Maximin and minimax strategies in two-players game with two strategic variables

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21 January 2017

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Abstract

We examine maximin and minimax strategies for players in two-players game with two strategic variables x and p. We consider two patterns of game; one is the x-game in which strategic variables of players are x's, and the other is the p-game in which strategic variables of players are p's. We call two players Players A and B, and will show that the maximin strategy and the minimax strategy in the x-game, and the maximin strategy and the minimax strategy in the p-game are all equivalent for each player. However, the maximin strategy for Player A and that for Player B are not necessarily equivalent, and they are not necessarily equivalent to their Nash equilibrium strategies in the x-game nor the p-game. But, in a special case, where the objective function of Player B is the opposite of the objective function of Player A, the maximin strategy for Player A and that for Player B are equivalent, and they constitute the Nash equilibrium both in the x-game and the p-game.

Keywords: two-players game; two strategic variables; maximin strategy; minimax strategy

JEL Classification: C72; D43.

1 Introduction

We examine maximin and minimax strategies for players in two-players game with two strategic variables. We consider two patterns of game; the x-game in which strategic variables of players are x's, and the p-game in which strategic variables of players are p's. The maximin strategy for a player is its strategy which maximizes its objective

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function that is minimized by a strategy of the other player. The minimax strategy for a player is a strategy of the other player which minimizes its objective function that is maximized by its strategy. We call two players Players A and B, and will show that the maximin strategy and the minimax strategy in the x-game, and the maximin strategy and the minimax strategy in the p-game for each player are all equivalent. However, the maximin strategy (or the minimax strategy) for Player A and that for Player B are not necessarily equivalent (if the game is not symmetric), and they are not necessarily equivalent to their Nash equilibrium strategies in the x-game nor the p-game[3]. But in a special case, where the objective function of Player B is the opposite of the objective function of Player A, the maximin strategy (or the minimax strategy) for Player A and that for Player B are equivalent, and they constitute the Nash equilibrium both in the x-game and the p-game. Thus, in the special case the Nash equilibrium in the x-game and that in the p-game are equivalent. This special case corresponds to relative profit maximization by firms in duopoly with differentiated goods in which two strategic variables are the outputs and the prices.

In Section 5 we consider a mixed game in which one of players chooses p and the other player chooses x as their strategic variables, and show that the maximin and the minimax strategies for each player in the mixed game are equivalent to those in the x-game and the p-game.

2 The model

There are two players, Players A and B. Their strategic variables are denoted by x_A and p_A for Player A, and x_B and p_B for Player B. They are related by the following functions.

$$p_A = f_A(x_A, x_B) \text{ and } p_B = f_B(x_A, x_B). \tag{1}$$

They are continuous, differentiable and invertible. The inverses of them are written as

$$x_A = x_A(p_A, p_B),\ x_B = x_B(p_A, p_B).$$

Differentiating (1) with respect to p_A given p_B yields

$$\frac{\partial f_A}{\partial x_A}\frac{dx_A}{dp_A} + \frac{\partial f_A}{\partial x_B}\frac{dx_B}{dp_A} = 1$$

and

$$\frac{\partial f_B}{\partial x_A}\frac{dx_A}{dp_A} + \frac{\partial f_B}{\partial x_B}\frac{dx_B}{dp_A} = 0.$$

From them we get

[3] If the game is symmetric, the maximin strategy (or the minimax strategy) for Player A and that for Player B are equivalent. But even if the game is symmetric, they are not necessarily equivalent to their Nash equilibrium strategies.

$$\frac{dx_A}{dp_A}=\frac{\frac{\partial f_B}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}} \tag{2}$$

and

$$\frac{dx_B}{dp_A}=-\frac{\frac{\partial f_B}{\partial x_A}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}}. \tag{3}$$

Symmetrically,

$$\frac{dx_B}{dp_B}=\frac{\frac{\partial f_A}{\partial x_A}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}} \tag{4}$$

and

$$\frac{dx_A}{dp_B}=-\frac{\frac{\partial f_A}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}}. \tag{5}$$

We assume

$$\frac{\partial f_A}{\partial x_A}\neq 0, \frac{\partial f_B}{\partial x_B}\neq 0, \frac{\partial f_A}{\partial x_B}\neq 0, \frac{\partial f_B}{\partial x_A}\neq 0 \text{ and } \frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}\neq 0. \tag{6}$$

The objective functions of Players A and B are

$$\pi_A(x_A,x_B) \text{ and } \pi_B(x_A,x_B).$$

They are continuous and differentiable. We consider two patterns of game, the x-game and the p-game. In the x-game strategic variables of the Players are x_A and x_B; in the p-game their strategic variables are p_A and p_B. We do not consider simple maximization of their objective functions. Instead we investigate maximin strategies and minimax strategies for the Players.

3 Maximin and minimax strategies

3.1 *x*-game

3.1.1 Maximin strategy

First consider the condition for minimization of π_A with respect to x_B. It is

$$\frac{\partial \pi_A}{\partial x_B} = 0. \tag{7}$$

Depending on the value of x_A we get the value of x_B which satisfies (7). Denote it by $x_B(x_A)$. From (7)

$$\frac{dx_B(x_A)}{dx_A} = -\frac{\frac{\partial^2 \pi_A}{\partial x_A \partial x_B}}{\frac{\partial^2 \pi_A}{\partial x_B^2}}.$$

We assume that it is not zero. The maximin strategy for Player A is its strategy which maximizes $\pi_A(x_A, x_B(x_A))$. The condition for maximization of $\pi_A(x_A, x_B(x_A))$ with respect to x_A is

$$\frac{\partial \pi_A}{\partial x_A} + \frac{\partial \pi_A}{\partial x_B}\frac{dx_B(x_A)}{dx_A} = 0.$$

By (7) it is reduced to

$$\frac{\partial \pi_A}{\partial x_A} = 0.$$

Thus, the conditions for the maximin strategy for Player A are

$$\frac{\partial \pi_A}{\partial x_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial x_B} = 0. \tag{8}$$

3.1.2 Minimax strategy

Consider the condition for maximization of π_A with respect to x_A. It is

$$\frac{\partial \pi_A}{\partial x_A} = 0. \tag{9}$$

Depending on the value of x_B we get the value of x_A which satisfies (9). Denote it by $x_A(x_B)$. From (9) we obtain

$$\frac{dx_A(x_B)}{dx_B} = -\frac{\frac{\partial^2 \pi_A}{\partial x_B \partial x_A}}{\frac{\partial^2 \pi_A}{\partial x_A^2}}.$$

We assume that it is not zero. The minimax strategy for Player A is a strategy of Player B which minimizes $\pi_A(x_A(x_B), x_B)$. The condition for minimization of $\pi_A(x_A(x_B), x_B)$ with respect to x_B is

$$\frac{\partial \pi_A}{\partial x_A}\frac{dx_A(x_B)}{dx_B} + \frac{\partial \pi_A}{\partial x_B} = 0.$$

By (9) it is reduced to

$$\frac{\partial \pi_A}{\partial x_B} = 0.$$

Thus, the conditions for the minimax strategy for Player A are

$$\frac{\partial \pi_A}{\partial x_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial x_B} = 0.$$

They are the same as conditions in (8). Similarly, we can show that the conditions for the maximin strategy and the minimax strategy for Player B are

$$\frac{\partial \pi_B}{\partial x_B} = 0 \text{ and } \frac{\partial \pi_B}{\partial x_A} = 0. \tag{10}$$

3.2 *p*-game

The objective functions of Players A and B in the p-game are written as follows.

$$\pi_A(x_A(p_A, p_B), x_B(p_A, p_B)) \text{ and } \pi_B(x_A(p_A, p_B), x_B(p_A, p_B)).$$

We can write them as

$$\pi_A(p_A, p_B) \text{ and } \pi_B(p_A, p_B)$$

because $\pi_A(x_A(p_A, p_B), x_B(p_A, p_B))$ and $\pi_B(x_A(p_A, p_B), x_B(p_A, p_B))$ are functions of p_A and p_B. Interchanging x_A and x_B by p_A and p_B in the arguments in the previous subsection, we can show that the conditions for the maximin strategy and the minimax strategy for Player A in the p-game are

$$\frac{\partial \pi_A}{\partial p_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial p_B} = 0. \tag{11}$$

We can rewrite them as follows.

$$\frac{\partial \pi_A}{\partial x_A}\frac{dx_A}{dp_A} + \frac{\partial \pi_A}{\partial x_B}\frac{dx_B}{dp_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial x_A}\frac{dx_A}{dp_B} + \frac{\partial \pi_A}{\partial x_B}\frac{dx_B}{dp_B} = 0.$$

By (2), (3), (4) and (5), and the assumptions in (6), they are further rewritten as

$$\frac{\partial \pi_A}{\partial x_A}\frac{\partial f_B}{\partial x_B} - \frac{\partial \pi_A}{\partial x_B}\frac{\partial f_B}{\partial x_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial x_A}\frac{\partial f_A}{\partial x_B} - \frac{\partial \pi_A}{\partial x_B}\frac{\partial f_A}{\partial x_A} = 0.$$

Again by the assumptions in (6), we obtain

$$\frac{\partial \pi_A}{\partial x_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial x_B} = 0.$$

They are the same as conditions in (8).

The conditions for the maximin strategy and the minimax strategy for Player B in the p-game are

$$\frac{\partial \pi_B}{\partial p_B} = 0 \text{ and } \frac{\partial \pi_B}{\partial p_A} = 0.$$

They are rewritten as

$$\frac{\partial \pi_B}{\partial x_B}\frac{dx_B}{dp_B}+\frac{\partial \pi_B}{\partial x_A}\frac{dx_A}{dp_B}=0 \text{ and } \frac{\partial \pi_B}{\partial x_B}\frac{dx_B}{dp_A}+\frac{\partial \pi_B}{\partial x_A}\frac{dx_A}{dp_A}=0.$$

By (2), (3), (4) and (5), and the assumptions in (6), they are further rewritten as

$$\frac{\partial \pi_B}{\partial x_B}\frac{\partial f_A}{\partial x_A}-\frac{\partial \pi_B}{\partial x_A}\frac{\partial f_A}{\partial x_B}=0 \text{ and } \frac{\partial \pi_B}{\partial x_B}\frac{\partial f_B}{\partial x_A}-\frac{\partial \pi_B}{\partial x_A}\frac{\partial f_B}{\partial x_B}=0.$$

Again by the assumptions in (6), we obtain

$$\frac{\partial \pi_B}{\partial x_A}=0 \text{ and } \frac{\partial \pi_B}{\partial x_B}=0.$$

They are the same as conditions in (10). We have proved the following proposition.

Proposition 1

1. The maximin strategy and the minimax strategy in the x-game, and the maximin strategy and the minimax strategy in the p-game for Player A are all equivalent.

2. The maximin strategy and the minimax strategy in the x-game, and the maximin strategy and the minimax strategy in the p-game for Player B are all equivalent.

4 Special case

The results in the previous section do not imply that the maximin strategy (or the minimax strategy) for Player A and that for Player B are equivalent (if the game is not symmetric), and they are equivalent to their Nash equilibrium strategies in the x-game or the p-game. But in a special case the maximin strategy (or the minimax strategy) for Player A and that for Player B are equivalent, and they constitute the Nash equilibrium both in the x-game and the p-game.

The conditions for the maximin strategy and the minimax strategy for Player A are

$$\frac{\partial \pi_A}{\partial x_A}=0 \text{ and } \frac{\partial \pi_A}{\partial x_B}=0. \tag{8}$$

Those for Player B are

$$\frac{\partial \pi_B}{\partial x_B}=0 \text{ and } \frac{\partial \pi_B}{\partial x_A}=0. \tag{10}$$

(8) and (10) are not necessarily equivalent. The conditions for Nash equilibrium in the x-game are

$$\frac{\partial \pi_A}{\partial x_A}=0 \text{ and } \frac{\partial \pi_B}{\partial x_B}=0. \tag{12}$$

(8) and (12) are not necessarily equivalent.

The conditions for Nash equilibrium in the p-game are

$$\frac{\partial \pi_A}{\partial p_A}=0 \text{ and } \frac{\partial \pi_B}{\partial p_B}=0. \tag{13}$$

(11) and (13) are not necessarily equivalent.

However, in a special case those conditions are all equivalent. We assume

$$\pi_A + \pi_B = 0, \text{ or } \pi_B = -\pi_A. \quad (14)$$

Then, (10) is rewritten as

$$\frac{\partial \pi_A}{\partial x_B} = 0 \text{ and } \frac{\partial \pi_A}{\partial x_A} = 0. \quad (15)$$

They are equivalent to (8). Therefore, the maximin strategy and the minimax strategy for Player A and those for Player B are equivalent. $\frac{\partial \pi_B}{\partial x_A} = 0$ and $\frac{\partial \pi_B}{\partial x_B} = 0$ in (10) mean, respectively, minimization of π_B with respect to x_A and maximization of π_B with respect to x_B. On the other hand, $\frac{\partial \pi_A}{\partial x_A} = 0$ and $\frac{\partial \pi_A}{\partial x_B} = 0$ in (8) and (15) mean, respectively, maximization of π_A with respect to x_A and minimization of π_A with respect to x_B.

In the special case (12) is rewritten as

$$\frac{\partial \pi_A}{\partial x_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial x_B} = 0. \quad (16)$$

(16) and (8) are equivalent. Therefore, the maximin strategy (Player A's strategy) and the minimax strategy (Player B's strategy) for Player A constitute the Nash equilibrium of the x-game. $\frac{\partial \pi_B}{\partial x_B} = 0$ in (12) means maximization of π_B with respect to x_B. On the other hand, $\frac{\partial \pi_A}{\partial x_B} = 0$ in (16) means minimization of π_A with respect to x_B.

(13) is rewritten as

$$\frac{\partial \pi_A}{\partial p_A} = 0 \text{ and } \frac{\partial \pi_A}{\partial p_B} = 0. \quad (17)$$

(17) and (11) are equivalent. Therefore, the maximin strategy (Player A's strategy) and the minimax strategy (Player B's strategy) for Player A in the p-game constitute the Nash equilibrium of the p-game. Since the maximin strategy and the minimax strategy for Player A in the x-game and those in the p-game are equivalent, the Nash equilibrium of the x-game and that of the p-game are equivalent.

Summarizing the results, we get the following proposition.

Proposition 2 *In the special case in which (14) is satisfied:*

1. *The maximin strategy and the minimax strategy in the x-game and the p-game for Player A and the maximin strategy and the minimax strategy in the x-game and the p-game for Player B are equivalent.*
2. *These maximin and minimax strategies constitute the Nash equilibrium both in the x-game and the p-game.*

This special case corresponds to relative profit maximization by firms in duopoly with differentiated goods in which two strategic variables are the outputs and the prices[4]. Let $\bar{\pi}_A$ and $\bar{\pi}_B$ be the absolute profits of Players A and B, and denote their relative profits by π_A and π_B. Then,

$$\pi_A = \bar{\pi}_A - \bar{\pi}_B \text{ and } \pi_B = \bar{\pi}_B - \bar{\pi}_A.$$

From them we can see

$$\pi_B = -\pi_A.$$

5 Mixed game

We consider a case where Player A's strategic variable is p_A, and that of Player B is x_B. Differentiating (1) with respect to p_A given x_B yields

$$\frac{\partial f_A}{\partial x_A}\frac{dx_A}{dp_A} = 1$$

and

$$\frac{\partial f_B}{\partial x_A}\frac{dx_A}{dp_A} = \frac{dp_B}{dp_A}.$$

Differentiating (1) with respect to x_B given p_A yields

$$\frac{\partial f_A}{\partial x_A}\frac{dx_A}{dx_B} + \frac{\partial f_A}{\partial x_B} = 0$$

and

$$\frac{\partial f_B}{\partial x_A}\frac{dx_A}{dx_B} + \frac{\partial f_B}{\partial x_B} = \frac{dp_B}{dx_B}.$$

From them we obtain

$$\frac{dx_A}{dp_A} = \frac{1}{\frac{\partial f_A}{\partial x_A}}, \frac{dp_B}{dp_A} = \frac{\frac{\partial f_B}{\partial x_A}}{\frac{\partial f_A}{\partial x_A}},$$

$$\frac{dx_A}{dx_B} = -\frac{\frac{\partial f_A}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}} \text{ and } \frac{dp_B}{dx_B} = \frac{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B} - \frac{\partial f_B}{\partial x_A}\frac{\partial f_A}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}}.$$

[4] About relative profit maximization under imperfect competition, please see Matsumura, Matsushima and Cato(2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo(1997).

We assume $\frac{dx_A}{dp_A}\neq 0$ and $\frac{\partial f_A}{\partial x_B}\neq 0$, and so $\frac{dx_A}{dx_B}\neq 0$.

We write the objective functions of Players A and B as follows.

$$\varphi_A(p_A,x_B)=\pi_A(x_A(p_A,p_B),x_B) \text{ and } \varphi_B(p_A,x_B)=\pi_B(x_A(p_A,p_B),x_B).$$

Then,

$$\begin{cases} \frac{\partial\varphi_A}{\partial p_A}=\frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dp_A}, \\ \frac{\partial\varphi_A}{\partial x_B}=\frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_A}{\partial x_B}, \\ \frac{\partial\varphi_B}{\partial p_A}=\frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dp_A}, \\ \frac{\partial\varphi_B}{\partial x_B}=\frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_B}{\partial x_B}. \end{cases} \tag{18}$$

By similar ways to the arguments in Section 3, we can show that the conditions for the maximin strategy and the conditions for the minimax strategy for Player A are equivalent, and they are

$$\frac{\partial\varphi_A}{\partial p_A}=0 \text{ and } \frac{\partial\varphi_A}{\partial x_B}=0. \tag{19}$$

The conditions for the maximin strategy and the minimax strategy for Player B are

$$\frac{\partial\varphi_B}{\partial p_A}=0 \text{ and } \frac{\partial\varphi_B}{\partial x_B}=0. \tag{20}$$

By (18), (19) is rewritten as

$$\frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dp_A}=0 \text{ and } \frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_A}{\partial x_B}=0.$$

Similarly, (20) is rewritten as follows.

$$\frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dp_A}=0 \text{ and } \frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_B}{\partial x_B}=0.$$

By the assumptions $\frac{dx_A}{dp_A}\neq 0$ and $\frac{dx_A}{dx_B}\neq 0$, then we obtain

$$\frac{\partial\pi_A}{\partial x_A}=0 \text{ and } \frac{\partial\pi_A}{\partial x_B}=0,$$

and

$$\frac{\partial\pi_B}{\partial x_A}=0 \text{ and } \frac{\partial\pi_B}{\partial x_B}=0.$$

They are the same as the conditions for the maximin and minimax strategies for Players A and B in the x-game. We have shown the following result.

Proposition 3 *The maximin strategy and the minimax strategy for each player in the mixed game are equivalent to those in the* x *-game and the* p *-game.*

6 Concluding Remark

We have analyzed maximin and minimax strategies in two-players game with two strategic variables. We assumed differentiability of objective functions of players. In the future research we want to extend the arguments of this paper to a case where objective functions of players are not assumed to be differentiable[5] and to a case of symmetric game with more than two players. In an asymmetric multi-person game with two strategic variables the equivalence results of this paper do not hold.

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[5] One attempt along this line is Satoh and Tanaka (2016).