Simplifying the Kohlberg Criterion on the Nucleolus: A Disproof by Oneself

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Nguyen (2016) claimed that he has developed a simplifying set of the Kohlberg criteria that involves checking the balancedness of at most $(n-1)$ sets of coalitions. This claim is not true. Analogous to Nguyen and Thomas (2016), he has incorrectly applied the indirect proof by $(\phi \Rightarrow \perp) \Leftrightarrow \neg\phi$. He established in his purported proofs of the main results that a truth implies a falsehood. This is a wrong statement and such a hypotheses must be rejected (cf. Meinhardt (2015, 2016a,b)). Executing a logical correct interpretation ought immediately lead him to the conclusion that his proposed algorithms are deficient. In particular, he had to detect that the imposed balancedness requirement on the test condition $(\cup_{j=0}^{k} T_j)$ within his proposed methods cannot be appropriate. As a consequence, either a nucleolus with a weakly balanced set will be dismissed by the implemented algorithms or a solution which is not a nucleolus will be selected as a nucleolus. Hence, one cannot expect that one of these algorithms makes a correct selection. The supposed algorithms are wrongly designed and cannot be set in any relation with Kohlberg.

Keywords: Transferable Utility Game, Nucleolus, Balancedness, Kohlberg Criteria, Convexity, Affine Hull, Propositional Logic, Circular Reasoning (circulus in probando), Indirect Proof, Proof by Contradiction.

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1 Introduction

Nguyen (2016) vaunts himself that he has developed a simplifying set of the Kohlberg criteria that involves checking the balancedness of at most $(n-1)$ sets of coalitions. In the course of this note, the inclined reader will find out that this statement is meaningless and has no substance due to misguided logic and an abuse of fundamental concepts from game theory and convex analysis. Analogous to Nguyen and Thomas (2016), he has incorrectly applied the indirect proof by $(\phi \Rightarrow \perp) \Leftrightarrow \neg\phi$. This is done while concluding that a valid premise ϕ, which implies a falsum $\perp$, is a valid statement, hence $(\phi \Rightarrow \perp)$ is true according to the author. However, if ϕ is set to be true, then $\neg\phi$ must be false, which implies that $(\phi \Rightarrow \perp)$ must be wrong either due to the above equivalence relation. Hence, he established in his purported proofs for the Theorems 2 and 5 that a truth implies a falsehood. This is a wrong hypotheses and must therefore be refused (cf. Meinhardt (2015, 2016a,b)). Rather than that the author preferred to prove perverse results though he has at least complete knowledge of the first two mentioned notes.

Performing a logical correct interpretation leads to the immediate and unique conclusion that the algorithms must be erroneous. In particular, we have to realize that the imposed balancedness requirement on the test condition $(\cup_{j=0}^{k} T_j)$ within author's proposed methods cannot be consistent with the criteria of Kohlberg. Implementing an incorrect test condition in his algorithms was a result of a misuse and misinterpretation of those. The author has not recognized that a correct test condition had implied positive coefficients for all coalitions in the set $(\cup_{j=1}^{k} T_j)$ under the condition that $(\cup_{j=0}^{k} T_j)$ is weakly balanced. As a consequence, either a nucleolus with a weakly balanced set $(\cup_{j=0}^{k} T_j)$ but with positive coefficients for all coalitions in $(\cup_{j=1}^{k} T_j)$ will be dismissed by the implemented algorithms or a solution which is not a nucleolus will be selected as a nucleolus. Hence, one cannot expect that one of these algorithms makes a correct selection. The supposed algorithms are complete misspecifications and cannot be granted to have any relation to Kohlberg.

These findings support our view from Meinhardt (2016a) that the author has already misused the criteria of Kohlberg in the article Nguyen and Thomas (2016). In this article, the stopping criterion of all proposed methods is wrong, because it does not satisfy one of those properties. Consequently, none of these algorithms of Nguyen and Thomas (2016) is robust. In particular, that which ought to compute the nucleoli of games with more than 50 players using nested linear programs (LP) must be unstable while computing wrong solutions.

Apart from an incorrect application of the indirect proof and an abuse of the Kohlberg properties, we have to recognize that the author has confound the balancedness concept with that of convexity (see Theorem 4 of Nguyen (2016)). There he argues while relying on a convex combination in connection with the Caratheodory Theorem that if T is a balanced set, one can find a proper subcollection of sets R of T satisfying $1 \leq rank(R) \leq rank(T)$ which is balanced, though it was not shown that the considered set is actual convex. Notice that in this context, he also misused Caratheodory's Theorem. Moreover, that the supposed theorem cannot be correct either, can be immediately observed by the fact that the author has no knowledge about the notion of a minimal balanced set. In general, we have to assert that it is very hard to find any argument in Nguyen (2016) which can be applied in favor of the author or which is not false. Besides, the paper is full of notational typos, which we have reproduced in our quotation.

The remaining part of this note is organized as follows. In Section 2 we present in a first step the misuse of the Kohlberg properties by the author while focusing in the next step of his Algorithms 1, 2, and 4 as well as on their associated theorems in order to establish how defective he has applied the proof by contradiction. Our arguments will be underpinned by some counter-examples. All of this illustrate that the main results of this article are flawed and one cannot guarantee that the proposed methods make a correct choice. We close this note by some final remarks in Section 3.

2 Simplified Kohlberg Criterion: A Disproof

In the sequel, we rely on the same notation and definitions as they can be found in Nguyen (2016). However, for the understanding of our arguments these are not really needed. Only some basics from formal logic is needed, a refresher of this important topic can be found in Meinhardt (2015, 2016a,b).

To start with our analysis, we have to quote in a first step some crucial definitions which are proposed by Nguyen (2016) and that should capture the criteria of Kohlberg in the view of the author.

For any set $\mathcal{C}$ of coalitions, let us define

$$Y(\mathcal{C}) = \{\mathbf{y} \in \mathbb{R}^n \;:\mathbf{y}(S) \geq 0 \;\forall S \in \mathcal{C}, \mathbf{y}(N) = 0\}.$$

Definition 2.1 (Nguyen (2016))**.** *$(Q_0, Q_1, \ldots)$ has Property I if for all $k \geq 1$, the following claim holds: $\mathbf{y} \in Y(\cup_{j=0}^{k} Q_j)$ implies $\mathbf{y}(S) = 0, \forall S \in \cup_{j=0}^{k} Q_j$.*

Definition 2.2 (Nguyen (2016))**.** *$(Q_0, Q_1, \ldots)$ has Property II if for all $k \geq 1, \cup_{j=0}^{k} Q_j$ is balanced.*

Note that the proposed definitions of the Kohlberg properties (cf. Kohlberg (1971)) are not correctly quoted, and can therefore not be credited to him. The correct definition of Property I states that

Definition 2.3. *A collection of sets $\mathcal{C} = (Q_0, Q_1, \ldots, Q_p)$ has Property I if for all $k = 1, \ldots, p$*

$$Y(\cup_{j=0}^{k} Q_j) = \{\mathbf{y} \in \mathbb{R}^n \;:\mathbf{y}(S) \geq 0 \;\forall S \in \cup_{j=0}^{k} Q_j, \mathbf{y}(N) = 0\},$$

implies $\mathbf{y}(S) = 0, \forall S \in \cup_{j=1}^{k} Q_j$.

Whereas the definition of Property II states that

Definition 2.4. *A collection of sets $\mathcal{C} = (Q_0, Q_1, \ldots, Q_p)$ has Property II if for all $k = 1, \ldots, p$ there exists $\omega \in 2^{|\mathcal{C}|}$ with $\omega \geq \mathbf{0}$ s.t.*

$$e(N) = \sum_{j=0}^{k} \omega_j e(Q_j) \qquad \textit{and} \qquad \omega_S > 0 \quad \forall\; S \in \bigcup_{j=1}^{k} Q_j.$$

Thus, we observe from the above definitions of Kohlberg's properties that $(\cup_{j=0}^{k} Q_j)$ is a weakly balanced collections of sets, i.e., some coefficients but not all are zero, whereas the collection $(\cup_{j=1}^{k} Q_j)$ has positive coefficients for all coalitions in $(\cup_{j=1}^{k} Q_j)$.

In the sequel of this note we will observe that the incorrect specification of the Kohlberg criteria has some severe consequences on the imposed test condition of the solution within his Algorithms 1, 2, and 4, that is to say, a nucleolus will be discarded by all of these algorithms. To see this, let us introduce Algorithm 1 below, but let us conduct first some significant considerations.

Comparing the definitions 2.3 and 2.4 of the properties of Kohlberg with the test condition $(\cup_{j=0}^{k} T_j)$ on the solution $\mathbf{x}$ to step 3 within Algorithm 2.1 (see Nguyen (2016, Algorithm 1; p. 4)), we realize that claiming balancedness is too much. This implies that a nucleolus with a weakly balanced set $(\cup_{j=0}^{k} T_j)$ will be dismissed from the proposed algorithm. Hence, the implemented test condition of the author is wrong. Due to Kohlberg's criteria it must be imposed that the collection of sets $(\cup_{j=1}^{k} T_j)$ has positive coefficients for all its coalitions under the constraint that $(\cup_{j=0}^{k} T_j)$ is weakly balanced. The algorithm is wrongly designed and has nothing to do with the criteria of Kohlberg. To summarize: it is neither guaranteed that Algorithm 2.1 makes a correct selection nor it is a real implementation of one of those properties.

Algorithm 2.1: Kohlberg Algorithm for verifying if a solution is the nucleolus of a cooperative game (Nguyen (2016, Algorithm 1; p. 4))

```
Data: Game ⟨ N, v ⟩, solution x.
Result: Conclude if x is the nucleolus.
begin
    Initialization: Set H_0 = {e_N, ∅}, T_0 = {{i}, i = 1, ..., n : x_i = v({i})}, and
    k = 1
    while H_{k-1} ≠ 2^N do
        Set T_k = argmax_{S ∉ H_{k-1}} {v(S) − x(S)};
        if (∪_{j=0}^{k} T_j) is balanced then
            Set H_k = H_{k-1} ∪ T_k, k = k + 1 and continue;
            else
            Stop the algorithm and conclude that x is not the nucleolus.
        end
    end
end
Conclude that x is the nucleolus.
```

Now let us have a closer look on author's so-called simplified Kohlberg algorithm which we have reproduced in Algorithm 2.2 (see Nguyen (2016, Algorithm 2; p. 6)). It is obvious that he made here again the same mistake while introducing the inappropriate test condition $(\cup_{j=0}^{k} T_j)$. As a consequence, it is hardly imaginable that this method will always make a correct selection. This will be scrutinized in the course of our discussion.

Algorithm 2.2: Simplified Kohlberg Algorithm for verifying if a solution is the nucleolus of a cooperative game (Nguyen (2016, Algorithm 2; p. 6))

```
Data: Game ⟨ N, v ⟩, solution x.
Result: Conclude if x is the nucleolus or not.
begin
    Initialization: Set H_0 = {e_N, ∅}, T_0 = {{i}, i = 1, ..., n : x_i = v({i})}, and
    k = 1
    while rank(H_{k-1}) < n do
        Find T_k = argmax_{S ∉ span(H_{k-1})} {v(S) − x(S)};
        if (∪_{j=0}^{k} T_j) is balanced then
            Set H_k = H_{k-1} ∪ T_k, k = k + 1 and continue;
            else
            Stop the algorithm and conclude that x is not the nucleolus.
        end
    end
end
Conclude that x is the nucleolus.
```

We quote now the main Theorem 2 from Nguyen (2016, p. 6) and discuss his proof in order to observe how deficient this author has applied the indirect proof. We cite in the sequel only the essential parts and conclusions of the author, and set his wrong arguments in italic and highlighted them by a red coloring.

Theorem 2.1 (Nguyen (2016)). *Algorithm 2.2 terminates after at most $(n-1)$ iterations and it correctly decides whether a solution is the nucleolus.*

Proof. To prove (...) Thus, $k \leq n-1$; i.e.,there are at most $(n-1)$ iterations.

Proving that the algorithm correctly decides whether a solution is the nucleolus is equivalent to showing that (a) if $\mathbf{x}$ is the nucleolus then the algorithm terminates at step 5, and (b) it terminates at step 4 otherwise.

Part (a): *If $\mathbf{x}$ is the nucleolus*, then T_1 must be balanced as a direct result from the Kohlberg criterion (described in Theorem 1). Thus $(T_1 \cup H_0)$ is balanced and the algorithm goes through to step 3 at $k=1$. *Suppose, as a contradiction that the algorithm goes through to step 4 at some index $k>1$; that is $(T_k \cup H_{k-1})$ is not balanced*. By Lemma 1, there exists (...) *which means x is not the nucleolus. Contradiction!*.

Part (b): If the algorithm went to step 5 and bypassed step 4, then $(T_k \cup H_{k-1})$ is balanced for all k until $rank(H_{k-1}) = n$. Let z be the nucleolus, then by its definition, (...). Given that $rank(H_{k-1}) = n$, we must have $\mathbf{x} = \mathbf{z}$ or $\mathbf{x}$ is the nucleolus. (cf. Nguyen (2016, pp. 6-8)) □

The proof of the Theorem 2.1 (Nguyen (2016, Theorem 2; p. 6)) has to proceed the following cases to get logical correct statements:

(Ha1) If $\mathbf{x}$ is the nucleolus (A), then the algorithm goes every time through step 3 (B);

(Ha2) If $\mathbf{x}$ is not the nucleolus $(\neg A)$, then the algorithm goes at most once through step 4 $(\neg B)$;

in contrast to that what was claimed by the author's guidance of the supposed proof.

The author tries only to establish the first case in his purported proof of Theorem 2.1, which is logically flawed due to an incorrect application of the indirect proof. The author has incorrectly applied $(\phi \Rightarrow \perp) \Leftrightarrow \neg\phi$. To see this, note that the author wants to prove *if* $A \Rightarrow B$ so he relies on the equivalent statement $(A \wedge \neg B \Rightarrow A \wedge \neg A)$ to prove this. Since, if we set $\phi := (A \wedge \neg B)$, and notice that $(A \wedge \neg A)$ is a falsum, i.e., $(A \wedge \neg A) = \perp$ as well as that $\neg\phi := (A \Rightarrow B)$, we have $(A \wedge \neg B \Rightarrow A \wedge \neg A) \equiv (A \Rightarrow B)$. As already mentioned in Meinhardt (2015) he established that a truth $(A \wedge \neg B)$ implies a falsehood $(A \wedge \neg A)$, from which he concludes that $A \Rightarrow B$ follows. This is a wrong statement as it was worked out in Meinhardt (2015, 2016a,b). In fact, the statement $A \Rightarrow B$ is an invalid implication, disproving his own theorem. Obviously, an implication is not a deduction (cf. Meinhardt (2016b)).

When we inspect the purported proof of the author in more details, we realize that he applies exactly this kind of logical incorrect argumentation. He first assumes that $\mathbf{x}$ is the nucleolus, that is to say (A) holds, and then he introduces that (B) is invalid while supposing that the algorithm goes through to step 4 at some $k>1$, hence he sets $(\neg B)$. From these prerequisites he derives after some manipulations of terms a contradiction, namely that $\mathbf{x}$ is not the nucleolus, that is, $(\neg A)$ is satisfied. He obtains his desired falsum $(A \wedge \neg A) = \perp$, from which he concludes that the algorithm passes every time through step 3. Of course, this is a fallacy, since one can never conclude that from a truth something false happens. This is a wrong statement! Consequently, it cannot be guaranteed that even though $\mathbf{x}$ is the nucleolus, the algorithm will pass every time through step 3. This is due to the incorrect application of the Kohlberg criteria. Hence, we infer $A \not\Rightarrow B$ caused by the wrong test condition on the solution that $(\cup_{j=0}^{k} T_j)$ is balanced.

Recall that Property II of Kohlberg is equivalent to condition: Given $Q_0 \subseteq \{S \subseteq N \mid |S| = 1\}$ and for each $k = 1, \ldots, p$, there is a subset q_0^k of Q_0 such that $\big(q_0^k \cup (\cup_{j=1}^k Q_j)\big)$ is balanced.

By the discussion from above we know that this expression could be weakly balanced if $\mathbf{x}$ is the nucleolus. Implying that a nucleolus will be dismissed when $(\cup_{j=0}^k T_j)$ is weakly balanced, but $(\cup_{j=1}^k T_j)$ has positive weights for all its coalitions. To see this, assume that $\mathbf{x}$ is the nucleolus, which has induced the following sets $T_0 = \{\{2\}, \{4\}\}$ and $T_1 = \{\{1,2\}, \{3,4\}\}$. Here, we realize that T_0 is not a balanced set, whereas T_1 it is. We can check that $(T_0 \cup T_1)$ is not a balanced set, however, it exists a proper subset t_0 of T_0 i.e., $(t_0 \subset T_0)$, namely the empty set, such that $(t_0 \cup T_1)$ is a balanced set. Thus, $\omega_S = 0$ for all $S \in T_0$ and $\omega_S = 1/2$ for all $S \in T_1$. Hence, the nucleolus will be discarded, though the collection of sets $T_0 \cup T_1$ is weakly balanced. All of this follows immediately from a logical correct interpretation of the derived implications.

As already indicated, the above supposed proof is not complete. The author has not checked the case whenever $\mathbf{x}$ is not the nucleolus. To see that in this case the Algorithm 2.2 does not correctly decide that a solution $\mathbf{x}$ is not the nucleolus, can be observed from the following four person game:

Table 2.1: Counter Example to Theorem 2.1[a,b,c]

Game	$\{2\}$	$\{4\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
v	2	4	11	16	18	17	14

[a] Nucleolus: $(3,2,5,4)$
[b] Pre-Nucleolus: $(3,2,8,1)$
[c] Remaining coalitions get two.

By this example, the nucleolus is given by $(3,2,5,4)$, and the pre-nucleolus is equal to $(3,2,8,1)$. Now let us use as a solution the pre-nucleolus to demonstrate incorrect choices of the proposed algorithm, and denote this solution as $\mathbf{x}$.

The selected solution $\mathbf{x}$ is a related to the nucleolus, however, it is not the nucleolus. We, therefore, get $T_0 = \{\emptyset\}$, which is not a balanced set, and if the algorithm goes through to iteration $k = 3$, we obtain

$$(\cup_{j=0}^3 T_j) = \{\{2\}, \{4\}, \{1,2\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}.$$

In the next step let us scrutinize whether the set $(\cup_{j=0}^3 T_j)$ is balanced or not. To see this, consider the set $\mathcal{T} = \{\{4\}, \{1,2\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$, this set is balanced but not minimal balanced, since the weight system ω is not unique. We can find, for instance, weights $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{1,1,1,1,1\}/3$ as well as $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{3,4,3,4,4\}/11$. From these weights, we also realize that $\mathcal{T}$ cannot be a convex set. Moreover, this system has rank 4, and we can conclude that $\mathbf{e}(\{2\}) \in span(\mathcal{T})$, which implies balancedness for $\mathcal{T} \cup \{2\} = (\cup_{j=0}^3 T_j)$ (cf. Peleg and Sudhölter (2007, Lemma 6.1.2)). Thus, the collection $(\cup_{j=0}^3 T_j)$ is balanced, and the Algorithm 2.2 goes through to step 3 with $H_3 = (T_3 \cup H_2) = \{\{2\}, \{4\}, \{1,2\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, N\}$. The collection H_3 is balanced as well with $rank(H_3) = 4$. The proposed algorithm stops, and it concludes that $\mathbf{x}$ is the nucleolus, even though it is the pre-nucleolus. Hence, the algorithm makes a wrong choice in contrast to that what was claimed by Theorem 2.1.

What is in this case really astonishing, is the fact, that even though Nguyen was absolutely aware of the work of Meinhardt (2015, 2016a) that he still draws incorrect conclusion from $(A \wedge \neg B \Rightarrow A \wedge \neg A)$. He proved perverted results by a logical incorrect application of the indirect proof though all available information points toward a rejection of the scrutinized hypotheses (Ha1). Executing a logical correct interpretation must immediately lead him to the conclusion that the algorithm is misspecified. In particular, that his test condition $(\cup_{j=0}^{k} T_j)$ is incompatible with Kohlberg's properties.

Apart from the fact that the author is obviously confused by propositional logic, he also confounds the concepts of balancedness and convexity as can be seen next. For this purpose, we cite his Theorem 4 (cf. Nguyen (2016, p. 9)), and analyze some crucial parts of his supposed proof.

Theorem 2.2 (Nguyen (2016))**.** *The following results hold*

(a) If $T \in 2^N$ is a balanced set then there exists $R \subset T$ with $1 \leq rank(R) \leq rank(T)$ that is balanced.

(b) For nonempty $P, Q \in 2^N$ with $P \cup Q$ is a balanced set, there exists a subset $Q' \subset Q$ with $1 \leq rank(Q') \leq rank(Q)$ such that $Q' \cup P$ is balanced.

Proof. (a) Let $T = \{S_1, \ldots, S_m\}$ and let $\mathbf{e}(N) = \alpha_1 \mathbf{e}(S_1) + \ldots + \alpha_m \mathbf{e}(S_m)$, *then* $\frac{1}{\sum_{j=1}^{m} \alpha_j}\mathbf{e}(N)$ *belongs to to the convex combination of* $\mathbf{e}(S_1), \ldots, \mathbf{e}(S_m)$. Applying the *Caratheodory theorem*, there exists a subset $U \subset T$ with $rank(U) = |U| = dim(T)$ such that $\frac{1}{\sum_{S \in U} \alpha_j}\mathbf{e}(N) = \sum_{S \in U} \beta_S \mathbf{e}(S)$.

By removing those coefficients $\beta_S = 0$, we obtain a subset $R \subset U \subset T$ with $rank(R) \leq rank(U)$ that is balanced. Note also that, since $e(N) \neq \mathbf{0}$, there exists at least a coalition S with $\beta_S > 0$. Thus, $1 \leq rank(R) \leq rank(T)$ and R is balanced. Nguyen (2016, p. 9)

□

One immediately recognizes that also this purported proof is full of mistakes. First of all, choose, for instance, $T = \{N\}$ for the player set $N = \{1, 2, 3\}$, then $T \in 2^N$ by a minor abuse of notation, and T is a balanced set. However, we cannot find any subset R of T s.t. R is balanced. Exemplarily, consider $R = \{\{1, 2\}\}$ which is a proper subset of T, but it is not balanced. This holds true for every other proper subset R of T. This contradicts the first statement of Theorem 2.2. Furthermore, even if we allow T to be a collection of sets rather than an element from the power set 2^N as the author does in his supposed proof, we run into difficulties. To see this, take a collection of sets $T = \{S_1, \ldots, S_m\}$ and assume that it is a minimal balanced set. This collection is obviously not an element of 2^N rather than a subset, therefore $T \subset 2^N$. Then one cannot find any subcollection of sets R of T s.t. R is balanced, otherwise the collection T cannot be minimal balanced. A further contradiction to Theorem 2.2. In addition, the author confounds balancedness with convexity. Though the collection of sets is assumed to be balanced, it is, of course, not convex. The author failed to give any indication that this set can be convex. Moreover, if so, we would get $\sum_{j=1}^{m} \alpha_j = 1$, and by the arguments of the author $\mathbf{e}(N)$ would belong to the convex combination of $\mathbf{e}(S_1), \ldots, \mathbf{e}(S_m)$. However, this is impossible if for such a set T, we have $N \notin T$. Thus, for some $T \subset 2^N$, we get $\mathbf{e}(N) \notin conv(\{\mathbf{e}(S_1), \ldots, \mathbf{e}(S_m)\})$. Hence, we realize that Caratheodory's Theorem is not applicable.

Notice that the flawed Theorem 2.2 was used in Algorithm 2.3 (see Nguyen (2016, Algorithm 4; p. 10)) to construct set R_k for steps $k > 1$ as well as in the forthcoming Theorem 2.3. Therefore, it is barely conceivable that the proposed algorithm will always deliver the desired results to implement one of the Kohlberg criteria. Again the implemented test condition is given by $(\cup_{j=0}^{k} T_j)$, which imposes incorrect selections as we have already noticed.

In view of Theorem 2.3 (cf. Nguyen (2016, Theorem 5; p.10)) the author relies again on misguided

Algorithm 2.3: Improved Kohlberg Algorithm for verifying if a solution is the nucleolus of a cooperative game (Nguyen (2016, Algorithm 4; p. 10))

```
  Data: Game ⟨N, v⟩, solution x.
  Result: Conclude if x is the nucleolus or not.
  begin
1     Initialization: Set H_0 = T_0 = e_N, T_0 = {{i}, i = 1,...,n : x_i = v({i})}, and
      k = 1
      while rank(H_{k-1}) < n do
2         Find T_k = argmax_{S ∉ span(H_k)} {v(S) − x(S)};
          if (∪_{j=0}^{k} T_j) is balanced then
3             Set R_k = rep(T_k; H_{k-1}), H_k = H_{k-1} ∪ R_k, k = k + 1 and continue;
          else
4             Stop the algorithm and conclude that x is not the nucleolus.
          end
      end
  end
5 Conclude that x is the nucleolus.
```

logic to conduct the indirect proof as well as on flawed results from Theorem 2.1 and 2.2. As a consequence, this theorem cannot be correct either as we will discover in a little while.

Theorem 2.3 (Nguyen (2016))**.** *Algorithm 2.3 terminates after at most $(n-1)$ iterations and it correctly decides whether a solution is the nucleolus.*

Proof. After each iteration, (...) We use results from Lemma 2 and Theorem 4 for this.
The proof for part (a) is still the same as that for Theorem 2 since the key property used in that proof was to keep H_k balanced. This is summarized as follows. *If* $\mathbf{x}$ *is the nucleolus* then $(T_1 \cup H_0)$ is balanced and the algorithm gets through to step 3 at $k = 1$. *Suppose, on contradiction, that the algorithm gets to step 4 at some* $k > 1$ *with* $(T_k \cup H_{k-1})$ *not balanced* while H_{k-1} is balanced by the construction in step 3 of the previous iteration. Then by Lemma 1, (...) *which means* $\mathbf{x}$ *is not the nucleolus.* Contradiction. Nguyen (2016, pp. 10-11) □

Similar to Theorem 2.1, the proof of the Theorem 2.3 (Nguyen (2016, Theorem 5; p. 10)) has to proceed the following cases to get logical correct statements:

(Hb1) If $\mathbf{x}$ is the nucleolus (A), then the algorithm goes every time through step 3 (B);

(Hb2) If $\mathbf{x}$ is not the nucleolus $(\neg A)$, then the algorithm goes at most once through step 4 $(\neg B)$.

We immediately perceive that the author has again erroneously applied the indirect proof. Thus, he applies $(A \wedge \neg B \Rightarrow A \wedge \neg A)$ while supposing that $\mathbf{x}$ is the nucleolus, that is to say (A) holds, and then he introduces that (B) is invalid while supposing that the algorithm runs through to step 4 at some $k > 1$, hence he sets $(\neg B)$. From this, he derives after some manipulations of terms a contradiction that $\mathbf{x}$ is not the nucleolus. He infers $(\neg A)$. He obtains his desired falsum $(A \wedge \neg A) = \perp$, from which he wrongly concludes that the algorithm passes every time through step 3. In fact he disproved himself. The logical correct interpretation of this statement simply says that it cannot be guaranteed that if $\mathbf{x}$ is the nucleolus, the Algorithm 2.3 runs through to step 3. Therefore, $A \not\Rightarrow B$ and the hypotheses (Hb1) must be dismissed.

Furthermore, an inspection of the elementary arguments of the proof reveals that the supposed proof for Theorem 2.3 cannot be correct by the following reasons: Firstly, the author uses results from Theorem 2.2 (cf. Nguyen (2016, Theorem 4)). However, as have been established in the discussion of the proof after this theorem, these results are not correct either. Hence, the selection of R_k by $rep(T_k; H_{k-1})$ and $H_k = H_{k-1} \cup R_k$ in step 3 of Algorithm 2.3 will not deliver the desired results to implement one of Kohlberg's criteria. Furthermore, he applies the same logical flawed arguments as in proof of Theorem 2.1 (Nguyen (2016, Theorem 2)) in connection with the wrong test condition $(\cup_{j=0}^{k} T_j)$. Again, a nucleolus will be discarded when $(\cup_{j=0}^{k} T_j)$ is weakly balanced, but $(\cup_{j=1}^{k} T_j)$ has positive coefficients for all its sets. However, due to the wrong Theorem 2.2 one can even expect that for a nucleolus the constructed set $(\cup_{j=1}^{k} T_j)$ through step 3 of Algorithm 2.3 is not a balanced set indicating that it will be dismissed. Moreover it is also conceivable that a proposed solution which is not the nucleolus will be selected as a nucleolus either due to a wrong construction of set $(\cup_{j=1}^{k} T_j)$ by Algorithm 2.3 or due to $rank(H_{k^*}) = n$ and a balanced collection of $(\cup_{j=0}^{k^*} T_j)$ whereas some T_j are not balanced.

3 Concluding Remarks

Analogous to Nguyen and Thomas (2016), we have detected in the article of Nguyen (2016) severe deficiencies so that the reported results and algorithms become invalid, and cannot be granted to have any connection with Kohlberg as the author did. First of all, the proofs of his main Theorems 2 and 5 are logically flawed due to an incorrect application of the proof by contradiction. Secondly, the imposed test condition within the supposed algorithms is incorrect, because of a misuse of the Kohlberg properties. And finally, the author confounds the concept of balancedness with that of convexity; has no knowledge on the notion of a minimal balanced set; and he misused Caratheodory's Theorem. All of these faults lead to wrong results and they invalidate the paper.

These facts strongly support our view from Meinhardt (2016a) that the author has already wrongly implemented Kohlberg's criteria in the article Nguyen and Thomas (2016) in which the authors vaunt themselves that they found a method to compute the nucleoli of games with more than 50 players using nested linear programs (LP). This implies that also this article is invalidated.

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