Neoclassical theories of stationary relative prices and the supply of capital

Fratini, Saverio M.

January 2017

Neoclassical theories of stationary relative prices and the supply of capital

Saverio M. Fratini

Roma Tre University

Abstract

In the traditional versions of the neoclassical theory of value and distribution, the stock of existing capital—understood as either an amount of value or an endowment of capital goods—was taken as given, together with the available quantities of labour and natural resources. This characteristic of the early neoclassical theories is analysed by the comparison with the modern neo-Walrasian models of stationary equilibrium, in which the stock of capital is not considered among the data.

We show that the attempt to put capital on the same footing as labour and land—i.e. to present it as a factor of production—led the early neoclassical author to write the zero net-accumulation condition, which was required by the stationarity of relative prices, in the form of a market clearing condition between supply of and demand for capital. The rate of interest was then understood as the price to determine by this market. However, as is well known, the conception of capital as a factor of production—and of the rate of interest as the price for its use—did not work and involved several problems, some of which are discussed in this paper.

JEL Codes: B13; B21; D24; D51; D91

Keywords: stationary relative prices, capital, net accumulation, Wicksell, Walras

1. The rate of interest in neoclassical theory

In early neoclassical theories, the distributive variables: wage rate, rent rate and rate of interest, were understood as the prices firms have to pay for the employment of the factors of production: labour, land and capital.[1] Income distribution was thus conceived as a market phenomenon. According to these theories, as the relative prices of commodities are determined by the equilibrium between their supply and demand, so income distribution comes out of the equilibrium between supply of and demand for factors of production.

[1] There is no need to say, here, that the conception of capital as a factor of production, i.e. as an input demanded by firms, did not work. It was made definitively clear that, in the production processes, capital does not play the same role as labour, natural resources and the commodities employed as means of production. As a consequence, there is, in general, "no unambiguous way of characterizing different processes as more 'capital-intensive'" (Samuelson 1966, p. 582) and, therefore, the demand for capital by firms cannot be understood as the demand for an input, whose elasticity relative to the price system derives from factor substitutability.

The rate of interest was accordingly intended as a variable reacting to discrepancies between supply of and demand for capital, in more or less the same way as the price of a commodity reacts to the difference between its supply and demand. Therefore, the equilibrium level of the rate of interest was thought to involve the equality between the quantity of capital demanded by firms and the stock of capital supplied by households.[2] Moreover, since the quantities of available labour and land were considered exogenous magnitudes within the theory, similarly, the existing stock of capital was understood as a given amount. According to Marshall (1920, p. 534), for instance, "it is only slowly and gradually that the rise in the rate of interest will increase the total stock of capital", so that, for the purposes of the theory of value and distribution, capital accumulation could be neglected.

More recently, the given stock of capital that characterized the initial versions of the neoclassical theory has been interpreted as due to a 'missing equation' in those equilibrium systems. In particular, re-reading those early attempts from a neo-Walrasian standpoint, various scholars identified the missing equation with a condition of zero net savings—we refer in particular to Hirshleifer (1967), but also Negishi (1982) and Malinvaud (2003)—which is required by the stationarity of the system and which relates intertemporal households' decisions about current and future consumption with firms' choices of the optimal production plans.[3]

Here, we set out to address the issue of the neoclassical conception of the market for capital, with its given existing stock, in reverse order. We shall start from the neo-Walrasian stationary equilibrium model (sec. 2) in order to use it as a benchmark in the analysis of the early equilibrium models. In particular, we shall consider here both a Wicksellian model (sec. 3), in which the existing stock of capital is taken in value terms, and Walras's theory without and with capital formation (sec. 4), in which, instead, the endowments of capital goods are included among the data on the same footing as the endowments of non-producible inputs. The comparison with the neo-Walrasian stationary model will allow us to analyse the logical working of Walras's and Wicksell's theories, which therefore are not merely considered from point of view of the history of economic thought, but also from a methodological perspective.

We believe that the neo-Walrasian standpoint, which is at the basis of our analysis, can help to shed new light on well-known features of the treatment of capital within traditional neoclassical theories. Moreover, since the conception of capital as a factor of production and of the rate of interest as the price for its use do not pertain just to the old versions of neoclassical theory, but crop

[2] Marshall wrote that "interest, being the price paid for the use of capital in any market, tends towards an equilibrium level such that the aggregate demand for capital in that market, at that rate of interest, is equal to the aggregate stock forthcoming there at that rate" (1920, p. 534).

[3] For a critical discussion of the literature on the so-called 'Wicksell's missing equation', see in particular Kompass (1992), Kurz (2000) and Fratini (2013b).

up, implicitly, every time the rate of interest is understood as corresponding to a supposed 'marginal product of capital', we hope that our analysis will prevent the perpetuation of such bad habits.

2. The stationary equilibrium model

The model we shall consider in this section is an adapted version of the stationary models discussed by various neo-Walrasian authors. We can mention, for instance, the 'semi-stationary' model studied by Malinvaud (1953, section IV) and Bliss (1975, chapter 4).[4] It is a recursive production model in which every period is identical to both the previous and the following period.

Assuming there are N different commodities, a vector of outputs $\mathbf{q} \in \Re_+^N$ emerges at each date from the production processes started in the previous period. Part of these outputs, namely a vector $\mathbf{c} \in \Re_+^N$, is consumed by households during the period. The other part $\mathbf{x} = \mathbf{q} - \mathbf{c}$, with $\mathbf{x} \in \Re_+^N$, is made up of the commodities employed as inputs[5] together with the available labour force $\bar{L}$. The employment of $\mathbf{x}$ and $\bar{L}$ will give a vector of outputs $\mathbf{q}$ in the subsequent period.

We shall present two versions of this model, corresponding to two different hypotheses about production technology. We shall first consider the case with linear production methods and secondly consider the case with a differentiable transformation function.

On the consumption side, for the sake of simplicity, we shall simply refer to a standard overlapping generation model. In particular, we assume that $\bar{L}$ identical individuals are born at each date[6] and that they live for two periods: youth and old age. At birth, each individual has no other endowment than a unit of labour services to perform during youth. Accordingly, consumption during the second period of life depends on saving decisions taken in the first period.

In particular, let $\mathbf{p} \in \Re_+^N$ be a stationary price vector and w and r be the wage and the interest rate respectively, an individual i, with $i = 1, 2, \ldots, \bar{L}$, decides the consumption plan so as to maximise her or his intertemporal utility subject to the budget constraint:

$$\mathbf{c}_{i1}^{\mathrm{T}}\, \mathbf{p} + \mathbf{c}_{i2}^{\mathrm{T}}\, \mathbf{p}\, (1 + r)^{-1} \leq w \qquad (1)$$

---

[4] A neo-Walrasian stationary model similar to the one studied here is considered in Bloise and Reichlin (2009).

[5] For the sake of simplicity, all the capital goods considered in our model are circulating. We completely disregard fixed capital goods.

[6] It is not necessary to assume that the individuals born in period t are identical among themselves; rather, what is needed for stationarity is that the generation of individuals born in period t is identical to the generation born in period $t - 1$.

where $\mathbf{c}_{ij} \in \Re_{+}^{N}$, with $j = 1, 2$, is the bundle of commodities consumed by the individual during her or his j-th period of life.

Let us use $\mathbf{c}_{i1}(\mathbf{p},w,r)$ and $\mathbf{c}_{i2}(\mathbf{p},w,r)$ to denote the demand functions for commodities arising from the solution of this constraint maximisation problem. The demand for consumption goods in each period will be the sum of the demand from the young generation born in that period and the one from the generation born in the previous period, who are now in old age:

$$\mathbf{c}(\mathbf{p},w,r) := \Sigma_i\, \mathbf{c}_{i1}(\mathbf{p},w,r) + \Sigma_i\, \mathbf{c}_{i2}(\mathbf{p},w,r). \tag{2}$$

Moreover, since, for clear reasons, there is no saving by elderly people, in each period, the total amount of gross savings corresponds to the difference between income and consumption expenditure of the young generation:

$$s(\mathbf{p},w,r) := \overline{L}\, w - [\Sigma_i\, \mathbf{c}_{i1}(\mathbf{p},w,r)]^{\mathrm{T}}\, \mathbf{p}. \tag{3}$$

2.1 The stationary equilibrium model with linear production methods

Let us use $\mathbf{A}$ and $\boldsymbol{\ell}$ to denote, respectively, the $N\times N$ matrix of commodity input coefficients and the $N\times 1$ vector of labour input coefficients, according to a standard notation we can write: $\mathbf{A} \oplus \boldsymbol{\ell} \rightarrow \mathbf{I}$, where $\mathbf{I}$ is the $N\times N$ identity matrix.

Given the technical coefficients $(\mathbf{A}, \boldsymbol{\ell})$, the labour force of the economy $\overline{L}$ and the functions $\mathbf{c}(\mathbf{p},w,r)$ and $s(\mathbf{p},w,r)$ defined by equations (2) and (3), we are able to set the equilibrium conditions for our stationary model. We can start from the market-clearing conditions for commodities and labour services:

$$[\mathbf{c}(\mathbf{p},w,r)]^{\mathrm{T}} = \mathbf{q}^{\mathrm{T}}(\mathbf{I} - \mathbf{A}) \tag{4}$$

$$\mathbf{q}^{\mathrm{T}}\boldsymbol{\ell} = \overline{L}\,. \tag{5}$$

Then, referring to a competitive equilibrium, we add the zero-profit conditions for the N production activities:

$$\mathbf{p} - \mathbf{A}\mathbf{p}\,(1 + r) - \boldsymbol{\ell}\, w = 0. \tag{6}$$

Finally, since our equilibrium model must be stationary, net capital accumulation must be nil, that is aggregate gross savings must correspond exactly to what is necessary to finance the replacement of the (circulating) capital goods employed:[7]

[7] The tendency toward a zero-net-saving equilibrium is outlined by Hicks in the following terms:

> A fall in the rate of interest would encourage the adoption of longer processes, requiring the use (at any moment) of larger quantities of intermediate products. But since we are in a stationary state, there can be no tendency for the

$$s(\mathbf{p},w,r) - \mathbf{q}^{\mathrm{T}}\mathbf{A}\mathbf{p} = 0. \tag{7}$$

Equations (4)-(7) form a system of $2N + 2$ equations with $2N + 2$ unknown equilibrium variables $(\mathbf{q},\mathbf{p},w,r)$. However, on the one hand, the equations are not independent since, due to Walras's law, one of them is always satisfied when the other $2N + 1$ equations are satisfied. On the other hand, commodity prices and the wage rate must be expressed in terms of a numéraraire commodity. So, at the end, there are only $2N + 1$ independent equations, but the relative prices to determine are just $N - 1$, and then the number of unknowns is $2N + 1$ as well.[8]

2.2 The stationary equilibrium model with a differentiable transformation function

If we continue use $\mathbf{q}$ to denote the vector of commodity outputs and $\mathbf{x}$ and L to denote the inputs of commodities and labour services, then $(\mathbf{q},\mathbf{x},L) \in \Re_+^{2N+1}$ is a production plan. We assume there is a differentiable[9] function $\varphi : \Re_+^{2N+1} \to \Re$ such that $Y := \{(\mathbf{q},\mathbf{x},L) \in \Re_+^{2N+1} : \varphi(\mathbf{q},\mathbf{x},L) = 0\}$ is the set of technically feasible production plans[10]—which we assume to be a convex cone (constant returns to scale).

Given the set Y, the labour force of the economy $\bar{L}$ and the functions $\mathbf{c}(\mathbf{p},w,r)$ and $s(\mathbf{p},w,r)$, we are able to set the equilibrium conditions for our stationary model. We can start, as in the previous case, from the market-clearing conditions for commodities and labour services:

$$\mathbf{c}(\mathbf{p},w,r) = \mathbf{q} - \mathbf{x} \tag{8}$$

$$L = \bar{L}. \tag{9}$$

stock of capital to increase or diminish; constancy of the stock thus gives us one relation between its size and the rate of interest. Also, if entrepreneurs do not desire to increase or diminish their stock, their net borrowing must be nil. If the demand and supply for loans are to be in equilibrium, net saving must therefore also be nil. The rate of interest must therefore be fixed at a level which offers no incentive for net saving or dis-saving. What this level is depends partly upon the propensities to save of the individuals composing the community, partly upon their real incomes—and these depend again upon the size of the stock of intermediate products. (Hicks, 1946, pp. 118-119).

[8] The well-known problems of "re-switching" and "reverse capital deepening" are not addressed in the present paper. However, as has been proved (cf. Fratini, 2013a), if alternative methods of production for the same commodity are allowed for, then re-switching can be a source of equilibrium instability for the model just outlined.

[9] The differentiability of the transformation function $\varphi(\mathbf{q},\mathbf{x},L)$ can be questioned. In fact, since capital goods are specialized inputs, different methods for the production of the same commodity typically employ different sorts of capital goods, so a change in the method in use involves the complete substitution of capital goods of a certain kind with those of a different kind, and not just a substitution 'at the margin'. If instead the method is not changed, then the inputs are complementary with each other. As a result, the marginal product of capital goods of a certain kind is unconceivable in the first case and zero in the second. For a discussion of this point, see Dvoskin and Fratini (2016).

[10] In the case of linear production methods discussed in the previous sub-section, the set of technically feasible production plans is $Y := \{(\mathbf{q},\mathbf{x},L) \in \Re_+^{2N+1} : \mathbf{x} = \mathbf{q}^{\mathrm{T}}\mathbf{A} \text{ and } L = \mathbf{q}^{\mathrm{T}}\boldsymbol{\ell}\}$.

Then, since firms want to adopt the profit maximizing production plan among those in the convex set Y,[11] the following first order conditions must be satisfied in equilibrium:

$$p_n - \lambda \frac{\partial \varphi(\mathbf{q}, \mathbf{x}, L)}{\partial q_n} = 0 \qquad n = 1, 2, \ldots, N \tag{10}$$

$$-p_n\,(1 + r) - \lambda \frac{\partial \varphi(\mathbf{q}, \mathbf{x}, L)}{\partial x_n} = 0 \qquad n = 1, 2, \ldots, N \tag{11}$$

$$-w - \lambda \frac{\partial \varphi(\mathbf{q}, \mathbf{x}, L)}{\partial L} = 0 \tag{12}$$

$$\varphi(\mathbf{q},\mathbf{x},L) = 0 \tag{13}$$

in which λ is the usual Lagrange multiplier. Finally, we have the zero net-accumulation condition:[12]

$$s(\mathbf{p},w,r) - \mathbf{x}^{\mathrm{T}}\mathbf{p} = 0. \tag{14}$$

Equations (8)-(13) form a system of $3N + 4$ equations with $3N + 4$ unknown equilibrium variables $(\mathbf{q},\mathbf{x},L,\lambda,\mathbf{p},w,r)$. As in the previous case, however, there are $3N + 3$ independent equations, due to Walras's law, and the prices must be expressed in terms of a numéraire commodity.

3. Capital as a factor of production

In both the cases considered in the previous section, given: (i) consumers' behaviour described by the functions $\mathbf{c}(\mathbf{p},w,r)$ and $s(\mathbf{p},w,r)$; (ii) the endowment of labour services $\overline{L}$; (iii) the technical conditions of production—represented by either the technical coefficients $(\mathbf{A},\ell)$ or the transformation function $\varphi(\mathbf{q},\mathbf{x},L) = 0$—it has been possible to write a consistent set of stationary equilibrium conditions. Among these equations there are market-clearing conditions for the commodities produced, namely consumption and capital goods, and for labour, the only non-produced input. There is not a market-clearing condition for capital. Actually, there is not a market for capital since it is neither a commodity nor a non-produced input. What we have is just a zero net-accumulation condition: equation (7) in the first system and (14) in the second.

---

[11] The problem consists in finding the production plan which maximizes the profits $\mathbf{q}^{\mathrm{T}}\mathbf{p} - \mathbf{x}^{\mathrm{T}}\mathbf{p}\,(1 + \mathrm{r}) - L\,w$, with the constraint $\varphi(\mathbf{q},\mathbf{x},L) = 0$.

[12] Typically, in the semi-stationary models mentioned at the beginning of this section, the zero net-accumulation is referred to the amount of capital goods per unit of labour, while the total quantity of each capital good employed can grow at the same rate as the labour force employed. For the sake of simplicity, this growth of the labour force is excluded in the present analysis.

This is not the way in which the founders of the marginalist/neoclassical theory of value and distribution conceived their general equilibrium systems. In particular, there are two main differences. First, these authors understood capital as a factor of production on the same footing as labour (and land), so that capital and labour are even substitutable at the margin in the production processes.[13] There must be, therefore, a demand for capital from firms that is similar to and connected with their demand for labour. Second, the zero net-accumulation condition, which characterizes the stationary models, is re-interpreted in terms of a constant stock of capital available $\overline{K}$. The analogy between capital and labour, whose availability is $\overline{L}$, is then completed.

The basic idea behind the given endowment of capital $\overline{K}$ is well known. Each individual, and accordingly the economy, is endowed with a stock of existing capital goods, which is a legacy of the past. Since capital accumulation is assumed to be a very slow and gradual process, the amount of capital in value terms can be approximately considered as a given magnitude and this makes the model stationary. Nonetheless, since the vector **x** of capital goods employed in equilibrium does not correspond, in general, to the stock inherited from the past, the quantities of the capital goods have to change while their total value remains constant and equal to $\overline{K}$, namely the value of the existing capital goods.[14] As this change is typically a long-run phenomenon, its result is called long-run equilibrium.

Accordingly, in every period, individual endowments are made up of a certain quantity of labour $\overline{L}_i$ and a certain amount of capital $\overline{K}_i$. A flow of net income $\overline{L}_i w + \overline{K}_i r$ springs from these endowments and is used to finance the consumption expenditure of each period, while individual gross savings are $\overline{K}_i$ by assumption. The single-period budget constraint is then:

$$\mathbf{c}_i^{\mathrm{T}}\, \mathbf{p} \leq \overline{L}_i w + \overline{K}_i r. \tag{15}$$

The individual demand function for commodities $\mathbf{c}_i(\mathbf{p},w,r)$ arises from the solution of the single-period utility maximization problem subject to the constraint (15). So the aggregate demand for consumption goods is $\mathbf{c}(\mathbf{p},w,r) := \Sigma_i\, \mathbf{c}_i(\mathbf{p},w,r)$.

Now, given the set $Y := \{(\mathbf{q},\mathbf{x},L) \in \Re_+^{2N+1} : \varphi(\mathbf{q},\mathbf{x},L) = 0\}$, the factor endowments $\overline{L}$ and $\overline{K}$, and the function $\mathbf{c}(\mathbf{p},w,r)$, we have the following equilibrium conditions for the modified stationary model:

---

[13] On the conception of capital as a factor of production and its role within the marginalist explanation of income distribution see also Trabucchi (2011) and Dvoskin and Fratini (2016).

[14] As is well known, the given amount of capital in value terms that appears, for instance, in Wicksell's theory involves logical difficulties. In particular, the point is that the value of the existing capital goods cannot be known until the price system is determined and, therefore, if this value plays a role for the determination of the price system, this ends up in circular reasoning. For a closer examination of this point, see Garegnani (1990, pp. 10 and 84), Potestio (1999) and Kurz and Salvadori (2001).

$$\mathbf{c}(\mathbf{p},w,r) = \mathbf{q} - \mathbf{x} \tag{16}$$

$$L = \overline{L} \tag{17}$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{p} = \overline{K} \tag{18}$$

$$p_n - \lambda \frac{\partial \varphi(\mathbf{q},\mathbf{x},L)}{\partial q_n} = 0 \qquad n = 1, 2, \ldots, N \tag{19}$$

$$-p_n\,(1+r) - \lambda \frac{\partial \varphi(\mathbf{q},\mathbf{x},L)}{\partial x_n} = 0 \qquad n = 1, 2, \ldots, N \tag{20}$$

$$-w - \lambda \frac{\partial \varphi(\mathbf{q},\mathbf{x},L)}{\partial L} = 0 \tag{22}$$

$$\varphi(\mathbf{q},\mathbf{x},L) = 0 \tag{23}$$

Needless to say, equilibrium conditions (16)-(22) are very close to equations (8)-(14). There are however two relevant differences. The first concerns the demand function $\mathbf{c}(\mathbf{p},w,r)$, which is based, in this section, on the single-period utility maximization, and not on the intertemporal utility maximization as in the case of the overlapping-generation model discussed in the previous section.

The second and most important difference concerns the replacement of equation (14) with equation (18). In the equilibrium system of section 2.2, equations (14), as is clear, is not a market-clearing condition. It is rather a stationarity condition: gross savings allow only the financing of the replacement of the capital goods employed, without any net capital accumulation. Now, equation (18) is still a stationarity condition, despite the fact that it seems symmetrical to equation (17), namely the equilibrium condition between demand for and supply of labour. Nonetheless, it responds to a slightly different idea of stationary state.[15]

In the model discussed in the previous section, the individual gross savings come from intertemporal utility maximization; they are $s_i(\mathbf{p},w,r) := w - [\mathbf{c}_{i1}(\mathbf{p},w,r)]^{\mathrm{T}}\,\mathbf{p}$. Accordingly, we know for sure due to equation (14), that the equilibrium levels of $\mathbf{p}$, w and r do not involve any inducement to net capital accumulation. Here, by contrast, the amount of individual gross savings

[15] In particular, Petri distinguishes between a 'static' and a 'secular' stationary state, defined in the following terms:

> one should be clear that what was traditionally assumed was a *static* stationary state, i.e. the constancy of the *given* amounts of factors, among which a given amount of 'capital'. This assumption has been occasionally confused […] with the assumption of a *secular* stationary equilibrium, in which tastes are assumed given and the amount of 'capital' is endogenously determined at the level determining such levels of income and of the rate of interest (assuming they exist) as to induce zero net accumulation […]. This second kind of stationary equilibrium does not need a given endowment of 'capital'; and the same holds for the non-stationary steady-growth equilibria determinable when there are no scarce natural resources and there is population growth at a constant rate. (Petri 1999, p. 23)

In this paper, referring to the terms used by other sciences, we call the former a 'quasi-stationary state' and the latter simply a 'stationary state'.

$\overline{K}_i$ is just a given magnitude, i.e. it does not result from any utility maximization, and therefore we cannot exclude that, at the equilibrium levels of **p**, *w* and *r*, a tendency to net capital accumulation will emerge when intertemporal decisions—and not just single-period decisions—are allowed for. The long-run equilibrium can accordingly be considered a *quasi-stationary* state: capital accumulation is not rigidly excluded, but simply neglected.[16]

Finally, since equation (18) looks like a market-clearing condition, it seems to suggest that there is a market for capital as well as a market for labour, corroborating the idea that capital is another factor of production, on the same footing as labour, which accordingly receives a payment that reflects its productive contribution. All that, however, is just the result of the misinterpretation toward which the founders of the neoclassical approach pointed their followers.

4. Walrasian equilibrium

The idea of capital as a factor of production and the rate of interest as the price for its use are rather common among the founders of the neoclassical approach. The most important exception is represented, as is well known, by Walras's theory, in which the endowment of capital $\overline{K}$ does not appear among the data, being replaced by the endowments of the capital goods.

It is also well known[17] that considering the quantities of capital goods employed and reproduced as given magnitudes created a contradiction in Walras's theory of prices and income distribution, at least in the stationary case. In order to elucidate this point, we shall begin with an analysis of Walrasian equilibrium without capital formation, and we shall then pass to the case with capital formation in order to show that, in building the latter, Walras made an improper generalization of the former.

4.1 Walrasian equilibrium without capital formation

Walras did not begin his analysis of economies with production by studying a stationary model. Instead, he first considered a model we nowadays call 'atemporal economy'. In this framework, the

[16] For this reason, Garegnani (1976) and some other scholars maintained that the quasi-stationary equilibrium—unlike the real stationary state of the previous section—cannot be considered as a special and practically irrelevant case. According to these authors, the quasi-stationary equilibrium must be regarded as a neoclassical re-interpretation of Adam Smith's idea of a theoretical position toward which actual prices and distribution variables tend to gravitate.

[17] Cf. in particular Garegnani (1990) pp. 11-19.

economy lasts for just one period, which is the time necessary to organize the transformation of given endowments of inputs into consumption goods.[18]

Let us assume that the endowments of the economy comprise a given quantity of labour $\overline{L}$ and a given vector of capital goods $\overline{\mathbf{k}} \in \Re_+^N$. Since, in the only period in which the economy is considered, the endowments of capital goods are neither produced nor reproduced, there is no actual difference—form the point of view of the theory—between them and natural resources.[19] Accordingly, capital goods, in this framework, do not have prices or costs of production, but there are just prices for their productive services and they are listed in a vector $\boldsymbol{\pi} \in \Re_+^N$.

Given the individual endowments made up of a certain quantity of labour $\overline{L}_i$ and a certain vector of capital goods $\overline{\mathbf{k}}_i$, the consumer's budget constraint is:

$$\mathbf{c}_i^{\mathrm{T}}\, \mathbf{p} \leq \overline{L}_i w + \overline{\mathbf{k}}_i^{\mathrm{T}}\, \boldsymbol{\pi}. \tag{23}$$

The individual demand function for commodities $\mathbf{c}_i(\mathbf{p},w,\boldsymbol{\pi})$ arises from the solution of the utility maximization problem subject to the constraint (23) and the aggregate demand for consumption goods is $\mathbf{c}(\mathbf{p},w,\boldsymbol{\pi}) := \Sigma_i\, \mathbf{c}_i(\mathbf{p},w,\boldsymbol{\pi})$.

Given the technical coefficients $(\mathbf{A}, \boldsymbol{\ell})$, the endowments of the economy $\overline{L}$ and $\overline{\mathbf{k}}$, and the function $\mathbf{c}(\mathbf{p},w,\boldsymbol{\pi})$, we are able to set the equilibrium conditions for the atemporal model. We can start from the market clearing conditions for consumption goods, labour services and capital goods:

$$\mathbf{c}(\mathbf{p},w,\boldsymbol{\pi}) = \mathbf{q} \tag{24}$$

$$\mathbf{q}^{\mathrm{T}}\boldsymbol{\ell} = \overline{L}. \tag{25}$$

$$\mathbf{q}^{\mathrm{T}}\mathbf{A} = \overline{\mathbf{k}}^{\mathrm{T}}. \tag{26}$$

Finally, we need to add the zero-profit condition following from the hypothesis of free competition:

$$\mathbf{p} - \mathbf{A}\,\boldsymbol{\pi} - \boldsymbol{\ell}\, w = 0. \tag{27}$$

Therefore, there are $3N + 1$ equations—but only $3N$ can be independent—and $3N + 1$ unknowns before the adoption of a numéraire commodity. It is worth stressing that the interest rate

---

[18] Among the possible representations of models with production, the atemporal economy is the closest to the pure exchange economy. In a pure exchange economy, in general time is irrelevant because, at this level of theoretical abstraction, it is only the length of the production process that can measure the logical time. The atemporal economy can be regarded as a sort of pure exchange model in which there are agents (firms) that exchange inputs for consumption goods (cf. also Rader 1989).

[19] Despite the fact that Walras also considered natural resources (landed capital) in his analysis of equilibrium without capital formation, for the sake of simplicity we continue to keep them aside.

does not appear among the unknowns to be determined in this model because of its atemporal shape: at least two periods are required for interest to exist.

4.2 Walrasian equilibrium with capital formation

In addressing the study of the case with capital formation, Walras tried to put together parts of analysis belonging to two completely different frameworks, such as the atemporal model outlined here above and the stationary model of section 2.1. On the one hand, he kept the assumption of given initial endowments of labour and capital goods. On the other hand, in his analysis: i) outputs include both consumption goods and 'newly produced capital goods', whose cost is financed by a flow of gross savings; ii) the same system of relative prices applies both to the commodities employed as inputs and to those obtained as outputs. The combination of these heterogeneous elements may entail not only an ambiguity, but even an inconsistency.

Once a temporal dimension is introduced, the capital goods are no longer similar to natural resources,[20] namely inputs without a cost of production. In particular, because of the stationarity assumption, the same vector of prices $\mathbf{p}$ refers to both the commodities obtained as outputs and those employed as capital goods. Hence, if r is the rate of interest, that is the rate of return on invested savings, then the prices of services of capital goods $\boldsymbol{\pi}$ must be equal to $\mathbf{p}(1 + r)$.[21]

Accordingly, the individual budget constraint is

$$\mathbf{c}_i^{\mathrm{T}}\,\mathbf{p} + s_i = \overline{L}_i w + \overline{\mathbf{k}}_i^{\mathrm{T}}\,\mathbf{p}(1 + r) \tag{28}$$

where s_i is the 'excess of income over consumption', namely gross savings.[22]

By the usual argument about consumers' utility maximization and aggregation across individual decisions, we get the aggregate demand for consumption goods $\mathbf{c}(\mathbf{p},w,r)$ and the gross saving function $s(\mathbf{p},w,r)$.

---

[20] Wicksell maintained that Walras treated durable capital goods as natural resources in the case with capital formation, too. Since, for the sake of simplicity, we do not include durable capital goods in the models discussed in the present paper, for Wicksell's critique of Walras's theory of capital we refer the reader to Imperia, Maffeo and Ravagnani (2014).

[21] Walras put this point in a peculiar way. He regarded (or at least he said we can regard) each kind of capital good as an asset with a specific rate of return. For instance, under the hypotheses we are adopting, the (gross) rate of return of a capital good n would be π_n/p_n, and it could be different from $1 + r$. According to this view, the kind and quantity of the capital goods produced would be decided by savers, because they decide the assets in which they invest their savings. Therefore, a further adjustment is needed in order to make the capital goods purchased by savers equal to the capital goods that firms wish to employ. This adjustment is driven by variations in the prices of services of capital goods $\boldsymbol{\pi}$—which, in turn, involve variations in the overall price system—and is completed when π_n/p_n, $= 1 + r$, for every $n = 1, 2, \ldots, N$. Cf. Walras 1977, pp. 267-272.

[22] Walras only used the word 'savings' for net savings, while gross savings were called 'excess of income over consumption' (cf. Walras 1977, pp. 273-274).

The system of equilibrium conditions can now be written. Following Walras, there are market clearing conditions for produced commodities, labour services and the productive services of available capital goods, whose endowments are given:

$$\mathbf{c}(\mathbf{p},w,r) = \mathbf{q} - \mathbf{x} \tag{29}$$

$$\mathbf{q}^{\mathrm{T}}\boldsymbol{\ell} = \bar{L} \tag{30}$$

$$\mathbf{q}^{\mathrm{T}}\mathbf{A} = \bar{\mathbf{k}}^{\mathrm{T}} \tag{31}$$

then there are the zero-profit conditions for the production activities:

$$\mathbf{p} - \mathbf{Ap}\,(1 + r) - \boldsymbol{\ell}\, w = 0. \tag{32}$$

and, finally, the equality between gross savings and the value of the newly produced capital goods:

$$s(\mathbf{p},w,r) - \mathbf{x}^{\mathrm{T}}\mathbf{p} = 0. \tag{33}$$

Before a numéraire is adopted, there are $3N + 2$ unknowns in the system (29)-(33), which has $3N + 2$ equations, but at least one of them is not independent from the others.

The discussion of the system must start from equation (33), which is not properly, at least at a first sight, a zero net-accumulation condition. In fact, Walras—who was probably aware[23] of the problems we are discussing—tried to depart from the stationarity hypothesis and refer his analysis to a 'progressive economy' in which there is net capital accumulation (cf. Walras 1977, pp. 269 and 276). However, he was forced to keep an ambiguous position. He wrote that

> equilibrium in capital formation […] will be established *effectively* by reciprocal exchange between savings to be accumulated and new capital goods to be supplied *within a given period of time*, during which *no change in the data is allowed.* Although the economy is becoming *progressive*, it remains [for the time being] *static* because of the fact that the new capital goods play no part in the economy until later in a period subsequent to the one under consideration (pp. 282-3, emphasis in the original).

Walras's ambiguity emerges rather clearly from the passage above: "the economy is becoming progressive", but "it remains static", in particular "no change in the data is allowed".[24]

As a matter of fact, in Walras's analysis, agents' decisions—including those about the production of new capital goods—are based on the assumption that the same system of relative prices remains stationary period after period. Hence, as Walras himself stressed, no change in the

---

[23] For Walras's perception of the difficulties that can arise because of the given vector of endowments of capital goods in the economy with capital formation, see also Petri (2016).

[24] Walras's ambiguity has opened the door to different interpretations of his model with capital formation: some scholars maintain that Walras's one is a temporary equilibrium model, other scholars claim he referred to a long-run equilibrium model as well as all the other economists of his time. On this controversy, see Dvoskin and Lazzarini (2013).

data on which the prices are determined is allowed, and these data, according to his theory, include the available quantities of the capital goods. Therefore, as in the neo-Walrasian case discussed in section 2, in order to have stationary relative prices, the amount of gross savings must correspond exactly to what is necessary for the replacement of the capital goods in use. In other words, the vector of produced capital goods $\mathbf{x}$ must equal the vector of capital goods employed $\bar{\mathbf{k}}$. Accordingly, N unknowns disappear from the system, which becomes over-determined.[25]

The over-determinacy of the Walrasian system under the assumption of zero net accumulation is sufficient[26] to prove the incompatibility of a given vector of endowments of capital goods with the determination of stationary relative prices.

If, instead, the vector of capital goods produced $\mathbf{x}$ is allowed, *ceteris paribus*, to be different from $\bar{\mathbf{k}}$, then the relative prices of the commodities cannot stay constant. In this case, agents that take their decisions on the assumption that prices will stay stationary are surely making mistakes. Accordingly, the model should have a different structure. Either by the introduction of agents expectations about future prices, as in the temporary equilibrium models, or assuming that the prices of commodities delivered at different dates are determined simultaneously, as in the Arrow-Debreu equilibrium.[27] But these possibilities are not particularly important here, because we are focussing our attention on neoclassical theories of stationary relative prices.

In conclusion, the stationary model is recursive by nature: each period is identical to both the preceding and the subsequent one. An initial period is logically inconceivable in this framework (as well as a final period). Therefore, the idea of initial endowment of inputs is totally extraneous to the stationary model. Of course, non-producible resources are available in given amounts, but they are not properly *initial* endowments.

5. Concluding remarks

In the present paper we have used the stationary neo-Walrasian model as a benchmark for the analysis of the early version of neoclassical theory of value and distribution. In particular, we have

[25] On this point, see also Napoleoni (1965, p. 114) and Garegnani (2008, p. 375, footnote 21).

[26] For a complete and in-depth discussion of the problems arising from Walras's system of equations for the economy with capital formation, the reader is referred to Garegnani (2008).

[27] If we drop the assumption of stationarity, then both the market-clearing conditions for initial endowments and the zero-profit conditions must be written with the inequality sign. In fact, some initially available inputs can be in excess supply even when their price is zero and the processes with nil equilibrium activity levels could entail losses if activated.

The use of inequalities in equilibrium systems aimed at determining stationary relative prices is, instead, rather doubtful.

focused on the discussion of two different versions of the traditional general equilibrium theory: the one in which the existing stock of capital is regarded as a given amount of value—as in Wicksell's theory—and Walras's approach, in which it is instead understood as an endowment of capital goods. The comparison of these theories with the neo-Walrasian model has allowed us to shed further light on the issues concerning the conception of capital as a factor of production and the rate of interest as the price for its use. Therefore, on the basis of the analysis developed in the previous sections, a number of remarks can be made.

First, we can start by stressing that in the neo-Walrasian model presented in section 2, the market for capital does not exist. There can be markets just for outputs (consumption goods, tools and raw materials) and original inputs (labour and natural resources). Capital belongs neither to the first group nor to the second, being simply an amount of value needed to finance the costs that firms bear in advance of the attainment of revenues. The amount of value required to finance the production of the capital goods in use appears, instead, in a zero net-accumulation condition that must be imposed in order to keep the system stationary.

Second, contrarily to what some scholars have maintained,[28] this zero net-accumulation condition is not missing in the traditional model with a given stock of capital (section 3). However, it is modified so as to appear similar to a market clearing condition. In particular, it is presented as the market clearing condition of a factor of production: capital. So that, accordingly, the rate of interest is regarded as the price for the use of this particular factor.[29]

Third, once the zero net-accumulation condition is reformulated as the equality between the given stock of 'existing capital' and the value of capital goods employed by firms, capital accumulation is not rigidly excluded, but simply ignored. In fact, savings correspond to the value of the existing capital by assumption and not as a result of households' utility optimization. However, as we know, this way of conceiving the zero net-accumulation condition leads the neoclassical theory of value to a circular reasoning.[30]

Fourth, in his analysis of the model with capital formation (section 4), Walras tried to keep together two different and incompatible elements: the given initial endowment of capital goods that belongs to the atemporal setting—i.e. Walras's equilibrium without capital formation—and the stationarity of relative prices. As a result, Walras's system of equations is unable to determine its unknowns, at least in the case of a stationary economy.

[28] We refer, for instance, to Hirshleifer (1967), Sandelin (1980), Negishi (1982), Samuelson (1982) and Malinvaud (2003).
[29] Needless to say that, in so doing, the neoclassical theory provided a justification for capital income that contrasted the potion of Ricardo and Marx, who instead understood capital income (profits) as a residuum or a surplus-value, coming from the difference between the value of the output and its cost of production.
[30] See also footnote 14.

Finally, the analysis put forward in this paper contributes to showing that the conception of capital as a factor of production—available in a given quantity, symmetrical and substitutable (at the margin) with labour or land—is untenable. We already know from the 'two-Cambridges debate' that there is no convincing way of characterizing different production processes as involving a greater (or lower) capital intensity. We can now add that the alleged market-clearing condition for capital that appeared in the traditional general equilibrium system, and which seems similar to the market-clearing condition on the labour market, actually has a different meaning: it is a stationarity condition. Therefore, since it is crystal clear that capital is not a factor of production—as stated also by authoritative neoclassical scholars as Samuelson (1966) and Hahn (1975)—it would be great progress for this science if economists could get rid of the idea of the rate of interest as the price for the use of this factor, or even of its equality with an alleged 'marginal product of capital'.

References

Bliss, C. J. 1975. *Capital Theory and Distribution of Income*. Amsterdam/New York: North-Holland/Elsevier.

Bloise, G. and Reichlin, P. 2009. 'An obtrusive remark on capital and comparative statics.' *Metroeconomica*, 60(1), 54-76.

Dvoskin, A. and Fratini, S.M. 2016. 'On the Samuelson–Etula Master Function and the capital controversy.' *The European Journal of the History of Economic Thought*, 23(6): 1032-1058.

Dvoskin, A and Lazzarini, A. 2013. 'On Walras's concept of equilibrium.' *Review of Political Economy*, 25(1), 117-138

Fratini, S. M. 2007. 'Reswitching of techniques in an intertemporal equilibrium model with overlapping generations.' *Contributions to Political Economy*, 26(1): 43-59.

Fratini, S.M. 2013a. 'Real Wicksell effect, demand for capital and stability.' *Metroeconomica* 64(2): 346-360.

Fratini, S. M. 2013b. 'Malinvaud on Wicksell's legacy to capital theory: Some critical remarks.' In: *Sraffa and the Reconstruction of Economic Theory*, edited by E. S. Levrero, A. Palumbo and A. Stirati. New York: Palgrave-Macmillan.

Garegnani, P. 1960. *Il Capitale nelle Teorie della Distribuzione*. Milano: Giuffré.

Garegnani, P. 1976. 'On a change in the notion of equilibrium in recent work on value and distribution.' In: *Essays in Modern Capital Theory*, edited by M. Brown, K. Sato and P. Zarembka. Amsterdam: North-Holland.

Garegnani, P. 1990. ‘Quantity of capital.’ In: *Capital Theory*, edited by J. Eatwell, M. Millgate and P. Newman. London: Macmillan.

Garegnani, P. 2008. ‘On Walras's theory of capital (Provisional draft 1962).’ *Journal of the History of Economic Thought*, 30(1): 367-384.

Hahn, F.H. 1975. ‘Revival of political economy: the wrong issues and the wrong argument.’ *Economic Record*, 51(3): 360-364.

Hirshleifer, J. 1967. ‘A note on Böhm-Bawerk/Wicksell theory of interest.’ *Review of Economic Studies*, 34(2): 191-9.

Imperia, A., Maffeo, V. and Ravagnani, F. 2014. ‘Wicksell on Walras’s early treatment of capital and interest.’ Mimeo.

Kompas, T. 1992. *Studies in the History of Long-Run Equilibrium Theory*. Manchester and New York: Manchester University Press.

Kurz, H.D. 2000. ‘Wicksell and the problem of the “missing” equation.’ *History of Political Economy*, 32(4): 765-88.

Kurz, H.D. and Salvadori, N. 2001. ‘The aggregate neoclassical theory of distribution and the concept of a given value of capital: A reply.’ *Structural Change and Economic Dynamics* 12(4): 479-485.

Malinvaud, E. 1953. ‘Capital accumulation and efficient allocation of resources.’ *Econometrica*, 21(2): 233-268.

Malinvaud, E. 2003. ‘The legacy of Knut Wicksell to capital theory.’ *Scandinavian Journal of Economics*, 105: 507-25.

Marshall, A. 1920. *Principles of Political Economy*, 8th ed. London: Macmillan.

Napoleoni, C. 1965. *L’Equilibrio Economico Generale*. Torino: Boringhieri.

Negishi, T. 1982. ‘Wicksell’s missing equation and Bohm-Bawerk’s three causes of interest in a stationary state.’ *Zeitschrift für Nationalokonomie*, 42: 161-74.

Petri, F. 1999. ‘Professor Hahn on the “neo-Ricardian” criticism of neoclassical economics.’ In: *Value, Distribution and Capital*, edited by G. Mongiovi and F. Petri. London: Routledge.

Petri, F. 2016. ‘Walras on capital: Interpretative insights from a review by Bortkiewicz.’ *Contributions to Political Economy*, 35 (1): 23-38.

Potestio, P. 1999. ‘The aggregate neoclassical theory of distribution and the concept of a given value of capital: Toward a more general critique.’ *Structural Change and Economic Dynamics* 10(3–4): 381-394.

Rader, T. 1989. ‘Production as indirect exchange.’ In: *General equilibrium*, edited by J. Eatwell, M. Millgate and P. Newman. London: Macmillan.

Samuelson, L. 1982. 'On Wicksell's missing equation.' *History of Political Economy*, 14(3): 301–7.

Samuelson, P.A. 1966. 'A summing up.' *Quarterly Journal of Economics*, 80(4), 568-583.

Sandelin, B. 1980 'Wicksell's missing equation, the production function, and the Wicksell effect.' *History of Political Economy*, 12(1): 29–40.

Trabucchi, P. 2011. 'Capital as a single magnitude and the orthodox theory of distribution in some writings of the early 1930s.' *Review of Political Economy*, 23: 169-88.

Walras, L. 1977 [1926]. *Elements of Pure Economics or the Theory of Social Wealth*, translated by W. Jaffé. New York: Augustus M. Kelley Publishers.

Wicksell, K. 1967 [1901]. *Lectures on Political Economy*, vol. I. New York: Augustus M. Kelley Publishers.