &= P(z_{ik} < b_{ik} \mid s) \\ &= P[\varepsilon_{ik} < (b_{ik} - s\sqrt{\rho_{ik}})/\sqrt{1-\rho_{ik}}\,] \\ &= \Phi[(b_{ik} - s\sqrt{\rho_{ik}})/\sqrt{1-\rho_{ik}}\,] \end{aligned} \tag{2.3}$$

where Φ denotes the standard normal cumulative distribution. Let

$$r_{ik} = \sqrt{\rho_{ik}}/\sqrt{1-\rho_{ik}} \tag{2.4}$$

$$\Rightarrow \rho_{ik} = r_{ik}^{\,2}/(1+r_{ik}^{\,2}),\ 1/\sqrt{1-\rho_{ik}} = \sqrt{1+r_{ik}^{\,2}} \tag{2.5}$$

By (2.3) and (2.5), we have

$$p_{ik}(s) = \Phi(b_{ik}\sqrt{1+r_{ik}^{\,2}} - r_{ik}s) \tag{2.6}$$

We can interpret the quantity r_{ik} as the risk sensitivity for the k^{th} forward PD, namely $p_{ik}(s)$, with respect to the systematic risk factor s.

Given a non-default rating R_i at the initial time t_0 and a forward term k, the risk sensitivity r_{ik} can be estimated by maximizing the likelihood given by (2.6) for observing the default frequency for the rating and the forward term, using for example, the SAS procedure PROC NLMIXED ([10], [12], [13]).

2.2. The proposed forward PD models

Let $E_s()$ denote the expectation with respect to *s*. The threshold value b_{ik} can be derived from the through-the-cycle average of the k^{th} forward PDs, as shown in the statement (b) below:

Lemma 2.1. (a) ([11]) $E_s[\Phi(a_0 + a_1 s)] = \Phi(a_0 / \sqrt{1+a_1^2})$, where $s \sim N(0,1)$

(b) $\Phi(b_{ik}) = E_s[p_{ik}(s))]$

Proof of Lemma 2.1 (b). This follows from (2.6) by applying Lemma 2.1 (a). □

Given a list of macroeconomic variables $x_1, x_2, ..., x_m$ with means $u_1, u_2, ..., u_m$, let $w(x)$ be a linear combination:

$$w(x) = a_1 x_1 + a_2 x_2 + ... + a_m x_m \tag{2.7}$$

Let $\tilde{x}_i = (x_1 - u_i)$. Normalize $w(x)$ by setting the credit index for the portfolio to be

$$ci(x) = [w(x) - u]/v = (a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + ... + a_m \tilde{x}_m)/v \tag{2.8}$$

where u and v denote respectively the mean and standard deviation of $w(x)$. We assume that, given the list of macroeconomic variables, the systematic risk factor s splits into two parts as in (2.9) below:

$$\begin{aligned} s &= -\lambda\, ci(x) - e\sqrt{1-\lambda^2}, \quad e \sim N(0,1), \;\; 0 < \lambda < 1 \\ &= -[\lambda(\tilde{a}_1 \tilde{x}_1 + \tilde{a}_2 \tilde{x}_2 + ... + \tilde{a}_m \tilde{x}_m) + \sigma e] \end{aligned} \tag{2.9}$$

where

$$\tilde{a}_i = a_i / v, \;\; \sigma = \sqrt{1-\lambda^2}$$

By (2.6) and (2.9), we have:

$$p_{ik}(s) = \Phi[b_{ik}\sqrt{1+r_{ik}^2} + r_{ik}(\lambda ci(x) + \sigma e)] \tag{2.10}$$

$$= \Phi[b_{ik}\sqrt{1+r_{ik}^2} + r_{ik}\lambda(\tilde{a}_1 \tilde{x}_1 + \tilde{a}_2 \tilde{x}_2 + ... + \tilde{a}_m \tilde{x}_m) + r_{ik}\sigma e] \tag{2.11}$$

Let $p_{ik}(x) = E[p_{ik}(s) \mid x]$ be the expected value of $p_{ik}(s)$ given macroeconomic variables $x = (x_1, x_2, ..., x_m)$. We call $p_{ik}(x)$ the forward PD given the scenario *x*.

Applying Lemma 2.1 (a) to (2.10) and (2.11), we have the following theorem for forward PDs:

Theorem 2.2. Given a list of macroeconomic variable $x_1, x_2, ..., x_m$, assume that the residual e in (2.9) is independent of $x_1, x_2, ..., x_m$. Under (2.1), we have:

$$p_{ik}(x) = \Phi[b_{ik}\sqrt{1+\tilde{r}_{ik}^{\,2}} + \tilde{r}_{ik}ci(x)] \quad (2.12)$$

$$= \Phi[b_{ik}\sqrt{1+\tilde{r}_{ik}^{\,2}} + \tilde{r}_{ik}(\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + ... + \tilde{a}_m\tilde{x}_m)] \quad (2.13)$$

where

$$\tilde{r}_{ik} = r_{ik}\lambda / \sqrt{1+r_{ik}^{\,2}\sigma^2} = r_{ik}\lambda / \sqrt{1+r_{ik}^{\,2}(1-\lambda^2)} \quad (2.14)$$

Proof. By (2.14), the definition of $\tilde{r}_{ik}$, we have:

$$\sqrt{1+\tilde{r}_{ik}^{\,2}} = \sqrt{1+r_{ik}^{\,2}} / \sqrt{1+r_{ik}^{\,2}(1-\lambda^2)}$$

$$\Rightarrow b_{ik}\sqrt{1+r_{ik}^{\,2}} / \sqrt{1+r_{ik}^{\,2}(1-\lambda^2)} = b_{ik}\sqrt{1+\tilde{r}_{ik}^{\,2}}$$

We need only to show (2.12). Applying Lemma 2.1 (a) to (2.10), we have:

$$p_{ik}(x) = E[p_{ik}(s) \mid x]$$

$$= \Phi[b_{ik}\sqrt{1+r_{ik}^{\,2}} / \sqrt{1+r_{ik}^{\,2}(1-\lambda^2)} + ci(x)\, r_{ik}\lambda / \sqrt{1+r_{ik}^{\,2}(1-\lambda^2)}]$$

$$= \Phi[b_{ik}\sqrt{1+\tilde{r}_{ik}^{\,2}} + \tilde{r}_{ik}ci(x)]$$

□

There are a lot of choices for (2.8). Given the asset correlations $\{\rho_{ik}\}$ in (2.1) (thus $\{r_{ik}\}$), we define the credit index for a portfolio to be the $ci(x)$ by (2.8) satisfying the following conditions:

(a) The residual e in (2.9) is independent of $x_1, x_2, ..., x_m$.

(b) $ci(x)$ is normalized from a linear combination $a_1\tilde{x}_1 + a_2\tilde{x}_2 + ... + a_m\tilde{x}_m$ with which the model $\{\tilde{p}_i(x)\}$ best predicts (via maximum likelihood as stated more precisely in section 5) the default probability of the portfolio, where

$$\tilde{p}_i(x) = \Phi[c_i + \tilde{r}_{i1}(a_1\tilde{x}_1 + a_2\tilde{x}_2 + ... + a_m\tilde{x}_m)] \quad (2.15)$$

is a model predicting the default probability for the initial rating R_i in one-term horizon, and the corresponding risk sensitivity $\tilde{r}_{i1}$ is driven by (2.14). No constraint is imposed for $\{a_1, a_2, ..., a_m\}$ and the intercept parameters $\{c_{ik}\}$.

Remark 2.3. Forward PDs in models (2.12)-(2.13) are given after the portfolio credit index is determined. The fact that no constraint is imposed for intercepts $\{c_{ik}\}$ ensures the full optimization is possible for parameters $\{a_1, a_2, ..., a_m\}$.

Remark 2.4. The portfolio credit index is fitted targeting the portfolio default risk for one-term horizon only. It can be extended to cover a longer horizon when data sparsity is not an issue and the risk pattern is persistent for the extended horizon.

Similarly to the quantities r_{ik} and ρ_{ik}, which are defined under (2.1) with respect to the systematic risk factor s, the quantity $\tilde{r}_{ik}$ can be interpreted as the risk sensitivity for the k^{th} forward PD with respect to the credit index $ci(x)$, and a quantity $\tilde{\rho}_{ik}$ can be defined by:

$$\tilde{\rho}_{ik} = \rho_{ik}\lambda^2$$

Proposition 2.5. The following three equations hold:

$$\tilde{r}_{ik} = \sqrt{\tilde{\rho}_{ik}} / \sqrt{1-\tilde{\rho}_{ik}}, \quad \tilde{\rho}_{ik} = \tilde{r}_{ik}^{\,2} / (1+\tilde{r}_{ik}^{\,2}), \; 1/\sqrt{1-\tilde{\rho}_{ik}} = \sqrt{1+\tilde{r}_{ik}^{\,2}}$$

Proof of Proposition 2.5. We show only the first relation. Notice that $\sigma^2 = 1-\lambda^2$. By (2.14), we have:

$$\begin{aligned}
&\tilde{r}_{ik} = r_{ik}\lambda / \sqrt{1+r_{ik}^{\,2}\sigma^2} \\
&\Rightarrow 1+\tilde{r}_{ik}^{\,2} = 1+r_{ik}^{\,2}\lambda^2/(1+r_{ik}^{\,2}\sigma^2) = (1+r_{ik}^{\,2})/(1+r_{ik}^{\,2}\sigma^2) \\
&\Rightarrow \tilde{r}_{ik}^{\,2}/(1+\tilde{r}_{ik}^{\,2}) = r_{ik}^{\,2}\lambda^2/(1+r_{ik}^{\,2}) \qquad (2.16)
\end{aligned}$$

By (2.4), we have:

$$\begin{aligned}
&r_{ik} = \sqrt{\rho_{ik}} / \sqrt{1-\rho_{ik}} \\
&\Rightarrow \rho_{ik} = r_{ik}^{\,2}/(1+r_{ik}^{\,2}) \qquad (2.17)
\end{aligned}$$

By (2.16) and (2.17), we have

$$\begin{aligned}
&\tilde{r}_{ik}^{\,2}/(1+\tilde{r}_{ik}^{\,2}) = \rho_{ik}\lambda^2 = \tilde{\rho}_{ik} \\
&\Rightarrow \tilde{r}_{ik} = \sqrt{\tilde{\rho}_{ik}} / \sqrt{1-\tilde{\rho}_{ik}}
\end{aligned}$$

□

Consequently, by (2.12) and (2.13), for the determination of the forward PDs $\{p_{ik}(x)\}$, the following parameters are required:

(a) Parameters $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ for macroeconomic variables in credit index $ci(x)$, common to all non-default ratings and all forward terms
(b) Risk sensitivities $\{\tilde{r}_{ik}\}$, with one sensitivity for each non-default risk rating and each forward term
(c) Threshold values $\{b_{ik}\}$, with one value for each non-default risk rating and each forward term

The threshold values $\{b_{ik}\}$ can be estimated separately by using Lemma 2.1(b). Therefore, the key to the probabilities $\{p_{ik}(x)\}$ is the determination of parameters: $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ and $\{\tilde{r}_{ik}\}$.

Remark 2.6. When the number of ratings is large and data sparsity is an issue, fitting the rating level sensitivities $\{\tilde{r}_{ik}\}$ could be a problem. In practice, we can re-group the risk ratings into fewer classes, for example, into grades of investment, sub-investment, and problematic. While the forward term numbers can be re-grouped, based on the risk patterns observed from the historical term structure. For example, forward term numbers can be re-grouped into $\{1\},\{2\},\{3,4\}$, and one group for every four consecutive terms after time t_4.

2.3. A review of the benchmark point-in-time rating transition probability models

Point-in-time rating transition probability model is proposed by Miu and Ozdemir ([8]), and extended by Yang and Du ([12], [13])) to facilitate rating level asset correlation.

Let $t_{ij}(x)$ denote the expected value of transition probability from an initial rating R_i at t_0 to rating R_j at the end of horizon, given macroeconomic variables $x=(x_1,x_2,...,x_m)$. Under the Merton model framework (with the k in (2.1) being set to 1), it can be shown ([13]), similarly to (2.12)-(2.13), that

$$\begin{aligned} t_{ij}(x) &= \Phi(\bar{q}_{i(k-j+1)}+\tilde{r}_i ci(x)) - \Phi(\bar{q}_{i(k-j)}+\tilde{r}_i ci(x)) \\ &= \Phi[\bar{q}_{i(k-j+1)}+\tilde{r}_i(\tilde{a}_1\tilde{x}_1+\tilde{a}_2\tilde{x}_2+...+\tilde{a}_m\tilde{x}_m)] \\ &\quad -\Phi[\bar{q}_{i(k-j)}+\tilde{r}_i(\tilde{a}_1\tilde{x}_1+\tilde{a}_2\tilde{x}_2+...+\tilde{a}_m\tilde{x}_m)] \end{aligned}$$

where $\bar{q}_{ih}=q_{ih}\sqrt{1+\tilde{r}_i^2}$, and $ci(x)$ is the portfolio credit index defined similarly using (2.15). The quantities $\{q_{ij}\}$ are the threshold values with $q_{ij}=\Phi^{-1}(\bar{p}_{ij})$, where $\bar{p}_{ij}$ is the through-the-cycle transition probability from rating R_i to rating R_j, which can be estimated from the historical sample. The key parameters to this rating transition probability model are $\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$ and $\{\tilde{r}_i\}$, which can be estimated ([13]) by an approach similar to the algorithm described in section 5.

3. The Derived PD Term Structure and Multi-Period Loss Projection

In this section, we describe how a point-in-time PD structure can be derived from the forward PDs, and how loss can be projected over a multi-period scenario given the PD term structure or given a point-in-time rating migration model.

3.1. Point-in-time PD term structure derived from forward PDs

Let $x(t_k)$ denote the vector of values of macroeconomic variables $x_1,x_2,...,x_m$ at time t_k. Let $p_{ik}[x(t_k)]$ be the forward PD for the k^{th} forward term given the scenario $x(t_k)$. For a borrower with a non-default initial risk rating R_i at t_0, the cumulative probability of default $c_{ik}(t_k)$ over the period $(t_0,t_k]$ can be derived from the forward PDs as follows:

$$\begin{aligned} &c_{i1}(t_1)=p_{i1}[x(t_1)] \\ &c_{i2}(t_2)=c_{i1}(t_1)+[1-c_{i1}(t_1)]\times p_{i2}[x(t_2)] \\ &\quad ..., \\ &c_{ik}(t_k)=c_{ik-1}(t_{k-1})+[1-c_{ik-1}(t_{k-1})]\times p_{ik}[x(t_k)] \end{aligned}$$

Note that the quantity $(1-c_{ik}(t_k))$ is the survival probability for the period $[t_0, t_k]$. The following proposition demonstrates the relationship between the forward PD and survival probability:

Proposition 3.1. The factorization (3.1) holds for the survival probability:

$$1-c_{ik}(t_k)=(1-p_{i1}[x(t_1)])(1-p_{i2}[x(t_2)])...(1-p_{ik}[x(t_k)]) \tag{3.1}$$

Proof. Factorization (3.1) follows from the equation below by induction:

$$1-c_{ik}(t_k)=[1-c_{ik-1}(t_{k-1})]\times(1-p_{ik}[x(t_k)])$$

□

3.2. Multi-period scenario loss projection

Given the point-in-time PD term structure, the expected credit loss for the period $(t_0, t_k]$ for a loan of a borrower, with initial rating R_i at t_0, can be evaluated as follows (using the notation of (1.1)):

$$\begin{aligned} Loss_i(t_k) = {} & p_{i1}[x(t_1)]f_1(t_1)+[1-c_{i1}(t_1)]p_{i2}[x(t_2)]f_2(t_2)+... \\ & +[1-c_{ik-1}(t_k)]p_{ik}[x(t_k)]f_k(t_k) \end{aligned} \tag{3.2}$$

The marginal PD for the period $(t_{k-1}, t_k]$ is given by $[1-c_{i\,k-1}(t_{k-1})]p_{ik}[x(t_k)]$.

Given the point-in-time rating transition probability and a scenario $x(t_k)$, let $T[x(t_k)]=\{t_{ij}(t_k)\}$ denote the rating migration matrix, and $t_{ij}(t_k)$ the probability that a rating R_i will migrate to R_j in one-term horizon. Assume that higher index rating carries higher default risk and there are, for example, 21 ratings with R_{21} the default rating. Then the last column of the matrix contains the point-in-time PDs for all risk ratings, and the last row of the matrix is set as:

$$\begin{aligned} & v_{21\,j}(t_k)=0 \text{ if } 1\le j\le 20 \\ & v_{21\,21}(t_k)=1. \end{aligned}$$

With these notations, the cumulative PD for the period $(t_0, t_k]$ for a loan of a borrower, whose initial risk rating is R_i, can be derived by the matrix multiplication as below:

$$u_i T[x(t_1)]T[x(t_2)]...T[x(t_k)] \tag{3.3}$$

where u_i is a row vector with all components equal to zero except for i^{th} component, which is 1. Consequently, marginal PDs can be derived and multi-period scenario loss can be evaluated using a methodology similar to (3.2).

4. Log-Likelihood Functions for Observing Term Default Frequency

In this section, we introduce a concept called forward log-likelihood, corresponding to the forward PD for a forward term. We show how the log-likelihood, by observing the multistage term default frequency, can be formulated, using the forward log-likelihoods. The log-likelihood function expressions (4.1) and (4.3) below will be used later in section 5 for parameter fitting.

Recall from section 2.1 the following notations:

(a) $n_{ik}(t_k)$ - The number of borrowers who survived the period $[t_0, t_{k-1}]$ with an initial risk rating R_i at the initial time t_0

(b) $d_{ik}(t_k)$ - The number of borrowers who defaulted in $(t_{k-1}, t_k]$.

Given the historical data for a risk-rated portfolio, a time series of the form $\{n_{ik}(t_k), d_{ik}(t_k)\}$ can be derived. The forward log-likelihood is defined for each pair (i,k) as in (4.1) below, for the k^{th} forward term and the initial rating R_i at time t_0, using $p_{ik}[x(t_k)]$, i.e., the forward PD for the term $(t_{k-1}, t_k]$:

$$FL_{ik} = \sum_{t_k} \{[n_{ik}(t_k) - d_{ik}(t_k)] \log(1 - p_{ik}[x(t_k)]) + d_{ik}(t_k) \log(p_{ik}[x(t_k)])\} \qquad (4.1)$$

with t_k sliding through the sample time window. Here we assume that the term default count follows a binomial distribution. The binomial coefficient, which is independent of the parameters for $p_{ik}[x(t_k)]$ (as given by (2.12) or (2.15)), has been dropped. Expression (4.1) is the actual log-likelihood over the conditional probability space given that borrowers have survived the period $[t_0, t_{k-1}]$.

In general, we are interested in the log-likelihood for a forward period $[t_h, t_{h+k}]$ with k terms. We assume that there is no withdrawal in the sample, and a borrower either defaults or survives at the end of a period.

Let $L_i(h, h+k)$ denote the log-likelihood for a borrower with initial rating R_i at t_0 over the period $[t_h, t_{h+k}]$ given that the borrower survived the period $[t_0, t_{h-1}]$, where the time window $[t_h, t_{h+k}]$ slides through the sample time window as in (4.1). Similarly, let $L(h, h+k)$ be the log-likelihood over the period $[t_h, t_{h+k}]$ for all borrowers of the portfolio with a non-default initial risk rating at t_0 given that the borrowers survived the period $[t_0, t_{h-1}]$, where the time window $[t_h, t_{h+k}]$ slides through the sample time window.

Proposition 4.1. Under the assumption of no withdrawal, the following equations hold (up to a constant independent of the parameters for $\{p_{ij}[x(t_k)]\}$ as given by (2.12) or (2.15)):

$$L_i(h, h+k) = FL_{i\,h+1} + FL_{i\,h+2} + \ldots + FL_{i\,h+k} \qquad (4.2)$$

$$L(h, h+k) = \sum_i L_i(h, h+k) \qquad (4.3)$$

Expression (4.2) demonstrates an additive property of the log-likelihood function: the log-likelihood for a forward period of consecutive forward terms is the sum of the individual forward log-likelihoods for the forward terms. This is expected because of the multiplicative property of the conditional probability for a multistage event.

Proof of Proposition 4.1. Equation (4.3) follows directly from (4.2). We show only (4.2), and the case when $h = 0$. For the simplicity, we write

$$n_{ij}(t_j),\ d_{ij}(t_j),\ p_{ij}[x(t_j)],\ c_{ij}(t_j)$$

respectively by:

$$n_j,\ d_j,\ p_j, \text{and}\ c_j.$$

Note that the marginal probability that a borrower with an initial rating R_i defaults in the period $(t_{j-1}, t_j]$ is: $(1-c_{j-1})p_j$. Thus the likelihood for observing d_j defaults in period $(t_{j-1}, t_j]$ is:

$$(1-c_{j-1})^{d_j}\, p_j^{\,d_j}$$

(up to a factor given by binomial coefficient of choosing d_j defaulters from n_j borrowers). Consequently, the likelihood for observing a sequence $\{d_j\}_{j=1,2,\ldots,k}$ of defaults in the period $(t_0, t_k]$, with d_j defaults in each period $(t_{j-1}, t_j]$, is:

$$\Delta(t_k) = p_1^{\,d_1} p_2^{\,d_2} \ldots p_k^{\,d_k} (1-c_1)^{d_2} (1-c_2)^{d_3} \ldots (1-c_{k-1})^{d_k} (1-c_k)^{n_1-(d_1+d_2+\ldots+d_k)} \qquad (4.4)$$

(up to a constant factor given by binomial coefficients) where the last factor $(1-c_k)^{n_1-(d_1+d_2+\ldots+d_k)}$ is the likelihood of those surviving the entire period $[t_0, t_k]$ at the end.

Because of the no-withdrawal assumption, the following equation holds:

$$n_i = n_1 - (d_1 + d_2 + \ldots + d_{i-1}) \qquad (4.5)$$

By equation (3.1) of Proposition 3.1, we have:

$$\begin{aligned}
&(1-c_1)^{d_2} (1-c_2)^{d_3} \ldots (1-c_{k-1})^{d_k} (1-c_k)^{n_1-(d_1+d_2+\ldots+d_k)} \\
&= (1-p_1)^{d_2} [(1-p_1)(1-p_2)]^{d_3} \ldots [(1-p_1)(1-p_2)\ldots(1-p_{k-1})]^{d_k} \\
&\quad [(1-p_1)(1-p_2)\ldots(1-p_k)]^{n_1-(d_1+d_2+\ldots+d_k)} \\
&= (1-p_1)^{d_2+d_3+\ldots+d_k} (1-p_2)^{d_3+d_4+\ldots+d_k} \ldots (1-p_{k-1})^{d_k} [(1-p_1)(1-p_2)\ldots(1-p_k)]^{n_1-(d_1+d_2+\ldots+d_k)} \\
&= (1-p_1)^{n_1-d_1} (1-p_2)^{n_1-(d_1+d_2)} \ldots (1-p_{k-1})^{n_1-(d_1+d_2+\ldots+d_{k-1})} (1-p_k)^{n_1-(d_1+d_2+\ldots+d_k)} \\
&= (1-p_1)^{n_1-d_1} (1-p_2)^{n_2-d_2} \ldots (1-p_{k-1})^{n_{k-1}-d_{k-1}} (1-p_k)^{n_k-d_k} \qquad (4.6)
\end{aligned}$$

The last equality (4.6) follows from (4.5). By (4.4), we have the following log-likelihood for the period $[t_0, t_k]$:

$$\begin{aligned}
\log(\Delta(t_k)) &= [d_1 \log(p_1) + (n_1 - d_1)\log(1-p_1)] + [d_2 \log(p_2) + (n_2 - d_2)\log(1-p_2)] + \ldots \\
&\quad + [d_k \log(p_k) + (n_k - d_k)\log(1-p_k)]
\end{aligned}$$

Letting the period $[t_0, t_k]$ slide through the sample time window, we obtain the following log-likelihood:

$$L_i(0,k) = \sum_{t_k} \log(\Delta(t_k)) = FL_{i1} + FL_{i2} + \ldots + FL_{ik}$$

□

A function is log concave if its logarithm is concave. If a function is concave, a local maximum is actually a global maximum, and the function is unimodal. This property is important for searching of the maximum likelihood estimates.

Proposition 4.2. The log likelihood function (4.1), where $p_{ik}[x(t_k)]$ is given by (2.15), is concave as a function of c_i, $a_1, a_2, \ldots, a_m$, and it is concave as a function of $\tilde{r}_{ik}$, where $p_{ik}[x(t_k)]$ is given by (2.12). This concavity of (4.1) holds when the cumulative standard normal distribution Φ is replaced by any cumulative probability distribution which is log concave (e.g., the cumulative distribution for logistic distribution).

Proof. It is well-known that the cumulative standard normal distribution is log concave, and the sum of concave functions is again concave. It is also known that, if $f(x)$ is log concave, then so is $f(Az+b)$, where $Az+b: R^m \to R^1$ is any affine transformation from the *m*-dimensional Euclidean space to the 1-dimensional Euclidean space. This means both the cumulative distribution $\Phi(x)$ and $F(x) = \Phi(-x)$ are log concave, and (4.1) is concave as a function of c_i, $a_1, a_2, \ldots, a_m$, where $p_{ik}[x(t_k)]$ is given by (2.15).

For the concavity of (4.1) as a function of $\tilde{r}_{ik}$, where $p_{ik}[x(t_k)]$ is given by (2.12), it suffices to show that the 2nd derivative of the function

$$L(r) = \log[\Phi(b\sqrt{1+r^2} + ra)] \tag{4.7}$$

is non-positive for any constants a and b. The 2nd derivative $d^2[L(r)]/dr^2$ is given by:

$$\begin{aligned}
&(br/\sqrt{1+r^2}+a)^2\{-[\phi(b\sqrt{1+r^2}+ra)]^2/[\Phi(b\sqrt{1+r^2}+ra)]^2 + \phi'(b\sqrt{1+r^2}+ra)/\Phi(b\sqrt{1+r^2}+ra)\} \\
&\qquad + \phi(b\sqrt{1+r^2}+ra)(b)(1+r^2)^{-3/2}/\Phi(b\sqrt{1+r^2}+ra) \\
&= I + II
\end{aligned} \tag{4.8}$$

where ϕ and ϕ' denote the 1st and 2nd derivatives of Φ. Because the factor in the 1st term of (4.8) below

$$\{-[\phi(b\sqrt{1+r^2}+ra)]^2/[\Phi(b\sqrt{1+r^2}+ra)]^2 + \phi'(b\sqrt{1+r^2}+ra)/\Phi(b\sqrt{1+r^2}+ra)\}$$

corresponds to a 2nd derivative of $\log\Phi(x)$, it is non-positive. Thus the 1st term in (4.8) is non-positive. The 2nd term in (4.8) is non-positive if $b \le 0$. For the case $b > 0$, we can change b back to the negative case using the function $F(x) = \Phi(-x)$ and repeat the same discussion to have non-positivity of the 2nd derivative of (4.7). □

5. Parameter Estimation by Maximum Likelihood Approaches

In this section, we assume that the threshold values $\{b_{ik}\}$ are known and so are $\{r_{ik}\}$, where r_{ik} is the risk sensitivity given by (2.1) and (2.4) for the initial rating R_i and term k with respect to the latent systematic risk factor s. Note that both $\{b_{ij}\}$ and $\{r_{ik}\}$ are defined before observing any macroeconomic condition $x=(x_1,x_2,...,x_m)$ (see section 2.1 for the estimation of $\{r_{ik}\}$, and Lemma 2.1 (b) in section 2.2 for $\{b_{ik}\}$).

As indicated at the end of section 2.2, the key to the forward PDs $\{p_{ik}(x)\}$ is the determination of the coefficients $\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$ for the credit index, and rating level risk sensitivities $\{\tilde{r}_{ik}\}$. The credit index enters the model via (2.9) and is defined by parameters: $\lambda,\ \tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$. Recall that by (2.14) the following relation is satisfied:

$$\tilde{r}_{ik}=r_{ik}\lambda/\sqrt{1+r_{ik}{}^2(1-\lambda^2)} \tag{5.1}$$

Given $\{b_{ik}\}$ and $\{r_{ik}\}$, recall that the coefficients $\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$ for the credit index are derived from a normalization of a linear combination $a_1\tilde{x}_1+a_2\tilde{x}_2+...+a_m\tilde{x}_m$, with which the model $\{\tilde{p}_i(x)\}$ best predicts the default probability of the portfolio for initial ratings for one-term horizon, where $\tilde{p}_i(x)$ is by (2.15) as:

$$\tilde{p}_i(x)=\Phi[c_i+\tilde{r}_{i1}(a_1\tilde{x}_1+a_2\tilde{x}_2+...+a_m\tilde{x}_m)] \tag{5.2}$$

This can be implemented by using the log likelihood function (4.1) with $p_{ik}(x)$ being replaced by $\tilde{p}_i(x)$ above. Maximize the corresponding total log likelihood for parameters $\lambda,\ \tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$.

When $\lambda,\ \tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$ are known, $\{\tilde{r}_{ik}\}$ can be determined by a calibration for each $\tilde{r}_{ik}$ at rating level by maximizing the total log likelihood by (4.1) for the initial rating R_i and term k with $p_{ik}(x)$ being given by (2.12), that is, the final term structure model is given by (2.12).

We thus propose the following two-step approach:

Step 1. Estimate $\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$ for the credit index

Get the first estimates for $\lambda,\ a_1,a_2,...,a_m$ by maximizing the total log likelihood by (4.1) and (5.2) for all initial ratings for one forward term as a function of $\lambda,\ a_1,a_2,...,a_m$. To ensure these first estimates are the global maximum likelihood estimates, a series of additional searches are performed: Let $\lambda\in(0,1)$ vary through the set of values $\{i/N\,|1\le i<N\}$ for large integer N (e.g., $N\ge 10$). For each value of λ, calculate $\{\tilde{r}_{ik}\}$ using (5.1). Find the maximum likelihood estimates for $a_1,a_2,...,a_m$ using the total log likelihood by (4.1) and (5.2) for all initial ratings and one term. By the concavity of (4.1) as a function of $a_1,a_2,...,a_m$, any of these local maximum likelihood estimates $a_1,a_2,...,a_m$ are the global maximum likelihood estimates for a given λ. Use these estimates as the initial values for $\lambda,\ a_1,a_2,...,a_m$, and re-maximize the total log likelihood by (4.1) and (5.2) as a function of $\lambda,\ a_1,a_2,...,a_m$. Repeat this process to obtain the global maximum likelihood estimate for $\lambda,\ a_1,a_2,...,a_m$. Normalize the linear combination $a_1\tilde{x}_1+a_2\tilde{x}_2+...+a_m\tilde{x}_m$ to obtain the estimate for $\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m$.

Step 2. Estimate $\tilde{r}_{ik}$ for each initial rating R_i and term k separately

Calculate credit index $ci(x)$ as:

$$ci(x) = (\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + \ldots + \tilde{a}_m\tilde{x}_m)/v$$

where v is the standard deviation of $\tilde{a}_1\tilde{x}_1 + \tilde{a}_2\tilde{x}_2 + \ldots + \tilde{a}_m\tilde{x}_m$. We then calibrate and estimate $\tilde{r}_{ik}$ by maximizing the total log-likelihood using (4.1) for the initial rating R_i and term k with $p_{ik}[x(t_k)]$ being given by (2.12) as:

$$p_{ik}(x) = \Phi[b_{ik}\sqrt{1+\tilde{r}_{ik}^{\,2}} + \tilde{r}_{ik}ci(x)]$$

We implemented the above two-step optimization process by using SAS PROC NLMIXED procedure.

6. An Empirical Example: The PD Term Structure for a Corporate Portfolio

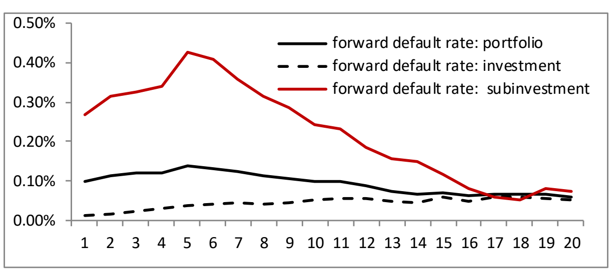
The sample is created synthetically from a historical dataset of a corporate portfolio containing quarterly rating level default frequency (The sample default rate does not represent the original portfolio default rate). There are 21 ratings for the portfolio, with rating R_1 being the best quality rating and R_{21} the default rating. The higher the rating index, the higher default risk.

The chart below depicts the trend of forward default rates, averaged over the sample time window, for 20 forward terms (i.e., 20 forward quarters):

1. At portfolio level
2. For investment rating
3. For sub-investment rating

It is observed that the simple average forward default rates tend to converge after about 20 terms:

Figure 1. Simple average forward default rate



For this reason, we can focus on terms covering a period of 4 years (16 quarters). For terms beyond 16 quarters, a constant forward rate is assumed for all ratings. This constant rate can be estimated, for example, by the portfolio level average forward default rate for the 5th year.

Macroeconomic data is sourced from the Federal Reserve. It is merged with the term default frequency sample by matching the end quarter of a term to the calendar quarter of the macroeconomic data. Inclusion of macroeconomic variables is subject to a governance review process. All variables should pass the unit root tests. We consider four lag-versions for each macroeconomic variable: lag 0 (current), lag 1 (lag 1 quarter), lag 2 (lag two quarters), lag 3 (lag three quarters). Each lag-version variable is named by prefixing to the original name by a label "L" together with its lag number.

The table below shows the 9 macroeconomic variables we use. The Pearson correlation to the quarterly portfolio level default rate is reported in the last 4 columns for the four lag-versions of each variable:

Table 1. Macroeconomic variables

#	Variable	Description	L0	L1	L2	L3
1	GDP_GQOQ_COM	Growth Rate of US Gross Domestic Product (quarter over quarter annualized by compounding)	-0.38	-0.24	-0.39	-0.53
2	LURC_DQOQ	Increase of US Civilian Unemployment Rate (quarter over quarter annualized)	0.56	0.59	0.64	0.56
3	PCREPI_GQOQ_COM	Growth Rate of US Commercial Real Estate Price (quarter over quarter annualized by compounding)	-0.32	-0.49	-0.48	-0.35
4	PPSDJT_GQOQ_COM	Growth Rate of Dow Jones Total Stock Market Index (quarter over quarter annualized by compounding)	-0.06	0.01	-0.16	-0.38
5	RCBBB_DQOQ	Increase of US BBB 10-Year Corporate Yield (quarter over quarter annualized)	0.13	-0.15	0.16	0.14
6	RCBBB_RT10Y	US 10-year BBB Corporate Credit Spread	0.53	0.51	0.58	0.51
7	RT10Y_DQOQ	Increase of US Constant Maturity Treasury Yield, 10 Yrs (quarter over quarter annualized)	0.14	0.02	0.03	-0.28
8	RTB_DQOQ	Increase of US 3-Month Treasury Bill: Secondary Market Rate (quarter over quarter annualized)	-0.16	0.06	-0.31	-0.43
9	VIX_FED	US Implied Volatility (Maximum of daily values per quarter)	0.52	0.36	0.43	0.47

In the remainder of this section, we focus on model fitting, as described by (a)-(c) below:

(a) Variable selection for term models

Let m denote the number of variables in a model. Due to the limited number of data points in the time series sample, we consider only models with $m = 2$ or 3. A preliminary model selection process is performed via SAS logistic regression with model selection option being set to "Score", targeting portfolio level default frequency over the sample. The top best 5000 models for each value of m are selected for further processing.

This SAS selection option calculates the score statistics for each possible variable combination without performing a full regression analyses, then select the specified number of models based on the higher score statistics.

(b) Forward PD model fitting

For each list of macroeconomic variables $x_1, x_2, ..., x_m$ from step (a), follow the steps proposed in section 5 to fit for coefficients $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m$ and sensitivities $\{\tilde{r}_{ik}\}$.

The table below shows the top 10 forward PD models ranked by RSQ for portfolio level quarterly default rate (e.g., 1st term), and the average RSQ for portfolio level forward default rates for forward terms in the 1st, 2nd, 3rd, and 4th years.

Table 2. Top 10 models

#	Risk Factors for Credit Index Models			Average Portfolio Level RSQ				
	V1	V2	V3	1st Quart	1st Year	2nd Year	3rd Year	4th Year
1	L0_PPSDJT_GQOQ_CO	L1_RTB_DQOQ	L0_LURC_DQOQ	0.66	0.63	0.59	0.71	0.32
2	L3_PCREPI_GQOQ_CO	L1_RTB_DQOQ	L0_LURC_DQOQ	0.61	0.70	0.76	0.79	0.36
3	L0_VIX_FED	L1_RTB_DQOQ	L0_LURC_DQOQ	0.60	0.61	0.61	0.73	0.35
4	L1_RTB_DQOQ	L1_RT10Y_DQOQ	L0_LURC_DQOQ	0.60	0.69	0.75	0.78	0.34
5	L0_GDP_GQOQ_COM	L1_RTB_DQOQ	L0_LURC_DQOQ	0.59	0.61	0.61	0.72	0.24
6	L3_GDP_GQOQ_COM	L1_RTB_DQOQ	L0_LURC_DQOQ	0.58	0.69	0.79	0.80	0.40
7	L2_PCREPI_GQOQ_CO	L1_RTB_DQOQ	L0_LURC_DQOQ	0.58	0.69	0.77	0.78	0.36
8	L3_RCBBB_RT10Y	L1_RTB_DQOQ	L0_LURC_DQOQ	0.57	0.68	0.78	0.78	0.38
9	L1_PCREPI_GQOQ_COM	L1_RTB_DQOQ	L0_LURC_DQOQ	0.56	0.68	0.77	0.73	0.40
10	L0_RCBBB_RT10Y	L1_PCREPI_GQOQ_COM	L1_RTB_DQOQ	0.56	0.68	0.77	0.83	0.42

(c) Benchmarking and back tests for the selected forward PD model

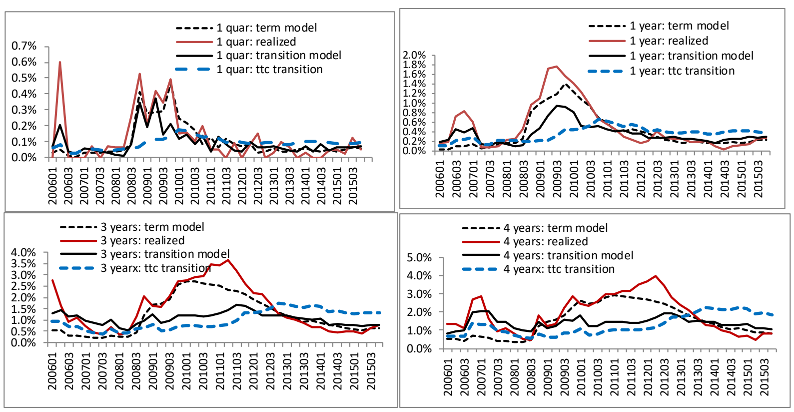
The top term model selected from Table 2 is scored over the development sample. Two rating migration models are used for benchmarking:

(1) The through-the-cycle rating transition model
(2) A point-in-time rating transition model, using the same list of macroeconomic variables as for the selected top forward term model.

The through-the-cycle transition matrix is calculated from the rating migration frequency across time (Lemma 2.1 (b), or see [12] for detailed calculation). The point-in-time transition model is developed following the approaches as reviewed in section 2.3.

The figure below plots the quarterly portfolio level default rate (e.g., one-term horizon), and portfolio level cumulative default rate for 1, 3, and 4 year horizons.

Figure 2. Predicted vs. realized cumulative portfolio default rate



Model RSQ is given as in the tables 3 for back-test for the selected point-in-time term structure model and point-in-time transition model, to predict the portfolio cumulative default rates over the entire sample:

Table 3. Backtest - model RSQ for predicting portfolio cumulative default rate

PIT Model	1 Quar	1 Year	2 Years	3 Years	4 Years
Term	0.66	0.88	0.88	0.76	0.77
Transition	0.67	0.81	0.72	0.56	0.38

Table 4. Out sample test - model RSQ for predicting portfolio cumulative default rate

	Training					Training+Validation				
PIT Model	1 Quar	1 Year	2 Years	3 Years	4 Years	1 Quar	1 Year	2 Years	3 Years	4 Years
Term	0.62	0.91	0.91	0.90	0.54	0.63	0.86	0.87	0.82	0.83
Transition	0.74	0.83	0.69	0.76	0.21	0.70	0.76	0.58	0.33	0.11

For the out-of-sample test for the model methodologies, we face a limitation for the availability of the number of data points (number of quarters) and the number of downturn periods in the time series sample. We split the time series sample into two parts: the training and validation. The term and transition models are refitted over the training sample consists data points up to year of 2010, based on the end quarter of a forward term, using the same list of macroeconomic variables for both models as the top selected term model. The sample after 2010 is used as the validation sample. Model RSQ over the training sample is reported in table 4. Since the training sample contains the stress period (2008Q1-2009Q4) when default rate is very high, while the validation sample covers a period after 2010 when default rate is very low (a period of 5 years after 2010), for a fair comparison between training and validation, we report only the model RSQ over the combined sample.

The following are observed from this empirical example:

(a) The point-in-time term model slightly underperforms the point-in-time transition model for predicting portfolio default rate for one-quarter horizon. Both models demonstrate strong strengths in predicting portfolio quarterly default rate, particularly, for the downturn period 2008Q1-2010Q1.
(b) The point-in-time term model outperforms in general the point-in-time transition model when predicting horizon extends to longer periods, due to the fact that the term model is calibrated over a longer horizon.
(c) Through-the-cycle transition model is weak in picking up the trends during the economic recession or expansion.

Conclusions. Models that directly fit the forward term default rate are proposed in this paper. The proposed models are structured via a credit index, representing the part of systematic risk for the portfolio explained by a list of given macroeconomic variables, together with the risk sensitivity, for each non-default initial risk rating and each forward term. An algorithm for parameter fitting is proposed by using the maximum likelihood for observing the term default frequency. We believe the proposed model and approaches will provide practitioners a new and robust tool to the modeling of PD term structure for multi-period scenario loss projection, for CCAR stress testing and IFRS 9 expected credit loss estimation.

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