Networks, Frictions, and Price Dispersion

Donna, Javier and Schenone, Pablo and Veramendi, Gregory

The Ohio State University, Arizona State University, Arizona State University

24 February 2017

Networks, Frictions, and Price Dispersion*

Javier Donna — The Ohio State University

Pablo Schenone — Arizona State University

Gregory Veramendi — Arizona State University

First version: January, 2014

This version: February 24, 2017

*Discussions with Jim Albrecht, Matt Backus, David Blau, Hector Chade, Eleanor Dillon, Laura Doval, Domenico Ferraro, John Hatfield, John Kagel, Shachar Kariv, Rasmus Lentz, Dan Levin, Jim Peck, Tiago Pires, Rob Porter, Andrew Rhodes, Alan Sorensen, Steve Tadelis, Rune Vejlin, Susan Vroman, Bruce Weinberg, Huanxing Yang as well as seminar participants at Arizona State University, Cycles, Adjustment, and Policy conference (Aarhus University), the Southwest SaM conference (University of California Riverside), 2014 and 2016 annual SaM Conferences, Mainz Workshop in Labour Economics, the 13th International Industrial Organization conference (Boston), and anonymous referees have greatly benefited this work. All errors are our own.

Abstract

This paper uses networks to study price dispersion in seller-buyer markets where buyers with unit demand interact with multiple, but not all, sellers; and buyers and sellers compete on prices after they meet. Our approach allows for *ex post* indirect competition, where a buyer who is not directly linked with a seller affects the price obtained by that seller. Indirect competition generates the central finding of our paper: price dispersion depends on both the number of links in the network, and how these links are distributed. Networks with very few links can have no price dispersion, while networks with many links can still support significant price dispersion. We present three main theoretical results. First, for any given network we characterize the pairwise stable matchings and the prices that support them. Second, we characterize the set of all graphs where price dispersion is precluded. Third, we use a theorem from Frieze (1985) to show that the graphs where price dispersion is precluded arise asymptotically with probability one in random Poisson networks, even as the probability of each individual link goes to zero. We also provide quantitative results on the finite sample properties of price dispersion in random networks. Finally, we present an application to eBay to show that: (i) a calibration of our model reproduces the price dispersion documented in eBay quite well, and (ii) the amount of price dispersion in eBay would decrease substantially (35-45 percent as measured by the coefficient of variation) in a counterfactual analysis, where we change eBay's network structure so that links are drawn with equal probability for all sellers and buyers.

JEL Codes: L11; J31; J64.

Keywords: Price Dispersion; Frictions; Networks.

Javier Donna
Department of Economics
The Ohio State University
1945 N High St
425 Arps Hall
Columbus, OH 43210
Phone: 614-688-0364
Email: donna.1@osu.edu

Pablo Schenone
Department of Economics
W.P. Carey School of Business
Arizona State University
501 E. Orange Street
Tempe, AZ 85287
Phone: 480-965-5596
Email: pablo.schenone@asu.edu

Gregory Veramendi
Department of Economics
W.P. Carey School of Business
Arizona State University
501 E. Orange Street
Tempe, AZ 85287
Phone: 480-965-0894
Email: gregory.veramendi@asu.edu

1 Introduction

This paper studies price dispersion in seller-buyer markets where buyers with unit demand interact with multiple, but not all, sellers (meetings are many-to-many); and buyers and sellers compete on prices after they meet (competition is *ex post*). Examples include labor markets, where firms interview multiple applicants for a given vacancy and then bargain over wages; eBay auctions, where buyers participate in multiple auctions and prices are determined when the auctions end; and automobile markets, where consumers visit multiple dealerships and negotiate over prices. These markets exhibit significant price dispersion. In labor markets, similar workers are paid different wages (Mortensen 2005); in eBay, identical goods are sold by the same seller at different prices (Einav, Kuchler, Levin, and Sundaresan 2015); and in markets for automobiles, identical automobiles are sold at different prices by the same dealer (Morton, Zettelmeyer, and Silva-Risso 2001). One important channel for generating price dispersion is the presence of frictions. We define frictions as anything that limits opportunities for trade between buyers and sellers in a market (*e.g.* search costs, transaction costs, advertising costs, *etc.*). A number of questions arise: When do frictions lead to price dispersion? In particular, is it necessary to have buyers interact with every seller to preclude price dispersion?

The central finding of this paper is that price dispersion in these markets is determined by both the number of seller-buyer meetings and how these meetings are distributed in the market. The number of meetings is the counterpart to the level of search frictions in a frictional search model. We show how the distribution of meetings determines whether or not price dispersion can occur. An implication is that there exists markets with few meetings that preclude price dispersion, while markets with many meetings can still support significant price dispersion.[1] The absence of price dispersion in markets with few meetings is caused by "indirect competition." To illustrate what indirect competition means, consider the case of two sellers and many buyers, where there is only one common buyer meeting with both sellers. Buyers meeting with one seller *indirectly compete* with buyers meeting with the other seller because the two sellers are connected through the common buyer. Indirect competition results in an interdependence in the prices between these two sellers. Even if buyers do not meet with every seller, indirect competition can equalize the prices in the market. How meetings are distributed determines the extent of the indirect competition and, hence, whether price dispersion can occur.

In this paper we use networks to model seller-buyer markets. Buyers have unit demand, sellers offer one unit of an indivisible homogeneous good, and a buyer can obtain a good from the seller only if the two are linked. On one hand, when all buyers are linked to all sellers, the market is frictionless. On the other hand, whenever there is at least one seller that is not linked to every buyer, the market has frictions. Hence, the level of frictions in the network is

[1]See example 1 in Section 2.

determined by the total number of links in the network. Our theoretical results take networks as exogenous. As we do not model the link formation process in the networks, these results apply to the network resulting from any link formation process (*i.e.* to any realized network). We study pairwise stable matchings in these networks and characterize the prices that sustain them. Pairwise stability is a specification of whom buys from whom and at what prices, such that two properties hold: (1) trades are individually rational, and (2) there are no pairwise Pareto improvements restricted to the network. When we say "restricted to the network," we mean that a buyer can propose an improvement to a seller (or vice versa) only if they are linked in the network.[2]

For any given network, we characterize the pairwise stable matchings and the prices that support them. To do this, we construct a decomposition of the network, which we call an *abstraction* of the network. Abstractions decompose the original network into *cliques*, *i.e.*, fully connected subnetworks of the original network, and the links between these cliques. For any given matching (stable or not), we consider abstractions where matched pairs belong to the same clique. We then make two observations. First, within each clique all buyers pay the same price because cliques are frictionless submarkets of the original. Second, if a buyer in one clique is linked to a seller in another clique, this must mean that the price consistent with pairwise stability in the first clique is weakly lower than the price in the second clique, otherwise the buyer would not have purchased from the seller in its own clique. Our propositions prove that these two observations completely characterize pairwise stable matchings and their supporting prices (see example 2).

We use the characterization result outlined above to characterize the set of all graphs where price dispersion is precluded. To do this, we define two concepts: the Law of One Price, and the Strong Law of One Price. Consider a set of buyers, a set of sellers, and a set of links between buyers and sellers; we call this a graph. Suppose we also consider a valuation profile, that is, a function that assigns a valuation to each buyer and seller. A graph endowed with a valuation profile is called a network. Consider a network with the following property: for any pairwise stable matching, the only prices that make the matching stable are those where all buyers acquiring a good pay the same price. For example, if the valuation of all sellers and buyers is the same, then the network satisfies the Law of One Price. While networks that satisfy the Law of One Price are those where price dispersion is precluded, the driving force behind this result is the valuation profile, rather than it being a property of the underlying graph. Now, suppose that we have a graph with the following property: for all valuation profiles, and for any pairwise stable matching (given the valuations), the only way to make the matching stable is that all buyers acquiring a good pay the same price. We say that this graph satisfies the Strong Law of One Price. We call it "Strong" because in such graphs the Law of One Price holds across all valuation profiles (see example 3). For example,

[2]This is different from Kircher (2009), where the only dimension along which a Pareto improvement can be suggested is through the allocation, not the prices, because prices are fixed *ex ante*.

a graph where every buyer is linked to every seller satisfies the Strong Law of One Price. Theorem 1 characterizes all graphs that satisfy the Strong Law of One Price. A corollary of Theorem 1 is that sparse graphs—that is, graphs with a low proportion of existing links to total possible links—may still satisfy the Strong Law of One Price. This result implies that the level of frictions, represented by the sparsity of a graph, is not the only cause of price dispersion. Price dispersion is also a function of the network structure represented by the graph, and we can characterize all network structures that lead to (and preclude) price dispersion.

Having studied price dispersion for any given network, we look at the price dispersion that arises in the special case of random poisson networks.[3] We study random poisson networks because they are the natural counterpart of the random search model. We show that, as networks become large, the Strong Law of One Price holds asymptotically almost surely even as the probability of each individual link goes to zero. We do this in three steps. First, we use a theorem from Frieze (1985) to show that in a balanced random network (*i.e.* a random network with the same number of agents on both sides) the probability of a Hamiltonian cycle (that is, a cycle that visits each node once, and only once, and ends on the same node as it began) goes to one as the the market grows. Second, we show that for graphs with Hamiltonian cycles, the set of pairwise stable matchings is the set of perfect matchings. That is, matchings where all agents are matched. Lastly, we apply the results from the previous paragraph to show that the Strong Law of One Price holds asymptotically almost surely, even when the number of agents on both sides is different. To understand the relevance of the previous result in finite networks, we perform a numerical analysis and simulate a large number of random poisson networks.[4] We develop a deferred acceptance algorithm for finding pairwise stable matches and the full set of prices that supports them. On one hand, we show that in a network of 10,000 sellers, over 99 percent of the sellers are paid the same price when less than 0.1 percent of the possible links are active. On the other hand, only 5 percent of the sellers are paid the same price when 0.01 percent of the possible links are active. These correspond to the cases where, on average, each buyer has ten links and one link.

Are the network structures discussed above relevant from an empirical standpoint? The online trading platform eBay provides a natural application of our model with *ex post* competition for identical products. One of the selling mechanisms in eBay are competitive auctions. Einav, Kuchler, Levin, and Sundaresan (2015) report substantial price dispersion in auction prices of identical goods sold by the same seller (mean coefficient of variation 10-15 percent). At the same time, links are not distributed uniformly at eBay; the platform maximizes

[3] A random poisson network is one where each link is formed with a given probability, λ, and the formation is independent across links. We also consider networks where each link is formed with a different probability.

[4] From a practical standpoint, numerical analysis is necessary for this problem because solving finite random networks analytically is intractable for all but the simplest networks. We are not the first to rely on numerical methods for analyzing finite random networks. See the discussion in Jackson (2008, Chapter 4, Section 2).

revenue by giving certain listings more prominance in search results than others. In contrast, conditional on a given search query, eBay does not tailor search results based on the characteristics of the buyers. In the quantitative analysis applied to eBay we calibrate our model using the network structure reported in Backus, Podwol, and Schneider (2013) and the search behavior reported by Blake, Nosko, and Tadelis (2016).[5] We then compare the summary statistics (*i.e.* moments) generated by our model and the ones reported by Einav, Kuchler, Levin, and Sundaresan (2015). We show that: (i) our model reproduces the price dispersion observed in eBay quite well (Table 1) and (ii) the amount of price dispersion, as measured by the mean coefficient of variation, would decrease substantially (35-45 percent) if links are drawn with equal probability for all sellers and buyers (this is done in a counterfactual analysis).

This paper is related to the price dispersion literature. Most of the literature does not study frictional markets with many-to-many meetings and *ex post* competition. The literatures closest to our paper are the simultaneous search and competing sellers literatures.[6] These models feature many-to-many meetings and a subset of these papers have some form of *ex post* competition. Most models in simultaneous search allow workers to make multiple applications and vacancies (firms) post prices *ex ante*.[7] In these models, pricing and link formation are simultaneous because they are decisions that are made *ex ante* to network formation. Since vacancies cannot adjust their wages after the network has been formed, these models often feature inefficiencies where linked workers and firms leave the market unmatched. Albrecht, Gautier, and Vroman (2006) allow for *ex post* competition between two vacancies that want to hire the same worker (many-to-one meetings) and find that this *ex post* competition corrects for this linked-but-not-matched inefficiency. Although agents in these models do indirectly compete when choosing *ex ante* prices and search intensities, the *ex post* competition is constrained by the price-posting. In contrast, this paper investigates the role of *ex post* indirect competition where agents have many-to-many meetings and where the price competition occurs after agents meet. In the competing sellers literature, either buyers are allowed to interact with only one seller (*e.g.* Wolinsky 1988; McAfee 1993; Julien, Kennes, and King 2000; Albrecht, Gautier, and Vroman 2014) or there are no frictions (*e.g.* Peters and Severinov 1997, 2006).[8] Restricting buyers to interact with only one seller restricts *ex post* competition in settings with many-to-many meetings. By allowing buyers to be linked to many sellers, our model generates *ex post* competition among sellers absent in

[5] Backus, Podwol, and Schneider (2013) show that more "visible" listings (*i.e.* ranked higher in eBay search results) are more likely to result in a sale, have more bidders, and have higher prices. Blake, Nosko, and Tadelis (2016) use eBay's clickstream data to report detailed statistics on consumer search behavior.

[6] A full review of the literature on price dispersion, search, and matching is outside the scope of this paper. See Baye, Morgan, and Scholten (2004) and Chade, Eeckhout, and Smith (2015) for detailed surveys.

[7] See *e.g.* Stigler (1961); Butters (1977); Burdett and Judd (1983); Acemoglu and Shimer (2000); Chade and Smith (2006); Kircher (2009); Galenianos and Kircher (2009); Gautier, Moraga-González, and Wolthoff (2014); and Wolthoff (2015).

[8] Backus, Podwol, and Schneider (2013) consider the case of two sellers with frictions using the framework of Peters and Severinov (2006).

competing sellers models with frictions.

Finally, our paper is related to the literature that uses networks to study seller-buyer markets with *ex post* competition and many-to-many meetings. There are two main strands of literature that do this: a literature that is concerned with the network formation process, and the properties of the networks that arise endogenously (*e.g.* Kranton and Minehart 2001; Elliott 2014; Elliott 2015; Gautier and Holzner 2016b), and a literature that studies properties of exogenously given networks, regardless of how they were formed (*e.g.* Corominas-Bosch 2004; Manea 2011; Polanski and Vega-Redondo 2013; Gautier and Holzner 2016a). Our paper is closest to the latter literature because we do not have a network formation model. Within this literature, authors generally propose a game to be played by the agents, and they study properties of the equilibrium strategy profiles. Such properties include efficiency of the final allocation, and whether or not there is price dispersion in the supporting prices. In our paper, since we want to study the effect of frictions on price dispersion, allowing for potential inefficiencies in the final allocation results in a confounding effect: we would be unable to tell whether price dispersion is generated by the presence of frictions, by the presence of inefficiencies in the equilibrium allocations, or as a function of the strategic details of the game under study. To bypass this problem, we do not posit an explicit game, focusing instead on pairwise stable matchings. Thus, the main contribution of our paper to this literature is that we characterize the set of graphs that satisfy the Stong Law of One Price in a game-free environment, assuming only that trades and prices form a pairwise stable matching.

Our analysis is also related to the literatures on matching, models of financial markets, and computer science. Please see section A in the online appendix for a discussion of these other related literatures.

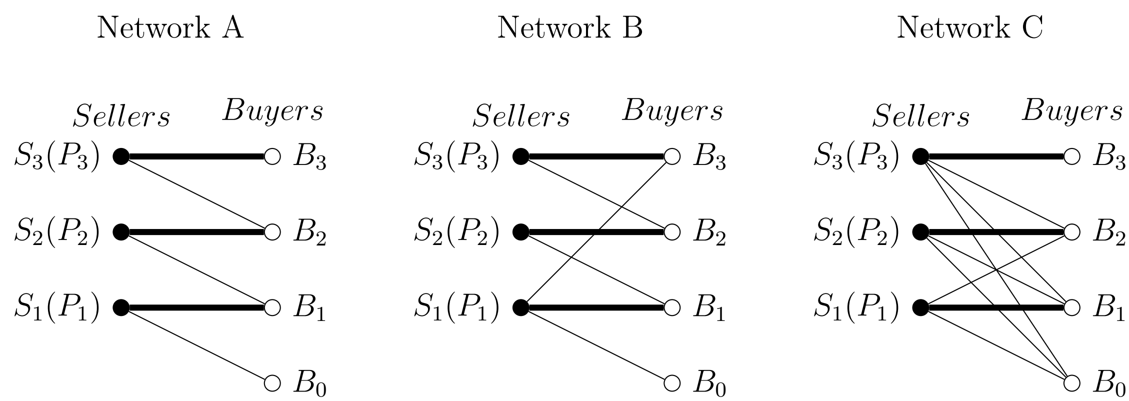
In summary, we develop a model whereby, for any given network, we characterize the set of pairwise stable matchings, and the prices that support them. We then characterize the set of graphs where the only prices that support pairwise stable matchings are those where each matched buyer pays the same price. Such graphs can never exhibit price dispersion. We then use tools from the random networks literature to derive conditions under which random graphs have no price dispersion. We use simulations to understand the relevance of our asymptotic results in large, but finite, networks. Finally, we calibrate our model to the online trading platform eBay and show that our model replicates the price dispersion documented at eBay quite well. We use the calibrated model to provide predictions on counterfactual network structures at eBay.

The rest of the paper is organized as follows. In Section 2, we present two motivating examples. In Section 3, we describe the model and our theoretical results. In Section 4, we describe the application of our model to eBay. In Section 5, we present our quantitative results on price dispersion in finite random networks. Finally, in Section 6, we discuss how our results can be interpreted in the context of labor markets and conclude. All proofs are in the appendix.

2 Two Motivating Examples

Example 1 illustrates two core concepts of this paper: 1) that indirect competition implies that only a subset of a network's links are relevant for price determination, 2) that price dispersion depends on both the sparsity and the structure of the network. To show this second point, we show a sparse network where price dispersion can never arise, and a dense network where price dispersion arises.

Example 1. Assume that sellers sell identical goods. Assume that buyer B_0 has the lowest valuation ($\min\{\mu(B_1), \mu(B_2), \mu(B_3)\} \geq \mu(B_0) > 0$), and sellers S_1, S_2, and S_3 have the same valuation (normalized to 0). Let P_1 be the price that seller S_1 receives, P_2 the price that S_2 receives, and P_3 the price seller S_3 receives. Consider three networks, where thick lines indicate a pairwise stable matching:[9]



First consider the subnetwork of Network A formed by S_1, B_0, and B_1. Buyer B_1 cannot pay less than $\mu(B_0)$ because buyer B_0 will poach seller S_1. So any price that supports the given matching must satisfy $P_1 \geq \mu(B_0)$. Now consider the full network. Buyer B_2 cannot pay less than P_1 to seller S_2 because buyer B_1 will poach seller S_2. Likewise for the price paid by buyer B_3. In summary, the set of prices that sustain this pairwise stable matching must satisfy $P_3 \geq P_2 \geq P_1 \geq \mu(B_0)$. In this example, buyers B_3 and B_2 are indirectly competing with buyer B_0. For buyer B_3, indirect competition forces him to pay at least $\mu(B_0)$ even though buyer B_0 is not linked to seller S_3. For this reason, adding a link between buyer B_0 and seller S_3 is redundant. The link (B_0, S_3) would impose the constraint $P_3 \geq \mu(B_0)$, but this constraint is already required by pairwise stability. Likewise, links (B_0, S_2) and (B_1, S_3) are redundant. To further highlight the role of indirect competition, notice that if buyer B_0 dropped out of the market, then $P_1 = P_2 = P_3 = 0$ would support the given matching.

[9]Pairwise stability is a specification of whom buys from whom and at what prices, such that two properties hold: (1) trades are individually rational, and (2) there are no pairwise Pareto improvements restricted to the network. When we say "restricted to the network", we mean that a buyer can propose an improvement to a seller (or vice versa) only if they are linked in the network.

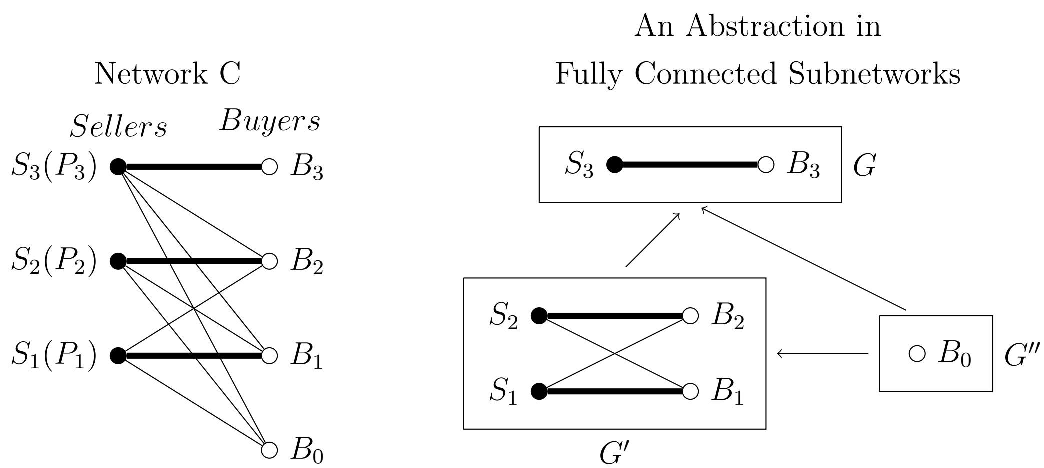
Next consider Network B, obtained from Network A by adding the link (B_3, S_1). Adding this link has two related effects. First, it makes price dispersion impossible. Second, it generates what we later call an *alternating cycle*. Because of the new link, Buyer B_1 cannot pay less than P_3 to seller S_1; otherwise, B_3 will poach seller S_1. So in addition to $P_3 \geq P_2 \geq P_1 \geq \mu(B_0)$, prices that support this matching must satisfy $P_1 \geq P_3$. Together, these conditions imply that any prices that support the given matching must satisfy $P_1 = P_2 = P_3 \geq \mu(B_0)$. By adding this link, price dispersion is precluded. Moreover, this graph has the alternating cycle property. Intuitively, an alternating cycle is a cycle that alternates between links that represent trades, with links that represent outside options. For example, seller S_1 trades with buyer B_1, who can use seller S_2 as an outside option. Seller S_2 trades with buyer B_2, who can use S_3 as an outside option. Finally, seller S_3 trades with buyer B_3, who can use S_1 as an outside option, thus closing the cycle. That the path travels through the pair of links (S_i, B_i), (B_i, S_{i+1}) implies that $P_i \leq P_{i+1}$. That this path traces a cycle implies that all buyers must pay the same price. This foreshadows our main theoretical result, theorem 1: the alternating cycles property is necessary and sufficient for price dispersion to be precluded.

Finally consider Network C, where we have added the redundant links to Network A as well as link (B_2, S_1). Crucially, we did not add (B_3, S_1). The set of prices that sustain this pairwise stable matching now satisfies $P_3 \geq P_2 = P_1 \geq \mu(B_0)$. Thus, price dispersion is no longer precluded. For example, $P_3 = \mu(B_3) > P_1 = P_2 = \mu(B_0)$ support this matching.

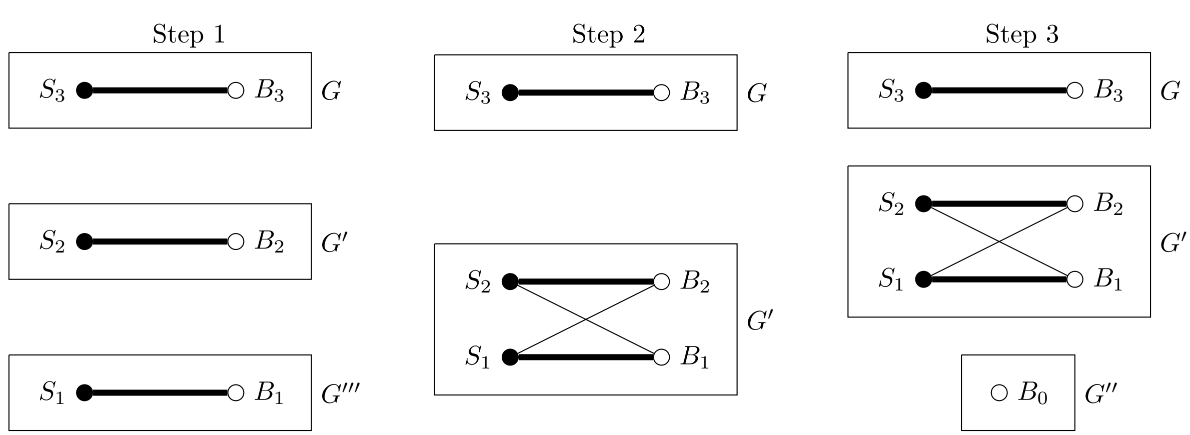
Comparing Network B to Network C shows that the density of the network is not the sole driving force behind the presence (or absence) of price dispersion. Network B is relatively sparse (7 out of 12 links are active) yet, for all valuation profiles that satisfy the constraint in the example $(\min\{\mu(B_1), \mu(B_2), \mu(B_3)\} \geq \mu(B_0) \geq 0)$, the proposed matching can only be supported through constant prices. Network C is considerably more dense than Network B (10 links out of 12 are active), but the same matching can be supported with non-constant prices. In this way, a single link can be the difference between supporting prices that exhibit price dispersion (Networks A and C), or supporting prices where dispersion is precluded (Network B).

Example 2 demonstrates how we use a network decomposition (which we call *abstractions*) to highlight the importance of indirect competition and to characterize the prices that sustain pairwise stable matchings. An abstraction in fully connected networks is a decomposition of a network into fully connected subnetworks that satisfies the following properties: (1) each node in the abstraction is a fully-connected subnetwork of the original network, (2) each link in the original network is either a link within a subnetwork in the abstraction or a link that connects two distinct nodes in the abstraction, and (3) there is a directed link from one subnetwork (say, G') to another (say, G) if there is at least one buyer in G' that has a link to at least one seller in G.[10] This construction uses that fully connected subnetworks are frictionless markets where the Law of One Price holds. The following example shows one possible abstraction of a network. We then show how this abstraction helps us characterize the prices that support a given pairwise stable matching.

Example 2. Consider Network C from example 1. Assume that sellers sell identical goods. Assume that buyer B_0 has the lowest valuation ($\min\{\mu(B_1), \mu(B_2), \mu(B_3)\} \geq \mu(B_0) > 0$) and sellers S_1, S_2, and S_3 have the same valuation (normalized to 0). Let P_1 be the price that seller S_1 receives, P_2 the price that S_2 receives, and P_3 the price seller S_3 receives. Thick lines indicate a pairwise stable matching:



[10]One way to construct an abstraction is to follow four steps (see example 2 for the original network): (1) form a subnetwork around each stable match, (2) combine subnetworks that are fully-connected, (3) form a separate subnetwork for each unmatched buyer (seller), and (4) form a directed link between subnetworks if there is a buyer in one subnetwork that is connected to a seller in another subnetwork. The direction of the link will point from the subnetwork that contains the buyer to the subnetwork that contains the seller. Although there may not be a unique assignment in step 2, any assignment will characterize the same set of pairwise stable matches and their supporting prices.



Even though many prices sustain it, there is essentially a unique pairwise stable matching: Buyer B_3 buys from seller S_3, buyer B_2 buys from seller S_2, and buyer B_1 buys from seller S_1.[11]

Abstractions are useful to highlight how indirect competition affects price formation. We proceed with 3 observations. First, consider subnetwork G' as an independent subnetwork. In this case, a pairwise stable matching corresponds to a frictionless allocation. Buyers B_1 and B_2 buy a good from sellers S_1 and S_2, they both pay the same price (say, $p(G')$), and this price is between 0 and $\min\{\mu(B_1), \mu(B_2)\}$. Similarly, when viewed as an independent subnetwork, G is a frictionless economy, so buyer B_3 must pay a price (say, $p(G)$) between 0 and $\mu(B_3)$. Second, subnetworks G and G' are not independent. Since buyer B_0 is linked to at least one seller in G and at least one seller in G', then those sellers must receive at least $\mu(B_0)$. Thus, $p(G') \geq \mu(B_0)$ and $p(G) \geq \mu(B_0)$. Finally, since at least one buyer in G' is linked to S_3, then S_3 must receive at least $p(G')$. Together, these observations imply that any prices that support the given matching must satisfy $p(G) \geq p(G') \geq \mu(B_0)$. Mapped back to the original network, this yields the prices in example 1. More generally, trading partners that belong to the same node in an abstraction pay the same price, and directed links in an abstraction indicate the relationship between the prices that prevail in each subnetwork. This foreshadows the result in Proposition 1: there is a one to one relationship between the prices that support a given matching and the prices induced by abstractions of a network.

We call this an abstraction because the identity of the buyer in subnetwork G' linked to the seller in subnetwork G is irrelevant. Similarly, the identity of the seller to which B_0 is connected is irrelevant. Abstractions also help clarify which links are redundant. For example, the link between buyer B_2 and seller S_3 establishes a directed link between G' and G, as does the link between buyer B_1 and seller S_3. Thus, one of these links matters for price formation, while the second is redundant. Likewise the link between buyer B_0 and seller S_3 is redundant. Link (B_0, S_3) generates a directed link between G'' and G. However, because there is already a path from G'' to G (going through G'), a direct link between G'' and G is redundant. Intuitively, the directed path from G'' to G reflects that buyer B_3 is indirectly competing with buyer B_0, and this makes direct competition between them redundant. In this way, abstractions facilitate our understanding of the competition in the network. What is relevant is the existence of the links and paths between subgraphs, not the specific identity nor the number of buyers and sellers that generate those links.

Abstractions formalize the effect we previously called "indirect competition."

[11] Nothing changes if buyers B_1 and B_2 are switched, so that B_1 buys from S_2 and B_2 buys from S_1.

In the example above, imagine the link between B_0 and S_3 is removed. We make two observations. First, since B_0 is no longer linked to S_3, B_0 and B_3 are not directly competing for seller S_3. However, B_0 and B_3 are indirectly competing for S_3. Indeed, S_3 and S_2 are connected though buyer B_2, and B_0 is linked to S_2. This means that B_0 is indirectly competing with all buyers linked to S_3; in paticular, B_0 is indirectly competing with B_3. The indirect competition between B_0 and B_3, and the subsequent effect of $\mu(B_0)$ on P_3, is manifested in the directed path from G'' to G', and from G' to G. In general, consider a node A that points to a node A'. Any buyer in A that is not connected to a seller in A' indirectly competes with every buyer in A'. In this way arrows that connect nodes in abstractions encode the indirect competition structure of the network.

3 The Model

3.1 Seller-Buyer Model

We consider a market for a homogeneous good. Sellers differ in their valuation (or cost) and offer a single unit for sale. Buyers differ in their valuation and have single unit demand. A buyer with valuation μ that buys from a seller at price p has utility $\mu - p$, and 0 otherwise. A seller with cost c that sells a good at price p has utility $p - c$, and 0 otherwise. These utility functions assume that buyers and sellers only care about trading at the best possible price, irrespective of whom they trade with. We do this because the focus of our paper is to study, in the context of *ex post* competition and many-to-many meetings, the effect of frictions on price dispersion, unconfounded by any other forces that might also generate price dispersion.

Trading takes place in exogenous seller-buyer graphs. A graph is a set of nodes connected by links, or edges. An edge is represented by the pair of nodes it joins. We say the graph is undirected if the direction of the link does not matter; otherwise, we say the graph is directed. We say that the graph is bipartite if the set of nodes can be partitioned into two subsets such that no two nodes in the same set are connected to each other. In our framework, buyers and sellers constitute a bipartite undirected graph. First, the set of nodes is partitioned into a set of buyers and a set of sellers; second, a buyer is linked to a seller if, and only if, that seller is linked to that buyer; third, no buyer (respectively seller) is connected to another buyer (respectively seller). We say a graph is fully connected (or complete) whenever each buyer is linked to every seller, and vice versa. We denote the set of sellers with $\mathcal{I}$, the set of buyers with $\mathcal{J}$, and the set of edges with E.

Definition (graph). *Given a finite set V of* nodes *and a set $E \subset V^2$ of* edges *we say (V, E) is a* graph. *Moreover,*

- *We say a graph (V, E) is trivial if $E = \emptyset$ and V is a singleton.*

- *Given two nodes $v, v' \in V$, a path from v to v' is a sequence $(e_1, ..., e_N) \in E^N$ where $e_t = (v_t, v_{t+1})$ for each $t \in \{1, ..., N\}$, $v_1 = v$, and $v_{N+1} = v'$.*

- *We say that a graph is connected if, for any pair of nodes $v, v' \in V$, there is a path from v to v', and a path from v' to v.*

- *We say the graph is* undirected *when, for each $v, v' \in V$, $(v, v') \in E$ if and only if $(v', v) \in E$. Otherwise, we say it is* directed.

- *We say (V, E) is* a bipartite graph *if there exists two disjoint sets, $V_1, V_2 \subset V$, such that $V = V_1 \cup V_2$ and $(v, v') \in E$ only if $v \in V_i \Rightarrow v' \in V_j$, for $i \neq j$. We write these graphs explicitly as (V_1, V_2, E).*

- *We say a bipartite graph (V_1, V_2, E) is* fully connected *if for each $v_1 \in V_1$, $(v_1, v_2) \in E$ for each $v_2 \in V_2$.*

Given a graph, a matching is any subset of the set of links such that three properties hold: each buyer is matched to at most one seller, each seller is matched to at most one buyer, and a seller is matched to a buyer if, and only if, the buyer is matched to the seller. We typically denote matchings with M, and we use the expression "i and j are matched" to mean that the links (i, j) and (j, i) are in M. Given a matching M, we define $i^* : \mathcal{J} \to \mathcal{I} \cup \{\emptyset\}$ as the function that maps each buyer to the seller with whom it is matched, or to the symbol $\emptyset$ if the buyer is unmatched. Likewise, $j^* : \mathcal{I} \to \mathcal{J} \cup \{\emptyset\}$ is the function that maps each seller to the buyer with whom it is matched, or to the symbol $\emptyset$ if the seller is unmatched. Finally, we say a matching is *maximal* if, whenever a seller an a buyer are linked, at least one of them is matched.

Definition (Matching). *Let $\mathcal{G} = (\mathcal{I}, \mathcal{J}, E)$ be a graph. We say $M \subset E$ is a* matching *if the following hold:*

1 If $(i, j), (i', j) \in M$ then $i = i'$,

2 if $(i, j), (i, j') \in M$ then $j = j'$,

3 $(i, j) \in M \Leftrightarrow (j, i) \in M$

We say a matching M is maximal if for all $(i, j) \in E$, either $i^*(j) \neq \emptyset$, $j^*(i) \neq \emptyset$, or both.

Since graphs tell us which buyers are connected to which sellers, but they do not tell us the valuation of buyers nor the valuation of the sellers, we extend the definition of the graph to the definition of a network. A network is a graph where each node is given a numerical value. This value is interpreted as the valuation of the buyer or seller. For the rest of the paper, even if not explicitly mentioned, $\mu(\cdot)$ denotes the valuation profiles of both buyers and sellers: for a seller $i \in \mathcal{I}$, $\mu(i)$ denotes i's valuation, and for a buyer $j \in \mathcal{J}$, $\mu(j)$ denotes j's valuation. Throughout the paper, unless explicitly mentioned, we assume valuation profiles are such that there are always positive gains from trade amongst linked agents. That is, if a seller i is linked to a buyer j, then $\mu(j) > \mu(i)$.[12]

Definition (networks). *Let $\mathcal{G} = (\mathcal{I}, \mathcal{J}, E)$ be an undirected bipartite graph, and let $\mu : \mathcal{I} \cup \mathcal{J} \to \mathbb{R}$. A* network *is a tuple $\mathcal{N} = (\mathcal{I}, \mathcal{J}, E; \mu)$.*

The distinction between a graph and a network is important for the results in sections 3.3 and 3.4. Because networks contain information about valuation profiles, characterizing properties of networks that accommodate price dispersion confounds two effects: the effect that frictions have on price dispersion (as encoded by the missing links in the underlying graph), and the effect that valuation profiles have of price dispersion. To avoid this confounding effect, the results in section 3.4 characterize all graphs—not networks—that accommodate price dispersion. Doing this requires first understanding how prices are determined for each given network; we do this in section 3.3.

Given a network, we define a price function for the network. For any set of edges, S, a price function, p_S, is a function that maps edges in S into real numbers, with the property that $p_S(i,j) = p_S(j,i)$ whenever $(i,j),(j,i) \in S$. This real number is interpreted as the price that would prevail if the buyer was to buy the good from the seller. The price function is individually rational if, for each pair of agents, it specifies a price that lies between the seller's valuation and the buyer's valuation. Finally, given a matching, M, and a price function, p_M, the function $v(M,p_M)(\cdot)$ summarizes the price each agent pays, or is paid, under matching, M, at prices, p_M. Likewise, $u(M,p_M)$ is the utility each agent receives under matching M at prices p_M. For notational convenience, if a buyer j is unmatched, we let $v(M,p_M)(j) = \mu(j)$; likewise, if a seller i is unmatched, we let $v(M,p_M) = \mu(i)$. Also for notational convenience, we simply write $v(\cdot)$ and $u(\cdot)$ whenever the matching and prices that determine v and u are clear from context. Formally, for each buyer j and seller i,

$$\begin{aligned} v(M,p_M)(j) &= \begin{cases} \mu(j) & \text{if} \quad i^*(j) = \emptyset \\ p_M(i^*(j),j) & \text{if} \quad i^*(j) \neq \emptyset, \end{cases} \\ u(M,p_M)(j) &= \mu(j) - v(M,p_M)(j), \end{aligned}$$

and

$$\begin{aligned} v(M,p_M)(i) &= \begin{cases} \mu(i) & \text{if} \quad j^*(i) = \emptyset \\ p_M(i,j^*(i)) & \text{if} \quad j^*(i) \neq \emptyset, \end{cases} \\ u(M,p_M)(i) &= v(M,p_M)(i) - \mu(i). \end{aligned}$$

Next, we define pairwise stability of a matching M with respect to a price function p_M.

[12]See Appendix D for further discussion of the gains from trade assumption.

Pairwise stability means that the edges in M are priced such that individual rationality holds, and there are no mutually beneficial blocks by pairs of agents that are linked but are not matched. In other words, any extension of p_M to more edges cannot yield pairwise Pareto improvements over the match M executed at prices p_M.

Definition (Blocking). *Let M be a matching and p_M be a price function. Suppose i is linked, but not matched, to j;* i.e. $(i,j) \in E \setminus M$. *We say the pair (i,j)* blocks (M, p_M) *if $v(M, p_M)(i) < v(M, p_M)(j)$.*

Definition (Pairwise Stability). *Given a network $\mathcal{N}$ and a matching M, we say M is* pairwise stable *in $\mathcal{N}$ at prices p_M if the following hold:*

- No blocking: *no pair $(i,j) \in E \setminus M$ blocks (M, p_M),*
- Individual rationality: *for all pairs $(i,j) \in M$, $p_M(i,j) \in [\mu(i), \mu(j)]$.*

In this case, we say that p_M supports M. *Moreover, we say M is* pairwise stable *if there exist prices p_M at which M it is pairwise stable.*

We use pairwise stability as our solution concept because we want to understand how frictions, and frictions alone, affect price dispersion. In our framework, pairwise stable matchings are those where all possible pairwise gains from trade are exhausted. A solution concept where pairwise gains from trade are not exhausted could lead to price dispersion, but this price dispersion would be driven by the solution concept, not the frictions themselves.

An important property that a network can have is the Law of One Price (LOP) property. Consider a network $\mathcal{N} = (\mathcal{I}, \mathcal{J}, E; \mu)$, and assume that the following property holds: for any pairwise stable matching, M, if p_M supports M, then p_M is a constant function. We then say that network, $\mathcal{N}$, satisfies the Law of One Price; for such networks, pairwise stability is incompatible with price dispersion.

Given our assumptions, pairwise stable matchings in fully connected networks are characterized by the Law of One Price. To see this, assume i is matched to j, i' is matched to j'. Since (j, i') does not block (j, i), then $v(j) \leq v(i') = v(j')$, and since (j', i) does not block (j', i'), then $v(j') \leq v(i) = v(j)$. Thus, $v(i) = v(j) = v(i') = v(j')$. Thus, if M is pairwise stable at prices p_M, then p_M must be a constant function: given any two distinct trading pairs, those pairs must be trading at the same price. As a corollary, all pairwise stable matchings can be characterized by whether there are more buyers than sellers or vice versa. Pairwise stable matchings are those matchings where every agent in the short side of the market is matched, they can be sustained by individually rational prices that price out the long side of the market, and each matched buyer pays the same price. Finally, any time a matching is pairwise stable at prices p_M, where p_M is a non-constant function, it must be because the network is not fully connected. Because we are interested in the effect that frictions have on price dispersion, we set the model up so that price dispersion can only arise in networks that are not fully connected. We summarize this in the following remark.

Remark 1. *Let $(\mathcal{I}, \mathcal{J}, E; \mu)$ be a fully connected network, where E is the set of edges, and $\mu : \mathcal{J} \cup \mathcal{I} \to \mathbb{R}$ be the valuation profile. Let $J = \#\mathcal{J}$, $I = \#\mathcal{I}$. Assume that $\bar{b} = \max\{\mu(i) : i \in \mathcal{I}\} \leq \min\{\mu(j) : j \in \mathcal{J}\} = \underline{\mu}$. Let $M \subset E$ be a matching.*

- *If $I > J$, M is pairwise stable if, and only if,*
  - *All buyers are matched: For each $j \in \mathcal{J}$ there is $i \in \mathcal{I}$ such that $(j, i) \in M$.*
  - *Only lowest valuation sellers are matched: If $i \in \mathcal{I}$ is such that $\#\{i' : \mu(i) > \mu(i')\} \geq J$ then there is no $j \in \mathcal{J}$ such that $(j, i) \in M$.*
  - *Seller valuations determine matching prices: For each $(j, i) \in E$, $p(j, i) = p$ where $p \in [\max\{\mu(i) : (\exists j \in \mathcal{J})$ such that $(j, i) \in M\}, \{\min\{\mu(i) : (\nexists j \in \mathcal{J})$ such that $(j, i) \in M\}]$.*
- *If $I = J$, M is pairwise stable if, and only if,*
  - *All buyers are matched: For each $j \in \mathcal{J}$ there is $i \in \mathcal{I}$ such that $(j, i) \in M$.*
  - *All sellers are matched: For each $i \in \mathcal{I}$ there is $j \in \mathcal{J}$ such that $(j, i) \in M$.*
  - *Sellers sell at an intermediate price: For each $(j, i) \in E$, $p(j, i) = p$ where $p \in [\bar{b}, \underline{\mu}]$.*
- *If $I < J$, M is pairwise stable if, and only if,*
  - *Only highest valuation buyers are matched: For each $j \in \mathcal{J}$ if $\{j' : \mu(j') > \mu(j)\} \geq I$ then there is no $i \in \mathcal{I}$ such that $(j, i) \in M$.*
  - *All sellers are matched: For each $i \in \mathcal{I}$ there is $j \in \mathcal{J}$ such that $(j, i) \in M$.*
  - *Buyer valuations determine matching prices: For each $(j, i) \in E$, $p(j, i) = p$ where $p \in [\max\{\mu(j) : (\nexists i \in \mathcal{I})$ such that $(j, i) \in M\}, \min\{\mu(j) : (\exists i \in \mathcal{I})$ such that $(j, i) \in M\}]$.*

The LOP implies that, for all pairwise stable matchings, supporting price functions must be constant, in the sense that any two trading pairs must trade at the same price. However, there may be many such functions. To avoid semantic ambiguity, we use the term "constant price," rather than "unique price," to reference situations where all matched buyers pay the same price.

3.2 An overview of theoretical results

In this framework, we present three main results. First, for any given network, $\mathcal{N}$, and any given matching, M, we characterize the set of price functions that support M. This set may be empty, in which case M is not pairwise stable; therefore, this result also characterizes the set of matchings that are pairwise stable. We view this characterization as a tool for understanding what network structures can accommodate price dispersion. Concretely, suppose that a graph

$\mathcal{G}$ satisfies the following property: for all valuation profiles, for all matchings M in $\mathcal{G}$, and for all supporting prices, p_M, if M is pairwise stable given the valuation profile then p_M is constant. Such a graph has the property that, for all valuation profiles, the corresponding network satisfis the LOP. Thus, we say that such a graph satisfies the *Strong Law of One Price*, or SLOP. Our second result uses the first result to characterize the set of all graphs that satisfies the SLOP. This second result conveys the main contribution of our paper: price dispersion is jointly determined by both the level of frictions, as measured by the number of existing links out of total possible links, and also the structure of the market, as encoded by how these links are distributed in the network. Both of these results hold for any exogenously given network, so they are independent of the network formation process. For our third result we consider a simple network formation process: for any seller-buyer pair, (i, j), the links (i, j) and (j, i) are drawn according to a Poisson parameter $\lambda > 0$. This network formation process is analogous to the assumption that search is random and follows a Poisson process. Our third result provides conditions such that, as the number of both buyers and sellers grows to infinity and as λ converges to 0, the asymptotic probability that the realized graph satisfies the SLOP converges to 1. The result highlights that the LOP is compatible with high frictions, provided the market is large enough. This is consistent with our second result, and highlights that price dispersion depends on both the level of frictions, and the structure of whom met with whom.

3.3 Step 1: Characterizing Prices

In this section we present two propositions, which we use to understand how indirect competition affects the degree of price dispersion in the market. Given a seller-buyer network and a pairwise stable matching in such network, Proposition 1 states that only a subset of links is relevant for determining the prices that sustain that matching. To identify this set of links, we define the *abstraction* of a network. This is a construction that abstracts away from links that are irrelevant for determining the prices that sustain a given pairwise stable matching. We use a special class of abstractions, which we denote *maximal abstractions*, to characterize the full set of prices that support any given pairwise stable match. Proposition 1 shows the characterization.

We start by defining the abstraction of a network. An abstraction of a network is a directed graph with nodes and edges defined as follows: each node in the abstraction is a subnetwork of the original network, these subnetworks are disjoint, and there is an link pointing from a node a to a node a' if there is a buyer a whom, in the original network, is linked to a seller in a'.

Definition (Abstraction). *Let $\mathcal{G}$ be a seller-buyer graph. From $\mathcal{G}$ construct a directed graph, $\mathcal{A} = (A, E^*)$, as follows:*

- *Each node in $a \in A$ is associated with a subgraph of $\mathcal{G}$: $(\forall a \in A)$, $a = (\mathcal{I}_a, \mathcal{J}_a, E_a)$*

where $E_a = \{(x, y) \in E : x, y \in \mathcal{I}_a, \cup \mathcal{J}_a\}$,

- *These subgraphs are disjoint:* $(\forall a, a' \in A)$, $\mathcal{I}_a \cap \mathcal{I}_{a'} = \emptyset$, $\mathcal{J}_a \cap \mathcal{J}_{a'} = \emptyset$,

- *A node* a' *in* $\mathcal{A}$ *is linked to a node* a'' *in* $\mathcal{A}$ *if the subnetwork associated to* a' *contains a buyer,* j*, and the subnetwork associated with* a'' *contains a seller,* i*, such that, in* $\mathcal{N}$*,* i *and* j *are linked:* $(\forall a', a'' \in A)$*,* $(a', a'') \in E^*$ *if* $(\exists j \in \mathcal{J}_{a'})(\exists i \in \mathcal{I}_{a''})$ *such that* $(j, i) \in E$*.*

We say the abstraction is in fully connected subgraphs *if each* a' *is fully connected.*
To make terminology easier, we say an agent, x*, belongs to a node* $a' \in A$ *if* $x \in \mathcal{I}_{a'} \cup \mathcal{J}_{a'}$*. We also say an edge* $e \in E$ *belongs to* a' *if* $e \in E_{a'}$*.*

While the notion of an abstraction is independent from the notion of a matching, since we use abstractions as a tool to characterize all price functions that support any given matching, it is convenient to build abstractions in a manner consistent with the matching under consideration. This yields the definition of a *maximal abstraction*. We say $\mathcal{A}$ is a *maximal abstraction for* M if two conditions hold. First, for every unmatched buyer j (respectively, seller i), the subnetwork of $\mathcal{A}$ that contains j (respectively, i), contains only j (respectively, i). Second, matched pairs belong to the same node in the abstraction. We call these abstractions "maximal" because they allow us to characterize the full set of price functions that support any given matching. Abstractions that are not maximal generally characterize strict subsets of the set of prices that support any given matching (this point is further elaborated in appendix A). A formal definition of maximal abstractions follows.

Definition (Maximal Abstractions). *Let* $\mathcal{N}$ *be a network and* M *be a matching. We say* $\mathcal{A} = (A, E^*)$ *is a* maximal abstraction for M *if the following conditions hold:*

- *If* $i^*(j) = \emptyset$*, then there exists* $a \in A$ *such that* $a = (\emptyset, \{j\}, \emptyset)$*,*

- *If* $j^*(i) = \emptyset$*, then there exists* $a \in A$ *such that* $a = (\{i\}, \emptyset, \emptyset)$*,*

- *If* $(i, j) \in M$*, then there exists* $a \in A$ *such that* $(i, j) \in E_a$ *and* $(j, i) \in E_a$*.*

When the last condition holds, we say the abstraction does not break the matching.

Maximal abstractions always exist. Indeed, given a graph $\mathcal{G} = (\mathcal{I}, \mathcal{J}, E)$, and a matching M, define $\mathcal{A} = (A, E^*)$ as follows. First, for each pair $(i, j) \in M$, let $a_{i,j} = (\{i\}, \{j\}, \{(i, j), (j, i)\})$, and for each seller i and buyer j that are unmatched, define a_i and a_j as in the definition of a maximal abstraction. This defines the set of nodes, A. Second, define the set of links as in the definition of an abstraction. The resulting construction is a maximal abstraction. Maximal abstractions are generally non-unique, and the above construction is the one that employs the maximum number of nodes. However, Proposition 1 only requires existence, not uniqueness.

Given an abstraction of a graph, we define a price function for the abstraction. A price function for the abstraction is a function, ρ, that assigns a number (*i.e.* a price) to each node in the abstraction. The following remark defines a natural way in which price functions in an abstraction induce price functions in the original network, and vice-versa.

Remark 2. *Consider a network, $\mathcal{N}$, and a matching, M. Let $\mathcal{A} = (A, E^*)$ be an abstraction in fully connected networks. Assume $\mathcal{A}$ does not break M; that is, if $(i, j) \in M$, i belongs to $a \in A$ and j belongs to $a' \in A$, then $a = a'$.*

Given a price function ρ for the abstraction, ρ induces a price function, p_M, for the original network as follows:

- *For each $(i, j) \in M$, if (i, j) belongs to $a \in A$, then $p_M(i, j) = \rho(a)$.*

Conversely, let p_M be a price function for $\mathcal{N}$. If p_M is such that M is pairwise stable in $\mathcal{N}$ at prices p_M, then p_M induces a price function ρ for the abstraction as follows:

- *If a is a trivial subnetwork that contains only seller i, $\rho(a) = \mu(i)$,*
- *If a is a trivial subnetwork that contains only buyer j, $\rho(a) = \mu(j)$,*
- *If a contains a matched pair $(i, j) \in M$, then $\rho(a) = p_M(i, j)$. This is well defined because subnetworks are fully connected; by pairwise stability, p_M must be constant when restricted to matched pairs within the same fully-connected subnetwork.*

We say a matching $M \subset E$ is pairwise stable in an abstraction $\mathcal{A}$ at prices ρ when three conditions hold. First, the abstraction does not break M: if a buyer j is matched to a seller i, then i and j belong to the same node in the abstraction. Second, prices $\rho(\cdot)$ induce pairwise stability in each node of the abstraction. Suppose p_M is the price function for $\mathcal{N}$ induced by ρ, and suppose a is a node in the abstraction. Viewing a as a network on its own, consider the restrictions of M and p_M to a. Then, M restricted to a should be stable at the prices p_M, also restricted to a. The last condition for stability with respect to an abstraction is the *cheapest sorting* condition: if node a in the abstraction has a directed link to node a', then $\rho(a) \leq \rho(a')$. That a is linked to a' implies that some buyer in the subnetwork associated to a is linked to some seller in the subnetwork associated to a'; thus, cheapest sorting is a natural arbitrage requirement. As before, if M is pairwise stable in $\mathcal{A}$ at prices ρ, we say ρ supports M in $\mathcal{A}$.

Definition (stability abstraction). *Let $(\mathcal{I}, \mathcal{J}, E; \mu)$ be a seller-buyer network and the directed graph $\mathcal{A} = (A, E^*)$ be an abstraction of it in fully connected graphs. Let $\rho : A \to \mathbb{R}$. We say that M is stable with respect ρ in $\mathcal{A}$ if three conditions hold:*

- *A does not break M: for each $e \in M$, $e \in E_a$ for some $a \in A$.*
- *Prices $\rho(\cdot)$ induce pairwise stability in each subnetwork:*

- *For each non-trivial* $a \in \mathcal{A}$*, M restricted to a is stable at prices* $p_M(j,i) = \rho(a)$ *for all* $(j,i) \in M \cap E_a$*,*
- *If* $a = (\{i\}, \emptyset, \emptyset)$ *for some j, then* $\rho(a) = \mu(i)$*,*
- *If* $a = (\emptyset, \{j\}, \emptyset)$ *for some j, then* $\rho(a) = \mu(j)$*.*

- *Cheapest sorting: for each directed link* $(a, a') \in E^*$*,* $\rho(a) \leq \rho(a')$*.*

As before, if M is stable with respect to ρ in $\mathcal{A}$, we say ρ supports M in $\mathcal{A}$.

With these definitions we can state our first proposition.

Proposition 1. *Let $\mathcal{N}$ be a network and M be a matching. Let $\mathcal{A}$ be an abstraction of $\mathcal{N}$ in fully-connected networks that is maximal for M. Then, the following statements are true:*

1 *If p_M supports M, there exists $\rho : A \to \mathbb{R}$ such that ρ induces p_M, and ρ supports M in $\mathcal{A}$.*

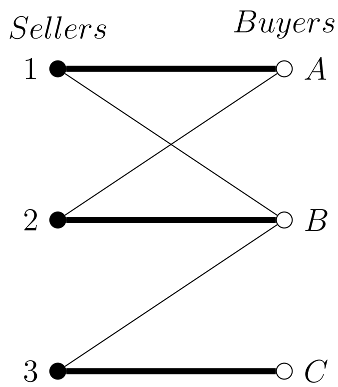
2 *If ρ supports M in $\mathcal{A}$, there exists $p_M : M \to \mathbb{R}$ such that p_M induces ρ, and p_M supports M in $\mathcal{N}$.*

3.4 Step 2: Characterizing the Strong Law of One Price

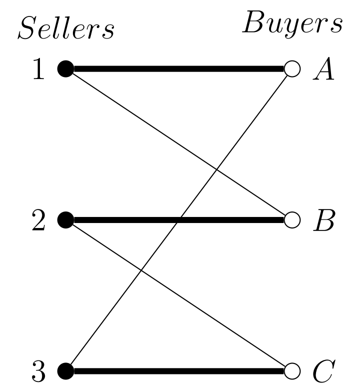
In this section we define the SLOP, and characterize the graphs such that the SLOP holds. Consider a network $\mathcal{N} = (\mathcal{I}, \mathcal{J}, E; \mu)$, and suppose it satisfies the LOP: given the valuation profile μ, any pairwise stable matching can only be supported by a constant price function. However, if the same graph is endowed with a different valuation profile, the new network may no longer satisfy the LOP (see example 3 below); in these cases, the presence of price dispersion depends on how valuations are distributed in the network, rather than a property of the underlying graph. Now, consider a graph $(\mathcal{B}, \mathcal{S}, F)$ with the property that, for all valuation profiles μ, the network $(\mathcal{B}, \mathcal{S}, F; \mu)$ satisfies the LOP. In this graph, regardless of how valuations are assigned to the nodes, the LOP must hold. Hence, for all valuation profiles, pairwise stable matchings can only be supported with constant price functions. When a graph has this property we say it satisfies the SLOP. Notice that a complete graph satisfies the SLOP.

Example 3. First, consider the graph on the left. There are valuations such that the matching indicated in bold can only be supported with a constant price function. For example, if all sellers have 0 cost and all buyers have 0 valuation. However, there also exist valuations such that the matching indicated in bold can be supported with non constant prices: if $\mu(1) = \mu(2) = 0$, $\mu(A) = \mu(B) = \mu(C) = 2$, $\mu(3) = 1$, prices $p_M(A, 1) = p_M(B, 2) = 0$, $p_M(C, 3) = 1$ support this matching. Thus, if price dispersion is precluded, it is an artifact of the valuation profile, but not a structural property of the graph itself. Now, consider the graph on the right, and the matching indicated in bold. For all valuation profiles such that linked pairs have positive gains from trade, the only prices that can support the given matching are constant prices: for all supporting prices, A will pay no more than C, C will pay no more than B, and B will pay no more than A.

Matching That Supports Price Dispersion



Matching That Does Not Support Price Dispersion



To characterize the graphs that satisfy the SLOP we proceed in two steps. First, for any graph $\mathcal{G} = (\mathcal{I}, \mathcal{J}, E)$, we provide necessary and sufficient conditions on the maximal abstractions of $\mathcal{G}$ so that $\mathcal{G}$ satisfies the SLOP. For each maximal matching, M, $\mathcal{G}$ must have a maximal abstraction that is connected and does not break M. Because abstractions are directed graphs, being connected means that for any pair of nodes in the abstraction, there is a directed path from the first to the second, and a directed path from the second to the first. In particular, this is equivalent to saying the abstraction has a cycle that visits every node at least once. We now provide an intuition for why this result holds. Proposition 1 says that we may interpret an economy with frictions (as represented by a graph) as a collection of interrelated, frictionless sub-economies, as represented by different nodes in an abstraction of the original graph. Because each of these sub-economies is frictionless, they satisfy the SLOP, so all trades within each sub-economy occur at the same price. However, trades in different sub-economies might occur at different prices. In order for the economy as a whole to satisfy the SLOP, we need that trades in each of these sub-economies occur at the same price. For this to be true, arbitrage opportunities across any two sub-economies (represented through directed paths that connect nodes in the abstraction) must be eliminated. This can only happen if, and only if, for any two sub-economies (say, a and a'), there is a directed path from a to a', and a directed path from a' to a. The first of these paths implies the price at which agents in a trade must be lower than, or equal to, the price at which agents in a' trade, and the second path implies the opposite inequality. Thus, the SLOP holds if, and only if, for each maximal matching, M, $\mathcal{G}$ has a maximal abstraction that is connected and does not break M. The downside of this first result is that the primitive of the model is the graph, $\mathcal{G}$, not its abstractions. The second result provides necessary and sufficient conditions on $\mathcal{G}$ for it to have such an abstraction, and therefore satisfy the SLOP.

To state the results for this section, we require some auxiliary notation. If $\mathcal{G}$ is a graph, and M is a matching, then $\mathcal{G}|M$ is the subgraph of $\mathcal{G}$ that is restricted to the matching. Formally, $\mathcal{G}|M = (\mathcal{J}', \mathcal{I}', E')$, where $\mathcal{J}' = \{j \in \mathcal{J} : i^*(j) \neq \emptyset\}$, $\mathcal{I}' = \{j \in \mathcal{I} : j^*(i) \neq \emptyset\}$, and $E' = \{(x, y) \in E : x, y \in \mathcal{J}' \cup \mathcal{I}'\}$. Also, for this section, we assume that $\mathcal{G}$ is such that, for each maximal M, $\mathcal{G}|M$ has at least two buyers and at least two sellers. This assumption rules out the trivial case where only one buyer and one seller trade: if a graph $\mathcal{G}$ is such that for all maximal matchings, M, $\mathcal{G}|M = (\{i\}, \{j\}, \{(i, j), (j, i)\})$, then the law of one price holds trivially.

Below we define two properties, which we use to prove a theorem and a corollary. The first property is the M-Alternating cycles (M-AC) property. It states that, given a graph, $\mathcal{G}$, and a maximal matching, M, the graph $\mathcal{G}|M$ has a complete cycle with the property that the odd edges are in M and the even edges are not. We then generalize this property to the Strong Alternating Cycles (SAC) property: a graph $\mathcal{G}$ has the SAC property if, for each maximal matching M, $\mathcal{G}|M$ has the M-AC property. Thus, the SAC property is a global version of the M-AC property. The main theorem says that the SAC property holds if, and only if, the SLOP holds. As a corollary, we obtain a local version of this theorem: given a maximal matching, M, $\mathcal{G}$ has the M-AC property if, and only if, for all valuation profiles, μ, and for all prices, p_M, if p_M supports M given μ, then p_M is a constant function.

Definition (Alternating Cycles properties). *$\mathcal{G} = (\mathcal{I}, \mathcal{J}, E)$ be a graph, $M \subset E$ be a maximal matching in $\mathcal{G}$, and denote $\mathcal{G}|M = (\mathcal{I}', \mathcal{J}', E')$. We say $\mathcal{G}$ has the M-Alternating Cycles property (M-AC) if there is a vector of edges, $\mathcal{C} = (e_1, ..., e_T)$, that satisfies the following:*

- *$e_t \in M$ if, and only if, t is even,*
- *$\mathcal{C}$ is a complete cycle in $\mathcal{G}|M$: edges are adjacent, the path described by the vector starts and ends at the same node, and each node is visited at least once. Formally,*
    - *if $e_t = (x, y)$ for some $(x, y) \in E'$ and some $t < T$, then $e_{t+1} = (y, z)$ for some $(y, z) \in E'$,*
    - *if $e_1 = (x, y) \in E'$ for some $(x, y) \in E'$, then $e_T = (z, x)$ for some $(z, x) \in E'$,*
    - *for each $x \in \mathcal{I}' \cup \mathcal{J}'$, there is a $y \in \mathcal{I}' \cup \mathcal{J}'$ and a t such that $e_t = (y, x) \in E'$.*

We say that $\mathcal{G}$ satisfies the Strong Alternating Cycles property (SAC) if, for each maximal matching M, $\mathcal{G}$ satisfies the M-AC.

Definition (Laws of One Price). *Let $(\mathcal{I}, \mathcal{J}, E)$ be a graph.*

- *Let $\mu : \mathcal{I} \cup \mathcal{J} \to \mathbb{R}$ be some valuation profile. Let $\mathcal{M}$ be the set of pairwise stable matchings in network $(\mathcal{I}, \mathcal{J}, E; \mu)$. If for all $M \in \mathcal{M}$, and all p_M that supports M, p_M is constant, we say that the network $(\mathcal{I}, \mathcal{J}, E)$ satisfies the Law of One Price (LOP).*

- *If for all $\mu : \mathcal{I} \cup \mathcal{J} \to \mathbb{R}$, the network $(\mathcal{I}, \mathcal{J}, E)$ satisfies the Law of One Price, then we say the graph $(\mathcal{I}, \mathcal{J}, E)$ satisfies the Strong Law of One Price (SLOP).*

With the above definitions, we can state our main result for this section.

Theorem 1. *Let G be a graph. Then G satisfies the SLOP $\Leftrightarrow$ it satisfies the SAC.*

To provide intuition for why Theorem 1 is true we proceed in two steps. First, only maximal matchings can be pairwise stable. This is because all linked pairs have positive gains from trade; if two linked pairs were unmatched, then they would block the match by matching together. Second, each matched pair in a maximal matching, M, can be thought of as a two-agent node in an abstraction of the graph. Traveling through a link $(i, j) \in M$ is like traveling within the node that contains i and j, whereas traveling through a link $(j, i') \notin M$ is like traveling from the node that contains buyer j to the one that contains seller i'. Thus, the M-AC property is equivalent to finding a path that starts at a node in an abstraction, travels through all nodes, and finishes where it begun. When this holds, M can only be supported by constant price functions. Thus, the SAC is equivalent to the graph satisfying the SLOP.

Corollary 1. *Let $G = (\mathcal{I}, \mathcal{J}, E)$ be a graph, and $M \subset E$ be a maximal matching. $(\forall \mu : \mathcal{I} \cup \mathcal{J} \to \mathbb{R})$, $(\forall p_M)$, $[p_M$ supports M in $(\mathcal{I}, \mathcal{J}, E; \mu) \Rightarrow p_M$ is constant$] \Leftrightarrow G$ satisfies the M-AC.*

This result is simply a local version of Theorem 1. The SLOP quantifies over all matchings; however, one might be interested in whether a particular matching can be supported at non-constant prices, irrespective of the prices that support other matchings. The corollary says that the M-AC—the local version of the SAC property—is equivalent to this local version of the SLOP.

3.5 Step 3: Asymptotic Results

Proposition 1 and Theorem 1 apply for any given network, and no assumptions are made about the network formation process. However, price dispersion is often studied using random search models, where buyers and sellers meet following a Poisson process.[13] As a point of comparison, we now make an analogous assumption, and study the asymptotic properties of that network formation process. In particular, we look for conditions on meeting rates and market size so that asymptotically almost surely the realized graph satisfies the SLOP.

[13] See the surveys by Baye, Morgan, and Scholten (2004) and Chade, Eeckhout, and Smith (2015) and references therein.

The network formation process considered here follows the standard Erdos-Renyi model (Erdös and Rényi, 1959). Given a set of buyers, $\mathcal{J}$, and a set of sellers, $\mathcal{I}$, we assume that the probability with which any link $(i, j) \in \mathcal{I} \times \mathcal{J}$ is formed is $\lambda > 0$. If a link (i, j) is formed, we assume that (j, i) is also formed. This induces a natural probability distribution on the space of bipartite, undirected graphs with nodes in $(\mathcal{I}, \mathcal{J})$. Since each seller-buyer meeting occurs with a constant Poisson parameter, $\lambda > 0$, this is a natural point of comparison to random search models with Poisson arrival rates. Finally, let $\theta \equiv \frac{I}{J}$, and $t = \min\{J, \theta J\}$. That is, fix a constant market tightness and let the size of the market, t, be characterized by the short side of the market.

For this section, we assume that the only admissible valuation profiles are those that satisfy the following: for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$, $\mu(i) < \mu(j)$. This assumption allows us to refinee our result on pairwise stable matchings in the following way: for graphs with Hamiltonian Cycles, the set of pairwise stable matchings is the set of perfect matchings. That is, matchings where all agents are matched. This additional structure is useful to prove Proposition 2.

Proposition 2. *Let $\theta > 0$, let $\mathbb{G}(\theta J, J)$ be the set of all bipartite, undirected graphs with node sets of cardinality θJ and J, and let $t = \min\{\theta J, J\}$. For each $\lambda \in (0, 1)$ let P_λ be a probability distribution over $\mathbb{G}(\theta J, J)$ such that each graph with K undirected edges is drawn with equal probability $\lambda^K (1 - \lambda)^{\theta J^2 - K}$. Consider a sequence $(\lambda_t)_{t \in \mathbb{N}}$ such that $\lambda_t > \frac{\ln(t)+\ln(\ln(t))+c_t}{t}$ where $c_t \to \infty$ with t. Then $\lim_{t\to\infty} P_{\lambda_t}(\{G\colon G \text{ satisfies the SLOP}\}) = 1$.*

The above proposition is a simple corollary from Frieze (1985). Suppose, for a moment, that $\theta = 1$. Three things are true. First, under our assumptions on the valuation profiles, if a graph has a Hamiltonian cycle then the only matchings that are pairwise stable are the perfect matchings. Second, under the assumptions about λ and t, Frieze (1985) states that asymptotically almost surely the realized graph will have a Hamiltonian cycle. Third, for any graph $\mathcal{G}$, and any maximal matching M, it is simple to prove that if $\mathcal{G}|M$ has a Hamiltonian cycle then it satisfies the M-AC. Thus, when $\theta = 1$, asymptotically almost surely the realized graph, $\mathcal{G}$, will be such that for all pairwise stable matchings, M, $\mathcal{G}|M \equiv \mathcal{G}$ will have a Hamiltonian cycle. Thus, for all pairwise stable matchings matchings, M, $\mathcal{G}$ will satisfy the M-AC, so it will satisfy the SLOP. The above logic also holds when $\theta \neq 1$, and this generates the results stated above (see appendix B for a formal proof). Finally, that $\mathcal{G}|M$ has a Hamiltonian cycle is sufficient, but not necessary, for $\mathcal{G}$ to satisfy the M-AC.

Proposition 2 is relevant for two reasons. First, it provides a foundation with which to understand the results of our simulations in section 5. For any given $\lambda \in (0, 1)$, as market size grows without bound, eventually the realized graph will satisfy the SLOP with probability arbitrarily close to 1. This is reflected in our simulation results, where for each value of λ, price dispersion disappears as market size grows. Second, this result provides some asymptotic comparative statics on how the expected number of links affects the presence of price dispersion. If the expected number of links per agent increases at rate $\ln(t)$, asymptotically the SLOP will hold.

Finally, proposition 2 can be adjusted to accommodate the case where λ is not constant across agents. Suppose that, for a given set $\mathcal{I}$, we had a sequence $(\lambda_i)_{i\in\mathcal{I}}$, where each $\lambda_i > 0$. Furthermore, assume the probability with which a seller i meets a buyer j is λ_i. For instance, different sellers might have different exposure, so that, on average, some sellers receive more links than others. If we define $\lambda = \min\{\lambda_i : i \in \mathcal{I}\}$, proposition 2 remains valid. That is, proposition 2 remains true as long as all sellers increase their expected number of links at a rate $\ln(t)$.

4 Application: Quantitative Analysis Applied to eBay

The online trading platform eBay provides a natural application of our model with *ex post* competition for identical products. It is the largest consumer auction platform in the world. It had approximately 157 million active registered users and \$20 billion in gross merchandise volume in the second quarter of 2015. One of the selling mechanisms in eBay are competitive auctions. In a recent paper, Einav, Kuchler, Levin, and Sundaresan (2015) report substantial price dispersion in auction prices of identical goods sold by the same seller (mean coefficient of variation 10-15 percent). A number of questions arise: Is the model capable of delivering the amount of price dispersion observed in real-world markets? What would happen to the amount of price dispersion in such markets if all sellers are contacted by buyers with the same probability (*i.e.* a change in the structure of the network without changing its sparsity)?

To answer the questions above, we calibrate our model using the network structure from eBay as documented by Backus, Podwol, and Schneider (2013) and the search behavior documented by Blake, Nosko, and Tadelis (2016). We find that the model reproduces the amount of price dispersion in eBay documented by Einav, Kuchler, Levin, and Sundaresan (2015) quite well (Table 1 discussed below). We also find that the amount of price dispersion in eBay as measured by the mean coefficient of variation would decrease substantially (35-45 percent as reported on p. 30 in Step 3) under a "Uniform Network Structure," whereby links are drawn with equal probability for all sellers and buyers (as defined in Step 3 discussed below).

4.1 A Simple Link Formation Process

In this application we approximate the link formation at eBay using random networks, where sellers have different probabilities of receiving a link.[14] We believe that our network formation

[14] One way to define a link in eBay is to look at the listings "clicked" by the potential buyer. Using this definition, a buyer and a seller are linked if the buyer clicked on the seller's listing at least once. A buyer and a seller are not linked if the buyer never clicked on the seller's listing. We use this definition of a link in the remainder of this section.

process using random networks captures the salient network structure at eBay due to four observations. First, eBay displays search results using a ranking algorithm called "Best Match." The Best Match algorithm was created to display items to maximize eBay's expected revenue (*i.e.* the probability that a product is purchased times its sale price; see Blake, Nosko, and Tadelis 2016). One important feature of the Best Match algorithm is that it is not tailored to individual users (potential buyers), nor does it consider prices explicitly (Dinerstein, Einav, Levin, and Sundaresan 2017). In other words, the Best Match algorithm does not "target" search listings based on the characteristics of the users. Thus, if two different users perform the same search query, the Best Match algorithm will display identical search results, independent of the users' characteristics. Second, most users never go beyond the first page of search results (Richardson, Dominowska, and Ragno 2007), and are reluctant to use other than the default settings in a search (Chau, Fang, and Liu Sheng 2005; Cone, Franklin, Ryan, and Stalker 2005). While the choice of the listing among the first page of results is endogenous, the selection of the listings in the first page of results is done by the Best Match algorithm and, conditional on the search query, is exogenous to the buyer. Third, the Best Match algorithm creates incentives for the sellers to design their listings to maximize their prominence in search results. However, as discussed above, they cannot target certain types of buyers through the ranking algorithm. Based on, *e.g.*, the number of words in the title or their rating, sellers are more or less likely to have their listings shown in the first page of results after a search query. Finally, auction listings are displayed so that the auction that closes first is on top (Dinerstein, Einav, Levin, and Sundaresan 2017). Thus, the relative timing of the users' search, and the end of the auction are important to determine which listings are displayed in the first page of results by the Best Match algorithm. This adds an element of randomness to the search listings displayed in the first page, in that the auction's ending time is unknown to most of the users when performing the search query.

4.2 Quantitative Analysis

The quantitative analysis proceeds in three steps: (1) simulation of the model using eBay's network structure; (2) calibration of the model's parameters using the simulated model and eBay's market level data;[15] and (3) counterfactual policy using the calibrated model, whereby we simulate a change in eBay's network structure to a "Uniform Network Structure."

Step 1: Simulation of the Model using eBay's Network Structure

First, we approximate the network structure at eBay. Second, we simulate the seller-buyer model from section 3, conditional on the network structure. Finally, we use a deferred acceptance algorithm to find a matching in each of the simulated markets. Now we describe

[15] Our "data" consist of the market level summary statistics reported by Einav, Kuchler, Levin, and Sundaresan (2015). See Step 2 below for details.

these sub-steps.

Step 1.1: A Simple Description of the Network Structure from eBay

To reconstruct the seller-buyer network structure we use the results reported by Backus, Podwol, and Schneider (2013, henceforth BPS). BPS define six types of sellers based on their feedback score. BPS report their distribution in the population of DVD listings and the median number of words per title (see summary statistics in BPS, Table 2). More experienced sellers attract more buyers (and improve their search ranking) by using more words in the listing's title. BPS estimates how the number of bidders depends on the feedback score and the number of words in the title (see BPS, Table 5).[16] We use the results from regression (2) in Table 5 from BPS to calculate the relative number of bidders that each type of seller receives.[17] When simulating markets from eBay's network structure, we use the same distribution of sellers' types as in BPS (*i.e.* six types) and define the relative probability of receiving a link for each type as the relative number of bidders that each type receives, as reported by the BPS estimates in Table 5. For example, the type with the highest feedback score receives 3.59 times more links than the type with the lowest feedback score.[18] This procedure creates a network structure where sellers' types with the highest feedback score receive a greater percentage of the links.

Step 1.2: Simulation of the Model Conditional on the Network

There are three parameters in the seller-buyer model from section 3: the number of buyers (J), the number of sellers (I), and the number of expected links per buyer (ELB). Every seller begins with one unit of a good (so the number of goods is I). The market tightness, θ, is the ratio of the number of buyers to the number of sellers, $\theta = \frac{J}{I}$.

We start the baseline simulation with $I = 2,000$ identical sellers[19] and J heterogeneous buyers (we describe buyers' valuations in the next paragraph). We vary the number of buyers (J) from 2,200 to 20,000 so that $\theta \in (1.1, 10)$. We also consider markets with $ELB \in [0.5, 10]$. We assign the $I = 2,000$ sellers to six seller types (sellers' types with higher feedback score have higher probability of receiving a link) using the results reported by BPS as described

[16]Note that the number of observed bidders is only a proxy for the latent number of actual bidders (many of which may never actually hold the standing high bid).

[17]Rather than documenting links, BPS provide quantitative evidence that the network is not fully connected. To do that BPS exploit a discontinuity in the visibility of auctions due to eBay's search tool, what allows them to identify search costs. Using our terminology, BPS show that observables that affect the network structure, also explain price dispersion.

[18]We use Table 2 from BPS to define each seller type, the median words per title, and a mid-point in the seller rating ($2.5 \times 6^{i_{type}}$ where $i_{type} \in \{1, 2, ..., 6\}$). We then use the coefficients on seller rating and median words in title from regression (2) in Table 5 of BPS to predict relative number of bidders. We predict 1.08 bidders ($0.215 \times 5 + 0.989 \times {}^{2.5}/_{10000}$) for the lowest type and 3.86 bidders for the highest type. We take ratios of the predicted number of bidders for the different types to get the relative probability of receiving a link in our model.

[19]We obtain almost identical results using 200 or 20,000 sellers. Results are available upon request.

in Step 1.1.

Buyers' valuations are normalized to range between 0 and 100, which bounds the minimum and maximum prices between those values. In the baseline analysis (Table 1, Panel A) we use a uniform $[0, 100]$ distribution for the buyers' valuations. We have performed a number of robustness checks using other distributions for buyers' valuations (see, *e.g.*, Panel B in Table 1). As discussed in subsection 4.3 the distribution of private valuations is not identified with our data.

Given the parameters J, I, and ELB, a network is formed by drawing links between buyers and sellers as follows. First, we draw $ELB \times J$ links. Then, we assign these links to buyers and sellers: all buyers have equal probability of receiving links; sellers receive links according to their relative probability of their seller type from Step 1.3 (recall that we consider six seller types following BPS). Once the network is constructed, we apply the deferred acceptance algorithm described in Step 1.3 to find a matching in each simulated market.

Step 1.3: Algorithm for Finding the Matching in each Simulated Market

To find the matching (an allocation and the pairwise stable prices that sustain it) we use a deferred acceptance algorithm as described below.[20] The deferred acceptance algorithm and the related technical issues that arise are presented in smaller font and can be skipped without loss of continuity.

A Deferred Acceptance Algorithm

We describe the algorithm as a "first-price auction" to give intuition of how the algorithm works. A formal description of the algorithm can be found in Section C in the appendix. We denote the agents on the side of the market that are holding the "auctions" as sellers and the agents on the other side that are "bidding" as bidders. This is only for expositional reasons; recall that we are approaching this problem from the matching perspective, so we are not making any statement about the actual economic mechanisms or incentives of the agents that determine prices and matches. Bidders bid in increments of $\frac{\Delta}{2}$. The value of Δ is set so that the productivity of buyers lie in a Δ grid. Formally, for all j, $\mu(j) = b + k_j\Delta$ for some integer k_j that is randomly drawn at the start of the algorithm. We describe the algorithm for the case where the sellers hold the auctions. When buyers hold the auctions, the bidding starts at their valuation and prices decrease.

The algorithm has two stages. The first stage outputs an allocation and is motivated by the wage adjusting process in Crawford and Knoer (1981) and Kelso and Crawford (1982). (See Section C in the appendix for a detailed comparison of the first stage of our algorithm and the algorithms in Crawford and Knoer and Kelso and Crawford.) This allocation has the property that there exist prices for which it is pairwise stable. The second stage outputs two prices: the pointwise minimum price at which the stage 1 allocation is stable, and the pointwise maximum price at which the stage 1 allocation is stable.

[20] One way to interpret our algorithm is in terms of a competing auctions environment similar to Peters and Severinov (2006), but where buyers are linked with a subset of the sellers (*i.e.* when there are market frictions). The environment of Peters and Severinov (2006) is frictionless in the sense that any buyer may participate in any auction. The bidding rule proposed by Peters and Severinov (2006) for their frictionless competing auctions environment is not a Perfect Bayesian Equilibrium when frictions are present (see Donna, Schenone, and Veramendi 2016).

Stage 1: The Matching Determination Program

The algorithm starts in round $t = 1$ when none of the sellers has received any bid. All bidders are placed into a queue and arrive sequentially. The entering order of the bidders is determined randomly. The *standing bid* of a seller is the last bid accepted by the seller or b if the seller has not received any bids. The *winning bidder* is the bidder who placed the last standing bid. The "bids" in the first stage of the algorithm take place on a grid of possible prices with $2J$ grid points.

This is round t of the matching determination program.

1. Take the first bidder in the queue (for concreteness, call it bidder j). Bidder j selects the seller with the lowest standing bid among the linked sellers. If there is more than one such seller, the bidder selects one of these sellers at random. Call it seller i. If the lowest standing bid is greater than $\mu(j) - \frac{\Delta}{2}$, bidder j does nothing and leaves the queue. Otherwise, bidder j bids the standing bid of seller i plus $\frac{\Delta}{2}$.

2. If bidder j makes a bid, seller i accepts the bid from bidder j. The new standing bid of seller i is now the previous standing bid plus $\frac{\Delta}{2}$. Bidder j leaves the queue. If there was a bidder j' who was the winning bidder (before bidder j bid), bidder j' is placed at the end of the queue.

3. The algorithm continues from step 1 with the next bidder in the queue. The algorithm stops when there are no bidders left in the queue. In this case, each seller is matched to the winning bidder.

We now present the second stage, the price determination program. The key insight of this stage is that, if a seller i is matched to a buyer j, and is also linked to an unmatched buyer j', then the price j pays i must price j' out of the market. That is, $p_M(i, j) \geq \mu(j')$. Moreover, if seller i is matched to buyer j, and seller i is also linked to a buyer j' who is also matched (say, to a seller i') then i must be getting payed at least what i' is getting payed. Otherwise, j' would like to block with i.

Stage 2: The Price Determination Program (I)

The program starts in round $t = 1$ with $M \subset E$ produced from stage 1 as its input.

1. Set the "price" of all unmatched sellers to b.

2. For matched sellers, set the price of seller i for buyer j to the maximum $\mu(j')$ amongst all j' that are linked to i but are not matched.

3. We call these prices $(\rho_i^1)_{i \in \mathcal{I}}$.

This is round $t > 1$ of the price determination algorithm. We take $(\rho_i^{t-1})_{i \in \mathcal{I}}$ as inputs for this round.

1. Set the "price" of all unmatched sellers in round t to b.

2. For matched sellers, set the price of each seller i for buyer j to the maximum price in round $t - 1$ of the matched buyers that are linked to i. That is, amongst all matched j' that are linked to i, set ρ_i^t to the maximum $\rho_{i^*(j')}^{t-1}$. Note that one such j' is j itself, so these prices form a non-decreasing sequence.

3. If $\rho_i^t = \rho_i^{t-1}$ for all i, stop the algorithm and output these prices. Otherwise, start step $t + 1$.

As formally stated in Proposition 3, the Price Determination Program (I) captures the pointwise minimum price function at which M is stable. A modified version of this program, which we call Price Determination Program (II), generates the pointwise maximum price function at which M is stable. Rather than starting with ρ^1 at a low value, with successive iterations iterations rising it, the modified program starts with ρ^1 at high values and successive iterations lower it. Section C contains the formal algorithm, including both versions of the Price Determination Program.

Proposition 3. *The deferred acceptance algorithm has the following properties:*

1. *It stops after a finite number of rounds.*

2. *It outputs a pairwise stable allocation.*

3. *Price Determination program (I) outputs the pointwise minimum price function at which M is stable.*

4. *Price Determination program (II) outputs the pointwise maximum price function at which M is stable.*

The proof of Proposition 3 is in section C in the appendix.

Step 2: Calibration

Einav, Kuchler, Levin, and Sundaresan (2015, henceforth EKLS) report the distribution of several measures of price dispersion across different markets in eBay (*e.g.* different coefficient of variations for different "markets," where a market is defined as a set of identical products sold by the same seller). In our model, these markets are characterized by a combination of expected links per seller (ELB) and market tightness (θ). The econometric problem consists of finding which markets (*i.e.* combinations of ELB and θ) best rationalize price dispersion in eBay, conditional on the network structure from Step 1.1.

The joint distribution of ELB and θ is not identified in this setting. Our data consist of 3 measures of price dispersion reported by EKLS: coefficient of variation, $^{(75\text{th pctile}-25\text{th pctile})}/_{\text{median}}$, and $^{(90\text{th pctile}-10\text{th pctile})}/_{\text{median}}$. Note that these measures are aggregated at the market level, defined as identical products sold by the same seller. For example, consider the measure of coefficient of variation reported by EKLS that for the mean market corresponds to 11.1 percent (see EKLS, p. 223, Table 2). In our model, there is more than one market (defined as a combination of ELB and θ) that generates the same mean price dispersion conditional on the network structure from step 1.1.[21] Thus, ELB and θ are not jointly identified from the market level data reported by EKLS and cannot be estimated.

To show that the model can reproduce the amount of price dispersion in eBay we calibrate the parameters of the model, (ELB, θ), as follows. First, for the parameter ELB we use the search behavior documented by Blake, Nosko, and Tadelis (2016). Consider a buyer who is "linked" to a seller according to our model in section 3. A necessary condition to have a link in eBay is to have "clicked" on the listing (*i.e.* to have seen the complete listing according to the definition of the link in the model). So one can interpret the mean clicked items per buyer as an upper bound to the mean ELB. The mean clicked items per buyer in eBay is 5.25 per category (Blake, Nosko, and Tadelis 2016, p. 13, Table 1).[22]

Second, we find the value of θ that minimizes the distance between the predicted measure of price dispersion in eBay reported by EKLS and the corresponding prediction by our model, conditional on the network structure from Step 1.1 and $ELB = 5.25$. For each summary statistic of coefficient of variation (*e.g.* mean, 25th percentile, 75th percentile, *etc.*), this procedure outputs a θ that gives the best prediction of the model relative to the number reported by EKLS.

Table 1 displays the results for the coefficient of variation. The first line in Panel A reports the results from EKLS. The second line, "eBay Network Structure," reports the corresponding measure obtained by our model using the network structure from Step 1.1.

[21] For example, $(ELB_L, \theta_H) = (3, 4.4)$ and $(ELB_H, \theta_L) = (7, 1.1)$ generate the same mean price dispersion of 11.1 percent.

[22] Blake, Nosko, and Tadelis (2016) investigate returns to consumer search, so they do not focus on identical products. The mean click items per buyer across categories is 12.5 and the mean categories per buyer is 2.39 ($5.25 = {}^{12.55}/_{2.39}$).

Table 1: Price Dispersion at eBay with Different Network Structures

	Coefficient of Variation		
Panel A: Calibration	25th Percentile	Mean	75th Percentile
Einav, Kuchler, Levin, and Sundaresan (2015)	0.018	0.111	0.148
eBay Network Structure	0.018	0.109	0.149
Uniform Network Structure	0.010	0.063	0.097
Uniform Network Structure/eBay Network Structure	0.551	0.576	0.649
Panel B: Robustness			
Uniform Network Structure/eBay Network Structure	(0.306,0.598)	(0.545,0.835)	(0.618,0.857)

Notes: Price dispersion from simulations of the calibrated model. The first row shows three moments of the coefficient of variation (CV) for different seller-listing pairs in Einav, Kuchler, Levin, and Sundaresan (2015). We simulate a large number of markets using two network structures of the model where: (i) certain sellers' types have higher probabilities of receiving a link than others ("eBay Network Structure" as reported by Backus, Podwol, and Schneider (2013); see Step 1 in the text in section 4) and (ii) all sellers have the same probability of receiving a link ("Uniform Network Structure"). We find that markets with market tightness (θ) equal to 7.8, 2.0, and 1.6, reproduce the 25th percentile CV, the mean CV, and 75th percentile CV from Einav, Kuchler, Levin, and Sundaresan (2015), respectively. Under "Uniform Network Structure" the market is re-simulated so that all sellers have the same probability of receiving a link (*i.e.* representative seller), keeping the number of links and market tightness constant. The ratio (Uniform Network Structure/eBay Network Structure) shows the ratio of the CV for the two market structures. For the baseline results in Panel A, we use $ELB = 5.25$ (the mean clicked items per category documented by Blake, Nosko, and Tadelis 2016) and we assume that buyers' valuations are drawn from a uniform distribution $[0, 100]$. For robustness in Panel B, we repeat the exercise for different values of $ELB \in [0.5, 10]$ and for different distributions of buyer's valuations as discussed in subsection 4.3. Panel B shows the lower and upper bounds of the ratio (Uniform Network Structure/eBay Network Structure) when varying ELB and the distribution of valuations.

The model reproduces the amount of price dispersion observed in eBay quite well.[23]

Step 3: Change in the Network Structure as a Counterfactual Policy

With the calibrated model we consider the counterfactual policy where all sellers and buyers receive the same expected number of links (*i.e.* a change in the network structure from Step 1.1, with only one representative type). This policy can be thought as the result of an enhancement in eBay's search algorithm that reduces price dispersion in the network.[24]

For the counterfactual policy we simulate the network using the calibrated model (*i.e.* $ELB = 5.25$ and the calibrated θ for the relevant summary statistic of the measure of price dispersion) and a "Uniform Network Structure," whereby links are drawn with equal probability for all sellers, and all buyers receive the same expected number of links, $ELB = 5.25$.

[23] We obtain similar results for the other measures of price dispersion reported by EKLS. Results are available upon request.

[24] Intuitively one can think of the search results being displayed such that all identical products are pooled together automatically by the search algorithm. With individual level data one could also allow more realistic change in the network structure (*e.g.* allow sellers with high feedback score to provide amenities valued by consumers, such as fast shipping, returns, *etc.*).

The line labeled "Uniform Network Structure" in Table 1, Panel A shows the results. Under a "Uniform Network Structure" the amount of price dispersion measured by the mean coefficient of variation drops by 53 percent $\left({}^{(0.109-0.063)}/_{\frac{1}{2}(0.109+0.063)}\right)$ relative to the eBay's network structure. The network structure alone explains 42.2 percent $(1 - {}^{0.063}/_{0.109})$ of the price dispersion in eBay, as measured by the mean coefficient of variation. Table 1 shows that similar results are obtained using other statistics of the coefficient of variation, such as the 25th and 75th percentiles.[25]

4.3 Discussion

Robustness to Alternative Distribution of Valuations and ELB. Our data consists of one cross section of summary statistics (aggregate or market level data as it is typically referred in the industrial organization literature) on several standardized measures of price dispersion. It is impossible to identify the distribution of buyers' valuations from this data. For the calibration exercise we use a uniform distribution of buyers' valuations whose support we normalize between 0 and 100. For robustness, we have repeated the quantitative analysis using a lognormal distribution and a normal distribution truncated at 0 for the buyers' valuations,[26] and different values for ELB. Panel B in Table 1 summarizes the results of this robustness analysis. It shows the lower and upper bounds of the ratio $\left({}^{\text{Uniform Network Structure}}/_{\text{eBay Network Structure}}\right)$. That is, to obtain the lower and upper bounds in Panel B we proceed as follows. (1) For each distributions of buyers' valuations (uniform $[0, 100]$, lognormal, and normal truncated at zero, the last two distributions with the same mean and standard deviation as the uniform $[0, 100]$ distribution) and for each $ELB \in [0.5, 10]$, we repeat the quantitative analysis T times.[27] Each combination of distribution of buyers' valuations and ELB outputs a number for the ratio, $\left({}^{\text{Uniform Network Structure}}/_{\text{eBay Network Structure}}\right)$, for each summary statistic of the coefficient of variation. (2) The lower bound, for each summary statistic, is the minimum number obtained for the ratio, $\left({}^{\text{Uniform Network Structure}}/_{\text{eBay Network Structure}}\right)$, across $t = 1, \ldots, T$ from step 1. Similarly for the upper bound.

Link Formation Process in eBay. Our theoretical analysis provides a general framework to study price dispersion and *ex post* competition in any realized network (*i.e.* conditional on the realized network). It provides no guidance, however, to the link formation process

[25]We also obtain similar results using other measures of price dispersion (such as percentile differences ratios). Results are available upon request.

[26]The level of these distributions is not identified with our data, so we use the same mean and standard deviation as the uniform $[0, 100]$ distribution. This is allows us to compare the results in the two panels of Table 1.

[27]We use increments of 0.1 for $ELB \in [0.5, 10]$, so we repeat the quantitative analysis $T = 288$ times:

$$\underbrace{(3)}_{\text{3 Distribution of Buyers' Valuations}} \times \underbrace{(96)}_{ELB\in[0.5,10],\text{ with 0.1 increments}} = 288.$$

or the search behavior of buyers. Thus, our analysis in this subsection takes as given the network structure documented by Backus, Podwol, and Schneider (2013) and the search behavior documented by Blake, Nosko, and Tadelis (2016). Although our model is not intended to capture the specific ways in which the links in the network arise, nevertheless our approximation using random networks (see subsection 4.2) summarizes the main aspects of eBay that are relevant to model its network structure. Enriching the model in these dimensions is an avenue of future research.

5 Price Dispersion in Finite Random Networks

In this section, we investigate price dispersion in finite random networks. First we explain how we simulate random networks. Then we discuss simulation results.

5.1 Simulation

As in the eBay application, there are three parameters in the seller-buyer model: the number of buyers (J), the number of sellers (I), and the expected number of links per buyer (ELB).

We start the baseline simulation with $I = 10,000$ identical sellers and $J = 10,000 \times \theta$ heterogeneous buyers. Similar to the eBay application, we use a uniform $[0, 100]$ for the distribution of buyers' valuations, which bounds the minimum and maximum prices between those values. We consider markets with $J \in [1000, 50000]$, so $\theta \in [0.1, 5]$. We also consider markets with $ELB \in [1, 10]$.[28] The higher the ELB, the lower the frictions in the market. The product of the number of buyers and the ELB determines the number of active links in the market. The total number of possible links in the market is $J \times I$. The proportion of active links relative to the total number of possible links in a network is a measure of the sparsity of the network. Given the parameters J, I, and ELB, a network is formed by randomly drawing buyers and sellers to form links. Once the network is constructed, we apply the algorithm from Section 4.2 to the network. As in the eBay application, the "bids" in the first stage of the algorithm take place on a grid of possible prices with $2J$ grid points.

We compare the price distributions to the frictionless outcome (henceforth Walrasian outcome), where all buyers are linked to all sellers. The Walrasian outcome price, $p_{Walrasian}$, is given by:

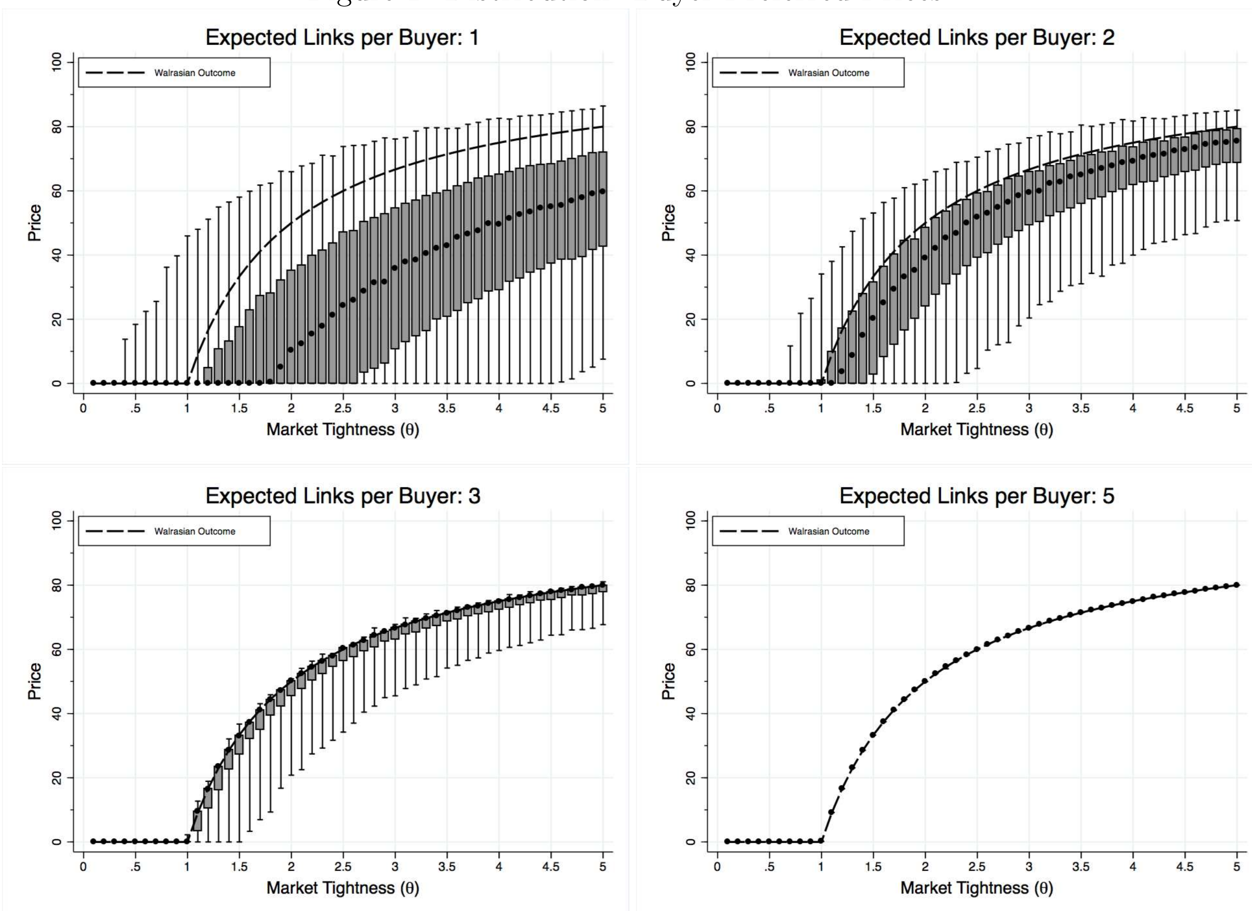
$$p_{Walrasian} = \begin{cases} 0 & if \quad \theta \leq 1 \\ (1 - \frac{1}{\theta}) \times 100 & if \quad \theta > 1. \end{cases}$$

Recall that the Walrasian outcome has a constant price (see Remark 1). When $\theta \leq 1$, there

[28] We obtain almost identical results: (i) using 1,000 or 100,000 sellers (instead of $I = 10,000$); (ii) varying expected links per seller, ELS (instead of varying the expected links per buyer, ELB); and using alternative distributions (*e.g.* normal distribution truncated at 0 or a lognormal distribution, both with the same mean and standard deviation as the uniform $[0, 100]$) for buyers' valuations (instead of uniform $[0, 100]$). Results are available upon request.

are more sellers than buyers and so there is always a seller who is indifferent between selling the good at 0 or not selling it at all. In other words, the reservation price of the marginal seller is zero, which is what determines the market price. When $\theta > 1$, there are more buyers than sellers. Only $\frac{1}{\theta}$ of the buyers buy the good. Hence the valuation of the marginal buyer is $(1 - \frac{1}{\theta}) \times 100$. This buyer is indifferent between paying $(1 - \frac{1}{\theta}) \times 100$ and leaving the market, and so the market price is $(1 - \frac{1}{\theta}) \times 100$.

Figure 1: Distribution: Buyer-Preferred Prices.



Notes: Starting in the top left, panels 1 to 4 figure display the empirical distribution of prices from the model using the buyer-preferred match disaggregated by: i) Market Tightness (which ranges from 0.1 to 5 in the horizontal axis in each graph) and ii) Expected Links per Buyer (1, 2, 3, and 5). Each vertical box corresponds to a simulated market characterized by those parameters. Each vertical box displays the 95th percentile (upper whisker), 75th percentile (upper hinge), median (black circle marker), 25th percentile (lower hinge), and 5th percentile (lower whisker). Note that buyers' valuation is normalized to range between 0 and 100 which, in turn, bounds the minimum and maximum prices between those values. If the 95th percentile coincides with the 5th percentile, then the figure shows only a dot (which corresponds to the median too). In addition, each panel displays the Walrasian outcome, $p_{Walrasian}$. We describe how to calculate the Walrasian outcome in subsection 5. The distributions for the seller-preferred prices, although stochastically dominating the buyer-preferred prices, exhibit similar distributions. See online appendix for figures showing both distributions.

5.2 Results

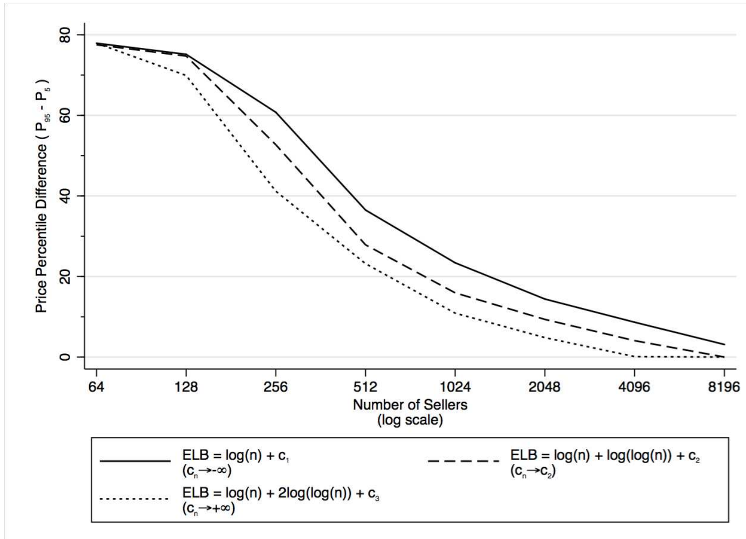
Distribution of Prices. Figure 1 displays the distribution of prices for the buyer-preferred match by market tightness (horizontal axis in each panel) and ELB (different panels). Each vertical box corresponds to a simulated market characterized by those parameters. Each panel shows the population distribution of prices for different levels of frictions in different markets. The top-left panel shows the price distribution for high frictions, where ELB equals 1. The top-right and bottom panels show what happens in markets with lower frictions (when ELB equals 2, 3, and 5, respectively). At low levels of θ there are many sellers for each buyer. So low numbers for θ indicate "loose" seller markets where sellers are at a disadvantage. In addition, each panel displays the Walrasian outcome.

For market tightness less than one, the market looks like a monopsony and nearly all sellers are paid their valuation. This is because it is unlikely for a seller to receive multiple links. Even if a seller receives two links, it is likely that at least one of the buyers has an outside option of zero. This happens if the buyer is also linked with another seller who has no other links.

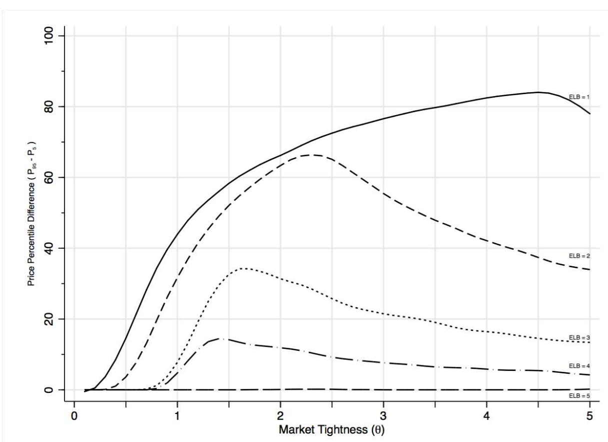
On the other hand, as market tightness is increased the market becomes more competitive between buyers and more favorable for sellers. The median price increases as does price dispersion. There are now many buyers linked to each seller and the buyers have worse outside options. Even if a buyer is linked to a second seller, it is likely that the second seller is linked to many other buyers. In markets with lower frictions, competition between buyers increases, thus increasing prices until they reach the Walrasian outcome.

Price Dispersion and the Walrasian Outcome. Price dispersion decreases when frictions decrease. There are many buyers linked to each seller, but there are also many sellers linked to each buyer, improving the outside options of both parties. These improved outside options reduces price dispersion (*i.e.* the likelihood that a seller has to take a low price is low, but at the same time the probability that a buyer has to pay a high price is also low). The top panel in Figure 2 shows the evolution of the price distribution for the buyer-preferred match. It shows the price percentile difference (95th percentile minus 5th percentile or $P_{95} - P_5$) when $\theta = 3$. Almost identical results are obtained for other values of θ. While there is price dispersion when $ELB \leq 4$, the price distribution begins to collapse for larger values of ELB. When $ELB = 5$, ninety percent of the prices are equal to the Walrasian outcome. Likewise, when $ELB = 8$, ninety-nine percent of the prices are equal to the Walrasian outcome. In other words, at least 90 percent or 99 percent of the sellers are paid the same price when the number of active links relative to the total number of links is only 5/10,000 or 8/10,000, respectively. The price distribution in the model collapses with less than 0.1 percent of the possible links in the network.

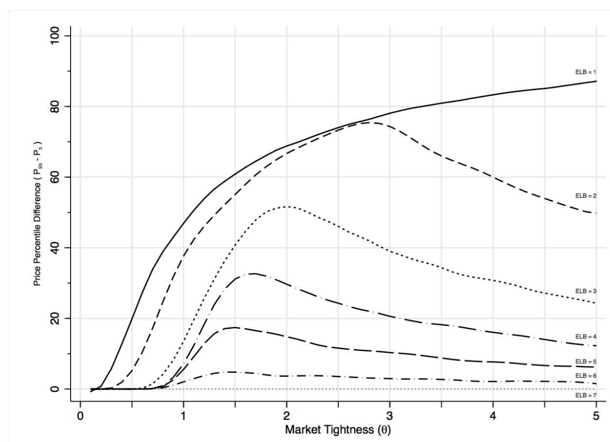
Figure 2: Price Dispersion in Finite-sized Random Networked Markets



Poisson Random Networks



Non-Poisson Random Networks



Notes: The figure displays the price dispersion for simulated random markets. The top panel shows the finite-network properties of proposition 2 in subsection 3.5, where the market tightness (θ) is set to 3 and the expected links per buyer is ELB$=\lambda * n = (\frac{\log n + \log\log n + c_n}{n}) * n$ and we use $c_n = \pm \log\log(N) + c$. Results are similar for other values of θ. The bottom panels show how the price dispersion in random markets with 10,000 sellers depends on θ, ELB, and the random network formation process. The figure on the bottom left shows the price dispersion in markets with Poisson random networks, where every seller has equal probability of drawing a link (λ). The figure on the lower right, shows the price dispersion in non-Poisson random networks, where sellers have different probabilities of receiving a link. Let $j \in \{1, 2, .., 10000\}$ index sellers from lowest probability to highest probability of receiving a link. The non-Poisson random networks are generated by setting the relative probability that seller j receives a link (compared to the highest probability seller $j = 10000$) to $\frac{\Pr_j}{\Pr_{10000}} = 0.25 + 0.75\frac{j}{10000}$.

Price Dispersion in Finite-Sized Random Networks. Figure 2 shows how price dispersion depends on the market size, market tightness, number of links, and the structure of the network (Poisson and non-Poisson random networks). We use the 95th-5th percentile difference in prices as our measure of price dispersion.[29] The top panel shows the finite-network properties of proposition 2 in subsection 3.5, where the market tightness is $\theta = 3$, the expected links per buyer is ELB=$\lambda \times n = (\frac{\log n + \log\log n + c_n}{n}) \times n$, and we use $c_N = \pm \log\log(N) + c$. Results are similar for other values of θ. The three lines correspond to three rates at which λ monotonically goes to zero under the conditions of the proposition in subsection 3.5:

$$\begin{aligned}
&\text{- } if\ c_N = -\log\log(N) + c, && then\ \mathbb{P}(Hamiltionian\,Cycle) = 0\ as\ N \to \infty,\\
&\text{- } if\ c_N = c, && then\ \mathbb{P}(Hamiltionian\,Cycle) \in (0,1)\ as\ N \to \infty,\\
&\text{- } if\ c_N = +\log\log(N) + c, && then\ \mathbb{P}(Hamiltionian\,Cycle) = 1\ as\ N \to \infty.
\end{aligned}$$

Our analysis of finite-sized random networks shows that: (1) price dispersion can disappear even in finite-sized markets; (2) price dispersion decreases even when the probability that a buyer meets a seller goes to zero too fast to guarantee a Hamiltonian cycle asymptotically.[30]

The bottom panels investigate how price dispersion depends on market tightness (θ), ELB, and the structure of the network. The bottom left panel shows the price dispersion in markets with Poisson random networks, where every seller has equal probability of drawing a link ($\lambda_{ij} = \lambda$). The bottom right panel shows the price dispersion in non-Poisson random networks, where sellers have different probabilities of receiving a link. We chose a distribution of probabilities such that the lowest probability seller receives links at half the rate than the average probability seller, and the highest probability seller receives links at twice the rate than the average probability seller.[31]

To summarize, our analysis of finite random networks shows the following. (1) Price dispersion in these networks is large when ELB is small. (2) Price dispersion decreases rapidly as buyers are linked to more sellers. This has implications for policies that reduce frictions as price dispersion decreases quickly with the number of links. (3) The structure of the network matters. The non-Poisson markets have substantially higher levels of price dispersion compared to the Poisson markets. This indicates that policies that affect how buyers and sellers meet can be important in decreasing price dispersion, as discussed in our eBay application in the previous section.

[29] Results are similar if we use other percentile differences or the fraction of prices that are equal to the Walrasian price. Results are available upon request.

[30] In the online appendix we show that price dispersion does not change with the size of the market when ELB is fixed.

[31] Specifically, let $j \in \{1, 2, \ldots, 10,000\}$ index sellers from lowest probability to highest probability of receiving a link. The non-Poisson random networks are generated by setting the probability that seller j receives a link with buyer i relative to the probability that seller $j = 10,000$ (the highest probability seller) receives a link with buyer i to: $\frac{\lambda_{ij}}{\lambda_{i10,000}} = 0.25 + 0.75\frac{j}{10,000}$.

6 Concluding Remarks

The defining characteristic of markets with frictions is that similar goods or services are traded at different prices, resulting in price dispersion. In this paper we use networks to characterize pairwise stable allocations and their supporting prices in seller-buyer markets with frictions. Next we characterize the set of graphs where the only prices that support pairwise stable matchings are constant prices, where each matched buyer pays the same price. Such graphs can never exhibit price dispersion. We then use tools from the random networks literature to derive conditions under which random graphs have no price dispersion. We use simulations to understand the relevance of our asymptotic results in large, but finite, networks. Finally, we calibrate our model to the online trading platform eBay and show that our model replicates the price dispersion documented at eBay quite well. We use the calibrated model to provide predictions on counterfactual network structures at eBay.

In addition to price dispersion, our model is informative on other aspects of markets, such as labor market dynamics. In the online appendix section C, we perform a quantitative analysis of the labor market that sheds light on the following question in the labor literature:[32] Why has worker mobility declined over the last three decades in the US labor market? To do the quantitative analysis, we build a discrete time on-the-job search model using our quantitative random network model (Section 5).[33] As in the seller-buyer model, indirect competition pushes wages and labor market dynamics toward the frictionless outcome even when frictions are present (*i.e.* even in sparse networks). This leads to a novel prediction about worker mobility relative to the search literature. Adding more links (increasing the job offer rate) can lead to *lower* wage growth and mobility. Again, in the frictionless outcome, workers and firms immediately find their best match and so there is no wage growth or mobility. Our model predicts that due to indirect competition workers do not always benefit from reducing frictions. Workers in tight markets will have higher wages, while workers in loose markets will have lower wages. These predictions give us new insights into the impact of technologies (*e.g.* the internet) and policies (*e.g.* job search assistance programs) that reduce frictions.

References

Acemoglu, D., and R. Shimer (2000): "Wage and technology dispersion," *The Review of Economic Studies*, 67(4), 585–607. (Cited on page 4.)

Albrecht, J., P. Gautier, and S. Vroman (2006): "Equilibrium Directed Search with Multiple Applications," *Review of Economic Studies*, 73(4), 869–891. (Cited on page 4.)

[32] See *e.g.* evidence in Molloy, Smith, Trezzi, and Wozniak (2016).

[33] Search models can be mapped into the corresponding network using the same firms and workers where each worker receives a link from a firm when they receive a job offer from that firm in the search model. Many search models are in continuous time. Hence, decreasing frictions in these models increases the offer arrival rates to workers. Yet, since these models are in continuous time, at any instant they only receive one offer.

——— (2014): "Efficient Entry in Competing Auctions," *American Economic Review,* forthcoming. (Cited on page 4.)

Backus, M., J. Podwol, and H. S. Schneider (2013): "Search Costs and Equilibrium Price Dispersion in Auction Markets," *Working Paper. Available at SSRN 2353444.* (Cited on pages 4, 23, 25, 29, and 31.)

Baye, M., J. Morgan, and P. Scholten (2004): "Price Dispersion in the Small and Large: Evidence from an Internet Price Comparison Site," *Journal of Industrial Economics*, 52, 463–496. (Cited on pages 4 and 21.)

Blake, T., C. Nosko, and S. Tadelis (2016): "Returns to Consumer Search: Evidence from eBay," Working Paper 22302, National Bureau of Economic Research. (Cited on pages 4, 23, 24, 28, 29, and 31.)

Burdett, K., and K. L. Judd (1983): "Equilibrium price dispersion," *Econometrica*, pp. 955–969. (Cited on page 4.)

Butters, G. R. (1977): "Equilibrium distributions of sales and advertising prices," *The Review of Economic Studies*, pp. 465–491. (Cited on page 4.)

Chade, H., J. Eeckhout, and L. Smith (2015): "Search and Matching Models in Microeconomics," *Mimeo*. (Cited on pages 4 and 21.)

Chade, H., and L. Smith (2006): "Simultaneous search," *Econometrica*, 74(5), 1293–1307. (Cited on page 4.)

Chau, M., X. Fang, and O. R. Liu Sheng (2005): "Analysis of the query logs of a web site search engine," *Journal of the American Society for Information Science and Technology*, 56(13), 1363–1376. (Cited on page 24.)

Cone, J., G. Franklin, G. Ryan, and W. Stalker (2005): "Adaptive search engine," US Patent App. 11/195,225. (Cited on page 24.)

Corominas-Bosch, M. (2004): "Bargaining in a Network of Buyers and Sellers," *Journal of Economic Theory*, 115, 35–77. (Cited on page 5.)

Crawford, V. P., and E. M. Knoer (1981): "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49, 437–450. (Cited on pages 26 and 52.)

Dinerstein, M., L. Einav, J. Levin, and N. Sundaresan (2017): "Consumer Price Search and Platform Design in Internet Commerce," Working Paper 20415, National Bureau of Economic Research. (Cited on page 24.)

Donna, J., P. Schenone, and G. Veramendi (2016): "Frictions in Internet Auctions with Many Traders: a Counterexample," *Economic Letters*, 138, 81–84. (Cited on page 26.)

Einav, L., T. Kuchler, J. Levin, and N. Sundaresan (2015): "Assessing sale strategies in online markets using matched listings," *American Economic Journal: Microeconomics*, 7(2), 215–247. (Cited on pages 1, 3, 4, 23, 24, 28, and 29.)

Elliott, M. (2014): "Heterogeneities and the Fragility of Labor Markets," *Mimeo*. (Cited on page 5.)

——— (2015): "Inefficiencies in networked markets," *American Economic Journal: Microeconomics,* forthcoming. (Cited on page 5.)

Erdös, P., and A. Rényi (1959): "On random graphs, I," *Publicationes Mathematicae (Debrecen)*, 6, 290–297. (Cited on page 22.)

Frieze, A. (1985): "Limit Distribution for the Existence of Hamiltonian Cycles in Random Bipartite Graphs," *European Journal of Combinatorics*, 6(4), 327–334. (Cited on pages , 3, 22, 49, and 50.)

Galenianos, M., and P. Kircher (2009): "Directed search with multiple job applications," *Journal of Economic Theory*, 144(2), 445–471. (Cited on page 4.)

Gautier, P. A., and C. Holzner (2016a): "Maximum matching in the labor market under incomplete information," Discussion paper, mimeo. (Cited on page 5.)

Gautier, P. A., and C. Holzner (2016b): "Simultaneous search and efficiency of entry and search intensity," Discussion paper, mimeo. (Cited on page 5.)

Gautier, P. A., J. L. Moraga-González, and R. P. Wolthoff (2014): "Structural Estimation of Search Intensity: Do Non-Employed Workers Search Enough?," *Mimeo*. (Cited on page 4.)

Jackson, M. O. (2008): *Social and Economic Networks*, vol. 3. Princeton University Press Princeton. (Cited on page 3.)

Julien, B., J. Kennes, and I. King (2000): "Bidding for Labor," *Review of Economic Dynamics*, 3(4), 619–649. (Cited on page 4.)

Kelso, A. S., and V. P. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504. (Cited on pages 26 and 52.)

Kircher, P. (2009): "Efficiency of Simultaneous Search," *Journal of Political Economy*, 117, 861–913. (Cited on pages 2 and 4.)

Kranton, E., and D. F. Minehart (2001): "A Theory of Buyer-Seller Networks," *American Economic Review*, 91, 485–508. (Cited on page 5.)

Manea, M. (2011): "Bargaining in Stationary Networks," *American Economic Review*, 101(5), 2042–80. (Cited on page 5.)

McAfee, R. P. (1993): "Mechanism Design by Competing Sellers," *Econometrica*, 61(6), 1281–1312. (Cited on page 4.)

Molloy, R., C. L. Smith, R. Trezzi, and A. Wozniak (2016): "Understanding declining fluidity in the US labor market," Discussion paper, FEDS Working Paper. (Cited on page 36.)

Mortensen, D. T. (2005): *Wage Dispersion: Why Are Similar Workers Paid Differently?* The MIT Press, Cambridge. (Cited on page 1.)

Morton, F. S., F. Zettelmeyer, and J. Silva-Risso (2001): "Internet car retailing," *The Journal of Industrial Economics*, 49(4), 501–519. (Cited on page 1.)

Peters, M., and S. Severinov (1997): "Competition among Sellers Who Offer Auctions Instead of Prices," *Journal of Economic Theory*, 75(1), 141–179. (Cited on page 4.)

——— (2006): "Internet auctions with many traders," *Journal of Economic Theory*, 130(1), 220–245. (Cited on pages 4 and 26.)

Polanski, A., and F. Vega-Redondo (2013): "Markets, bargaining, and networks with heterogeneous agents," *Available at SSRN 2218605*. (Cited on page 5.)

Richardson, M., E. Dominowska, and R. Ragno (2007): "Predicting clicks: estimating the click-through rate for new ads," in *Proceedings of the 16th international conference on World Wide Web*, pp. 521–530. ACM. (Cited on page 24.)

Stigler, G. J. (1961): "The economics of information," *The journal of political economy*, pp. 213–225. (Cited on page 4.)

Wolinsky, A. (1988): "Dynamic Markets with Competitive Bidding," *Review of Economic Studies*, 55(1), 71–84. (Cited on page 4.)

Wolthoff, R. (2015): "Applications and Interviews: A Structural Analysis of Two-Sided Simultaneous Search," *Working Paper*. (Cited on page 4.)

Appendix

A Non-Maximal Abstractions

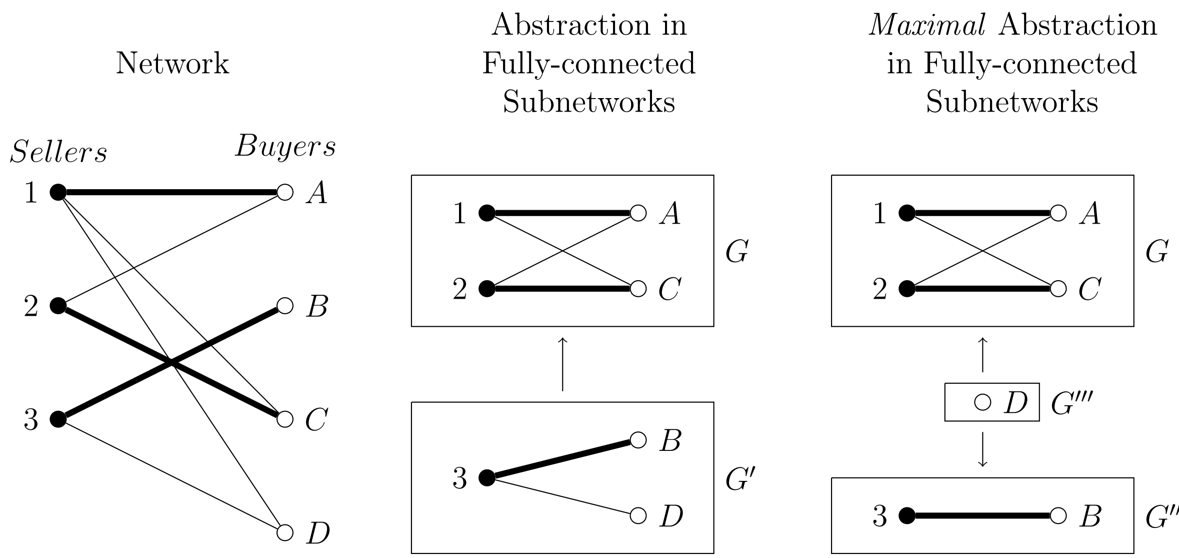
Proposition 4. *Let $\mathcal{N}$ be a network and M be a matching. Then the following are equivalent:*

1. *There exists a price function, p_M, such that M is pairwise stable in $\mathcal{N}$ at prices p_M.*

2. *There exists an abstraction of $\mathcal{N}$ in fully connected networks, $\mathcal{A}$, and a price function ρ for $\mathcal{A}$, such that ρ supports M in $\mathcal{A}$.*

While most of the intuition for our results comes from Proposition 4, this proposition does not characterize the full set of prices that support any given pairwise stable match. Proposition 4 tells us that any matching that is pairwise stable in a network, $\mathcal{N}$, is also pairwise stable with respect to some price function, ρ, for some abstraction of $\mathcal{N}$. Let $\mathcal{A}$ denote one such abstraction, and $\mathcal{P}^*$ denote the set of prices that support M in $\mathcal{A}$. Then, each $\rho \in \mathcal{P}^*$ induces a price function for $\mathcal{N}$ that supports M. Thus, given a matching, an abstraction that does not break the matching induces prices that support it. However, there might be other prices that also support the matching, which are not induced by a price that supports the matching in that particular abstraction. The example below shows this.

Example 4. In this example we present a network (left) and construct two abstractions from it. Assume that the valuations of the buyers are ordered as their labels ($\mu(A) > \mu(B) > \mu(C) > \mu(D)$), and costs are normalized to 0.



The abstraction in the middle imposes two constraints that prices need to satisfy in order to make the matching pairwise stable in the abstraction: stability in every subnetwork implies $\rho(G) \in [0, \mu(C)]$, $\rho(G') \in [\mu(D), \mu(B)]$; cheapest sorting implies $\rho(G') \leq \rho(G)$. Let $\mathcal{P}^*$ be set of price functions satisfying these conditions. Notice that any $\rho \in \mathcal{P}^*$ induces prices in the original network (say, p_M) that support M. However, there are other prices (say, p'_M) that also support M in the original network but are not induced by any $\rho \in \mathcal{P}^*$. For example, $p'_M(1, A) = p'_M(2, C) = \mu(C)$ and $p'_M(3, B) = \mu(B)$ support M in the original network, but is not induced by prices in $\mathcal{P}^*$.

Now, consider the abstraction on the right. The following are the constraints on prices that support M in the abstraction: $\rho(G) \in [\mu(D), \mu(C)]$, $\rho(G'') \in [\mu(D), \mu(B)]$. Now, any price function p_M that makes M pairwise stable in the original network is induced by a price function, ρ, that satisfies the above constraints.

Proposition 1 identifies a class of abstractions, which we call *maximal*, such that the constraints imposed by these abstractions are necessary and sufficient for stability. An example of a maximal abstraction is the rightmost abstraction in example 4. Thus, the maximality property of an abstraction buys us the full set of prices at which a matching is stable, not just a subset.

B Proofs of Propositions and Theorems

In this section we provide proofs for all Propositions and Theorems in section 3.

Proposition 1. Let $\mathcal{N}$ be a network and M be a matching. Let $\mathcal{A}$ be a maximal abstraction of $\mathcal{N}$ in fully-connected networks that does not break M.

1 If p_M supports M, there exists $\rho : A \to \mathbb{R}$ such that ρ induces p_M, and ρ supports M in $\mathcal{A}$.

2 If ρ supports M in $\mathcal{A}$, there exists $p_M : M \to \mathbb{R}$ such that p_M induces ρ, and p_M supports M in $\mathcal{N}$.

Then M is pairwise stable with respect to some price function p_M if, and only if, p_M induces prices, ρ, that support M in $\mathcal{A}$. Equivalently, M is pairwise stable with respect to some price function, p_M, if, and only, if p_M is induced by prices, ρ, that support M in $\mathcal{A}$.

Proof. For this proof, let $\mathcal{N} \equiv (\mathcal{I}, \mathcal{J}, E, \mu)$, M and $\mathcal{A} \equiv (A, E^*)$ be as in the statement of the theorem.

Item 1:

Let p_M support M. Because M is pairwise stable, it must be maximal. Define ρ as follows:

- If $a \in A$ is written as $a = (\{i\}, \emptyset, \emptyset)$, $\rho(a) = \mu(i)$,

- If $a \in A$ is written as $a = (\emptyset, \{j\}, \emptyset)$, $\rho(a) = \mu(j)$,
- Else, $\rho(a) = p_M(i, j)$ for some $(i, j) \in M$. This is well defined because $\mathcal{A}$ does not break M, and a is fully connected.

By definition ρ induces p_M. Also, ρ induces stability with respect to each subnetwork. Let $(a, a') \in E^*$. Because M is maximal, there is j in a and i in a' such that $(j, i) \in E \setminus M$. Thus $\rho(a) = p_M(j, i^*(j)) \leq p_M(j^*(i), i) = \rho(a')$, so cheapest sorting holds. Thus, ρ supports M in $\mathcal{A}$.

Item 2:

For each $i \in \mathcal{I}$, let $a_i \in A$ be such that i belongs to a_i (i.e. $i \in \mathcal{I}_{a_i}$). For each $j \in \mathcal{J}$, let $a_j \in A$ be such that i belongs to a_j (i.e. $i \in \mathcal{J}_{a_j}$). Since $\mathcal{A}$ does not break M, if $(i, j) \in M$, $a_i = a_j$. For each $(i, j) \in M$, let $p_M(i, j) = \rho(a_i)$. Because ρ supports M in $\mathcal{A}$, then p_M is individually rational. Assume that $(i, j) \in E \setminus M$ wants to block M. Then $v(M, p_M)(i) < v(M, p_M)(j)$. There are two cases. Case 1: $a_i \neq a_j$. Then, $(a_j, a_i) \in E^*$. Since, $v(M, p_M)(i) < v(M, p_M)(j)$, then $\rho(a_i) < \rho(a_j)$, which contradicts that ρ supports M in $\mathcal{A}$. Case 2: $a_i = a_j$. Then $v(M, p_M)(i) = v(M, P_M)(j)$, which contradicts that $v(M, p_M)(i) < v(M, p_M)(j)$. Therefore p_M supports M. This concludes the proof. $\square$

We now prove Theorem 1: a graph satisfies the SLOP if, and only if, it satisfies the Strong Alternating Cycles condition. To make the proof simple, we split it into four lemmas. After we present and prove the lemmas, we present and prove the theorem. As mentioned before, since we work with valuation profiles where linked agents have positive gains from trade, the set of pairwise stable matchings is the set of maximal matchings. We also use the following observation: for directed graphs, the property of being connected (*i.e.* given any two nodes, there is always a path from one to the other) and the property of having a complete cycle (*i.e.* there exists a directed path that starts and ends in the same node, and visits all nodes) are equivalent properties. Since abstractions are directed graphs, an abstraction is connected if, and only if, it has a complete cycle.

Lemma 1. *Let G be a graph that satisfies the SLOP. For all maximal matchings, M, there exists a maximal abstraction of $G|M$, that does not break M and is connected.*

Proof. Let $G = (\mathcal{I}, \mathcal{J}, E)$ be a graph, and let M be any maximal matching. Enumerate $M = \{(i_1, j_1), ...(i_K, j_K), (j_1, i_1), ..., (j_K, i_K)\}$. Define $\mathcal{A} = (\{a_1, ..., a_K\}, E^*)$ as follows: $a_k = (\{i_k\}, \{j_k\}, \{(i_k, j_k), (j_k, i_K)\})$, and E^* as in the definition of abstractions. By construction, $\mathcal{A}$ does not break M; it only remains to be shown that $\mathcal{A}$ is connected. Consider the following valuation profile, μ. First, for all sellers, normalize all costs to 0: that is, $\mu(i) = 0$ for each $i \in \mathcal{I}$. Second, let $a \in A$ be arbitrarily chosen, and let $\bar{A} = \{\hat{a} \in A : \hat{a} = a$ or there exists a directed path that connects a to $\hat{a}\}$. For each buyer j in $\hat{a} \in \bar{A}$ let $\mu(j) = p > 0$, for all other j let $\mu(j) = q \in (0, p)$. Given μ, define ρ as follows: $\rho(\hat{a}) = p$ if $\hat{a} \in \bar{A}$, and $\rho(\hat{a}) = q$ otherwise. Note that ρ induces stability with respect to each node. Moreover, cheapest sorting is verified for ρ: indeed, for all $a', a'' \in A$, if there is a path from a' to a'', then it cannot be that $a' \in \bar{A}$, and $a'' \notin \bar{A}$. Thus, for all $a', a'' \in A$, $\rho(a') \leq \rho(a'')$ with strict inequality if $a' \notin \bar{A}$ and $a'' \in \bar{A}$. Since M is pairwise stable with respect to ρ in $\mathcal{A}$, and since SLOP holds, this can only hold if $A = \bar{A}$. Since $a \in A$ was arbitrarily chosen, then $\mathcal{A}$ is connected abstraction. □

Lemma 2. *Let G be a graph. Assume that for all maximal matchings (M), there exists a connected maximal abstraction of $G|M$ (say, $\mathcal{A}$) that does not break M. Then G satisfies the SLOP.*

Proof. Pick G, M and $\mathcal{A}$ arbitrarily as in the statement of the Lemma. Let μ be any valuation profile such that M is pairwise stable given μ.[34] Let $\rho : A \to \mathbb{R}$ be such that M is pairwise stable with respect to ρ in $\mathcal{A}$, and let p_M be the corresponding prices in G. Then ρ is constant because $\mathcal{A}$ is connected. Thus, p_M is constant. Since μ and ρ where arbitrarily chosen, this implies SLOP holds. □

Lemma 3. *Let G be a graph, and M be a maximal matching in G. Suppose there exists a connected abstraction of $G|M$ that does not break M. Then G satisfies the M-AC.*

Proof. Let G, and M be as in the statement of the Lemma. Let $\mathcal{A}$ be the connected abstraction of $G|M$ that does not break M. If a connected abstraction that does not break M exists, then so does a connected abstraction where each node contains a single matched pair. Indeed, if a node in the abstraction contains T pairs, it can be split into T different nodes that each contain one pair, and where each has a directed link to the other. Thus, without loss of generality, we assume $\mathcal{A}$ is of this form. Enumerate $M = \{(i_1, j_1), ..., (i_K, j_K), (j_1, i_1), ..., (j_K, i_K)\}$ and $A = \{a_1, ..., a_K\}$ such that a_k is associated to the subgraph $(\{i_k\}, \{j_k\}, \{(i_k, j_k), (j_k, i_k)\})$. Since $\mathcal{A}$ is a directed connected graph, then it is cyclical. Let $\mathcal{C}^* = \{c_1^*, ..., c_T^*\}$ be a cycle in $\mathcal{A}$. Construct a cycle $\mathcal{C} = \{c_1, ..., c_{2T}\}$ in $G|M$ as follows:

- Let $c_1 = c_1^*$, $c_3 = c_2^*$, (...), $c_{2t-1} = c_t^*$,
- For each odd t, if $c_t^* = (a_{k'}, a_k)$ (with $k' \neq k$), let $c_{t+1} = (i_k, j_k)$.

In words, the odd links join buyers and sellers that belong to different nodes in the abstraction, the even links join buyers and sellers that belong to the same node in the abstraction. Because $\mathcal{A}$ does not break M, this is an alternating cycle: $e_t \notin M$ for all odd t, and $e_t \in M$ for all even t. Thus, G satisfies the M-AC. □

Lemma 4. *Let G be a graph that satisfies the SAC. Let M be any maximal matching in G. Then, there exists a maximal abstraction of $G|M$ (say, $\mathcal{A}$) such that M does not break $\mathcal{A}$, and $\mathcal{A}$ is connected.*

[34]Since M is maximal, such μ always exists: for example, $\mu(j) = 3$ if buyer j is matched, and $\mu(j) = 1$ otherwise; $\mu(i) = 0$ if seller i is matched, $\mu(i) = 2$ otherwise. Because M is maximal no two unmatched agents can be linked, so $\mu(j) > \mu(i)$ for all linked pairs (i, j).

Proof. Let G be a graph, and $M = \{(j_t, i_t) : 1 \leq t \leq K\} \cup \{(i_t, j_t) : 1 \leq t \leq K\}$ be any maximal matching in G. Define $\mathcal{A}$ as in Lemma 1, with $a_k = (\{j_k\}, \{i_k\}, \{(j_k, i_k), (i_k, j_k)\})$. Because G satisfies the SAC then there exits a cycle $\mathcal{C}$ in G such that $e_t \in M$ for all even t, and $e_t \notin M$ for all odd t. Without loss of generality, the cycle starts with a buyer: i.e. $c_1 = (j_k, i_{k'})$ for some $k, k' \in \mathbb{N}$. Since, by construction, $\mathcal{A}$ does not break M, cycle $\mathcal{C}$ induces a cycle in $\mathcal{A}$. Thus, $\mathcal{A}$ is connected, and this proves the lemma. □

Theorem 1 Let $\mathcal{G}$ be a graph. Then $\mathcal{G}$ satisfies the SLOP $\Leftrightarrow$ it satisfies the SAC.

Proof. This is a straightforward application of the previous four lemmas. Suppose G satisfies the SLOP. Let M be an arbitrarily selected maximal matching. By lemma 1, there exists a maximal abstraction of $G|M$ (say, $\mathcal{A}$) that is connected and does not break M. By Lemma 3, G satisfies the M-AC. Since M was arbitrarily selected, then G satisfies the SAC. Conversely, if G satisfies the SAC, then (by Lemma 4) for any maximal matching M there exists an abstraction $\mathcal{A}$ of $G|M$ such that $\mathcal{A}$ does not break M and $\mathcal{A}$ is connected. Therefore, by Lemma 2, G satisfies the SLOP. □

Corollary 2. *Let $G = (\mathcal{I}, \mathcal{J}, E)$ be a graph, and $M \subset E$ be a maximal matching. ($\forall \mu : \mathcal{I} \cup \mathcal{J} \to \mathbb{R}$), ($\forall p_M$), such that M is pairwise stable in $(\mathcal{I}, \mathcal{J}, E; \mu)$, and such that p_M supports M, p_M is constant $\Leftrightarrow$ G satisfies the MAC.*

Proof. Immediate from the proof of the previous theorem. □

We now prove Proposition 2. To do this, we need a series of lemmas. The first, states that if a graph, $\mathcal{G}$, is such that for all maximal matchings, M, $\mathcal{G}|M$ has a Hamiltonian cycle, then $\mathcal{G}$ satisfies the SAC. The second lemma states that, when we require all agents to have gains from trade, if a graph G is Hamiltonian, and M is a pairwise stable matching in G, then M has to be perfect. That is, M matches all agents. With these two lemmas we can prove Proposition 2 for balanced bipartite graphs. That is, for bipartite graphs where both sets of nodes have the same size. To generalize the result for when the number of buyers and sellers is different, we require an additional lemma, pertaining to the Hamiltonicity of balanced subgraphs of an unbalanced graph.

For the first of these lemma we use the notion of path concatenation. Let C and D be two paths, with the property that C ends at the same node where D begins. Then $C * D$ denotes the path that travels through C, and then continues as in D. Formally, let $C = (e_1, ..., e_K) \in E^K$ and $D = (d_1, ..., d_T) \in E^T$, where $e_K = (x, y)$ for some $x, y \in \mathcal{I} \cup \mathcal{J}$ and $d_1 = (y, z)$ for some $z \in \mathcal{I} \cup \mathcal{J}$; then $C * D = (e_1, ..., e_K, d_1, ..., d_T)$.

Lemma 5. *Let $\mathcal{G}$ be a graph, and M a maximal matching. If $\mathcal{G}|M$ has a Hamiltonian cycle then $\mathcal{G}|M$ satisfies the M-AC.*

Proof. Let $\mathcal{G}$ and M be as in the statement of the Lemma. Since the proof only involves $\mathcal{G}|M$, we abuse notation and use $\mathcal{I}$, $\mathcal{J}$, and E to denote the set of buyers, sellers and undirected links between buyers and sellers in $\mathcal{G}|M$ (as opposed to the set of sellers, buyers, and undirected links in $\mathcal{G}$.) If M contains a single matched pair (that is, $M = \{(i, j), (j, i)\}$ for some $i \in \mathcal{I}$ and some $j \in \mathcal{J}$) $\mathcal{G}|M$ satisfies the M-AC by convention, and this concludes the proof. Assume M contains more than a single matched pair. Let $\mathcal{C}$ be the Hamiltonian cycle. Enumerate $\mathcal{C} = ((i_1, j_1), (j_1, i_2), ..., (i_t, j_t), (j_t, i_{t+1}), ..., (j_T, i_1)\}$, where $T = \#\mathcal{J} = \#\mathcal{I} > 1$. Notice that because the cycle is Hamiltonian, $j_t \neq j_{t'}$ whenever $t \neq t'$; similarly, $i_t \neq i_{t'}$ whenever $t \neq t'$. Since in $\mathcal{G}|M$ all sellers are matched, then there exists a permutation $\sigma : \{1, ..., T\} \to \{1, ..., T\}$ with the following property: for all $t \in \{1, ..., T\}$, $(i_t, j_{\sigma(t)}) \in M$. This is well defined because $i_t \neq i_{t'}$ iff $t \neq t'$ iff $\sigma(t) \neq \sigma(t')$ iff $j_{\sigma(t)} \neq j_{\sigma(t')}$, so no agent is matched to two of his counterparts. In the remainder of this proof, given any number $k \in \mathbb{N}$, we use the convention that σ^k is the composition of σ a number k times. We say a set $L \subset \{1, ..., T\}$ is a *loop* if there exists a number $x \in \{1, ..., T\}$ such that $L = \{x, \sigma(x), ..., \sigma^{t-1}(x)\}$, where $t = \min\{\tau \in \mathbb{N} : \sigma^\tau(x) = x\}$.

Part 1:

First, assume that $\{1, .., T\}$ is a loop. That is, for all x, $\min\{\tau \in \mathbb{N} : \sigma^\tau(x) = x\} = T > 1$. In particular, this implies that σ has no fixed points, so $\sigma^t(x) \neq \sigma^{t-1}(x)$ for all x and all $t < T$.[35] Consider the following edges (should they be well defined):

- $e_1 = (i_1, j_{\sigma(1)}) \in M$.
- given e_{t-1}, if $t-1$ is odd, let $e_t = (j_{\sigma^{t-1}}, i_{\sigma^{t-1}(1)})$,
- given e_{t-1}, if $t-1$ is even, let $e_t = (i_{\sigma^{t-1}(1)}, j_{\sigma^t})$.

Before proceeding, we check these edges are well defined. First, all odd edges are well defined because $(i_{\sigma^{t-1}(1)}, j_{\sigma^t(1)}) \in M \subset E$ (by definition, all edges of the form $(i_t, j_{\sigma(t)})$ are elements of M). Second, all even edges are well defined because $(j_{\sigma^{t-1}(1)}, i_{\sigma^{t-1}(1)}) \in \mathcal{C}$ (by definition, all edges of the form (j_t, i_t) are elements of $\mathcal{C}$). By construction $\sigma^t(x) \neq \sigma^{t-1}(x)$ for all $t < T$, so $e_t \notin M$ for all even values of t.[36] Construct the cycle $\mathcal{C}_0 = (e_1, e_2, ..., (j_{\sigma^T(1)}, i_{\sigma^T(1)}))$. By construction $\mathcal{C}_0$ is an alternating cycle, so $\mathcal{G}|M$ satisfies the M-AC.

Part 2:

Now, assume that $\{1, ..., T\}$ has K loops, for some $K > 1$. Enumerate them $L_1, ..., L_K$, and without loss of generality $1 \in L_1$. Furthermore, let $\Xi = \{t \in \{1, ..., T\} : (\exists k \in \{1, ..., K\}) \quad t \in L_k, \quad t+1 \notin L_k\}$. Enumerate $\Xi = \{\Xi_1, ..., \Xi_N\}$ for some $N \leq T$. Without loss of generality, the enumeration is monotone increasing: $\Xi_1 < \Xi_2 < ... < \Xi_N$ The K loops define K disjoint subgraphs of $\mathcal{G}|M$: for each k, set $\mathcal{I}_k \equiv \{i_x : x \in L_k\} \subset \mathcal{I}$

[35] If there was x and $t < T$ such that $\sigma^{t-1}(x) = \sigma^t(x)$, then $\sigma^{t-1}(x)$ is a fixed point of σ, which contradicts that $\{1, ..., T\}$ is a loop.

[36] Indeed, if t is even and $e_t \in M$, then $(j_{\sigma^{t-1}(1)}, i_{\sigma^{t-1}(1)}) \in M$ and $(j_{\sigma^t(1)}, i_{\sigma^{t-1}(1)}) \in M$. This is a contradiction because $\sigma^t(1) \neq \sigma^{t-1}(1)$ and $i_{\sigma^t(1)} \neq i_{\sigma^{t-1}(1)}$.

and $\mathcal{J}_k \equiv \{j_x : x \in L_k\} \subset J$, and $E_k = E \cap ((\mathcal{I}_k \times \mathcal{J}_k) \cup (\mathcal{J}_k \times \mathcal{I}_k))$. Denote with G_k the k-th such subgraph. With these ingredients, we inductively construct an alternating cycle. Start with loop L_1, and construct an alternating cycle on G_1 as we did in Part 1: $\mathcal{C}_1 = ((i_1, j_{\sigma(1)}), (j_{\sigma(1)}, i_{\sigma(1)}), ..., (j_1, i_1))$. Augment path $\mathcal{C}_1$ so that it continues following $\mathcal{C}_1$ but ends in j_{Ξ_1}. Formally, $\bar{\mathcal{C}}_1 = \mathcal{C}_1 * ((i_1, j_{\sigma(1)}), (j_{\sigma(1)}, i_{\sigma(1)}), ..., (i_{\sigma^{-1}(\Xi_1)}, j_{\Xi_1}))$. By the same arguments as in part 1, this is an alternating path, and the last edge, $(i_{\sigma^{-1}(\Xi_1)}, j_{\Xi_1})$, is in M. Add to this path an edge that will travel across to another loop. Formally, $\mathcal{C}_1^{alt} = \bar{\mathcal{C}}_1 * ((j_{\Xi_1}, i_{\Xi_1+1}))$. Since $(i_{\sigma^{-1}(\Xi_1)}, j_{\Xi_1}) \in M$ and, by construction, $(j_{\Xi_1}, i_{\Xi_1+1}) \notin M$, then $\mathcal{C}_1^{alt}$ is an alternating path. Now, proceed inductively. Formally, given a number $n-1 < N$, and given $\mathcal{C}_{n-1}^{alt}$, define the following:

$$
\begin{aligned}
\mathcal{C}_n &= \mathcal{C}_{n-1}^{alt} * ((i_{\Xi_{n-1}+1}, j_{\sigma(\Xi_{n-1}+1)}), (j_{\sigma(\Xi_{n-1}+1)}, i_{\sigma(\Xi_{n-1}+1)}), ..., (j_{\Xi_{n-1}+1}, i_{\Xi_{n-1}+1}))) \\
\bar{\mathcal{C}}_n &= \mathcal{C}_n * ((i_{\Xi_{n-1}+1}, j_{\sigma(\Xi_{n-1}+1)}), (j_{\sigma(\Xi_{n-1}+1)}, i_{\sigma(\Xi_{n-1}+1)}), ..., (i_{\sigma^{-1}(\Xi_n)}, j_{\Xi_n}). \\
\mathcal{C}_n^{alt} &= \bar{\mathcal{C}}_n * ((j_{\Xi_n}, i_{\Xi_n+1}))
\end{aligned}
$$

This is well defined, because, by construction, $(\forall n)$ $(\exists k)$ $\{\Xi_{n-1}+1, \Xi_n\} \subset L_k$ for some loop L_k. [37] This process stops after N rounds. Notice $\mathcal{C}_N^{alt}$ is an alternating path, so it only remains to close the cycle. To do this, notice that $\Xi_N + 1 \in L_1$ (modulo $T + 1 = 1$).[38] Construct the following paths:

$$
\begin{aligned}
LAST &= ((i_{\Xi_N+1}, j_{\sigma(\Xi_N+1)}), (j_{\sigma(\Xi_N+1)}, i_{\sigma(\Xi_N+1)}), ..., (j_1, i_1)), \\
\mathcal{C}^{alt} &= \mathcal{C}_N * LAST.
\end{aligned}
$$

By construction, $\mathcal{C}^{alt}$ is an alternating cycle. Thus, $\mathcal{G}|M$ satisfies the M-AC. □

For the next lemma, we assume that the only admissible valuation profiles are those that satisfy the following: for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$, $\mu(i) < \mu(j)$. This assumption allows us to refine our result on pairwise stable matchings in the following way: for graphs with Hamiltonian Cycles, the set of pairwise stable matchings is the set of perfect matchings. That is, matchings where all agents are matched. This additional structure is useful to prove Proposition 2.

Lemma 6. *Let $G \in \mathbb{G}(n, n)$ be Hamiltonian. If M is pairwise stable then M is a perfect matching (*i.e. *every agent is matched).*

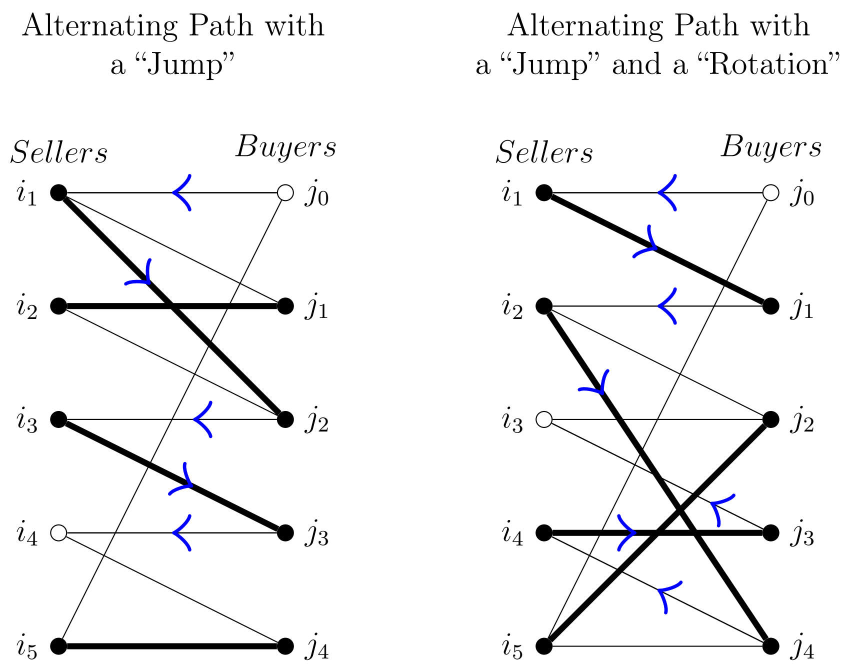
We prove this lemma by a process of *jumps* and *rotations*. In what follows we give an intuitive idea of how this process works, and then provide the formal proof of the lemma.

[37] If $\Xi_{n-1} + 1 = \Xi_n$ this is trivial. If not, given $\Xi_{n-1} \in L_k$, let $\Xi_n - \Xi_{n-1} \equiv l > 0$. Then (by construction) $\{\Xi_{n-1}+1, ..., \Xi_{n-1}+(l-1), \Xi_{n-1}+l\} \subset L_k$. Thus, $\Xi_n \in L_k$ also.

[38] If $\Xi_N = T$ then $\Xi_N + 1 = 1 \in L_1$. If $\Xi_N \neq T$, by definition, $\{\max\{\Xi\}+1, ..., T\} \subset L_k$ for some loop L_k. Since $T \neq \Xi_N$ then $\{T, T+1\} = \{T, 1\} \subset L_k$, so $k = 1$. Thus, $\Xi_N + 1 \in L_1$.

Suppose we are given a matching, M, and a Hamiltonian cycle
$\mathcal{C} = ((j_0, i_1), (i_t, j_t), (j_t, i_{t+1}), ..., (i_n, j_{n+1}))_{t=1}^{n}$. We want to consider the case where there is at least one unmatched buyer and at least one unmatched seller. Let buyer j_0 be unmatched. The objective is to find a path from the buyer j_0 to some unmatched seller, and this path must alternate edges not in M with edges in M. Showing that such a path always exists will be crucial to the proof of Lemma 6. Let t^* be the smallest index such that i_{t^*} is not matched. We say that edges of the form (i_t, j_t) or (j_t, i_{t+1}) move "forward" in $\mathcal{C}$, whereas edges of the form (j_t, i_t) or (i_{t+1}, j_t) move "backwards" in $\mathcal{C}$. Moreover, we say that a vertex $i_t \in \mathcal{I}$ is an inflection vertex if (along cycle $\mathcal{C}$) if it is adjacent to two vertexes with which it is not matched (formally, if $(j_{t-1}, i_t) \notin M$ and $(j_t, i_t) \notin M$). If starting at j_0 and moving forward in $\mathcal{C}$ does not generate an alternating path that ends in an unmatched seller, then eventually we will encounter an inflection vertex, i_t, that is itself matched (*i.e.* $j^*(i_t) \neq \emptyset$). The construction of the desired path works as follows: start at j_0 and move forward in $\mathcal{C}$ until i_{t^*} is reached, or we reach an inflection vertex. If we reach an infection vertex (say, i_τ), continue the path with edge $(i_\tau, j^*(i_\tau))$. Then, there are two cases. First, $j^*(i_\tau)$ has an index smaller than t^*. In this case, upon reaching $j^*(i_\tau)$, we can continue moving from $j^*(i_\tau)$ forward in $\mathcal{C}$ until we reach i_{t^*} or we encounter another inflection vertex. We call the operation of moving from i_τ to $j^*(i_\tau)$, and then moving forward in $\mathcal{C}$, a jump, because we are jumping forward in $\mathcal{C}$, but we are still moving forward towards i_{t^*}. The second case is when the index of $j^*(i_\tau)$ is larger or equal to t^*. Then, upon reaching $j^*(i_\tau)$, we continue by moving backwards in $\mathcal{C}$, until we reach an unmatched seller or another inflection vertex. We call the operation of moving from i_τ to $j^*(i_\tau)$, and then backwards in $\mathcal{C}$, a rotation, because we are rotating the direction in which we traverse $\mathcal{C}$. This process of combining jumps and rotations is illustrated in the figure below, and is the conceptual core of how the proof of the lemma works.

Example 5. We present two examples of alternating paths where there is one unmatched buyer and one unmatched seller.



Consider the following graphs, where the thick lines indicate a pairwise stable matching and the Hamiltonian Cycle is $\mathcal{C} = ((j_0, i_1), (i_1, j_1), (j_1, i_2), ..., (i_5, j_0))$. We show two examples of alternating paths as described above. In the left panel, the path begins with the unmatched buyer j_0 and follows the Hamiltonian cycle to seller node i_1. Node i_1 is an inflection vertex ($(j_0, i_1) \notin M$ and $(i_1, j_1) \notin M$) and the path "jumps" from node i_1 to node j_2. The path then follows the Hamiltonian Cycle until it reaches the unmatched seller at node i_4. In the right panel, the path again begins with the unmatched buyer j_0 and follows the Hamiltonian cycle until it reaches seller node i_2, which is an inflection vertex ($(j_1, i_2) \notin M$ and $(i_2, j_2) \notin M$). The path then "jumps" to buyer node j_4. Since the index is now larger than the index of the unmatched seller (3), we rotate and "move backwards" along the Hamiltonian cycle to seller node i_4. The path continues to move backwards along the Hamiltonian cycle until it reaches unmatched seller node i_3.

Proof. Let G be as in the Lemma. Let M be a pairwise stable matching. We proceed by contradiction: assume that there exists $j \in \mathcal{J}$ that is unmatched. Thus, there must be at least one vertex in $\mathcal{I}$ that is also unmatched. Furthermore, let $\mu : \mathcal{I} \cup \mathcal{J} \to \mathbb{R}$ be a valuation profile that makes M pairwise stable in $(\mathcal{I}, \mathcal{J}, E, \mu)$, and p_M be the supporting prices. Notice that, by assumption, $\mu(j) > \mu(i)$. Since G is Hamiltonian, we can find a cycle as follows:

$$\mathcal{C} = ((j_0, i_1), (i_t, j_t), (j_t, i_{t+1}), ..., (i_n, j_{n+1}))_{t=1}^{n},$$

where

- $j_0 = j_{n+1} = j$, and all other nodes are different ($t \neq t'$ implies $i_t \neq i_{t'}$ and $j_t \neq j_{t'}$).

- There exists t^* such that i_{t^*} is unmatched.
- Without loss of generality, t^* is the lowest index for which i_t is unmatched.

We now define a path $\mathcal{P} = ((j, i^1), (i^t, j^t), (j^t, i^{t+1}))_{t=1}^{K}$, with the following properties:

- i^{K+1} is unmatched (it may be i_{t^*} or some other unmatched vertex in $\mathcal{I}$),
- for all $t \geq 1$, $(t^t, i^t) \in M$.

We construct this path inductively. Let $\mathcal{P}^1 = ((j_0, i_1)) \equiv ((j^0, i^1))$.
Assume $\mathcal{P}^n = ((j^0, i^1), (i^1, j^1), ..., (j^{n-1}, i^{n-1}), (j^{n-1}, i^n))$ is given, where $\mathcal{P}^n$ satisfies that for all $t \in \{1, ..., n-1\}$, $(t^t, i^t) \in M$. Let $k \in \{1, ..., T\}$ be such that $i^n = i_k$. There are seven cases to consider.

- If $j^*(i^n) = \emptyset$, let $\mathcal{P}^n = \mathcal{P}$, and this concludes the construction of $\mathcal{P}$. Hence, for the remaining three cases we assume i^n is matched.
- Assume that $k < t^*$, and that $(i_k, j_k) \in M$. Then, let $\mathcal{P}^{n+1} \equiv \mathcal{P}^n * ((i^n, j^n), (j^n, i^{n+1}))$ where $j^n = j_k$, and $i^{n+1} = i_{k+1}$.
- Assume $k < t^*$, $(i_k, j_k) \notin M$, and $j^*(i_k) = j_{k'}$ where $k' < t^*$. Then, define $\mathcal{P}^{n+1} \equiv \mathcal{P}^n * ((i^n, j^n), (j^n, i^{n+1}))$ where $j^n = j_{k'}$, and $i^{n+1} = i_{k'+1}$.
- Assume $k < t^*$, $(i_k, j_k) \notin M$, and $j^*(i_k) = j_{k'}$ where $k' \geq t^*$. Then, define $\mathcal{P}^{n+1} \equiv \mathcal{P}^n * ((i^n, j^n), (j^n, i^{n+1}))$ where $j^n = j_{k'}$, and $i^{n+1} = i_{k'}$.
- Assume that $k > t^*$, and that $(i_k, j_{k-1}) \in M$. Then, let $\mathcal{P}^{n+1} \equiv \mathcal{P}^n * ((i^n, j^n), (j^n, i^{n+1}))$ where $j^n = j_{k-1}$, and $i^{n+1} = i_{k-1}$.
- Assume $k > t^*$, $(i_k, j_{k-1}) \notin M$, and $j^*(i_k) = j_{k'}$ where $k' < t^*$. Then, define $\mathcal{P}^{n+1} \equiv \mathcal{P}^n * ((i^n, j^n), (j^n, i^{n+1}))$ where $j^n = j_{k'}$, and $i^{n+1} = i_{k'+1}$.
- Assume $k > t^*$, $(i_k, j_{k-1}) \notin M$, and $j^*(i_k) = j_{k'}$ where $k' \geq t^*$. Then, define $\mathcal{P}^{n+1} \equiv \mathcal{P}^n * ((i^n, j^n), (j^n, i^{n+1}))$ where $j^n = j_{k'}$, and $i^{n+1} = i_{k'}$.

Since $\mathcal{C}$ is a cycle of length $2n$, then $\mathcal{P}$ utilizes at most $3n$ edges. Hence, this inductive process eventually stops. Let $i^T \in \mathcal{I}$ be the endpoint of $\mathcal{P}$; by construction, i^T is unmatched. Because $\mathcal{P}$ satisfies $(i^t, j^t) \in M$, and because $(j, i^1) \in E$, $(j^{T-1}, i^T) \in E$ then the following are true:

- $\mu(j) \leq p_M(i^1, j^1)$,
- $p_M(i^t, j^t) \leq p_M(i^{t+1}, j^{t+1})$ for all $t \in \{1, ..., T-2\}$,
- and $p_M(i^{T-1}, j^{T-1}) \leq \mu(i^T)$.

Then, $\mu(j) \leq \mu(i^T)$, a contradiction. Therefore, if a matching in a Hamiltonian graph is stable, it must not leave agents unmatched. $\square$

We now prove a version of proposition 2 for the case of balanced graphs (that is, when there are equal number of buyers and sellers). We then generalize to the case of unbalanced graphs.

Proposition 2 Let $\theta > 0$, let $\mathbb{G}(J, J)$ be the set of all bipartite, undirected balanced graphs with node sets of cardinality J. For each $\lambda \in (0, 1)$ let P_λ be a probability distribution over $\mathbb{G}(\theta J, J)$ such that each graph with K undirected edges is drawn with equal probability $\lambda^K(1-\lambda)^{\theta J^2 - K}$. Consider a sequence $(\lambda_t)_{t\in\mathbb{N}}$ such that $\lambda_t > \frac{\ln(t)+\ln(\ln(t))+c_t}{t}$ where $c_t \to \infty$ with t. Then $\lim_{t\to\infty} P_{\lambda_t}(\{\text{G: G satisfies the SLOP}\}) = 1$.

Proof. Fix $(\lambda_t)_t$ be as in the statement of the Proposition. Define the following sets:

$$\begin{aligned} S &= \{G \in \mathcal{G}(t, t) : G \text{ satisfies the SLOP.}\} \\ HAM &= \{G \in \mathcal{G}(t, t) : G \text{ has a Hamiltonian cycle.}\} \end{aligned}$$

The previous lemma shows that $HAM \subset S$. Indeed, pick any $G \in HAM$, and pick any matching M in G that is pairwise stable. By our previous lemma, M must be perfect. Thus, $G|M = G$. Since G has a Hamiltonian cycle, then $G|M$ has a Hamiltonian cycle. Thus, by Lemma 5, $G|M$ has an alternating cycle. Since M was arbitrarily selected, this implies that G has the M-AC property for all pairwise stable M. Therefore, G has the SAC property, so $G \in S$. This proves the claim that $HAM \subset S$. By Frieze (1985), we know $P_{\lambda_t}(HAM) \to 1$ as $t \to \infty$. Thus. $P_{\lambda_t}(S) \to 1$ as $t \to 1$, and this conclude the proof. $\square$

We now generalize the previous proposition when there can be different number of buyers and sellers (formally, when $\theta \neq 1$). Without loss of generality we assume $\theta > 1$. Before doing this we need an extra Lemma. In what follows, given any graph G, we use $E(G)$ to denote the set of edges of G.

Lemma 7. *Let $(\lambda_t)_t$ be as in Proposition 2. Define the set*

$$SubHAM = \{G \in \mathcal{G}(\theta t, t) : (\forall \textit{ subgraphs } \hat{G} \in \mathbb{G}(t, t) \textit{ of } G)\ \hat{G} \textit{ has a Hamiltonian cycle.}\}$$

Then, $P_{\lambda_t}(SubHAM) \to 1$ as $t \to \infty$.

Proof. For any given t, take $G \notin SubHAM$. Then, there exists a subgraph $\Gamma_G \in \mathbb{G}(t, t)$ of G such that Γ_G is not Hamiltonian. For each $G \notin SubHAM$ pick a Γ_G as above. Furthermore, we can factorize the probability of drawing G as follows:

$$P_{\lambda_t}(G) = \lambda_t^{|E(\Gamma_G|}(1-\lambda_t)^{t^2 - |E(\Gamma_G)|}\lambda_t^{|(E(G))| - |E(\Gamma_G)|}(1-\lambda_t)^{t^2(\theta-1) + |E(\Gamma_G)| - |E(G)|}$$

where $E(\Gamma_G)$ is the set of edges in Γ_G and $E(G)$ is the set of edges in G. Let $\lambda_t^k(1-\lambda_t)^{t^2(\theta-1)-k} \equiv R(k)$. Then taking the convention that a sum over an empty set of indices is 0,

$$\begin{aligned}
1 - P_{\lambda_t}(SubHAM) &= \sum_{\{\Gamma\in\mathbb{G}(t,t):\ \Gamma \text{ is not Hamiltonian}\}} \{ \sum_{\{G\notin SubHAM: \Gamma_G=\Gamma\}} P_{\lambda_t}(G)\}\} \\
&\leq \sum_{\{\Gamma\in\mathbb{G}(t,t):\ \Gamma \text{ is not Hamiltonian}\}} \{\lambda_t^{|E(\Gamma|}(1-\lambda_t)^{t^2-|E(\Gamma)|}\{ \sum_{k=0}^{t^2(\theta-1)} \binom{t^2(\theta-1)}{k} R(k)\}\} \\
&= \sum_{\{\Gamma\in\mathbb{G}(t,t):\ \Gamma \text{ is not Hamiltonian}\}} \lambda_t^{|E(\Gamma|}(1-\lambda_t)^{t^2-|E(\Gamma)|}\} \\
&\leq P_{\lambda_t}(\{\Gamma \in \mathbb{G}(t,t):\ \Gamma \text{ is not Hamiltonain})
\end{aligned}$$

Pick any $\varepsilon > 0$ and $T \in \mathbb{N}$ such that for all $t > T$, $P(\{\Gamma \in \mathbb{G}(t,t):\ \Gamma \text{ is not Hamiltonain}\}) \leq \varepsilon$. Such a T exists by Frieze (1985). Then, for all $t > T$, $P_{\lambda_t}(SubHAM) \geq 1-\varepsilon$, and this concludes the proof. □

We now prove Proposition 2 for the case with $\theta \neq 1$.

Proof. Fix $(\lambda_t)_t$ be as in the statement of the Proposition. Without loss of generality assume that $\theta > 1$, so that there are more sellers than buyers (the proof for the case $\theta < 1$ is analogous). Define the following sets:

$$\begin{aligned}
S &= \{G \in \mathcal{G}(\theta t, t) : G \text{ satisfies the SLOP.}\} \\
SubHAM &= \{G \in \mathcal{G}(\theta t, t):\ (\forall \text{ subgraphs } \hat{G} \in \mathbb{G}(t,t) \text{ of } G)\ \hat{G} \text{ has a Hamiltonian cycle.}\}
\end{aligned}$$

We now prove that $SubHAM \subset S$. Indeed, pick any $G \in SubHAM$, and pick any matching M in G that is pairwise stable. Pick any subgraph $\hat{G}$ of G such that $\hat{G} \in \mathbb{G}(t,t)$ and $G|M$ is a subgraph of $\hat{G}$.[39] Then, M is also pairwise stable in $\hat{G}$. Since M is pairwise stable in $\hat{G} \in \mathbb{G}(t,t)$ then Lemma 6 implies M is perfect in $\hat{G} \in \mathbb{G}(t,t)$. Thus, $|M| = t$, so $G = \hat{G} = G|M$. Since G has a Hamiltonian cycle, then $G|M$ has a Hamiltonian cycle. Thus, by Lemma 5, $G|M$ has an alternating cycle. Since M was arbitrarily selected, this implies that G has the M-AC property for all pairwise stable M. Therefore, G has the SAC property, so $G \in S$. This proves the claim that $SubHAM \subset S$. By Lemma 7, we know $P_{\lambda_t}(SubHAM) \to 1$ as $t \to \infty$. Thus, $P_{\lambda_t}(S) \to 1$ as $t \to 1$, and this conclude the proof. □

[39] Such a graph $\hat{G}$ always exists. Let $\hat{\mathcal{J}} = \mathcal{J}$. Since $|\{i \in \mathcal{I} : j^*(i) \neq \emptyset\}| \leq t$ then there exists $\hat{\mathcal{I}} \subset \mathcal{I}$ such that $|\hat{\mathcal{I}}| = t$ and $\{i \in \mathcal{I} : j^*(i) \neq \emptyset\} \subset \hat{\mathcal{I}}$. Let $\hat{G}$ be the graph spanned by $(\hat{\mathcal{J}}, \hat{\mathcal{I}})$.

C Formal algorithm and proof of Proposition 3

In this appendix we discuss the formal algorithm used in the main paper and prove some of its properties. We now present the basic notation we use in the match determination program. Let $s^t \in \mathbb{R}^{J\times I}$ be a matrix of prices for each seller-buyer pair. Each element, $s^t_{i,j}$, represents the price that buyer j would have to bid for seller i if j were to bid for i in round t. Vector q^t represents the bidding queue in period t: $q^t_n = j \in \mathcal{J}$ represents that in round t, buyer j is the n-th bidder in the queue. The algorithm ends when $l(q) = 0$, where $l(q)$ indicates the length of q. Quantity $D(j)$ indicates j's demand. Quantities with primes will indicate quantities that will carry over to the next round of the algorithm. Finally, for each seller j, we use the following payoff function to model that a buyer can only buy a good from a seller if the two are linked in the network: $u_j : \mathcal{I} \times \mathbb{R}^{I\times J} :\rightarrow \bar{\mathbb{R}}$, $u_j(i,s) = \mu(j) - s_{i,j}$ if $(i,j) \in E$ and $u_j(i,s) = -\infty$ otherwise.

Recall some notational conventions: given a matching M, $i^*(\cdot) : \mathcal{J} \rightarrow \mathcal{I} \cup \{\emptyset\}$ satisfies $(i^*(j), j) \in M$ for each M-matched j, and $i^*(j) = \emptyset$ if j is M-unmatched. Analogously, $j^*(\cdot) : \mathcal{I} \rightarrow \mathcal{J} \cup \{\emptyset\}$ satisfies $(i, j^*(i)) \in M$ for each M-matched i, and $j^*(i) = \emptyset$ if i is M-unmatched. Also, even if not explicitly stated, the network is denoted $\mathcal{N} = (\mathcal{I}, \mathcal{J}, E; \mu(\cdot))$, $I = \#\mathcal{I}$, $J = \#\mathcal{J}$, and $\mu(s) = b \in \mathbb{R}$ for all s. This last normalization is simply for convenience of the simulation.

Match Determination Program.

Input= $(\mathcal{N}, s^0, (u_1, ..., u_J), h^0, q)$ where:

- $s^0 = (s^0_1, ..., s^0_J) \in \mathbb{R}^{J\times I}$, $s^0_j = (b, ..., b) \in \mathbb{R}^I$,
- For each buyer j, and each $t \in \mathbb{N} \cup \{0\}$, $u_j(i, s^t) = \mu(j) - s^t_{i,j}$ if $(i,j) \in E$ and $u_j(i, s^t) = -\infty$ if $(i,j) \notin E$,
- $h^0 = (0, ..., 0) \in \mathbb{R}^{I\times J}$.
- $q^0 \in \mathcal{J}^J$ such that $q^0_n = q^0_m$ iff $m = n$.

Start step $R(1)$:

R(t). Set $h^t = h$, $s^t = s$, $q^t = q$, $j = q_1$.

1. If $\max\{u_j(i,s) : i \in \mathcal{I}\} < 0$ set $s' = s$ and $h' = h$, $q' = (q_2, ..., q_{l(q)})$.
   a. If $l(q') = 0$, stop, set $M = \{(i,j) : h_{i,j} = 1\}$, and *Output*= M.
   b. If $l(q') \neq 0$, set $q^{t+1} = q'$, $s^{t+1} = s'$, $h^{t+1} = h'$ and proceed to $R(t+1)$.
2. If $\max\{u_j(i,s) : i \in \mathcal{I}\} \geq 0$ let $D(j) \in \arg\max\{u_j(i,s) : i \in \mathcal{I}\}$.
   a. If $\arg\max\{u_j(i,s) : i \in \mathcal{I}\}$ has more than one element, select $D(j) \in \arg\max_{i\in\mathcal{I}}\{u_j(i,s)\}$ randomly.

3. Set the following parameters:

   a. $s'_{D(j),j} = s_{D(j),j}$; for all $j' \neq j$, $s'_{D(j),j'} = s_{D(j),j} + \frac{\Delta}{2}$; $s'_{i'',j''} = s_{i'',j''}$ elsewhere,

   b. If $h_{D(j),j'} = 0$ for all $j' \neq j$, set $q' = (q_2, ..., q_{l(q)})$; if $h_{D(j),j'} = 1$ for some $j' \neq j$, set $q' = (q_2, ..., q_{l(q)}, j')$,

   c. $h'_{D(j),j} = 1$; for all $j' \neq j$, $h'_{D(j),j'} = 0$; $h'_{i'',j''} = h_{i'',j''}$ elsewhere.

4. If $l(q') = 0$, stop. Set $M = \{(i,j) : h_{i,j} = 1\}$. *Output*$= M$.

   If $l(q) \neq 0$ set $h' = h^{t+1}$ and $s' = s^{t+1}$ and $q' = q^{t+1}$. Then start $R(t+1)$.

Although this algorithm is motivated by Crawford and Knoer (1981) and Kelso and Crawford (1982), there are three important differences. The first is that firm productivities increase in increments of Δ whereas bids increase in increments of $\frac{\Delta}{2}$. Since Crawford and Knoer (1981) and Kelso and Crawford (1982) work with a discrete core, the algorithm they run produces a stable match when both bid increments and productivities increase by the same amount. However, since we work with a continuous core, it is not true that the matching generated by such an algorithm is stable. One can construct examples where the matching generated by the algorithms in Crawford and Knoer (1981) and Kelso and Crawford (1982) (say, M) satisfies that there is no price function p_M such that M is stable with respect to p_M. We provide one example in section D (Figure A4) in the online appendix. The modification we introduce, that bids live in a finer grid than firm productivities, helps us bypass this problem. The second difference with the algorithms in Crawford and Knoer (1981) and Kelso and Crawford (1982) is that we only use their program to find the matching, but not the prices that make it stable. The reason is that their algorithm makes prices rise too quickly. While in some networks the price generated by the algorithms in Crawford and Knoer (1981) and Kelso and Crawford (1982) is the pointwise minimum price that makes the matching stable, this is not always guaranteed. This is because in our setting we violate the non-indifference assumptions made in Crawford and Knoer (1981) and Kelso and Crawford (1982). In order to capture, for each matching, the pointwise maximum and minimum prices at which that matching is stable we run two independent programs. We call these the Price Determination Programs, and we describe them below. The first Price Determination Program (I), outputs the pointwise minimum price function at which a matching is stable. The second Price Determination program (II), outputs the pointwise maximum price function at which a matching is stable. The third difference is that, when a seller i accepts a bid from a buyer j, then any future bid buyer j' submits to i must outbid j's bid. In symbols, if in round t seller i accepts bid $s^t_{i,j}$ from j, then at the end of round t all sellers j' linked to i have their bid price raised to $s^{t+1}_{i,j'} = s^t_{i,j} + \frac{\Delta}{2}$. This modification reduces the run time of the algorithm by a factor of four.

Price Determination Program (I).

Input$= (\mathcal{N}, M)$.

1. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^1 = b$.

2. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$ set $\rho_i^1 = \max\{\mu(j) : (i,j) \in E \text{ and } i^*(j) = \emptyset\}$.

3. Set $t = 1$. Start step 4(1).

4(t). Given $(\rho_1^t, ..., \rho_I^t)$:

   a. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^{t+1} = \rho_i^t$.

   b. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$, let $j \equiv j^*(i)$. Then, set

$$\rho_i^{t+1} = \max\{\rho_{i'}^t : (\exists j')(i', j') \in M,\ (i, j') \in E\}.$$

   c. If for all $i \in \mathcal{I}$ $\rho_i^{t+1} = \rho_i^t$:

      * For each i such that $j^*(i) \neq \emptyset$ set $p_M(i, j^*(i)) = \rho_i^{t+1}$.
      * *Output*$= (p_M(\cdot))$.

   d. Otherwise, start step $4(t+1)$.

Price Determination program (I) outputs the minimum price at which M can be made stable.

Price Determination Program (II).
Input$= (\mathcal{N}, M)$.

1. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^1 = b$.

2. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$ set $\rho_i^1 = \mu(j^*(i))$.

3. Set $t = 1$. Start step 4(1).

4(t). Given $(\rho_1^t, ..., \rho_I^t)$:

   a. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^{t+1} = \rho_i^t$.

   b. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$, let $j \equiv j^*(i)$. Then, set

$$\rho_i^{t+1} = \min\{\rho_{i'}^t : (i', j) \in E\}.$$

   c. If for all $i \in \mathcal{I}$ $\rho_i^{t+1} = \rho_i^t$:

      * For each i such that $j^*(i) \neq \emptyset$ set $p_M(i, j^*(i)) = \rho_i^{t+1}$.
      * *Output*$= (p_M(\cdot))$.

   d. Otherwise, start step $4(t+1)$.

Price Determination program (II) outputs the maximum price at which M can be made stable.

In Section 4.2 we claimed our algorithm has four properties: it ends in finite time, it selects a pairwise stable allocation, and for each allocation it selects the pointwise minimum and maximum prices that sustain it.

Proposition 3: The deferred acceptance algorithm has the following properties:

1. It stops after a finite number of rounds.

2. It outputs a pairwise stable allocation.

3. Price Determination program (I) outputs the pointwise minimum price function at which M is stable.

4. Price Determination program (II) outputs the pointwise maximum price function at which M is stable.

We now prove these items one at a time. In what follows, we use MDP and PDP to abbreviate the Matching Determination Program and the Price Determination Program respectively. Finally, if $(x_i)_{i\in\mathcal{I}}$ is a vector indexed by $\mathcal{I}$ we use the convenient shorthand notation $x_\cdot$ to denote the whole vector, whenever ambiguity is unlikely.

We need two lemmas: the first, shows that, given M produced by the MDP, there exist prices p_M such that M is stable with respect to M. The second shows that the prices generated by the PDP are weakly lower than any p_M such that M is stable with respect to M. To prove these Lemmas, recall that $(\rho_i^t)_{i\in\mathcal{I},t\geq 1}$ from the PDP(I) is defined as follows:

- If $j^*(i) = \emptyset$, $\rho_i^t = b$ for all t.

- If $j^*(i) = j$ for some $j \in \mathcal{J}$, $\rho_i^1 = \max\{\mu(j) : (i,j) \in E, i^*(j) = \emptyset\}$ for each $i \in \mathcal{I}$, and $\rho_i^t = \max\{\rho_{i'}^{t-1} : (\exists j', i') : (j', i') \in M,\, (j', i) \in E\}$ for all $t \geq 2$.

The following properties imply that there exists a value T such that, for all i and all $t \geq T$, $\rho_i^t = \rho_i^{t+1}$. That is, $(\rho_\cdot^t)_{t\geq 0}$ is eventually constant. We let $\rho_\cdot^\infty \equiv \lim_{t\to\infty} \rho_\cdot^t$.

1. For all i, $\rho_i^t \leq \rho_i^{t+1}$. This follows because $\rho_i^{t-1} \in \{\rho_{i'}^{t-1} : (\exists j') : (j', i') \in M,\, (j', i) \in E\}$ whenever $j^*(i) = j$ and $\rho_i^t = b$ whenever $j^*(i) = \emptyset$.

2. For all i, $\rho_i^t \leq \max\{\mu(j) : j \in \mathcal{J}\}$.

3. For all i, if $\rho_i^t \neq \rho_i^{t+1}$ then $\rho_i^{t+1} - \rho_i^t \geq \Delta$.

Finally, recall that $\Delta \in \mathbb{R}$ is chosen so that for all $j \in \mathcal{J}$, $\mu(j) = b + k_j\Delta$ for for $k_j \in \mathbb{N} \cup \{0\}$. In particular, $\mu(j) \geq b$ for all j. This normalization only rules out uninteresting cases where a buyer never places a bid and is never matched to a seller.

Lemma 8. *Let M be the matching produced by the MDP. Then, there exists p_M such that M is stable with respect to p_M.*

Proof. For each edge $(i, j) \in M$ define $p_M(i, j) = \rho_i^\infty$ where $(\rho_i^t)_{i\in\mathcal{J},t\in\mathbb{N}\cup\{\infty\}}$ is as defined in the PDP(I). Also, let T be the last round of the MDP and let $[s_{i,j}^T]_{i\in\mathcal{I},j\in\mathcal{J}}$ be the matrix of final prices generated by the MDP. We show that M is stable with respect to p_M. Assume first that $(i, j) \in E$ are such that $j^*(i) = i^*(j) = \emptyset$. Then i received no bids, so $s_{i,j}^T = b$. Since the algorithm ended, it must be that $u_j(i, s^T) < 0 \Leftrightarrow \mu(j) < b$, a contradiction. Thus, there does not exist an edge $(i, j) \in E$ such that $j^*(i) = i^*(j) = \emptyset$ so, a fortiori, no such edge $(i, j) \in E$ blocks M. Now let $(i, j) \in M$. Pick $j' \neq j$ such that $(i, j') \in E$. We show $(i, j') \in E$ does not block M. If $i^*(j') = \emptyset$ then $\mu(j') \leq \rho_i^1 \leq \rho_i^\infty = p_M(i, j)$. If $i^*(j') \neq \emptyset$ then $\rho_i^\infty \geq \rho_{i^*(j')}^\infty$ by construction. Thus, $p_M(i, j) \geq p_M(i^*(j'), j')$. Thus, (i, j') does not block M. Pick $i' \neq i$ such that $(i', j) \in E$. We show $(i', j) \in E$ does not block M. If $j^*(i') \neq \emptyset$ then $\rho_{i'}^\infty \geq \rho_i^\infty$ by construction. Thus, $p_M(i', j^*(i')) \geq p_M(i, j)$. Let $j^*(i') = \emptyset$. Then i' never received a bid. Let t be the last time j bids for i. Since bidders bid for the cheapest seller $s_{i,j}^t \leq s_{i',j}^t = b$. By definition of t, $s_{i,j}^t = s_{i,j}^T$ so $s_{i,j}^T = p_M(i, j) = b$. We use this to argue that $\rho_{\hat{i}}^1 = b$ for all matched $\hat{i}$ such that $(i, j^*(\hat{i})) \in E$ (note that i is one such $\hat{i}$). Pick $\hat{i}$ such that $j^*(\hat{i}) \neq \emptyset$ and $(i, j^*(\hat{i})) \in E$. Then, $s_{\hat{i},j^*(\hat{i})}^T \leq s_{i,j^*(\hat{i})}^T$.[40] Since $s_{i,j}^T = b$ and $s_{i,j^*(\hat{i})}^T \leq s_{i,j}^T + \frac{\Delta}{2}$, then $s_{\hat{i},j^*(\hat{i})}^T \leq b + \frac{\Delta}{2}$. If there exists $\tilde{j}$ such that $i^*(\tilde{j}) = \emptyset$ and $(\hat{i}, \tilde{j}) \in E$, then it must be that $\mu(\tilde{j}) = b$. Indeed, if $\mu(\tilde{j}) > b$ then $\mu(\tilde{j}) \geq b + \Delta$ which is a contradiction: since $s_{\hat{i},j^*(\hat{i})}^T \leq b + \frac{\Delta}{2}$ and $i^*(\tilde{j}) = \emptyset$, $u_{\tilde{j}}(\hat{i}, s^T) \geq 0$, which contradicts T being the last round of the MDP. Thus, $\mu(\tilde{j}) = b$. Hence, $\rho_{\hat{i}}^1 = b$. We now conclude the argument in an inductive manner: if $\rho_{\hat{i}}^k = b$ for some k and all $\hat{i}$ that satisfy $j^*(\hat{i}) \neq \emptyset$ and $(i, j^*(\hat{i})) \in E$, then by construction $\rho_i^{k+1} = b$. Thus, $\rho_i^\infty = b = p_M(i, j)$. Thus, (i', j) does not block M. Therefore, M is stable with respect to p_M. $\square$

Lemma 9. *Let M be the matching generated by the MDP. Let p_M be any price function such that M is stable with respect to p_M (which is well defined by lemma) and let v be the associated payment function. Let p_M^* be the price generated by the PDP(I) and v^* the corresponding payment function. Then, $v^* \leq v$.*

Proof. Let M, p_M, v, p_M^* and v^* be as in the statement of the lemma. Then, for all i, $v(i) \geq \rho_i^1$. Indeed, if $j^*(i) = \emptyset$ then $v(i) = b = \rho_i^1$. If $j^*(i) = j$ for some j then, by stability of M with respect to p_M, $v(i) \geq \mu(j')$ for each j' such that $i^*(j') = \emptyset$. Thus, $v(i) \geq \rho_i^1$.

[40] Indeed, let t be the last time $j^*(\hat{i})$ bids for $\hat{i}$. Then, $s_{\hat{i},j^*(\hat{i})}^t = s_{\hat{i},j^*(\hat{i})}^T$, and $s_{\hat{i},j^*(\hat{i})}^t \leq s_{i,j^*(\hat{i})}^t$, where the last inequality holds because buyers always bid for the cheapest sellers. By monotonicity of the matrix of prices, $s_{i,j^*(\hat{i})}^t \leq s_{i,j^*(\hat{i})}^T$. Thus, $s_{\hat{i},j^*(\hat{i})}^T \leq s_{i,j^*(\hat{i})}^T$.

We now show that if $v \geq \rho^t$ for some k, then $v \geq \rho^{t+1}$. Indeed, for all i such that $j^*(i) = \emptyset$, $v(i) = b = \rho_i^t = \rho_i^{t+1}$. For all i such that $j^*(i) = j$, we have the following:

$$\begin{aligned}\rho_i^{t+1} &= \max\{\rho_{i'}^t : (\exists j', i')(i', j') \in M \text{ and } (i, j') \in E\} \\ &\leq \max\{v(i') : (\exists j', i')(i', j') \in M \text{ and } (i, j') \in E\} \leq v(i),\end{aligned}$$

where the last inequality follows from stability of M with respect to p_M. Thus, for each t and each i, $\rho_i^t \leq v(i)$. Hence $\rho_\cdot^\infty \equiv v^*(\cdot) \leq v(\cdot)$. □

We now prove items 1 through 4 of Proposition 1.

1. The algorithm ends in finite time.

Proof. By the same arguments as Crawford-Knoer, the matching determination program ends in finite time. Furthermore, let $K \in \mathbb{N}$ satisfy $\max\{\mu(j) : j \in \mathcal{J}\} = b + K\Delta$. Then the price determination program ends in at most $2K$ rounds. □

2. The algorithm outputs a pairwise stable matching.

Lemma 9 proves this item.

3. Price Determination program (I) outputs the pointwise minimum price function at which M is stable.

Proof. Let p_M be the prices generated by the Price Determination program (I). By construction, M is stable ap_M. The rest follows from lemma 9 □

4. Price Determination program (II) outputs the pointwise maximum price function at which M is stable. The result then follows from Lemmas 10 and 11

Lemma 10. *Let M be the matching generated by the MDP, and let p_M be the prices generated by the PDP(II). M is stable with respect to p_M.*

Proof. Let M be the matching outputted by the matching determination program, and p_M be the prices generated by the price determination program. Assume $(i, j) \in E$, $j^*(i) = i^*(j) = \emptyset$. Since there exists $\hat{p}_M$ such that M is stable with respect to $\hat{p}_M$ then $\mu(j) \leq b$. Thus (i, j) do not block M at ρ^∞. Now consider $(i, j) \in M$. We show no seller and no buyer wishes to block (i, j):

a. *No Buyer blocks:* Let j' be such that $(i, j') \in E$. If $i^*(j') \neq \emptyset$ then, by construction, $\rho_{i^*(j')}^\infty \leq \rho_i^\infty$, so (i, j') does not block. Assume now that $i^*(j') = \emptyset$. We say a seller i' is indirectly connected to seller j if there exists sequences $(i_1, ..., i_k)$ and $(j_1, ..., j_{k-1})$ such that $(i_1, j) \in E$, $(i_1, j_1) \in E$, $(i_2, j_1) \in E$, ..., $(i_k, j_{k-1}) \in E$, with $i' = i_k$. That is, if a path can be constructed from j to i'. By construction, $\min\{\mu(j^*(i')) :$

i' is indirectly connected to j$\} \leq \rho_i^\infty$ where, by convention, $\mu(\emptyset) = b$. Now consider the abstraction used in Proposition 1: each matched pair $(\hat{i}, \hat{j}) \in M$ is assigned their own subgraph, and all unmatched buyers/sellers are assigned a trivial subgraph that contains only them. Because there exist prices $\hat{p}_M$ such that M is stable at $\hat{p}_M$, cheapest sorting implies that

$$\begin{aligned} \mu(j') &\leq \hat{p}_M(i,j) \leq \min\{v(i') : i' \text{ is indirectly connected to j}\} \\ &\leq \min\{\mu(j^*(i')) : i' \text{ is indirectly connected to j}\}. \end{aligned}$$

Thus, $\mu(j') \leq \rho_i^\infty$ so (i, j') does not block.

b *No Seller blocks:* Let i' be such that $(i', j) \in E$. By construction, $\rho_i^\infty \leq \rho_{i'}^\infty$. Thus, (i', j) does not block.

□

Lemma 11. *Let M be the matching generated by the MDP. Let p_M be any price function such that M is stable with respect to p_M (which is well defined by our previous lemma) and let v be the associated payment function. Let p_M^* be the price generated by the PDP(II) and v^* the corresponding payment function. Then, $v^* \geq v$.*

Proof. Let M, p_M, v, p_M^* and v^* be as in the statement of the lemma. Then, $v(i) \leq \rho_i^1$ for all i. Indeed, if $j^*(i) = \emptyset$ then $v(i) = b = \rho_i^1$. If $j^*(i) = j$ for some j then, by stability of M with respect to p_M, $v(i) \leq \mu(j) = \rho_i^1$.

We now show that if $v \leq \rho^t$ for some k, then $v \leq \rho^{t+1}$. Indeed, for all i such that $j^*(i) = \emptyset$, $v(i) = b = \rho_i^t = \rho_i^{t+1}$. For all i such that $j^*(i) = j$, we have the following:

$$\begin{aligned} \rho_i^{t+1} &= \min\{\rho_{i'}^t : (i', j) \in E\} \\ &\geq \min\{v(i') : (i', j) \in E\} \geq v(i), \end{aligned}$$

where the last inequality follows from stability of M with respect to p_M. Thus, for each t and eaxh i, $\rho_i^t \geq v(i)$. Hence $\rho_\cdot^\infty \equiv v^*(\cdot) \geq v(\cdot)$. □

D The case with negative gains from trade.

In the main text, we assume that linked agents in a network have positive gains from trade. While this assumption is natural, and has been used in related works, in this appendix we drop the assumption and obtain theorems analogous to 1 and 1. We do this in the interest of completeness.

For Proposition 1, the assumption is essentially without loss of generality. Intuitively, linked agents that can't engage in profitable trades (*i.e.* agents with negative gains from trade) do not affect the set of pairwise stable matching, nor do they affect the prices that support the pairwise stable matchings. The only agents that change the set of pairwise stable matchings, and their supporting prices, are those that have 0 gains form trade. More formally, we proceed in three steps. First, we define a matching that contains *trivial matches.* Second, for any given network, $\mathcal{N}$, we construct an alternative network, $\hat{\mathcal{N}}$, by eliminating the links between agents that have negative gains from trade. Third, we show that $\mathcal{N}$ and $\hat{\mathcal{N}}$ have the same set of stable matchings up to matchings that contain trivial matches. Regarding Theorem 1, this theorem is without loss of generality. That is, it is still true that a graph $\mathcal{G}$ satisfies the SLOP if, and only if, it satisfies the M-AC for all matchings M such that (for some valuation profile) M is stable in the network induced by $\mathcal{G}$ and the valuation profile. However, once we drop the assumption that linked pairs have positive gains from trade, the set of matchings M that are pairwise stable (given a suitable valuation profile) is now larger. When only positive gains from trade are allowed, only maximal matchings are pairwise stable (given a suitable valuation profile), whereas now any matching is pairwise stable (again, given a suitable valuation profile. In particular, this implies that the complete graph is (essentially) the only graph where SLOP holds.

We begin by defining a matching with *trivial matches.* Say $\mathcal{N}$ is a network, and M is a matching. We say M contains trivial matches if there exists a pair $(i, j) \in M$ such that $\mu(i) = \mu(j)$. We call this a trivial match because any price at which i and j could trade is a price that leaves them indifferent between trading and not trading.

Proposition 5. *Let $\mathcal{N} = (\mathcal{I}, \mathcal{J}, E; \mu)$ be a network. Let $\hat{\mathcal{N}} = (\mathcal{I}, \mathcal{J}, \hat{E}, \mu)$ be as follows: $(i, j) \in \hat{E}$ and $(j, i) \in \hat{E}$ if, and only if, $(i, j) \in E$ and $\mu(i) \leq \mu(j)$. Finally, let $M \subset E$ be a an arbitrary matching (in $\mathcal{N}$) that contains no trivial matches. Then, the following are true:*

1 Matching $M \subset E$ is pairwise stable in $\mathcal{N}$ if, and only, if M is pairwise stable in $\hat{\mathcal{N}}$,

2 p_M supports $M \subset E$ in $\mathcal{N}$ if, and only if, p_M supports M in $\hat{\mathcal{N}}$.

Proof. Let $\mathcal{N}$, $\hat{\mathcal{N}}$ be as in the statement of the Proposition. First, let M be pairwise stable in $\mathcal{N}$, and let p_M be any price function that supports M in $\mathcal{N}$. Notice that if $(i, j) \in M$, then $\mu(i) \leq \mu(j)$, because otherwise individual rationality of p_M would fail. Because M contains no trivial matches, $\mu(i) < \mu(j)$. Thus, $(i, j) \in \hat{E}$. Moreover, since there are no blocks to (i, j) in $\mathcal{N}$, and because $\hat{E} \subset E$, then there are no blocks in $\hat{\mathcal{N}}$. Thus, M is pairwise stable in $\hat{\mathcal{N}}$ at prices p_M. Now, let M be pairwise stable in $\hat{\mathcal{N}}$, and let p_M be any price function that supports M in $\hat{\mathcal{N}}$. For an arbitrary pair $(i, j) \in M$, no links in $\hat{E}$ block (i, j). Let $(i, j') \in E \setminus \hat{E}$. Then, $\mu(i) > \mu(j')$ so (i, j') does not block (i, j). Similarly, no link of the form $(i', j) \in E \setminus \hat{E}$ blocks (i, j). Thus, M is stable in $\mathcal{N}$, and p_M supports it. $\square$

Proposition 6. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{J}, E)$ be a graph. Let M be any matching (not necessarily maximal). Then, there is a valuation profile μ such that M is stable in network $\mathcal{N} = (\mathcal{I}, \mathcal{J}, E; \mu)$.*

Proof. Let $\mathcal{G}$ and M be as in the statement of the Proposition. For all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$ such that $i^*(j) = j^*(i) = \emptyset$, let $\mu(i) = 2$, $\mu(j) = 0$. For all other agents, let $\mu(i) = \mu(j) = 1$. Then, M is stable in $(\mathcal{I}, \mathcal{J}, E; \mu)$ at prices $p_M(e) = 1$ for all $e \in M$. □

The Proposition above highlights the conceptual role played by the assumption we placed on gains from trade. Assuming that linked pairs have positive gains from trade yields a particular structure to the set of matchings that can be made stable. Namely, only maximal matchings are such that there exists a valuation profile at which they are stable. When we drop the assumption, then any matching can be made pairwise stable, via choice of an appropriate valuation profile. In turn, this means that the only graphs that satisfy the slop are the graphs such that, after eliminating all agent that have no links, the remaining graph is complete.

Proposition 7. *A graph $\mathcal{G} = (\mathcal{I}, \mathcal{J}, E)$ satisfies the SLOP if, and only if, the following property holds: for all $i, i' \in \mathcal{I}$ $(i \neq i')$ and all $j, j' \in \mathcal{J}$ $(j, \neq j')$, if $(i, j) \in E$ and $(i', j') \in E$, then $(i', j) \in E$ and $(i, j') \in E$.*

Proof. Assume $\mathcal{G}$ is a graph that satisfies the SLOP. Let $(i, j) \in E$ and $(i', j') \in E$, with $i \neq i'$ and $j \neq j'$. Set $\mu(i) = \mu(i') = \mu(j) = \mu(j') = 1$ and $\mu(i'') = 2$ for all other $i'' \in \mathcal{I}$ and $\mu(j'') = 0$ for all other $j'' \in \mathcal{J}$. Set $M = \{(i, j), (i', j'), (j, i), (j', i')\}$; then M is stable in the network induced by $\mathcal{G}$ and μ. Thus, for all p_M that support M, p_M must be constant. Replicating the argument made in Theorem 1, the above is true if, and only if, $\mathcal{G}|M$ satisfies the M-AC property. Thus, $\mathcal{G}|M = (\{i, i'\}, \{j, j'\}, \hat{E})$ where $\hat{E} = (\mathcal{I} \times \mathcal{J}) \cup (\mathcal{J} \times \mathcal{I})$. □