Subjective expected utility representations for Savage preferences on topological spaces

Marcus Pivato and Vassili Vergopoulos

THEMA, Université de Cergy-Pontoise, Paris School of Economics, and University Paris 1 Panthéon-Sorbonne

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Marcus Pivato† and Vassili Vergopoulos‡

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Abstract

In many decisions under uncertainty, there are technological constraints on both the acts an agent can perform and the events she can observe. To model this, we assume that the set $\mathcal{S}$ of possible states of the world and the set $\mathcal{X}$ of possible outcomes each have a topological structure. The only feasible acts are continuous functions from $\mathcal{S}$ to $\mathcal{X}$, and the only observable events are regular open subsets of $\mathcal{S}$. In this environment, we axiomatically characterize a Subjective Expected Utility (SEU) representation of preferences over acts, involving a continuous utility function on $\mathcal{X}$ (unique up to positive affine transformations), and a unique probability measure on a Boolean algebra $\mathfrak{B}$ of regular open subsets of $\mathcal{S}$. With additional topological hypotheses, we obtain a unique Borel probability measure on $\mathcal{S}$, along with an auxiliary apparatus called a *liminal structure*, which describes the agent's informational constraints. We also obtain SEU representations involving subjective state spaces, such as the Stone-Čech compactification of $\mathcal{S}$ and the Stone space of $\mathfrak{B}$.
Keywords: Subjective expected utility; topological space; technological feasibility; continuous utility; regular open set; Borel measure.
JEL classification: D81.

Natura non facit saltum. —Linnaeus

1 Introduction

Economic decisions under uncertainty often face technological constraints. Consider a farmer who must plant crops in the early spring, without knowing the meteorological conditions for the rest of the year. The crop yields of his various planting strategies

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‡Paris School of Economics, and University Paris 1 Panthéon-Sorbonne.

are thus uncertain at the moment of choice. But slight variations of the meteorological conditions will only result in slight variations in yields. The constraints of the agricultural technology imply that the only strategies available to the farmer are those where crop yields depend *continuously* on the unpredictable meteorological conditions.

Continuity constraints manifest in many other decision problems under uncertainty; in particular, they arise in most economic activities which depend upon natural resource extraction, weather conditions, or any other interaction with unpredictable features of the natural environment. They also arise in medical decisions, where the uncertainty concerns the patient's medical condition, and the outcome is her prognosis. Anthropogenic climate change generates a plethora of such decision problems; there is uncertainty about the values of many parameters in climate models, which leads to uncertainty about the response of weather patterns (e.g. temperature, rainfall, floods, droughts) to rising CO_2 concentrations. There is also uncertainty about the social and economic impact of these weather patterns, as well as the proposed policies to reduce CO_2 emissions. Finally, there is hour-to-hour uncertainty about the electricity output of solar and wind-power facilities. But in all these cases, the outcome varies continuously with the unknown variables.

Continuity constraints also arise in many other financial and economic decision problems. For example, the income stream arising from an individual's investment in education or a firm's investment in physical capital is a continuous function of future economic conditions. The value of a portfolio is a linear combination of the values of its constituent assets. In most financial derivatives (e.g. futures, options), the payoff for both buyer and seller is a continuous function of the price of the underlying assets. In most insurance contracts, the indemnity is a continuous function of the loss. Finally, the future real value of a savings instrument is a continuous function of future real interest rates. In these examples, continuity restrictions can be interpreted as a kind of market incompleteness.

Feasibility considerations constrain not only the possible actions, but also the information available to an agent. The limitations of her measurement technology may restrict the events that an agent can observe, form beliefs about, or employ for Bayesian updating. In particular, measurement devices are often *robust*, meaning that they are relatively insensitive to small perturbations. This is especially common in devices which convert "analog" to "digital" signals. For example, a digital thermometer lacks the precision needed to detect small changes of temperature. Measurement devices are also *approximate*: the output is often not a specific value, but a *range* of possible values.

The Subjective Expected Utility (SEU) model is the standard paradigm to describe decision-making under uncertainty. A key feature of the classic axiomatic foundations of Savage (1954) is that the agent has preferences over *all* possible functions from states into outcomes, and can condition on *all* subsets of the state space. This makes sense in decision problems where the state space has a *discrete* topology (e.g. bets on coin flips, urn experiments, sports games, or Arrow-Debreu economies). One *could* apply the Savage approach to technologically constrained decision problems, but this would require the agent to rank infeasible acts and condition on unobservable events; this would undermine both the normative and the descriptive content of the preference relation, the axioms, and the resulting SEU representation. At a normative level, an agent might feel uncomfortable about formulating preferences over infeasible acts, or conditioning on unobservable events; hence she might be reluctant to apply the Savage axioms to such preferences. These axioms are supposed to impose some "internal consistency" on preferences. But why should preferences over *feasible* acts be consistent with, and sometimes even determined by, preferences over *infeasible* acts? At a descriptive level, it is impossible by definition to observe an agent's preference over infeasible acts, or her preferences conditional on unobservable events. Such acts and events might still play a role in thought experiments. But since they are impossible to properly incentivize, one can seriously question the empirical meaning of such preferences and their relevance to the elicitation of utility and beliefs. For these reasons, technological constraints make it desirable to depart from the Savage framework and restrict preferences to feasible acts and observable events.

This paper studies decision-making under uncertainty with technological constraints, and axiomatically characterizes SEU representations of *ex ante* preferences in such an environment. The consequences of the decision range over a topological space of *outcomes*; these may be crop yields, health status, production levels, income streams, or consumption bundles. The underlying uncertainty is represented by a topological space of *states of the world*; this encodes all the meteorological, physiological, geophysical, or financial variables on which the outcome (continuously) depends. The feasible alternatives are given by a set of continuous functions, or *acts*, from the state space onto the outcome space; these could be production plans, medical interventions, climate policies, financial portfolios, or insurance contracts. The observable events are given by a family of open subsets of the statespace —the *regular* subsets. These represent the information that can be obtained by measuring meteorological conditions, estimating unknown climate parameters, or performing medical tests. Furthermore, the domain of feasible acts need not contain *all* continuous acts, and the observable events need not consist of *all* regular subsets; our framework can incorporate further technological restrictions on acts and information. Finally, the agent's preferences only rank feasible acts conditional upon observable events.

Of course, we do not claim that *every* decision under uncertainty exhibits these sorts of topological constraints. We only claim that *some* uncertain decisions exhibit such constraints —including some which occur frequently in practical contexts. This raises the question: does the axiomatization of SEU depend upon *ignoring* these constraints? Our main results show that it does not. But these constraints do create some technical difficulties. For example, Savage's axioms (e.g. the *Sure Thing Principle*) depend on the ability to splice any two acts on any bipartition of the state space. Furthermore, Savage obtains the subjective probability measure and utility function by restricting preferences to two-valued acts and finitely-valued acts respectively. But both spliced acts and finitely-valued acts are typically discontinuous, and hence inadmissible in our framework. Thus, we must depart from Savage, and use a very different axiomatization. Furthermore, our axioms only invoke *observable* events, so initially they can only yield subjective probabilities for these events. It requires a further step to extend this to a *bona fide* probability measure.

Despite these obstacles, we obtain several SEU representations. In these representations, utility is a continuous function; thus, similar outcomes yield similar utility levels. This makes our representations particularly relevant to applications in economics and finance, which usually take continuity for granted (Gollier, 2001). Utility is is unique up to positive affine transformations. But the representation of beliefs depends on the topology of the state space. Our first representation (Theorem 1) is "classical": beliefs take the form of a probability measure (called a *residual charge*), and are updated via Bayes rule as the agent acquires more information. This representation applies to any *Baire* state space. Starting from a *compact* state space, we obtain a more informative *liminal* SEU representation (Theorem 2); in this case, beliefs consist not only of a Borel probability measure, but also a *liminal structure*, with which the agent compensates for her informational constraints. These structures provide dynamically consistent, consequentialist updating rules for the Borel probability measures that generalize the classical Bayes rule. Assuming only a *locally* compact state space, we obtain *compactification* SEU representations (Theorems 3 and 4); here, beliefs are given by a Borel probability measure and a liminal structure on a compactification —a "subjective" state space which can be used to model catastrophic risks, anomalous risk preferences, and infinite-horizon intertemporal choice, *inter alia*.

These representations require specific assumptions on the state space topology, and assume that the agents observation technology is minimally constrained. For a general topological state space, or when the agent faces further informational constraints, beliefs are described by what we call a *credence*, a structure like a finitely additive probability measure on a Boolean subalgebra $\mathfrak{B}$ of regular subsets of the state space (Theorem 9). All our other theorems are consequences of this result. Finally, the *Stonean SEU* representations use the *Stone space* of $\mathfrak{B}$, a subjective state space larger than a compactification (Theorem 10), which admits a representation of beliefs consisting only of Borel probability measure.

The restriction of preferences to feasible acts and observable events inevitably weakens the grasp of the SEU axioms. The resulting SEU representations are not always as restrictive as in Savage's classical axiomatization. Rather, they accommodate a greater diversity of behavior by incorporating nonclassical features like liminal structures or compactifications. These nonclassical features can have a significant impact in economic applications, such as models of differential information, risk sharing and portfolio choice.

An agent might also confront *cognitive* feasibility constraints in addition to *technological* feasibility constraints. The classic Savage framework contains complex acts and complex events that are difficult for the agent to fully comprehend. Can she really formulate meaningful preferences over such acts, or condition on such events? If so, should we really expect such preferences to satisfy Savage's axioms? In contrast, *continuous* acts and *regular* events are easy to visualize and understand. Likewise, in addition to measurement errors, an agent may be subject to cognitive errors in processing the information she receives. Regular sets are more robust in the face of such errors, and hence, form a more reliable basis for the agent's mental representation of her environment. Thus, our results can also be interpreted as describing a cognitively limited agent.

The remainder of this paper is organized as follows: Section 2 introduces notation and terminology. Section 3 introduces the six axioms used in all our results. Section 4 presents our first SEU representation, which uses residual charges on Baire state spaces. Sections 5 and 6 present liminal SEU representations; Section 5 gives a liminal SEU representation for *compact* state spaces, while Section 6 obtains one for *locally compact* state spaces, via their compactifications. Section 7 presents our most general SEU representation, in terms of a credence on a Boolean subalgebra $\mathfrak{B}$ of regular sets. Section 8 continues this level of generality with an SEU representation using a Borel probability measure on the Stone space of $\mathfrak{B}$. Section 9 reviews prior literature.

All the proofs are in the Appendices. Appendix A contains the proof of Theorem 9, which is the lynchpin result of the paper. Appendix B then derives the SEU representations of Sections 4, 5, 6 and 8, using Theorem 9. Appendix C extends the representation theorem of Section 5 to *normal* Hausdorff spaces. Finally Appendix D discusses the topological implications of some of our assumptions.

2 Basic framework

Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces. Elements of $\mathcal{S}$ are called *states of the world* and describe the various possible resolutions of uncertainty. Elements of $\mathcal{X}$ are called *outcomes* and represent the various possible consequences of decisions. We will assume $\mathcal{X}$ is connected.

Acts. Like Savage, we will suppose that the agent can choose from a menu of *acts*, where each act is a function from the state space onto the outcome space. This function describes the outcome that would result from the choice of this act at each possible state of the world. Unlike Savage, we will assume only *continuous* acts are feasible.

Recall that a subset $\mathcal{Y} \subseteq \mathcal{X}$ is ***relatively compact*** if its closure $\text{clos}(\mathcal{Y})$ is compact. (It follows that any continuous, real-valued function on $\mathcal{X}$ is bounded when restricted to $\mathcal{Y}$.) For example, if $\mathcal{X}$ is a metric space, then $\mathcal{Y}$ is relatively compact if and only if $\mathcal{Y}$ is a bounded subset of $\mathcal{X}$. A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is ***bounded*** if its image $\alpha(\mathcal{S})$ is relatively compact in $\mathcal{X}$. If $\mathcal{X}$ is a metric space, then this agrees with the usual definition of "bounded". But this definition makes sense even if $\mathcal{X}$ is nonmetrizable. Let $\mathcal{C}(\mathcal{S}, \mathcal{X})$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$, and let $\mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$ be the set of all *bounded* continuous functions from $\mathcal{S}$ into $\mathcal{X}$. Unlike Savage, we assume only *bounded* acts are feasible. Meanwhile, our SEU representations will have potentially *unbounded* utility functions, whereas Savage's utility functions were bounded.

There may be additional feasibility restrictions on acts, beyond continuity and boundedness. Thus, we introduce an exogenously given subset $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$; this is the set of *feasible acts*. If technological constraints only entail continuity and boundedness, then $\mathcal{A} = \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$. But in general, $\mathcal{A}$ could be much smaller. For instance, if feasible production plans must be infinitely differentiable, then we could define $\mathcal{A}$ to be the set of all infinitely differentiable functions from $\mathcal{S}$ to $\mathcal{X}$. However, the collection $\mathcal{A}$ cannot be *too* small; it must be large enough to satisfy structural condition (Rch) below, and must contain all constant acts; these represents *riskless* alternatives. The inclusion of such acts in $\mathcal{A}$ means that we can risklessly obtain any outcome by a feasible act.

Information and conditional preferences. Most axiomatizations of SEU assume the agent can form *conditional preferences* once she acquires some information about the state

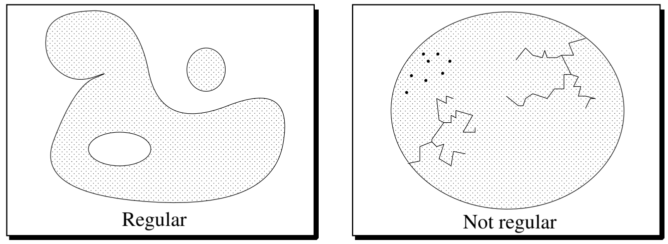

Figure 1: Regular and non-regular open subsets.

of the world. In some cases, these conditional preferences are defined implicitly through a separability axiom, such as Savage's P2. In other cases, the conditional preferences are explicitly built into the model (Hammond, 1988; Ghirardato, 2002). Even in these cases, however, we need a separability axiom such as *Dynamic consistency* to ensure that these conditional preferences are "consistent" with one another.

In all cases, the separability axiom says, roughly, "An agent's conditional preferences once she observes an event $\mathcal{E} \subseteq \mathcal{S}$ should completely ignore the behaviour of acts outside of $\mathcal{E}$." But if $\mathcal{S}$ is a topological space, and acts are *continuous*, then this statement is somewhat problematic, because the behaviour of an act on the *boundary* of $\mathcal{E}$ is completely determined by its behaviour on the *complement* of $\mathcal{E}$. Once we fix its value on the complement of $\mathcal{E}$, a continuous act can only vary freely on the *interior* of $\mathcal{E}$. So we can only meaningfully define $\mathcal{E}$-conditional preferences concerning the behaviour of acts on this interior. These preferences can only compare two acts which *agree* on the boundary, so they are very incomplete. Furthermore, if two events $\mathcal{E}_1$ and $\mathcal{E}_2$ have the *same* interior, then we must impute to them the *same* conditional preferences. Finally and most problematic: if the interior of $\mathcal{E}$ is empty, then $\mathcal{E}$-conditional preferences on continuous acts *cannot be defined.*

Furthermore, it is *unrealistic* to suppose an agent can form conditional preferences on arbitrary subsets of $\mathcal{S}$. In real life, observations are always noisy, fallible, and imprecise. The only part of an event $\mathcal{E}$ which is "robust"' against such noise and imprecision is its interior. If two events have the same interior, then they are "observationally indistinguishible" for the agent. In particular, if an event has an empty interior, then it is in *empirically unobservable.* For example, it would be absurd to suppose that an agent could consult a thermometer and observe the event, "The temperature is a rational number".

This suggests that we should only consider preferences conditional on *open* subsets of $\mathcal{S}$. But this creates a further problem, because the open subsets of $\mathcal{S}$ typically do not form a Boolean algebra: the complement of an open set is usually not open. Axiomatizations of SEU make heavy use of *partitions* of the state space (e.g. Savage's P6), and a topological space $\mathcal{S}$ does not admit nontrivial open partitions unless it is disconnected. (Indeed, $\mathcal{S}$ would not admit an open set version of P6 unless it was *totally* disconnected.)

However, there is a family of open sets which *do* form a Boolean algebra, albeit under slightly nonstandard operations: the *regular* subsets. Formally, a ***regular subset*** of a topological space $\mathcal{S}$ is an open subset $\mathcal{E} \subseteq \mathcal{S}$ which is the interior of its own closure.[1] For example, let $\mathcal{S} = \mathbb{R}$; then $(0, 1)$ is a regular subset, but $(0, 1) \cup (1, 2)$ is not. Heuristically,

[1]These are sometimes called "regular open" subsets.

an open set is regular if it does not have any "pinholes" or "cracks" (Figure 1). The intersection of two regular subsets is another regular subset. Given any two regular subsets $\mathcal{D}, \mathcal{E} \subseteq \mathcal{S}$, we define $\mathcal{D} \vee \mathcal{E} := \mathrm{int}[\mathrm{clos}(\mathcal{D} \cup \mathcal{E})]$ (the interior of $\mathrm{clos}(\mathcal{D} \cup \mathcal{E})$). This is the smallest regular set containing both $\mathcal{D}$ and $\mathcal{E}$. For example, if $\mathcal{S} = \mathbb{R}$, then $(0,1) \vee (1,2) = (0,2)$. Meanwhile, given a regular subset $\mathcal{D}$, we define $\neg\mathcal{D}$ to be the interior of $\mathcal{S} \setminus \mathcal{D}$ —another regular subset. The set $\mathfrak{R}(\mathcal{S})$ of all regular subsets of $\mathcal{S}$ forms a Boolean algebra under the operations $\vee$, $\cap$, and $\neg$ (Fremlin, 2004, Theorem 314P). We will only assume that the agent can observe *regular* subsets of $\mathcal{S}$ when defining her conditional preferences.

Now let $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. A *finite regular partition* of $\mathcal{R}$ is a collection $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ (for some $N \in \mathbb{N}$) of disjoint regular subsets of $\mathcal{S}$ such that $\mathcal{R} = \mathcal{R}_1 \vee \cdots \vee \mathcal{R}_N$. Equivalently: for all $m \in [1 \ldots N]$, the set $\mathcal{R}_m$ is the interior of $\mathcal{R} \setminus \bigcup_{n \neq m} \mathcal{R}_n$. This implies that $\bigcup_{n=1}^{N} \mathcal{R}_n$ is dense in $\mathcal{R}$. For instance, if $\mathcal{S} = \mathbb{R}$, then $\{(0,1), (1,2)\}$ is a regular partition of $(0,2)$. Regular partitions share some properties with the partitions used in standard Bayesian decision theory: cells must be nonempty and represent mutually exclusive events. But they depart from the standard model in that cells are not fully exhaustive: they only cover "almost all" of the space. A regular partition is nonetheless *observationally exhaustive*: the agent always *thinks* she observes some cell of the regular partition, even if the true state lies in the space between the cells. Suppose the agent makes an observation corresponding to the regular partition $\mathfrak{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, and let $s \in \mathcal{S}$ be the true state of the world. If $s \in \mathcal{R}_n$, then the agent will observe $\mathcal{R}_n$. But if $s \in \partial\mathcal{R}_n \cap \partial\mathcal{R}_m$, then the agent might "observe" $\mathcal{R}_n$ *or* $\mathcal{R}_m$ (or indeed, any other cell of $\mathfrak{R}$ whose closure contains s).

For example, suppose $\mathcal{S} = [-1, 1]$, let $\mathcal{L} := [-1, 0)$, and let $\mathcal{R} := (0, 1]$. Then, $\{\mathcal{L}, \mathcal{R}\}$ is a regular partition of $\mathcal{S}$. If the true state is negative, then the agent will observe $\mathcal{L}$, while if it is positive, she will observe $\mathcal{R}$. If the true state is zero, then she might "observe" either $\mathcal{L}$ or $\mathcal{R}$, even though neither event truly holds. Therefore, observing $\mathcal{L}$ (resp. $\mathcal{R}$) only provides the knowledge that the state is nonpositive (resp. nonnegative).

Regular partitions satisfy the aforementioned "robustness" property, because their cells are open sets: if $s \in \mathcal{R}_n$, then the agent will observe $\mathcal{R}_n$, and a sufficiently small perturbation of s will not modify this observation. By contrast, if s lies on the boundary between two cells, then a measurement of s will *not* be robust. This is precisely the reason for the "liminal structures" which appear in Sections 5 and 6; they describe how the agent copes with the "measurement instability" of states on the boundaries of regular sets.

Regular partitions also allow the possibility of *measurement errors*: the observation $\mathcal{R}_n$ does not mean that the true state lies in $\mathcal{R}_n$, but only that it lies inside the *closure* of $\mathcal{R}_n$. A measurement error occurs precisely when the true state lies in $\partial\mathcal{R}_n$ (and thus, outside of $\mathcal{R}_n$ itself), yet the agent nonetheless "observes" $\mathcal{R}_n$.

There may be additional technological constraints on information, beyond mere regularity. Let $\mathfrak{B}$ be some Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$; we can interpret $\mathfrak{B}$ as the algebra of "observable events". We will say that a regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of an event $\mathcal{R} \in \mathfrak{B}$ is a $\mathfrak{B}$-*partition* if $\mathcal{R}_n \in \mathfrak{B}$ for all $n \in [1 \ldots N]$. In Sections 4, 5 and 6, we will assume that *all* regular subsets are observable (i.e. $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$). But in the most general version of our model (Sections 7 and 8), we will suppose that the observable measurements arise from some Boolean subalgebra $\mathfrak{B}$ of $\mathfrak{R}(\mathcal{S})$. Because of this, we will formulate all the axioms of

Section 3 in terms of such a Boolean subalgebra. However, for the purposes of Sections 4, 5 and 6, the reader can just assume that $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ while reading Section 3.

Conditional preference structures. Savage (1954) started from a preference order on the set of unconditional acts. He then obtained conditional preferences via axiom P2 (the *Sure Thing Principle*). Axiom P2 assumes that, for any two feasible acts α and β, and any event $\mathcal{B}$, the "spliced" act $\alpha_{\mathcal{B}}\beta$ (which is equal to α on $\mathcal{B}$ and to β on the complement $\mathcal{B}^{\complement}$) is also feasible. But such "spliced" acts are often discontinuous, hence, inadmissible in our framework. So instead of defining conditional preferences *implicitly* via P2, we must assume they exist explicitly. But we will only assume that these preferences can rank *feasible acts*, and we only assume preferences conditional on *observable* events. Thus, in terms of its primitive behavioral data, our model is not directly comparable to the Savage (1954) theory: while Savage assumed a *single* preference order on the universal domain of acts, our approach relies on a *collection* of preference orders on a more restrictive domain. But compared to other conditional versions of SEU (e.g. Ghirardato, 2002), our approach requires less data, both in terms of the number of preference orders and their domain.

For any regular subset $\mathcal{B} \in \mathfrak{B}$, and any act $\alpha \in \mathcal{A}$, let $\alpha_{\upharpoonright\mathcal{B}}$ denote the restriction of α to a function on $\mathcal{B}$. Let $\mathcal{A}(\mathcal{B}) := \{\alpha_{\upharpoonright\mathcal{B}};\ \alpha \in \mathcal{A}\}$ be the set of acts conditional upon $\mathcal{B}$. For example, if $\mathcal{X}$ is compact and $\mathcal{A} = \mathcal{C}(\mathcal{S}, \mathcal{X})$, and $\mathcal{B}$ is a *retract* of $\mathcal{S}$, then $\mathcal{A}(\mathcal{B}) = \mathcal{C}(\mathcal{B}, \mathcal{X})$.[2] For a concrete example, suppose $\mathcal{S} = \mathbb{R}^N$ and $\mathcal{X} = [0, 1]^M$ (for some $N, M \geq 1$), and let $\mathcal{B}$ be the open unit ball in $\mathbb{R}^N$. If $\mathcal{A} = \mathcal{C}(\mathbb{R}^N, [0, 1]^M)$, then $\mathcal{A}(\mathcal{B}) := \mathcal{C}(\mathcal{B}, [0, 1]^M)$. Likewise, if $\mathcal{A}$ is the set of all differentiable (resp. Lipschitz, Hölder, or bounded variation) functions from $\mathbb{R}^N$ to $[0, 1]^M$, then $\mathcal{A}(\mathcal{B})$ is the set of all differentiable (resp. Lipschitz, Hölder, or bounded variation) functions from $\mathcal{B}$ to $[0, 1]^M$.

For all $\mathcal{B} \in \mathfrak{B}$, let $\succeq_{\mathcal{B}}$ be a preference order on $\mathcal{A}(\mathcal{B})$. We interpret $\succeq_{\mathcal{B}}$ as the *conditional preferences* over $\mathcal{A}(\mathcal{B})$ of an agent who has observed the event $\mathcal{B}$. We will therefore refer to the system $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ as a *conditional preference structure*; this will be the primitive data of the model. Our goal is to axiomatically characterize an SEU representation for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$.

3 Axioms

As already noted, the restriction to continuous acts means that we cannot rely on "spliced" acts the way that Savage did. Instead, we will require the set $\mathcal{A}$ of feasible acts to satisfy a "Richness" condition with respect to the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$.

The richness condition. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ be disjoint regular subsets of $\mathcal{S}$. For any $\alpha_1 \in \mathcal{A}(\mathcal{B}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{B}_2)$, say that α_1 and α_2 are *compatible* if there is some $\alpha \in \mathcal{A}$ with $\alpha_{\upharpoonright\mathcal{B}_1} = \alpha_1$ and $\alpha_{\upharpoonright\mathcal{B}_2} = \alpha_2$. We need $\mathcal{A}$ to satisfy the following condition:

(Rch) For any disjoint regular subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, and any $\alpha_1 \in \mathcal{A}(\mathcal{B}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{B}_2)$, there is an act $\beta_2 \in \mathcal{A}(\mathcal{B}_2)$ which is compatible with α_1, such that $\alpha_2 \approx_{\mathcal{B}_2} \beta_2$.

[2] If $\mathcal{B}$ is *not* a retract of $\mathcal{S}$, then not every continuous function on $\mathcal{B}$ extends continuously to $\mathcal{S}$. So $\mathcal{A}(\mathcal{B})$ is usually a proper subset of $\mathcal{C}(\mathcal{B}, \mathcal{X})$. The same holds for differentiable functions, Lipschitz functions, etc.

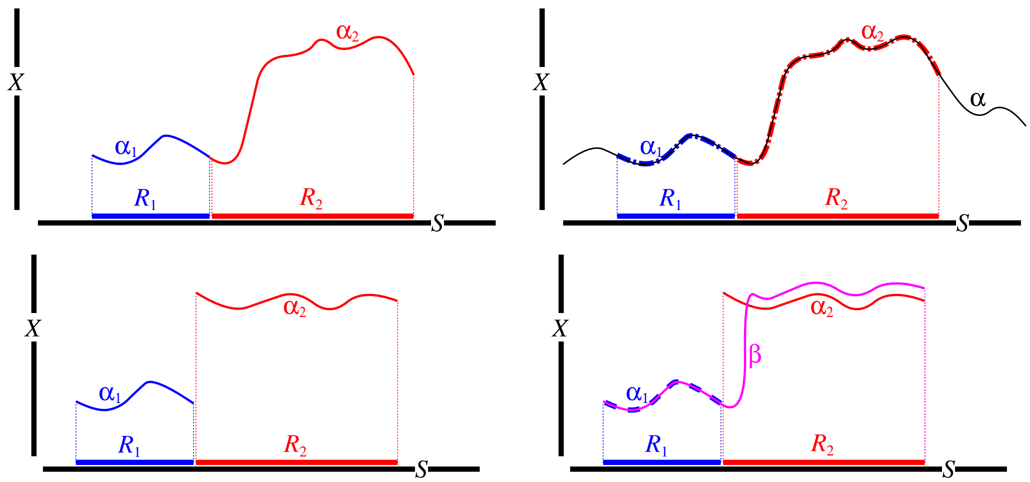

Figure 2: *Top row.* α_1 is compatible with α_2. *Bottom row.* The richness condition.

In other words, the values of an act on a regular subset $\mathcal{B}_1$ do not restrict the indifference class of that act conditional upon the disjoint regular subset $\mathcal{B}_2$, in spite of the continuity requirement on feasible acts. If there is a "gap" between $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathcal{S}$, then (Rch) is not very restrictive; often, *every* element of $\mathcal{A}(\mathcal{B}_2)$ is compatible with α_1. The nontrivial case of (Rch) is when $\mathcal{B}_1$ and $\mathcal{B}_2$ are "touching" –e.g. when $\mathcal{B}_1 = \neg\mathcal{B}_2$. In particular, (Rch) provides a weak version of Savage's act splicing: For any $\mathcal{B} \in \mathfrak{B}$, and any $\alpha, \beta \in \mathcal{A}$, there is some $\gamma \in \mathcal{A}$ that is equal to α on $\mathcal{B}$ and *indifferent* to $\beta_{\upharpoonright\neg\mathcal{B}}$ conditional on $\neg\mathcal{B}$. (Rch) is also similar to *solvability*, a condition often used in axiomatizations of additive utility.

$\mathcal{A}$ need not contain *all* bounded continuous functions from $\mathcal{S}$ to $\mathcal{X}$, as long as it satisfies (Rch) and contains all constant acts. For example, suppose $\mathcal{S}$ and $\mathcal{X}$ are differentiable manifolds (e.g. open subsets of Euclidean spaces $\mathbb{R}^N$ and $\mathbb{R}^M$, for some $N, M \geq 1$), and let $\mathcal{A}$ be the set of all *differentiable* functions from $\mathcal{S}$ to $\mathcal{C}$; then a conditional preference structure on $\mathcal{A}$ can easily satisfy (Rch) along with our other axioms.[3] Alternatively, let $\mathcal{S}$ and $\mathcal{X}$ be metric spaces, let $c \in (0, 1]$, and let $\mathcal{A}$ be the set of all *c-Hölder-continuous* functions from $\mathcal{S}$ to $\mathcal{X}$; then (Rch) is easily satisfied.[4] Or, let $\mathcal{S}$ be a bounded interval in $\mathbb{R}$, let $\mathcal{X}$ be a path-connected metric space, and let $\mathcal{A}$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$ having *bounded variation*; then again (Rch) is easily satisfied.[5] But if $\mathcal{S}$ and $\mathcal{X}$ are open subsets of Euclidean spaces, and $\mathcal{A}$ is a set of *analytic* functions from $\mathcal{S}$ to $\mathcal{X}$ (e.g. polynomials), then a conditional preference structure on $\mathcal{A}$ *cannot* satisfy (Rch).[6]

[3]The same is true if $\mathcal{A}$ is the set of N-times differentiable functions, for any $N \in [2 \ldots \infty]$.

[4]A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *c-Hölder-continuous* if there is some constant $K > 0$ such that $d[f(s_1), f(s_2)] \leq K \cdot d(s_1, s_2)^c$ for all $s_1, s_2 \in \mathcal{S}$. In the special case when $c = 1$, these are called *Lipschitz-continuous* functions. Any continuously differentiable function is Lipschitz.

[5]A function $\alpha : [0, S] \longrightarrow \mathcal{X}$ has *bounded variation* if its "total variation" $\sup\{\sum_{n=1}^N d[\alpha(s_n), \alpha(s_{n-1})]$; $N \in \mathbb{N}$ and $0 \leq s_0 < s_1 < \cdots < s_N \leq S\}$ is finite. Heuristically, this means that α does not oscillate too violently; it describes a path through $\mathcal{X}$ of finite total length.

[6]An infinitely differentiable function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *analytic* if it is the limit of its own Taylor series in a neighbourhood around each point in $\mathcal{S}$. An analytic function can be completely reconstructed from its

The ordering axiom. For the rest of this paper, we will assume that each order $\succeq_{\mathcal{B}}$ in the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ is *complete* (for any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, at least one of $\alpha \succeq_{\mathcal{B}} \beta$ or $\beta \succeq_{\mathcal{B}} \alpha$ holds), *transitive* (for any $\alpha, \beta, \gamma \in \mathcal{A}(\mathcal{B})$, if $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$, then $\alpha \succeq_{\mathcal{B}} \gamma$), and *nontrivial* (there exist $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ such that $\alpha \succ_{\mathcal{B}} \beta$).

These assumptions are more natural in our framework than in Savage's: they only require a transitive ordering on *feasible* acts, not on all logically possible acts. To understand the interplay between feasibility and transitivity, consider a case where an agent observes event $\mathcal{B} \in \mathfrak{B}$, and must choose between two feasible acts α and γ in $\mathcal{A}(\mathcal{B})$. Say momentarily that she has preferences over unfeasible acts, and that there is an unfeasible act β such that $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$. A blind application of transitivity would yield $\alpha \succeq_{\mathcal{B}} \gamma$. But the unfeasibility of β undermines the meaningfulness of both rankings $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$. Why should these two rankings influence the choice between α and γ? By restricting preferences to feasible acts, we eliminate such spurious influences.

The separability axioms. Additive separability over disjoint events is a characteristic feature of SEU theories. In a Savage framework, it is captured by P2. In Ghirardato's (2002) axiomatization, where an agent is endowed with conditional preferences, separability is captured by the axiom of Dynamic Consistency. Dynamic Consistency also plays a central role in Hammond's (1988) derivation of SEU maximization on decision trees; see also Hammond (1998, §6-§7). Our next axiom captures separability through a version of Dynamic Consistency that only applies to regular partitions of a regular event.

(Sep) For any event $\mathcal{B} \in \mathfrak{B}$, any disjoint events $\mathcal{D}, \mathcal{E} \in \mathfrak{B}$ such that $\mathcal{D} \vee \mathcal{E} = \mathcal{B}$, and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ with $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, we have $\alpha \succeq_{\mathcal{B}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \succeq_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$.

In (Sep), the "forward implication" (from $\alpha \succeq_{\mathcal{B}} \beta$ to $\alpha_{\upharpoonright\mathcal{E}} \succeq_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$) says that a feasible act that was deemed optimal conditional on $\mathcal{B}$ will be still be optimal conditional on $\mathcal{E}$. The "backward implication" says that a more-informed decision is more reliable than a less-informed decision; thus, decisions based on inferior information should be guided by the hypothetical decisions that *would* have been made with *superior* information. In this case, the agent should choose α over β given inferior information ($\mathcal{B}$), because she recognizes that she *would* be willing to choose α given superior information (either $\mathcal{D}$ or $\mathcal{E}$).

Just as the restriction to feasible acts strengthens the appeal of the ordering axiom, the restriction to regular events strengthens the appeal of (Sep) —more specifically, its "backward implication". To see this, suppose the agent must choose between two feasible acts α and β, conditional on some regular event $\mathcal{B}$. Say that she has preferences conditional upon nonregular (hence, unobservable) events $\mathcal{D}$ and $\mathcal{E}$, with $\mathcal{D} \vee \mathcal{E} = \mathcal{B}$, such that $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}} \succeq_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$. A naïve application of separability would then yield $\alpha \succeq_{\mathcal{B}} \beta$. But $\mathcal{D}$ and $\mathcal{E}$ are *unobservable* events, so it is not clear that the preferences conditional on $\mathcal{D}$ and $\mathcal{E}$ are even meaningful, much less that they should determine the choice between α and β. The restriction to regular events eliminates this problem.

behaviour in a tiny neighbourhood around any point in its domain. This means that an analytic function defined on an open subset $\mathcal{R} \subseteq \mathcal{S}$ has at most *one* extension to an analytic function on all of $\mathcal{S}$.

It is easy to see that the logical equivalence in Axiom (Sep) also holds for indifference and for strict preference: for any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ with $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, we have:

(i) $\alpha \succ_{\mathcal{B}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \succ_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$; and

(ii) $\alpha \approx_{\mathcal{B}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \approx_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$.

Statement (i) means that *no event in* $\mathfrak{B}$ *is null.* Thus, any SEU representation must give nonzero probability to all events in $\mathfrak{B}$. Conversely, statement (ii) says that *the boundary of any event in* $\mathfrak{B}$ *is null*: the behaviour of α and β on that small part of $\mathcal{B}$ that is not covered by $\mathcal{D} \cup \mathcal{E}$ is irrelevant for decisions conditional on $\mathcal{B}$. This seems to suggest that the SEU representation must give *zero probability* to the boundary of any regular set. But $\mathcal{A}$ is a set of *continuous* functions; thus, the behaviour of α and β on the open sets $\mathcal{D}$ and $\mathcal{E}$ entirely determines their behaviour on the common boundary $\partial\mathcal{D} \cap \partial\mathcal{E}$. Thus, statement (ii) does not mean that we *ignore* the behaviour of α and β on $\partial\mathcal{D} \cap \partial\mathcal{E}$, as if $\partial\mathcal{D} \cap \partial\mathcal{E}$ had zero probability; it just means that we have *already* implicitly accounted for this behaviour in our rankings of $\alpha_{\upharpoonright\mathcal{D}}$ versus $\beta_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}}$ versus $\beta_{\upharpoonright\mathcal{E}}$. This implicit account will become explicit with the "liminal structures" of Sections 5 and 6.

If $\mathcal{D}, \mathcal{E} \in \mathfrak{B}$ are disjoint and $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$, then Axiom (Sep) says that the $\succeq_{\mathcal{B}}$-ranking of two acts $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ is partly determined by the $\succeq_{\mathcal{D}}$-ranking of $\alpha_{\upharpoonright\mathcal{D}}$ versus $\beta_{\upharpoonright\mathcal{D}}$ and the $\succeq_{\mathcal{E}}$-ranking of $\alpha_{\upharpoonright\mathcal{E}}$ versus $\beta_{\upharpoonright\mathcal{E}}$. The next axiom says that this dependency is continuous.

(PC) Let $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$ as in axiom (Sep). Let $\underline{\beta}, \alpha, \overline{\beta} \in \mathcal{A}(\mathcal{B})$ be three acts with $\underline{\beta} \prec_{\mathcal{B}} \alpha \prec_{\mathcal{B}} \overline{\beta}$. Then there exist $\underline{\delta}, \overline{\delta} \in \mathcal{A}(\mathcal{D})$ and $\underline{\epsilon}, \overline{\epsilon} \in \mathcal{A}(\mathcal{E})$, with $\underline{\delta} \prec_{\mathcal{D}} \alpha_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha_{\upharpoonright\mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ such that, for any $\alpha' \in \mathcal{A}(\mathcal{B})$, if $\underline{\delta} \prec_{\mathcal{D}} \alpha'_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha'_{\upharpoonright\mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ then $\underline{\beta} \prec_{\mathcal{B}} \alpha' \prec_{\mathcal{B}} \overline{\beta}$.

The intuition here is that a small variation in $\alpha_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}}$ (relative to the order topologies on $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{E})$) should not affect the $\succeq_{\mathcal{B}}$- ranking of α versus $\underline{\beta}$ and $\overline{\beta}$.

Continuity of *ex post* preferences. For any $x \in \mathcal{X}$, let κ^x be the constant x-valued act on $\mathcal{S}$. Let $\mathcal{K} := \{\kappa^x;\ x \in \mathcal{X}\}$. We have assumed $\mathcal{K} \subseteq \mathcal{A}$, so the preference order $\succeq_{\mathcal{S}}$, restricted to $\mathcal{K}$, induces a preference order $\succeq_{\mathrm{xp}}$ on $\mathcal{X}$ as follows: for any $x, y \in \mathcal{X}$,

$$\Big(x \succeq_{\mathrm{xp}} y\Big) \quad \Longleftrightarrow \quad \Big(\kappa^x \succeq_{\mathcal{S}} \kappa^y\Big). \tag{1}$$

$\succeq_{\mathrm{xp}}$ describes the *ex post* preferences of the agent on $\mathcal{X}$ when there is no uncertainty. The next axiom says that these preferences are compatible with the underlying topology on $\mathcal{X}$.

(C) The *ex post* order $\succeq_{\mathrm{xp}}$ is continuous in the topology on $\mathcal{X}$. That is: for all $x \in \mathcal{X}$, the contour sets $\{y \in \mathcal{X};\ y \succeq_{\mathrm{xp}} x\}$ and $\{y \in \mathcal{X};\ y \preceq_{\mathrm{xp}} x\}$ are closed subsets of $\mathcal{X}$.

Certainty equivalents. For any $\mathcal{B} \in \mathfrak{B}$ and $x \in \mathcal{X}$, let $\kappa^x_{\mathcal{B}} := (\kappa^x)_{\upharpoonright\mathcal{B}}$; this is the constant x-valued act, conditional on $\mathcal{B}$. Given an act $\alpha \in \mathcal{A}(\mathcal{B})$, we say x is a *certainty equivalent* for α on $\mathcal{B}$ if $\kappa^x_{\mathcal{B}} \approx_{\mathcal{B}} \alpha$. The next axiom is a mild richness condition on $\mathcal{X}$.

(CEq) For any event $\mathcal{B} \in \mathfrak{B}$, any act $\alpha \in \mathcal{A}(\mathcal{B})$ has a certainty equivalent on $\mathcal{B}$.

Axiom (CEq) may appear somewhat implausible. But it is a logical consequence of the following axiom of "constant continuity" which may seem more natural.

(CC) For any event $\mathcal{B} \in \mathfrak{B}$ and any act $\alpha \in \mathcal{A}(\mathcal{B})$, the sets $\{x \in \mathcal{X};\ \kappa^x_{\mathcal{B}} \succeq_{\mathcal{B}} \alpha\}$ and $\{x \in \mathcal{X};\ \alpha \succeq_{\mathcal{B}} \kappa^x_{\mathcal{B}}\}$ are closed in $\mathcal{X}$.

If $\mathcal{X}$ is connected and $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, then (CC) is equivalent to the conjunction of (C) and (CEq). So we could state our results with a single axiom (CC) in place of (CEq) and (C).

The statewise dominance axiom. Our next axiom imposes some consistency between the agent's conditional preference structure and her *ex post* preferences. It says that the agent always prefers a statewise dominating act.

(Dom) For any $\mathcal{B} \in \mathfrak{B}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, if $\alpha(b) \succeq_{\mathrm{xp}} \beta(b)$ for all $b \in \mathcal{B}$, then $\alpha \succeq_{\mathcal{B}} \beta$. Furthermore, if $\alpha(b) \succ_{\mathrm{xp}} \beta(b)$ for all $b \in \mathcal{B}$, then $\alpha \succ_{\mathcal{B}} \beta$.

Recall that if the agent "observes" the event $\mathcal{B} \in \mathfrak{B}$, this does not mean the true state lies in $\mathcal{B}$ —it only lies in the *closure* of $\mathcal{B}$. But this does not problematize the first part of (Dom): since α and β are continuous functions, they have unique extensions to the closure of $\mathcal{B}$, and these extensions preserve weak statewise dominance. Thus, weak statewise dominance over $\mathcal{B}$ implies weak statewise dominance over all states that remain possible: those in the closure of $\mathcal{B}$. Of course, the extensions of α and β might not preserve *strict* dominance. So the second part of (Dom) requires some sub-event of $\mathcal{B}$ to be non-null. But as we have already observed, (Sep) implies that *all* events in $\mathfrak{B}$ are non-null.

(Dom) appears similar to (Sep), and thus to Savage's axiom P2. The difference is that (Sep) applies to regular partitions, while (Dom) applies to partitions into singleton sets, which, in general, are *not* regular. Thus, (Dom) cannot be obtained as a special case of (Sep). Axiom (Dom) is also related to Savage's axioms P3 and P7. Axiom P3 requires the ranking of outcomes to be independent of the events that yield the outcomes. (Dom) entails a similar form of state independence: it implies that $\succeq_{\mathcal{S}}$ can be replaced by $\succeq_{\mathcal{B}}$ for any $\mathcal{B} \in \mathfrak{B}$, in formula (1). Thus, the *ex post* preference orders obtained from different conditional preference orders must agree with one other. Finally, to see how (Dom) and P7 overlap, consider the special case of (Dom) where one of α or β is a constant act.

Tradeoff consistency. Our last axiom is a version of the *Cardinal Coordinate Independence* axiom used in Wakker's (1988) axiomatization of SEU. We need some preliminary definitions. Let $\mathcal{B} \in \mathfrak{B}$, and let $\mathcal{Q} := \neg\mathcal{B}$. Consider an outcome $x \in \mathcal{X}$ and an act $\alpha \in \mathcal{A}(\mathcal{Q})$. Structural condition (Rch) yields an act $(x_{\mathcal{B}}\alpha) \in \mathcal{A}$ with two properties:

$$\textbf{(B1)}\quad (x_{\mathcal{B}}\alpha)_{\upharpoonright\mathcal{B}} \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}, \qquad \text{and} \qquad \textbf{(B2)}\quad (x_{\mathcal{B}}\alpha)_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \alpha.$$

We will call $(x_{\mathcal{B}}\alpha)$ an (x, α)-*bet* for $\mathcal{B}$; if $\mathcal{B}$ obtains, this bet is indifferent to the outcome x, while it is indifferent to α conditional on the complement of $\mathcal{B}$. Note that $(x_{\mathcal{B}}\alpha)$ is not uniquely defined by (B1) and (B2). But if $(x_{\mathcal{B}}\alpha)$ and $(x_{\mathcal{B}}\alpha)'$ are two acts satisfying (B1) and (B2), then axiom (Sep) implies that $(x_{\mathcal{B}}\alpha) \approx_{\mathcal{S}} (x_{\mathcal{B}}\alpha)'$.

Fix now four outcomes $x, y, v, w \in \mathcal{X}$, and a regular subset $\mathcal{B} \in \mathfrak{B}$. Let $\mathcal{Q} := \neg\mathcal{B}$. We write $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ if there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q})$, an (x, α)-bet $(x_\mathcal{B}\alpha) \in \mathcal{A}$, a (y, β)-bet $(y_\mathcal{B}\beta) \in \mathcal{A}$, a (v, α)-bet $(v_\mathcal{B}\alpha) \in \mathcal{A}$ and a (w, β)-bet $(w_\mathcal{B}\beta) \in \mathcal{A}$ such that $(x_\mathcal{B}\alpha) \preceq_\mathcal{S} (y_\mathcal{B}\beta)$ while $(v_\mathcal{B}\alpha) \succeq_\mathcal{S} (w_\mathcal{B}\beta)$. By the remark in the previous paragraph, this implies that for *any* such bets $(x_\mathcal{B}\alpha), (y_\mathcal{B}\beta), (v_\mathcal{B}\alpha), (w_\mathcal{B}\beta) \in \mathcal{A}$, we have $(x_\mathcal{B}\alpha) \preceq_\mathcal{S} (y_\mathcal{B}\beta)$ and $(v_\mathcal{B}\alpha) \succeq_\mathcal{S} (w_\mathcal{B}\beta)$.

If $(x_\mathcal{B}\alpha) \preceq_\mathcal{S} (y_\mathcal{B}\beta)$, then the "gain" obtained by changing x to y on $\mathcal{B}$ is at least enough to compensate for the "loss" incurred by changing α to β on $\mathcal{Q}$. In contrast, if $(v_\mathcal{B}\alpha) \succeq_\mathcal{S} (w_\mathcal{B}\beta)$, then the gain obtained by changing v to w on $\mathcal{B}$ is at *most* enough to compensate for the loss incurred by changing α to β on $\mathcal{Q}$. Together, these two observations imply that the gain obtained from changing x to y on $\mathcal{B}$ is at least as large as the gain from changing v to w on $\mathcal{B}$; hence the notation $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$. If $\succeq_\mathcal{S}$ has an SEU representation with utility function u, then $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ means that $u(y) - u(x) \geq u(w) - u(v)$.

Conversely, we write $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ if there exist $\gamma, \delta \in \mathcal{A}(\mathcal{Q})$, an (x, γ)-bet $(x_\mathcal{B}\gamma) \in \mathcal{A}$, a (y, δ)-bet $(y_\mathcal{B}\delta) \in \mathcal{A}$, a (v, γ)-bet $(v_\mathcal{B}\gamma) \in \mathcal{A}$ and a (w, δ)-bet $(w_\mathcal{B}\delta) \in \mathcal{A}$ such that $(x_\mathcal{B}\gamma) \succeq_\mathcal{S} (y_\mathcal{B}\delta)$ while $(v_\mathcal{B}\gamma) \prec_\mathcal{S} (w_\mathcal{B}\delta)$. Again, this implies that $(x_\mathcal{B}\gamma) \succeq_\mathcal{S} (y_\mathcal{B}\delta)$ and $(v_\mathcal{B}\gamma) \prec_\mathcal{S} (w_\mathcal{B}\delta)$ for *any* such bets $(x_\mathcal{B}\gamma), (y_\mathcal{B}\delta), (v_\mathcal{B}\gamma), (w_\mathcal{B}\delta) \in \mathcal{A}$. If $\succeq_\mathcal{S}$ had an SEU representation, then this means that $u(y) - u(x) < u(w) - u(v)$. Here is our final axiom:

(TC) For any two regular subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, there are no $x, y, v, w \in \mathcal{X}$ such that $(x \overset{\mathcal{B}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}_1}{\rightsquigarrow} w)$ while $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$.

In the case $\mathcal{B}_1 = \mathcal{B}_2$, (TC) requires trading attitudes over outcomes to be well-defined, independently of the acts that are used to reveal them. In the case $\mathcal{B}_1 \neq \mathcal{B}_2$, this axiom requires trading attitudes at different regular subsets to be consistent with each other: they must be independent of the event over which outcomes are traded. Thus, trading attitides can be evaluated independently from the choice situation used to reveal them.

Previous axiomatizations of SEU using a tradeoff consistency axiom (e.g. Wakker (1988)) did not also require a separability axiom, because it was implied by tradeoff consistency. But our axiom (Sep) is not superseded by (TC); to the contrary, (Sep) is necessary to even *state* (TC). Axiom (TC) needs "bets"' satisfying conditions (B1) and (B2). To construct these bets, we use (Rch). But the resulting construction is non-unique. To show that this non-uniqueness doesn't matter in our formulation of (TC), we must invoke (Sep).

4 SEU representations on Baire state spaces

In this section and the next two sections, we will suppose that *all* regular subsets of $\mathcal{S}$ are observable. Thus, the agent's conditional preference structure takes the form $\{\succeq_\mathcal{R}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$. We will see that, with mild assumptions on the topology of the state space, the axioms described in Section 3 yield a unique subjective expected utility representation for this conditional preference structure. But first we need some topological preliminaries.[7]

[7]For further explanation of these and all other topological concepts in this paper, see Willard (2004) or Engelking (1989). Other sources are Aliprantis and Border (2006, Ch.2) and Royden (1988, Ch.8).

A subset $\mathcal{N} \subseteq \mathcal{S}$ is *nowhere dense* if $\mathrm{int}[\mathrm{clos}(\mathcal{N})] = \emptyset$. For example, for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, the boundary $\partial\mathcal{R}$ is nowhere dense in $\mathcal{S}$. A subset $\mathcal{M} \subseteq \mathcal{S}$ is *meager* if it is a countable union of nowhere dense sets. Heuristically, meager sets are "small". For example, the set $\mathbb{Q}$ of rational numbers is meager in the space $\mathbb{R}$.

A subset $\mathcal{B} \subseteq \mathcal{S}$ has the *Baire property* if $\mathcal{B} = \mathcal{O} \triangle \mathcal{M}$ for some open $\mathcal{O} \subseteq \mathcal{S}$ and some meager $\mathcal{M} \subset \mathcal{S}$. Heuristically, this means that $\mathcal{B}$ is "almost open" in $\mathcal{S}$. Let $\mathfrak{Bai}(\mathcal{S})$ be the collection of all subsets of $\mathcal{S}$ with the Baire property; then $\mathfrak{Bai}(\mathcal{S})$ is a Boolean algebra under the standard set-theoretic operations. Observe that $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{Bai}(\mathcal{S})$ as sets, but the Boolean algebra operations are different.

A *probability charge* on $\mathfrak{Bai}(\mathcal{S})$ is a function $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0,1]$ such that (1) $\nu[\mathcal{S}] = 1$ and (2) $\nu[\mathcal{A} \sqcup \mathcal{B}] = \nu[\mathcal{A}] + \nu[\mathcal{B}]$ for any disjoint $\mathcal{A}, \mathcal{B} \in \mathfrak{Bai}(\mathcal{S})$. We say that ν is a *residual charge* if, furthermore, $\nu[\mathcal{M}] = 0$ for any meager $\mathcal{M} \subset \mathcal{S}$. We say that ν has *full support* if $\nu(\mathcal{R}) > 0$ for any nonempty regular subset $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. In particular, if every nonempty open subset of $\mathcal{S}$ contains a nonempty regular subset (i.e. $\mathcal{S}$ is *quasiregular*), then ν has full support if and only if $\nu(\mathcal{O}) > 0$ for any nonempty open subset $\mathcal{O}$.

The topological space $\mathcal{S}$ is a *Baire space* if the intersection of any countable family of open dense sets is dense. For example, any open subset of the Euclidean space $\mathbb{R}^N$ is a Baire space. More generally, any completely metrizable space is Baire (Willard, 2004, Corollary 25.4).[8] In particular, every topological manifold is Baire.[9] Finally, any locally compact Hausdorff space is Baire.[10] Intuitively, *non*-Baire spaces are extremely "sparse" or "porous"; they are unlikely to arise naturally in economic models. (For example, a countable Hausdorff space is not Baire. Also, the product topology on $\mathbb{Q} \times \mathbb{R}$ is not Baire.) Finally, $\mathcal{S}$ is *nondegenerate* if it contains a nonempty open subset which is *not* dense —or equivalently, a proper closed subset with nonempty interior. This means that $\mathfrak{R}(\mathcal{S})$ is not trivial. Nondegeneracy is a very mild condition; for example, any nonsingleton Hausdorff space is nondegenerate (Lemma B2(a)). Our first result gives an SEU representation for conditional preference structures on any nondegenerate Baire state space.

Theorem 1 *Let $\mathcal{S}$ be a nondegenerate Baire space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ satisfying* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies* (CEq), (C), (Dom), (Sep), (PC) *and* (TC) *if and only if there is a residual charge ν on $\mathcal{S}$ with full support, and a continuous function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$, we have*

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \left(\int_{\mathcal{R}} u \circ \alpha \, \mathrm{d}\nu \geq \int_{\mathcal{R}} u \circ \beta \, \mathrm{d}\nu\right). \qquad (2)$$

Furthermore, ν is unique, and u is unique up to positive affine transformation.

[8] A topological space is *metrizable* if its topology can be generated by a metric. It is *completely metrizable* if this metric is *complete*, meaning that every Cauchy sequence converges.

[9] A Hausdorff space is an N-dimensional *topological manifold* if every point has an open neighbourhood which is homeomorphic to $\mathbb{R}^N$. Thus, it is "locally Euclidean" near every point. A sphere is an example.

[10] See Section 5 for the definition of "Hausdorff", and Section 6 for the definition of "locally compact".

Theorem 1 yields a "classical" SEU representation, quite similar to Savage. But there are three possible objections to this result: first, $\mathfrak{Bai}(\mathcal{S})$ is not the familiar Borel sigma algebra; second, residual charges are only finitely additive, and third, they are generally not *normal*. To respond to these objections, the next section provides an SEU representation with a normal Borel probability measure. This representation will also give an insight into how the agent copes with her informational constraints —an insight which is lacking in Theorem 1. Heuristically, this is because a residual charge assigns probability zero to the boundary of any regular set, as if the agent thinks that such boundary events "never" occur. But a better representation of the agent's beliefs shows that she recognizes that boundary events *can* occur; this is encoded in what we call a *liminal* structure.

5 SEU representations on compact state spaces

A topological space $\mathcal{S}$ is *Hausdorff* if any pair of points in $\mathcal{S}$ can be placed in two disjoint open neighbourhoods. For example, any metrizable space (e.g. any topological manifold) is Hausdorff. In particular, any subset of $\mathbb{R}^N$ is Hausdorff. The space $\mathcal{S}$ is *compact* if, for any collection $\mathfrak{O}$ of open sets whose union is $\mathcal{S}$, there is a finite subcollection $\{\mathcal{O}_1, \ldots \mathcal{O}_N\} \subseteq \mathfrak{O}$ such that $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_N = \mathcal{S}$. For example, any closed, bounded subset of $\mathbb{R}^N$ is compact.

Let $\mathfrak{Bor}(\mathcal{S})$ be the Borel sigma-algebra of $\mathcal{S}$ —that is, the smallest sigma-algebra containing all open sets. Let ν be a *Borel probability measure* on $\mathcal{S}$ —that is, a (countably additive) probability measure on $\mathfrak{Bor}(\mathcal{S})$. For any $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$, let $\nu_{\mathcal{B}}$ be the restriction of ν to a Borel measure on $\mathcal{B}$, and let $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}})$ be the Banach space of real-valued, $\nu_{\mathcal{B}}$-integrable functions on $\mathcal{B}$, modulo equality $\nu_{\mathcal{B}}$-almost everywhere. Finally, let $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}}; [0,1])$ be the set of $[0,1]$-valued functions in $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}})$. A *liminal density structure subordinate to* ν is a collection $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$, where, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{R})$, $\phi_{\mathcal{R}} \in \mathbb{L}^1(\partial\mathcal{R}, \nu_{\partial\mathcal{R}}; [0,1])$ is a function such that, for any regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, we have

$$\phi_{\mathcal{R}_1} + \cdots + \phi_{\mathcal{R}_N} \quad = \quad 1, \quad \nu\text{-almost everywhere on } \partial\mathcal{R}_1 \cup \cdots \cup \partial\mathcal{R}_N. \tag{3}$$

In the model we develop in this section, the agent's "beliefs" will be represented by a Borel probability measure ν, along with a liminal density structure $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ subordinate to ν. Heuristically, ν describes the agent's *ex ante* beliefs, while $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ describes how she copes with her informational constraints. Recall that, when the agent "observes" a regular event $\mathcal{R}$, there is a chance that the state of the world is not in $\mathcal{R}$, but instead on its boundary $\partial\mathcal{R}$. The density $\phi_{\mathcal{R}}$ describes the agent's beliefs about the latter possibility. To be precise, she assigns the following conditional probability to any event $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$:

$$\frac{\nu(\mathcal{B} \cap \mathcal{R}) + \displaystyle\int_{\mathcal{B} \cap \partial\mathcal{R}} \phi_{\mathcal{R}} \, \mathrm{d}\nu}{\nu(\mathcal{R}) + \displaystyle\int_{\partial\mathcal{R}} \phi_{\mathcal{R}} \, \mathrm{d}\nu}. \tag{4}$$

Note that this expression may leave some conditional probability *outside* $\mathcal{R}$. Thus, "observing" $\mathcal{R}$ increases the probability that $\mathcal{R}$ actually holds to at least $\nu(\mathcal{R})/\nu[\mathrm{clos}(\mathcal{R})]$, but not necessarily to certainty. However, this "spillover" probability is confined to the *closure* of $\mathcal{R}$; formula (4) implies that the probability of $\mathrm{clos}(\mathcal{R})$ given $\mathcal{R}$ always equals one. The intuition is straightforward: because of the technological constraints on her information, an agent observing $\mathcal{R}$ only knows for sure that the true state lies in $\mathrm{clos}(\mathcal{R})$.

For example, suppose the agent is about to make an observation corresponding to a regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$. For all $n \in [1 \ldots N]$, her *ex ante* probability of "observing" $\mathcal{R}_n$ is $\mu[\mathcal{R}_n] := \nu[\mathcal{R}_n] + \int_{\partial\mathcal{R}_n} \phi_{\mathcal{R}_n}\, \mathrm{d}\nu$. Now suppose that she *has* observed $\mathcal{R}_n$; then she believes that the true state is *really* in $\mathcal{R}_n$ with probability $\nu[\mathcal{R}_n]/\mu[\mathcal{R}_n]$, whereas it is on the *boundary* of $\mathcal{R}_n$ with probability $1-(\nu[\mathcal{R}_n]/\mu[\mathcal{R}_n])$. Furthermore, the density $\phi_{\mathcal{R}_n}$ tells her *where* the state is likely to be on $\partial\mathcal{R}_n$, given that this latter case occurs.

In particular, suppose $\mathcal{S} = [-1, 1]$, and consider the partition $\{\mathcal{L}, \mathcal{R}\}$, where $\mathcal{L} := [-1, 0)$ and $\mathcal{R} := (0, 1]$. Thus, $\partial\mathcal{L} = \partial\mathcal{R} = \{0\}$, so that $\phi_{\mathcal{L}}$ and $\phi_{\mathcal{R}}$ are entirely determined by their values at 0. Suppose that $\nu\{0\} > 0$. Formula (3) says that $\phi_{\mathcal{L}}(0) + \phi_{\mathcal{R}}(0) = 1$. Let $\mu[\mathcal{L}] := \nu(\mathcal{L}) + \phi_{\mathcal{L}}(0)\,\nu\{0\}$, while $\mu[\mathcal{R}] := \nu(\mathcal{R}) + \phi_{\mathcal{R}}(0)\,\nu\{0\}$; these are the agent's subjective probabilities of observing $\mathcal{L}$ and $\mathcal{R}$. The agent believes that, if the true state were $s = 0$, then she would observe the event $\mathcal{L}$ with probability $\phi_{\mathcal{L}}(0)$, whereas she would observe the event $\mathcal{R}$ with probability $\phi_{\mathcal{R}}(0)$. Once she has observed $\mathcal{L}$, she thinks that actually $s = 0$ with probability $p := \phi_{\mathcal{L}}(0)\,\nu\{0\}/\mu(\mathcal{L})$, whereas $s < 0$ with probability $1-p$. On the other hand, if she has observed $\mathcal{R}$, then she thinks that actually $s = 0$ with probability $q := \phi_{\mathcal{R}}(0)\,\nu\{0\}/\mu(\mathcal{R})$, whereas $s > 0$ with probability $1 - q$.

Liminal density SEU representation. Let $\mathcal{X}$ be another topological space, let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A *liminal density SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a Borel probability measure ν on $\mathfrak{Bor}(\mathcal{S})$ and a liminal density structure $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ subordinate to ν, along with a continuous utility function $u : \mathcal{X}\longrightarrow\mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \quad\Longleftrightarrow \tag{5}$$
$$\left(\int_{\mathcal{R}} u\circ\alpha\ \mathrm{d}\nu + \int_{\partial\mathcal{R}} (u\circ\alpha)\,\phi_{\mathcal{R}}\ \mathrm{d}\nu \ \geq\ \int_{\mathcal{R}} u\circ\beta\ \mathrm{d}\nu + \int_{\partial\mathcal{R}} (u\circ\beta)\,\phi_{\mathcal{R}}\ \mathrm{d}\nu\right).$$

(Here we use the fact $u \circ \alpha$ and $u \circ \beta$ have unique extensions to $\partial\mathcal{R}$, by continuity.) Note that the "boundary" terms in (5) do *not* violate consequentialism or dynamic consistency. To see this, let $\alpha, \beta \in \mathcal{A}$. Because α, β and u are all continuous, the values of $u \circ \alpha$ and $u\circ\beta$ on $\partial\mathcal{R}$ are completely determined by their values on $\mathcal{R}$. Thus, (5) satisfies *consequentialism*: if $\alpha_{\upharpoonright\mathcal{R}} = \beta_{\upharpoonright\mathcal{R}}$, then α and β must have the same expected utility, conditional on $\mathcal{R}$. Meanwhile, equation (3) ensures that the expected utility of α on $\mathcal{S}$ is a weighted average of the conditional-expected utilities of α on $\mathcal{R}$ and on $\neg\mathcal{R}$ (as expressed on the right side of (5)), and likewise for β. Thus, if the conditional-expected utility of α is greater than that of β on both $\mathcal{R}$ and also on $\neg\mathcal{R}$, then it must be greater on all of $\mathcal{S}$, in accord with the principle of dynamical consistency. If $\mathcal{S} = \mathcal{R}$, then (5) simplifies to

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \left(\int_{\mathcal{S}} u\circ\alpha\ \mathrm{d}\nu \geq \int_{\mathcal{S}} u\circ\beta\ \mathrm{d}\nu\right). \tag{6}$$

In other words, the unconditional preference order $\succeq_{\mathcal{S}}$ always has a "classical" SEU representation. The liminal density structure only emerges in the representation of the *conditional* preference orders, and essentially serves in the updating of preferences and beliefs as new information is acquired. Our second result provides a characterization of liminal density SEU representations in terms of the axioms of Section 3.

Theorem 2 *Let $\mathcal{S}$ be a (nonsingleton) compact Hausdorff space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ satisfying* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if it admits a liminal density SEU representation* (5)*, where ν is a normal Borel probability measure with full support. Furthermore, ν is unique, the elements of $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ are unique* (ν-almost everywhere)*, and u is unique up to positive affine transformation.*

At first glance, there appears to be a direct contradiction between the "uniqueness" claims in Theorems 1 and 2, since both theorems could be applied to the *same* conditional preference structure. But there is no contradiction: Theorem 2 is formulated in terms of a Borel probability measure, while Theorem 1 was formulated in terms of a *residual charge*. They are not the same type of representation. (Likewise, none of our later SEU representations will contradict each other's "uniqueness" claims.)

The liminal density SEU representation (5) might appear rather unwieldy, since it requires the agent to carry around a liminal density structure $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$, which seems like a very complicated information structure. But the same criticism could be made of Savage's original SEU representation, because a finitely additive probability measure is *already* a very complicated information structure. (For example, any ultrafilter defines a finitely additive probability measure). Countably additive measures are no less complex; we simply underestimate this complexity when we think about the (very small) set of probability measures that admit parsimonious descriptions (e.g. uniform distributions, normal distributions, and other parametric families). Indeed, as we have already noted, the same conditional preference structure can admit *both* a liminal density representation *and* a residual charge representation; thus, all of the informational complexity of the liminal density structure $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is *already* implicitly present in the residual charge in (2).

It might seem that the "boundary" integrals in the SEU representations (5) are likely to be zero in most economic applications, where probability is typically distributed "smoothly" over the state space. But this is not the case, for two reasons. First, the probability distributions describing many economic or strategic interactions naturally concentrate around certain values. For example, consider a principal-agent interaction, where the principal has imperfect monitoring technology, and the agent wants to extract as much surplus as possible without crossing a threshold which would trigger a reprisal from the principal. Likewise, consider a regulated market, in which firms must meet or exceed threshold values of certain metrics to keep their license or to receive some desired certification. For example, to comply with the Basel III requirements, a bank must maintain a "Common Equity Tier 1 Ratio" of at least 4.5%, and a "Leverage Ratio" of at least 3%. It is likely that many banks will seek to exceed these minimum thresholds by *as small a margin as possible*, thereby maximizing expected profits while technically complying with the rules.

Similar phenomena occur in games where a player's strategy ranges continuously over a topological space, but other players can only observe this strategy through a regular partition. For example, professional boxers must compete with other boxers in the same "weight class"; in particular, the weight range is 63.5 kg - 66.7 kg for "Welterweight" class, and 66.7 kg - 69.9 kg for "Super Welterweight". During a fight, more weight is better. So each boxer wants to be as close as possible to the upper weight limit for his class. Thus, if you learn that a boxer is in the "Welterweight" class, then your conditional probability should not be smoothly distributed over the interval $[63.5, 66.7]$; instead, most of the probability should be concentrated at 66.7. On the other hand, upon learning that a boxer is "Super Welterweight", most of your conditional probability should be concentrated at 69.9. A liminal structure can naturally accommodate this kind of reasoning.[11]

Similar examples arise in any economic interaction where an actor's true "quality" is an element of a continuum, but it is summarized publicly by some discrete "grade". For example, a complete description of the credit-worthiness of a debtor might require a huge array of real numbers, but it is summarized by Fitch, Standard & Poor and other rating agencies with an alphabetic code like AAA, AA-, or BB+. Likewise, the academic quality of students, the gastronomic quality of food products, and the energy efficiency of home appliances are summarized by letter grades like A, B+, or C-. It is likely that many debtors, food producers, appliance manufacturers and university students will find ways to just barely exceed the quality threshold necessary to achieve a certain grade.

Concentration of probability also occurs in natural systems *without* such strategic considerations. For example, the theory of *self-organized criticality* suggests that many systems evolve gradually but inexorably toward the boundary of a "critical" region, where a small perturbation can trigger an arbitrarily large "crisis" (Jensen, 1998). After the crisis has run its course, the system will find itself far from the critical boundary, and the process will begin anew. Standard examples include mountain snow packs, tectonic fault lines, forests, and financial markets; the resulting "crises" are, respectively, avalanches, earthquakes, forest fires, and financial crashes. If such a system is viewed as a stochastic process, then it will not *always* be on the critical boundary (for example, shortly after a crisis), but it will be very close to it, with high probability.

Aside from these concrete examples, there is a second, purely mathematical reason why the "boundary" integrals in the representations (5) are unavoidable: if the space $\mathcal{S}$ satisfies mild topological conditions, then the liminal density structure *cannot* be zero for *all* regular sets (Pivato and Vergopoulos, 2017b, Proposition 6.9). Thus, these boundary integrals should not be waved away as a mere technicality or annoyance; they are an essential part of the representation, and are both economically meaningful and mathematically inevitable.

This is important, because the liminal structure affects how an agent updates her preferences when she observes an event. For an agent with a classical SEU representation, Bayes rule is the unique preference-updating rule that satisfies dynamic consistency and consequentialism (Hammond, 1998; Ghirardato, 2002). But each liminal structure provides a *distinct*, dynamically consistent and consequentialist updating rule. Thus, two "rational" agents starting with the same *ex ante* preferences and observing the same regular subset

[11] Formally, in this example, we would have $\mathcal{S} = \mathbb{R}_+$, $\mathcal{R}_{\text{welt}} = (63.5, 66.7)$, $\mathcal{R}_{\text{superwelt}} = (66.7, 69.9)$, etc.

might end up with different conditional preferences.

To see the economic significance of this, consider a collection of fully insured and risk-averse agents with a common prior probability measure. It is well-known that it is Pareto efficient for such agents not to engage in trade. But now suppose that they all receive the same public information in the form of a regular subset $\mathcal{R}$. If their liminal structures differ on $\partial\mathcal{R}$, then they will now have *different* conditional preferences, so they may now want to bet against each other. This is in sharp contrast with the predictions of the classical SEU model, which asserts that an *ex ante* Pareto efficient allocation remains Pareto efficient conditional on any event. But it is consistent with the empirical findings of Kandel and Pearson (1995), which show "abnormally high" volumes of trade around public earnings announcements, which they precisely attribute to differences in the way traders interpret these public announcements.

A similar issue arises with the No-Trade Theorem of Milgrom and Stokey (1982), which relies on the fact that, for two agents with the same utility function, any divergence in conditional trading preferences must be due to a difference in private information. Is this result still true in our framework? This depends on how we interpret the so-called "Harsanyi Doctrine" or "common prior assumption": the assumption that all agents begin with the same unconditional beliefs (Morris, 1995). In the liminal SEU representation (5), an agent's "belief" has two components: a Borel measure ν, and a liminal structure. If we interpret the Harsanyi Doctrine as saying only that all agents start with the same ν, then the No-Trade Theorem fails, because two agents observing the same information may still end up with different trading preferences. If we interpret the Harsanyi Doctrine as saying that all agents start with the same ν *and* the same liminal structure, then the No-Trade Theorem may survive. *However*, this "strong" version of the Harsanyi Doctrine is impossible to verify *ex ante* in our framework: we can only observe the agent's *ex ante* preferences over continuous acts, and in the SEU representation (6), these preferences are entirely determined by ν, independent of the liminal structure. Thus, the two agents might initially behave as if they have the same beliefs; only after they update on some event will the difference in their beliefs become apparent.

6 SEU representations on compactifications

A Hausdorff space $\mathcal{S}$ is *locally compact* if every point in $\mathcal{S}$ has a compact neighbourhood. For example, every compact Hausdorff space is locally compact. Every topological manifold is locally compact. In particular, any open or closed subset of $\mathbb{R}^N$ is locally compact. (However, the set of rational numbers is *not* locally compact.) Every totally bounded, locally complete metric space is locally compact. In short: most topological spaces which would arise naturally in economic applications are locally compact.

For any other Hausdorff space $\mathcal{X}$, let $\mathcal{C}_{\mathrm{L}}(\mathcal{S},\mathcal{X})$ be the set of all continuous functions $\alpha:\mathcal{S}\longrightarrow\mathcal{X}$ which converge to some limit "at infinity" in the following sense: there exists $x\in\mathcal{X}$ such that, for any open neighbourhood $\mathcal{O}\subseteq\mathcal{X}$ around x, there is a compact subset $\mathcal{K}\subseteq\mathcal{S}$ such that $\alpha(\mathcal{S}\setminus\mathcal{K})\subseteq\mathcal{O}$. When it exists, this limit x is unique and denoted $\lim\limits_{\infty} f$.

Let $\mathcal{S}$ be a locally compact Hausdorff space. The *Alexandroff* (or *one-point*) *compactification* $\dot{\mathcal{S}}$ is the set $\mathcal{S} \sqcup \{\infty\}$ (where ∞ represents a "point at infinity") equipped with the smallest topology such that every open subset of $\mathcal{S}$ remains open in $\dot{\mathcal{S}}$, while the open neighbourhoods of ∞ are the sets $\dot{\mathcal{S}} \setminus \mathcal{K}$, where $\mathcal{K}$ is any compact subset of $\mathcal{S}$. For example, the Alexandroff compactification of $[0, \infty)$ is $[0, \infty]$. The Alexandroff compactification of $\mathbb{R}$ is homeomorphic to a circle. The Alexandroff compactification of $\mathbb{R}^2$ is homeomorphic to a sphere. In general, Alexandroff compactifications have the following properties:

(A1) $\dot{\mathcal{S}}$ is compact and Hausdorff. If $\mathcal{S}$ is noncompact, then $\mathcal{S}$ is a dense open subset of $\dot{\mathcal{S}}$. (Otherwise, if $\mathcal{S}$ is already compact, then ∞ is an isolated point of $\dot{\mathcal{S}}$.)

(A2) For any Hausdorff space $\mathcal{X}$, any function $\alpha \in \mathcal{C}_{\mathrm{L}}(\mathcal{S}, \mathcal{X})$ has a unique extension $\dot{\alpha} \in \mathcal{C}(\dot{\mathcal{S}}, \mathcal{X})$ defined by $\dot{\alpha}_{\restriction \mathcal{S}} = \alpha$ and $\dot{\alpha}(\infty) = \lim_{\infty} \alpha$.

(A3) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique $\dot{\mathcal{R}} \in \mathfrak{R}(\dot{\mathcal{S}})$ such that $\dot{\mathcal{R}} \cap \mathcal{S} = \mathcal{R}$.[12]

Let $\mathcal{X}$ be a Hausdorff space. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. An *Alexandroff SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ is given by a normal Borel probability measure $\dot{\nu}$ on $\mathfrak{Bor}(\dot{\mathcal{S}})$ and a liminal density structure $\{\dot{\phi}_{\dot{\mathcal{R}}}\}_{\dot{\mathcal{R}} \in \mathfrak{R}(\dot{\mathcal{S}})}$ subordinate to $\dot{\nu}$, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \tag{7}$$
$$\left(\int_{\dot{\mathcal{R}}} u \circ \dot{\alpha} \, \mathrm{d}\dot{\nu} + \int_{\partial \dot{\mathcal{R}}} (u \circ \dot{\alpha}) \cdot \dot{\phi}_{\dot{\mathcal{R}}} \, \mathrm{d}\dot{\nu} \ \geq \ \int_{\dot{\mathcal{R}}} u \circ \dot{\beta} \, \mathrm{d}\dot{\nu} + \int_{\partial \dot{\mathcal{R}}} (u \circ \dot{\beta}) \cdot \dot{\phi}_{\dot{\mathcal{R}}} \, \mathrm{d}\dot{\nu}\right).$$

A key difference between the SEU representations obtained so far and the Alexandroff SEU representation lies in the unconditional preference order $\succeq_{\mathcal{S}}$. While both the residual charge and liminal density SEU representations yield a "classical" representation of type (6), formula (7) yields the following representation, for any $\alpha, \beta \in \mathcal{A}$:

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \tag{8}$$
$$\left(\int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\dot{\nu} + \dot{\nu}\{\infty\} \cdot \lim_{\infty} (u \circ \alpha) \ \geq \ \int_{\mathcal{S}} u \circ \beta \, \mathrm{d}\dot{\nu} + \dot{\nu}\{\infty\} \cdot \lim_{\infty} (u \circ \beta)\right).$$

Thus, *ex ante* beliefs consist of two components: a Borel probability measure on $\mathcal{S}$, and an additional coefficient weighting the asymptotic utility of the acts "at infinity".

Theorem 3 *Let $\mathcal{S}$ be a noncompact, locally compact Hausdorff space, let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq), (C), (Dom), (Sep), (PC) *and* (TC) *if and only if it admits an Alexandroff SEU representation* (7)*, where $\dot{\nu}$ has full support on $\dot{\mathcal{S}}$. Furthermore, $\dot{\nu}$ is unique, the elements of $\{\dot{\phi}_{\dot{\mathcal{R}}}\}_{\dot{\mathcal{R}} \in \mathfrak{R}(\dot{\mathcal{S}})}$ are unique* ($\dot{\nu}$-almost everywhere)*, and u is unique up to positive affine transformation.*

[12] See Lemma 7.4(a) in Pivato and Vergopoulos (2017b).

The main advantage of Theorem 3 is that it applies when $\mathcal{S}$ is an *unbounded* space like $\mathbb{R}^N$, whereas Theorem 2 does not. The main *disadvantage* of Theorem 3 is that the conditional preference structure can only compare acts which converge "at infinity". The problem is that a conditional preference structure defined over a larger domain of acts may be sensitive to the asymptotic behaviour of these acts in a way which eludes an Alexandroff SEU representation. For example, if $\mathcal{S} = \mathbb{R}^2$, then the conditional preference structure could be sensitive in different ways to the asymptotic behaviour of acts along different curves, like $y = x^2$ and $y = x^3$. Intuitively, to capture such sensitivity with an SEU representation, we would need to introduce distinct "endpoints" for these two curves, and then assign different probabilities to these endpoints. But *no such distinct endpoints exist* in $\mathbb{R}^2$, or in its (spherical) Alexandroff compactification. To solve this problem, we must add a plethora of new states to $\mathcal{S}$, each acting like a distinct "point at infinity". To be precise, we must extend $\mathcal{S}$ to its *Stone-Čech compactification.*

Let $\mathcal{S}$ is a locally compact Hausdorff space. There is a unique compact Hausdorff space $\check{\mathcal{S}}$, called the ***Stone-Čech compactification*** of $\mathcal{S}$, with the following properties.

(SČ1) $\mathcal{S}$ is an open, dense subset of $\check{\mathcal{S}}$, and the native topology of $\mathcal{S}$ is the same as the subspace topology it inherits from $\check{\mathcal{S}}$.

(SČ2) For any compact Hausdorff space $\mathcal{K}$, and any continuous function $f : \mathcal{S} \longrightarrow \mathcal{K}$, there is a unique continuous function $\check{f} : \check{\mathcal{S}} \longrightarrow \mathcal{K}$ such that $\check{f}_{\restriction \mathcal{S}} = f$.

(SČ3) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique $\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ such that $\check{\mathcal{R}} \cap \mathcal{S} = \mathcal{R}$. Furthermore, the mapping $\mathfrak{R}(\mathcal{S}) \ni \mathcal{R} \mapsto \check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ is a Boolean algebra isomorphism.[13]

Let $\mathcal{X}$ be another Hausdorff space. For any $\alpha \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, assertion (SČ2) says there is a unique function $\check{\alpha} \in \mathcal{C}(\check{\mathcal{S}}, \mathcal{X})$ such that $\check{\alpha}_{\restriction \mathcal{S}} = \alpha$. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A ***Stone-Čech SEU representation*** for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ is given by a normal Borel probability measure $\check{\nu}$ on $\mathfrak{Bor}(\check{\mathcal{S}})$ and a liminal density structure $\{\check{\phi}_{\check{\mathcal{R}}}\}_{\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})}$ subordinate to $\check{\nu}$, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \tag{9}$$
$$\left(\int_{\check{\mathcal{R}}} u \circ \check{\alpha} \ \mathrm{d}\check{\nu} + \int_{\partial \check{\mathcal{R}}} (u \circ \check{\alpha}) \, \check{\phi}_{\check{\mathcal{R}}} \ \mathrm{d}\check{\nu} \ \geq \ \int_{\check{\mathcal{R}}} u \circ \check{\beta} \ \mathrm{d}\check{\nu} + \int_{\partial \check{\mathcal{R}}} (u \circ \check{\beta}) \, \check{\phi}_{\check{\mathcal{R}}} \ \mathrm{d}\check{\nu}\right).$$

Similarly to formula (8), the Stone-Čech representation of $\succeq_{\mathcal{S}}$ yields *ex ante* beliefs that may assign some weight outside of $\mathcal{S}$. More precisely, for any $\alpha, \beta \in \mathcal{A}$, we have

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \tag{10}$$
$$\left(\int_{\mathcal{S}} u \circ \alpha \ \mathrm{d}\check{\nu} + \int_{\check{\mathcal{S}} \setminus \mathcal{S}} u \circ \check{\alpha} \ \mathrm{d}\check{\nu} \ \geq \ \int_{\mathcal{S}} u \circ \beta \ \mathrm{d}\check{\nu} + \int_{\check{\mathcal{S}} \setminus \mathcal{S}} u \circ \check{\beta} \ \mathrm{d}\check{\nu}\right).$$

[13] Property (SČ2) holds if $\mathcal{S}$ is any *Tychonoff* space; see e.g. Theorem 19.5 of Willard (2004) or Theorem 2.79 of Aliprantis and Border (2006). But property (SČ3) only holds for the somewhat smaller class of locally compact Hausdorff spaces (Pivato and Vergopoulos, 2017b, Lemma 6.4(a)).

The set $\check{\mathcal{S}} \setminus \mathcal{S}$ is called the *corona* —intuitively, this is the set of "points at infinity". These points play an essential role in the Stone-Čech representation. It is straightforward to construct examples of SEU representations where much of the probability weight lies in the corona (Pivato and Vergopoulos, 2017b, Examples 4.13 and 6.6). The next theorem generalizes Theorem 3 by providing an SEU representation on locally compact state spaces without requiring feasible acts to have a limit at infinity.

Theorem 4 *Let $\mathcal{S}$ be a (nonsingleton) locally compact Hausdorff space, let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ satisfying* (Rch)*. Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies* (CEq)*,* (C)*,* (Dom)*,* (Sep)*,* (PC) *and* (TC) *if and only if it admits a Stone-Čech SEU representation* (9)*, where $\check{\nu}$ has full support on $\check{\mathcal{S}}$. Furthermore, $\check{\nu}$ is unique, the elements of $\{\check{\phi}_{\check{\mathcal{R}}}\}_{\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})}$ are unique* ($\check{\nu}$-almost everywhere)*, and u is unique up to positive affine transformation.*

Theorems 3 and 4 are special cases of a large family of results. Let $\overline{\mathcal{S}}$ be any *compactification* of $\mathcal{S}$ —that is, a compact Hausdorff space which contains $\mathcal{S}$ as a dense subset. Let $\mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$ be the set of continuous functions in $\mathcal{C}(\mathcal{S}, \mathcal{X})$ which can be extended to continuous functions in $\mathcal{C}(\overline{\mathcal{S}}, \mathcal{X})$. (For example, let $\mathcal{S} := \mathbb{R}$. Then $\overline{\mathbb{R}} := [-\infty, \infty]$ is a compactification, and $\mathcal{C}_{\overline{\mathbb{R}}}(\mathbb{R}, \mathcal{X})$ is the set of continuous functions from $\mathbb{R}$ to $\mathcal{X}$ which converge to (possibly different) limits at $\pm\infty$.) If $\mathcal{A} \subseteq \mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$, and $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ is a conditional preference structure on $\mathcal{A}$ which satisfies (Rch), (CEq), (C), (Dom), (Sep), (PC) and (TC), then we can obtain a normal Borel probability measure $\overline{\nu}$ on $\mathfrak{Bor}(\overline{\mathcal{S}})$, a liminal density structure $\{\overline{\phi}_{\overline{\mathcal{R}}}\}_{\overline{\mathcal{R}} \in \mathfrak{R}(\overline{\mathcal{S}})}$ subordinate to $\overline{\nu}$, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, which together yield an SEU representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ analogous to representations (7) and (9).

There are generally *many* compactifications which will yield such a representation for a given conditional preference structure on $\mathcal{A}$. We just need $\overline{\mathcal{S}}$ to be "large enough" that $\mathcal{A}$ is contained in $\mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$. But if $\mathcal{S}$ is a locally compact Hausdorff space, then the set of all compactifications of $\mathcal{S}$ is a complete lattice (Engelking, 1989, Theorems 3.5.9-11, p.169). Thus, for any $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, there exists a unique *minimal* compactification $\overline{\mathcal{S}}$ such that $\mathcal{A} \subseteq \mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$; this is the smallest compactification on which we can construct SEU representations for conditional preference structures on $\mathcal{A}$. For example, if $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}(\mathcal{S}, \mathcal{X})$, then Theorem 3 says we can use the *smallest* compactification of $\mathcal{S}$, namely $\dot{\mathcal{S}}$. At the opposite extreme, if $\mathcal{A} = \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, then we must use the *largest* compactification, which is $\check{\mathcal{S}}$. Other collections of feasible acts lead to other choices of compactification.

Such an enlargement of the state space through the addition of "ideal points" has many precedents in decision theory. For example, Stinchcombe (1997) proposed such a state space enlargement to solve certain paradoxes which arise from the failure of countable additivity in the Savage SEU representation. In many models of ambiguity aversion, the agent's beliefs are not even finitely additive. But these failures of additivity in the original state space are sometimes consistent with finite additivity in some extended state space (Gilboa and Schmeidler, 1994). Other authors have used extended state spaces to distinguish between objective reality and the agent's internal representation of that reality. For example, Lipman (1999) augments the original state space with "impossible possible worlds" to model the agent's lack of logical omniscience. Jaffray and Wakker (1993) and

Mukerji (1997) introduce "two-tiered" state spaces; in the model of Jaffray and Wakker, the agent has probabilistic beliefs about one tier and total ignorance about the other, whereas in Mukerji's model, one tier represents the agent's internal epistemic state and the other tier represents objectively payoff-relevant information. In a similar way, we could interpret $\mathcal{S}$ as the "true" state space and $\overline{\mathcal{S}}$ as the agent's internal model of this space; in this view, the extra elements of $\overline{\mathcal{S}} \setminus \mathcal{S}$ would be like the "impossible possible worlds" of Lipman (1999). But there are at least four other interesting interpretations of compactification SEU representations, which we will now briefly describe.

Catastrophic risks. In some decisions, it is important to incorporate the risk of a "catastrophe": an event which has a very low probability of occuring, but which is so extreme that the agent has difficulty even imagining it, because it is completely beyond the scale of the events she normally considers. A catastrophe could be a major global disaster with few or no precedents (e.g. a nuclear bombardment, a large asteroid impact, or a Carrington-class geomagnetic storm), or some other event that we haven't even anticipated (an "unknown unknown"). The decision-maker cannot model such events in detail, because they are outside of her experience or understanding. But she can *approximately* model them as the asymptotic limits of events in her range of experience. The elements of $\overline{\mathcal{S}} \setminus \mathcal{S}$ can be interpreted as such asymptotic limits; thus, $\overline{\nu}_{\overline{\mathcal{S}}\setminus\mathcal{S}}$ (the restriction of $\overline{\nu}$ to $\overline{\mathcal{S}} \setminus \mathcal{S}$) could represent the agent's probabilistic beliefs about catastrophes that she can barely imagine.

There are similar models in previous literature. Chichilnisky (2000, 2009) has proposed a model where the agent's preferences are represented by a sum $\int_{\mathcal{S}} u \circ \alpha(s)\ \mathrm{d}\nu[s] + \Phi(u \circ \alpha)$. Here, the ν-integral represents subjective expected utility, while Φ is a linear functional that encodes sensitivity to catastrophic risks. One way to represent Φ is as an integral on the Stone-Čech compactification of the state space (Chichilnisky and Heal, 1997), so there is a clear similarity between Chichilnisky's representation and our Stone-Čech SEU representation (9) (although her axioms are very different to ours).

More recently, Alon (2015) has proposed a model of unawareness based on an augmented state space. In her model, the agent *knows* that the initial state space $\mathcal{S}$ is incomplete, but is unable to precisely describe the missing states. So she adds a single state s_0 to $\mathcal{S}$; the fact that s_0 obtains means that none of the states in $\mathcal{S}$ obtains. Each act α on $\mathcal{S}$ is extended into an act $\overline{\alpha}$ on $\overline{\mathcal{S}} = \mathcal{S} \cup \{s_0\}$ by defining $\overline{\alpha}(s_0)$ to be the *worst outcome* of α over $\mathcal{S}$. Thus, s_0 can again be interpreted as a "catastrophe". However, Alon's state space has no topology, so s_0 cannot be described in terms of a compactification.

Anomalous risk preferences and the equity premium puzzle. In the model of catastrophe aversion above, $\overline{\nu}_{\overline{\mathcal{S}}\setminus\mathcal{S}}$ represented the agent's beliefs about the catastrophes which could occur. But to have such "beliefs", she must at least have some mental image of what form these catastrophes could take; presumably this is what the points in $\overline{\mathcal{S}} \setminus \mathcal{S}$ represent. But the agent might also just have some inchoate anxiety about the future; in this case, we could interpret $\overline{\nu}_{\overline{\mathcal{S}}\setminus\mathcal{S}}$ as a description of this anxiety, without calling it a "belief". In this interpretation, only $\overline{\nu}_{\mathcal{S}}$ represents the agent's probabilistic beliefs about possible states of the world, whereas $\overline{\nu}_{\overline{\mathcal{S}}\setminus\mathcal{S}}$ is a sort of "psychological artifact". Nevertheless, the $\overline{\nu}$-weight of $\overline{\mathcal{S}} \setminus \mathcal{S}$ may affect the agent's apparent risk attitudes.

For example, suppose the agent has the Stone-Čech SEU representation (9). Suppose outcomes are monetary (that is, $\mathcal{X} = \mathbb{R}$) and consider an act $\alpha \in \mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$ with $\alpha = w + \epsilon$ where $w \in \mathbb{R}$ and ϵ is a "small" risk such that $\int_{\check{\mathcal{S}}} \check{\epsilon}\ \mathrm{d}\check{\nu} = 0$. The classical Arrow-Pratt approximation of the risk premium for α provides the following formula

$$\frac{1}{2}\ \frac{-u''(w)}{u'(w)} \left(\check{\nu}(\mathcal{S}) \cdot \int_{\mathcal{S}} \epsilon^2\ \mathrm{d}\check{\nu}_{\mathcal{S}}\ +\ \check{\nu}(\check{\mathcal{S}} \setminus \mathcal{S}) \cdot \int_{\check{\mathcal{S}} \setminus \mathcal{S}} \check{\epsilon}^2\ \mathrm{d}\check{\nu}_{\check{\mathcal{S}} \setminus \mathcal{S}} \right).$$

Consider now an increase in the probability $\check{\nu}(\check{\mathcal{S}} \setminus \mathcal{S})$ that leaves the Bayesian updates $\check{\nu}_{\mathcal{S}}$ and $\check{\nu}_{\check{\mathcal{S}} \setminus \mathcal{S}}$ unchanged. If the agent is risk-averse and $\int_{\check{\mathcal{S}} \setminus \mathcal{S}} \check{\epsilon}^2\ \mathrm{d}\check{\nu}_{\check{\mathcal{S}} \setminus \mathcal{S}} > \int_{\mathcal{S}} \epsilon^2\ \mathrm{d}\check{\nu}_{\mathcal{S}}$, this will increase her risk premium, in a way which appears be qualitatively similar to a change in the coefficient of absolute risk-aversion $-u''(w)/u'(w)$.

The standard portfolio problem with one riskless asset and one risky asset illustrates this point. Let π denote the excess return of the risky asset, and suppose that its expectation conditional on $\check{\mathcal{S}} \setminus \mathcal{S}$ is negative. Suppose also that the agent is risk-averse and invests a positive amount in the risky asset. Then, an increase of $\check{\nu}(\check{\mathcal{S}} \setminus \mathcal{S})$ reduces the optimal exposure to risk, thereby effectively operating as an increase in risk aversion (Gollier, 2001, §4.4). But $\check{\nu}(\check{\mathcal{S}} \setminus \mathcal{S})$ also leads to predictions that are qualitatively different from those of risk aversion. For instance, it is well-known that an agent with classical SEU preferences invests in the risky asset if and only if π is positive in expectation, independently of her level of risk-aversion (Gollier, 2001, §4.1). Thus, high levels of risk aversion cannot explain why many agents altogether avoid risky investements. Furthermore, assuming reasonable levels of risk aversion, most people should invest much more in risky assets than they actually do; this is sometimes called the *equity premium puzzle* (Gollier, 2001, §5.2). In contrast, an agent with Stone-Čech SEU representation (9) will invest a positive amount in the risky asset if and only if $\check{\nu}(\mathcal{S}) \cdot \int_{\mathcal{S}} \pi\ \mathrm{d}\check{\nu}_{\mathcal{S}}\ +\ \check{\nu}(\check{\mathcal{S}} \setminus \mathcal{S}) \cdot \int_{\check{\mathcal{S}} \setminus \mathcal{S}} \pi\ \mathrm{d}\check{\nu}_{\check{\mathcal{S}} \setminus \mathcal{S}} > 0$. Clearly, an increase in $\check{\nu}(\check{\mathcal{S}} \setminus \mathcal{S})$ makes this condition more restrictive. Thus, a negative expected value of π on the corona can explain the equity premium puzzle.

Intertemporal choice. Let $\mathcal{S} = \mathbb{R}_+$ or $\mathbb{N}$. Then elements of $\mathcal{S}$ can be interpreted as moments in time, and a function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ can be interpreted as a consumption stream. Rather than *ex ante* preferences under uncertainty, a conditional preference structure $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ now represents *intertemporal preferences* over an infinite time duration. Such infinite durations arise in intergenerational social choice theory, where the relevant planning horizon is the (hopefully) infinite lifetime of human civilization, rather than the lifetime of a single agent. In this context, a classical SEU representation like (6) can be interpreted as a discounted utility sum, where the measure ν encodes the intertemporal discount factors. However, such discounted utility sums have been criticized as giving insufficient weight to the extremely far future; Chichilnisky (1996) describes them as "dictatorships of the present". In contrast, a Stone-Čech SEU representation (9) is sensitive to the long-term asymptotic behaviour of consumption streams; this is similar to the *sustainable preferences* proposed by Chichilnisky (1996) and Chichilnisky and Heal (1997).

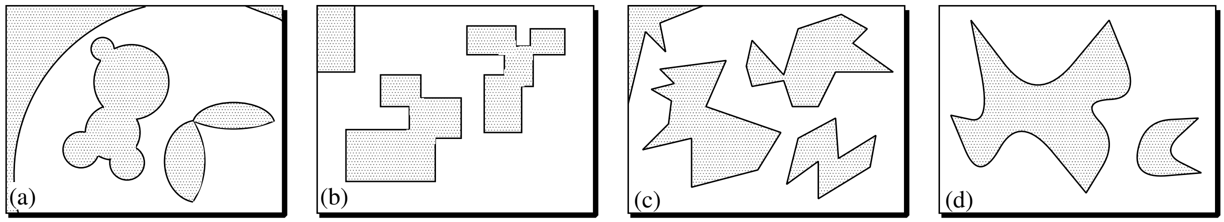

Figure 3: The Boolean subalgebras from Example 5. (a) A typical element of $\mathfrak{B}_{\rm ball}(\mathbb{R}^2)$. (b) A typical element of $\mathfrak{B}_{\rm box}(\mathbb{R}^2)$. (c) A typical element of $\mathfrak{B}_{\rm poly}(\mathbb{R}^2)$. (d) A typical element of $\mathfrak{B}_{\rm smth}(\mathbb{R}^2)$. (Note that in each case, the *negation* of the shaded set is also an element of the algebra in question.)

Vestigial uncertainty. Consider an agent with a liminal SEU representation like (5). As we explained at the end of Section 5, when the agent "observes" an event $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, all she knows for sure is that the true state lies in the *closure* of $\mathcal{R}$. The "boundary integral" in (5) reflects her awareness of the fallibility of her measurement technology.

But now suppose that $\mathcal{S}$ *itself* is the result of such a fallible observation. In other words, suppose that the agent's decision problem was originally described by some larger state space $\mathcal{T}$, a set of acts $\mathcal{A}(\mathcal{T}) \subseteq \mathcal{C}_{\rm b}(\mathcal{T}, \mathcal{X})$, and a conditional preference structure $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{T})}$ with a liminal density SEU representation like (5). Suppose $\mathcal{S}$ is a regular subset of $\mathcal{T}$, and at some point in the past, the agent "observed" the event $\mathcal{S}$; thus, she now knows that the true state of the world lies in the closure of $\mathcal{S}$ in $\mathcal{T}$, which we will call $\overline{\mathcal{S}}$. If $\mathcal{T}$ is compact, then $\overline{\mathcal{S}}$ is also compact; thus, it is a compactification of $\mathcal{S}$. Having observed $\mathcal{S}$, the agent forms conditional preferences $\succeq_{\mathcal{S}}$ on $\mathcal{A}(\mathcal{S})$. If we apply the liminal SEU representation (5) to $\succeq_{\mathcal{S}}$, then we will obtain a "compactification" SEU representation like (8) or (10), where the "points at infinity" are just the points on the *boundary* of $\mathcal{S}$ inside $\mathcal{T}$.

More generally, if $\mathcal{R} \subseteq \mathcal{S}$ and $\overline{\mathcal{R}}$ is its closure in $\mathcal{T}$, then $\overline{\mathcal{R}} \subseteq \overline{\mathcal{S}}$, so $\overline{\mathcal{R}}$ is also the closure of $\mathcal{R}$ in $\overline{\mathcal{S}}$. Now, $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{R}(\mathcal{T})$ (see Lemma B3), so we can restrict $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{T})}$ to $\mathfrak{R}(\mathcal{S})$ to obtain a new conditional preference structure $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$. If we take the liminal density SEU representation (5) for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{T})}$, and restrict it to $\mathfrak{R}(\mathcal{S})$, then we will obtain a compactification SEU representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$, like (7) or (9).

This interpretation raises a rather intriguing possibility. When we encounter the agent, she has already observed $\mathcal{S}$, so the original state space $\mathcal{T}$ is already gone, and we are left only with a conditional preference structure $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ on $\mathcal{A}(\mathcal{S})$. But by constructing a compactification SEU representation like (7) or (9), it seems that we can "reconstruct" the closure of $\mathcal{S}$ in $\mathcal{T}$; thus, we can learn something about the original state space $\mathcal{T}$ through the traces it leaves in $\mathcal{A}$, even though $\mathcal{T}$ was not explicitly specified in the model.

7 SEU representations via credences

This section contains our most general SEU representation result. First, we relax the assumptions on the state space topology made in Sections 4, 5 and 6. Second, we now allow further informational constraints on the agent, represented by two Boolean subalgebras $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} \subseteq \mathfrak{R}(\mathcal{X})$. We suppose that the agent can only observe events in the state space which are elements of $\mathfrak{B}$, and she can only observe properties of outcomes which correspond to elements of $\mathfrak{D}$. Such Boolean subalgebras arise from natural constraints on the agent's observational technology, as the next five examples show.

Example 5. In the following examples, we assume $\mathcal{S} = \mathbb{R}^N$ for some $N \in \mathbb{N}$.

(a) Any open ball in $\mathbb{R}^N$ is regular. So is the complement of any closed ball. Let $\mathfrak{B}_{\rm ball}(\mathbb{R}^N)$ be the collection of all regular subsets of $\mathbb{R}^N$ constructed by taking joins and/or intersections of finite collections of open balls and/or the complements of their closures. Then $\mathfrak{B}_{\rm ball}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is shown in Figure 3(a). This Boolean algebra describes the information available to an agent whose observational technology allows her to check whether the true state of the world is within some specified distance of some target state. (For example, she can check the statement, "The true state is within distance 1.6 of the point $(0, 0, 0)$".)[14]

(b) A subset of $\mathbb{R}^N$ is an ***open box*** if it is a Cartesian product of open intervals. Any open box is regular. The intersection of two open boxes is also an open box (if it is nonempty). Let $\mathfrak{B}_{\rm box}(\mathbb{R}^N)$ be the Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$ generated by open boxes. A typical element is shown in Figure 3(b). This Boolean algebra describes the information available to an agent whose observational technology allows her to check whether any particular *coordinate* of the true state satisfies some strict inequality. (For example, she can check the statement, "The horizontal coordinate of the state is strictly between 1.16 and 3.24.").[15]

(c) A subset $\mathcal{H} \subseteq \mathbb{R}^N$ is a ***hyperplane*** if there is a linear function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that $\mathcal{H} := \phi^{-1}\{r\}$ for some $r \in \mathbb{R}$. A regular subset $\mathcal{R} \subseteq \mathbb{R}^N$ is a ***polyhedron*** if there is a finite collection $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ of hyperplanes such that $\partial\mathcal{R} = (\mathcal{H}_1 \cap \partial\mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial\mathcal{R})$. (Heuristically, each of the sets $\mathcal{H}_n \cap \partial\mathcal{R}$ is one of the "faces" of the polyhedron. Note that we do not require these polyhedra to be convex, or even connected.) Let $\mathfrak{B}_{\rm poly}(\mathbb{R}^N)$ be the set of regular polyhedra; then $\mathfrak{B}_{\rm poly}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is shown in Figure 3(c). This Boolean algebra describes the information available to an agent whose observational technology allows her to check whether the state satisfies any finite collection of strict linear inequalities.[16]

(d) A subset $\mathcal{H} \subseteq \mathbb{R}^N$ is a ***smooth hypersurface*** if there is a differentiable function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that $\mathcal{H} := \phi^{-1}\{r\}$ for some $r \in \mathbb{R}$, and such that $\mathrm{d}\phi(h) \neq 0$ for all $h \in \mathcal{H}$. We will say that a regular subset $\mathcal{R} \subseteq \mathbb{R}^N$ has a ***piecewise smooth boundary*** if there is a finite collection $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ of smooth hypersurfaces such that $\partial\mathcal{R} = (\mathcal{H}_1 \cap \partial\mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial\mathcal{R})$. Let $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ be the set of regular subsets of $\mathbb{R}^N$ with piecewise smooth boundaries; then $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is

[14] This construction also works if we replace $\mathbb{R}^N$ with any metric space. But not all open balls in a general metric space are regular, so we must restrict our attention to the collection of *regular* open balls.

[15] This construction also works if we replace $\mathbb{R}^N$ with any Cartesian product of linearly ordered sets, each endowed with its order topology.

[16] This construction also works if we replace $\mathbb{R}^N$ with any topological vector space. But we must then stipulate that ϕ is *continuous* as well as linear.

shown in Figure 3(d).[17] The algebra $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ describes the information available to an agent whose observational technology allows her to check whether the state satisfies any finite collection of strict inequalities based on differentiable functions. We normally assume that the output of any scientific instrument is a differentiable function of the true state of the world; thus, $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ describes the information available through such scientific instruments. We can also construct Boolean subalgebras of $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ based on more restrictive classes of functions (e.g. polynomials).

(e) In the case $N = 1$, the Boolean algebras in parts (a)-(d) take a particularly simple form. Let us say that a subset $\mathcal{B} \subseteq \mathbb{R}$ is *basic* if $\mathcal{B} = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N)$ for some $-\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N \leq \infty$. Let $\mathfrak{Bas}(\mathbb{R})$ be the set of all basic open subsets of $\mathbb{R}$. Then $\mathfrak{Bas}(\mathbb{R})$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R})$, and it is easy to verify that $\mathfrak{B}_{\rm ball}(\mathbb{R}) = \mathfrak{B}_{\rm box}(\mathbb{R}) = \mathfrak{B}_{\rm poly}(\mathbb{R}) = \mathfrak{B}_{\rm smth}(\mathbb{R}) = \mathfrak{Bas}(\mathbb{R})$. ♢

We will say that a Boolean subalgebra $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{S})$ is *nontrivial* if it contains a proper regular subset of $\mathcal{S}$. We will consider only nontrivial Boolean subalgebras. In particular, all the statements in this section hold when $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$. So on a first reading, the reader may assume $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$ for simplicity.

Comeasurability. The Boolean subalgebras $\mathfrak{B}$ and $\mathfrak{D}$ and the set of feasible acts $\mathcal{A}$ must satisfy a natural consistency requirement. Consider a feasible act $\alpha \in \mathcal{A}$ and a regular subset $\mathcal{D} \in \mathfrak{D}$. Together, they define an observation of the state, obtained by first performing the act α, and then observing the outcome via the partition $\{\mathcal{D}, \neg\mathcal{D}\}$ to see whether the true state s has the property that $\alpha(s)$ lies in $\mathcal{D}$. But if such an observation of s is feasible, then there must be a $\mathfrak{B}$-partition representing it. This motivates the following requirement. A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *comeasurable* with respect to $\mathfrak{B}$ and $\mathfrak{D}$ if $\operatorname{int}(\alpha^{-1}[\operatorname{clos}(\mathcal{D})]) \in \mathfrak{B}$ for all $\mathcal{D} \in \mathfrak{D}$. For example, if α is any continuous function, then α is comeasurable with respect to $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$ (because the interior of any closed set is regular). Thus, all feasible acts in Sections 4, 5 and 6 automatically satisfied the comeasurability requirement. Let $\mathcal{C}_{\rm b}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$ denote the set of bounded, $(\mathfrak{B}, \mathfrak{D})$-comeasurable and continuous functions from $\mathcal{S}$ to $\mathcal{X}$. From now on, we assume that $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$ and that $\mathcal{A}$ contains all constant functions.

The measurability axiom. We will need one more axiom in addition to those which appeared in Section 3. For any $x \in \mathcal{X}$, the *open upper contour set* of x is $\{y \in \mathcal{X};\ y \succ_{\rm xp} x\}$; the *open lower contour set* of x is $\{y \in \mathcal{X};\ y \prec_{\rm xp} x\}$. Axiom (C) already ensures that these are open subsets of $\mathcal{X}$. The next axiom goes further.

(M) All open contour sets of $\succeq_{\rm xp}$ are elements of $\mathfrak{D}$.

The intuition here is straightforward. The agent is aware of her own *ex post* preferences. So for any $y \in \mathcal{X}$, she can discern whether y satisfies the properties "$y \succ_{\rm xp} x$" or "$y \prec_{\rm xp} x$" —i.e. whether y belongs to the open upper or lower contour set of x. In other words, these open contour sets must be "observable" subsets of $\mathcal{X}$. But then they must belong to $\mathfrak{D}$.

[17] This construction also works if we replace $\mathbb{R}^N$ with any differentiable manifold.

Beliefs. We will represent the probabilistic "beliefs" of the agent by a *credence* —a structure like a finitely additive probability measure on the Boolean subalgebra $\mathfrak{B}$. To be precise, a *credence* on $\mathfrak{B}$ is a function $\mu : \mathfrak{B} \longrightarrow [0,1]$ such that $\mu[\mathcal{S}] = 1$ and such that, for any finite collection $\{\mathcal{B}_n\}_{n=1}^N$ of disjoint elements of $\mathfrak{B}$, we have

$$\mu\left[\bigvee_{n=1}^{N} \mathcal{B}_n\right] \quad = \quad \sum_{n=1}^{N} \mu[\mathcal{B}_n]. \tag{11}$$

A credence μ behaves like an ordinary probability measure. For example, for any $\mathcal{B} \in \mathfrak{B}$ with $\mu[\mathcal{B}] > 0$, we can define the *conditional credence* $\mu_{\mathcal{B}}$ by setting $\mu_{\mathcal{B}}[\mathcal{R}] := \mu[\mathcal{B}\cap\mathcal{R}]/\mu[\mathcal{B}]$ for all $\mathcal{R} \in \mathfrak{B}$. Then $\mu_{\mathcal{B}}$ also satisfies equation (11). But note an important difference from the usual definition of a measure: additivity is defined with respect to the operation $\vee$, rather than ordinary union. For any regular subset $\mathcal{B} \in \mathfrak{B}$, the number $\mu(\mathcal{B})$ should be understood as the probability of *observing* $\mathcal{B}$, rather than the probability that $\mathcal{B}$ *actually obtains*. Since $\mathfrak{B}$-partitions are observationally exhaustive, additivity with respect to $\vee$ as captured by equation (11) becomes a natural requirement. We say that a credence μ has *full support* if $\mu[\mathcal{B}] > 0$ for all nonempty $\mathcal{B} \in \mathfrak{B}$.

Example 6. (a) Let $\mathcal{S} := (0,1)$, and let $\mathfrak{Bas}(0,1)$ be the Boolean algebra of basic open subsets of $(0,1)$, as defined in Example 5(e). For any $\mathcal{B} \in \mathfrak{Bas}(0,1)$, if $\mathcal{B} = (a,b)$ for some $a < b$, then let $\mu[\mathcal{B}] := b - a$. Next, if $\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_N$ for some disjoint open intervals $\mathcal{B}_1, \ldots, \mathcal{B}_N$, then define $\mu[\mathcal{B}] := \mu[\mathcal{B}_1] + \cdots + \mu[\mathcal{B}_N]$. Then μ is a credence on $\mathfrak{Bas}(0,1)$.

(b) Let $\mathfrak{Bai}(\mathcal{S})$ be the Boolean algebra of sets with the Baire property, and let $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0,1]$ be a residual charge, as defined in Section 4. Recall that $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{Bai}(\mathcal{S})$, but $\mathfrak{R}(\mathcal{S})$ is not a *subalgebra* of $\mathfrak{Bai}(\mathcal{S})$, because the Boolean algebra operations are different (i.e. $\vee$ vs. $\cup$). But if we restrict ν to $\mathfrak{R}(\mathcal{S})$, then we get a credence on $\mathfrak{R}(\mathcal{S})$.[18] Thus, ν also defines a credence when restricted to any Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$. $\diamondsuit$

We can also construct credences on $\mathfrak{R}(\mathcal{S})$ from the liminal density structures of Sections 5 and 6 (Pivato and Vergopoulos, 2017b, §6-§7).

Conditional expectation structures. From any credence, we can construct a system of *expectation functionals*, which assign "expected values" to the utility profiles of feasible acts. Recall that $\mathfrak{Bas}(\mathbb{R})$ is the Boolean algebra of all basic subsets of $\mathbb{R}$. A function $f : \mathcal{S} \to \mathbb{R}$ is *$\mathfrak{B}$-comeasurable* if it is comeasurable with respect to $\mathfrak{B}$ and $\mathfrak{Bas}(\mathbb{R})$. Equivalently, f is *$\mathfrak{B}$-comeasurable* if $\mathrm{int}\left(f^{-1}(-\infty, r]\right) \in \mathfrak{B}$ and $\mathrm{int}\left(f^{-1}[r,\infty)\right) \in \mathfrak{B}$ for all $r \in \mathbb{R}$.

Example 7. (a) If $f : \mathcal{S} \longrightarrow \mathbb{R}$ is continuous, then f is $\mathfrak{R}(\mathcal{S})$-comeasurable.

(b) Let $\mathcal{S} = \mathbb{R}$. A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is $\mathfrak{Bas}(\mathbb{R})$-comeasurable if there is a finite sequence of points $-\infty = r_0 < r_1 < r_2 < r_3 < \cdots < r_N = \infty$ such that for each

[18] *Proof.* For any disjoint $\mathcal{R}, \mathcal{Q} \in \mathfrak{R}(\mathcal{S})$, we have $\mathcal{R} \vee \mathcal{Q} = \mathcal{R} \sqcup \mathcal{Q} \sqcup \mathcal{M}$, where $\mathcal{M}$ is meager. Thus, $\nu[\mathcal{R} \vee \mathcal{Q}] = \nu[\mathcal{R}] + \nu[\mathcal{Q}] + \nu[\mathcal{M}] = \nu[\mathcal{R}] + \nu[\mathcal{Q}]$, because $\nu[\mathcal{M}] = 0$, because ν is a residual charge.

$n \in [1 \ldots N]$, either f is non-increasing on (r_{n-1}, r_n), or f is non-decreasing on (r_{n-1}, r_n). In particular, any polynomial is $\mathfrak{Bas}(\mathbb{R})$-comeasurable, as is any non-decreasing or non-increasing continuous function. But $f(x) = \sin(x)$ is not $\mathfrak{Bas}(\mathbb{R})$-comeasurable.

(c) Let $\mathcal{S} = \mathbb{R}^N$, and let $\mathfrak{B}_{\text{poly}}(\mathbb{R}^N)$ be the Boolean algebra of regular polyhedra, from Example 5(c). A function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is *affine* if $f = f_0 + r$ for some linear function $f_0 : \mathbb{R}^N \longrightarrow \mathbb{R}$ and some constant $r \in \mathbb{R}$. We say f is *piecewise affine* if there is a partition $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of $\mathbb{R}^N$ into regular polyhedra, and a collection $f^1, \ldots, f^N : \mathbb{R}^N \longrightarrow \mathbb{R}$ of affine functions, such that $f_{\restriction \mathcal{R}_n} = f^n_{\restriction \mathcal{R}_n}$ for all $n \in [1 \ldots N]$. Any continuous piecewise affine function is $\mathfrak{B}_{\text{poly}}(\mathbb{R}^N)$-comeasurable.

(d) Let $\mathcal{S} = \mathbb{R}^N$, and let $\mathfrak{B}_{\text{smth}}(\mathbb{R}^N)$ be the Boolean algebra of regular sets with piecewise smooth boundaries, from Example 5(d). If $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is any differentiable function such that $\mathrm{d}f(s) \neq 0$ for all $s \in \mathbb{R}^N$, then f is $\mathfrak{B}_{\text{smth}}(\mathbb{R}^N)$-comeasurable. $\diamondsuit$

Let $\mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$ be the Banach space of *bounded*, continuous, real-valued functions, with the uniform norm $\|\cdot\|_\infty$. Let $\mathcal{C}_{\mathfrak{B}}(\mathcal{S})$ be the set of all $\mathfrak{B}$-comeasurable functions in $\mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$. This set is not necessarily closed under addition. So, let $\mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ be the closed linear subspace of $\mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$ spanned by $\mathcal{C}_{\mathfrak{B}}(\mathcal{S})$. (If $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$, then $\mathcal{G}_{\mathfrak{B}}(\mathcal{S}) = \mathcal{C}_{\mathfrak{B}}(\mathcal{S}) = \mathcal{C}_{\text{b}}(\mathcal{S}, \mathbb{R})$.) For any subset $\mathcal{B} \subseteq \mathcal{S}$, let $\mathcal{G}_{\mathfrak{B}}(\mathcal{B}) := \{g_{\restriction \mathcal{B}};\ g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{S})\}$. This is a linear subspace of $\mathcal{C}_{\text{b}}(\mathcal{B}, \mathbb{R})$. An *expectation functional* on $\mathcal{B}$ is a linear functional $\mathbb{E} : \mathcal{G}_{\mathfrak{B}}(\mathcal{B}) \longrightarrow \mathbb{R}$ such that $\|\mathbb{E}\|_\infty = 1$, and such that, for any $f, g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, if $f(b) \leq g(b)$ for all $b \in \mathcal{B}$, then $\mathbb{E}[f] \leq \mathbb{E}[g]$. If $\mathbf{1}$ is the constant function with value 1, then it follows that $\mathbb{E}[\mathbf{1}] = 1$.

Now let μ be a credence on $\mathfrak{B}$. A *conditional expectation structure* for $\mathcal{S}$ that is *compatible with* μ is a collection $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$, where, for all $\mathcal{B} \in \mathfrak{B}$, $\mathbb{E}_{\mathcal{B}}$ is an expectation functional on $\mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, and furthermore, if $\mu[\mathcal{B}] > 0$, then for any $\mathfrak{B}$-partition $\{\mathcal{B}_n\}_{n=1}^N$ of $\mathcal{B}$, and any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, we require

$$\mathbb{E}_{\mathcal{B}}[g] \quad = \quad \frac{1}{\mu[\mathcal{B}]} \sum_{n=1}^{N} \mu[\mathcal{B}_n]\, \mathbb{E}_{\mathcal{B}_n}\left[g_{\restriction \mathcal{B}_n}\right]. \tag{12}$$

In particular, $\mathbb{E} = \mathbb{E}_{\mathcal{S}}$ is an expectation functional on $\mathcal{G}_{\mathfrak{B}}(\mathcal{S})$, and a version of equation (12) holds for every $\mathfrak{B}$-partition of $\mathcal{S}$. Equation (12) captures a key feature of Bayesianism: conditional expectations are additively separable over the disjoint events of a $\mathfrak{B}$-partition. Indeed, for any regular event $\mathcal{B} \in \mathfrak{B}$ with $\mu[\mathcal{B}] > 0$, the subcollection $\{\mathbb{E}_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{B}'}$, where $\mathfrak{B}'$ is the collection of sets in $\mathfrak{B}$ that are contained in $\mathcal{B}$, is itself a conditional expectation structure on $\mathcal{B}$, compatible with the conditional credence $\mu_{\mathcal{B}}$.

If $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ and $\mathcal{B} \in \mathfrak{B}$, we will abuse notation and write "$\mathbb{E}_{\mathcal{B}}[g]$" to mean $\mathbb{E}_{\mathcal{B}}[g_{\restriction \mathcal{B}}]$. We say $\mathbf{E}$ is *strictly monotonic* if, for all $\mathcal{B} \in \mathfrak{B}$ and $g \in \mathcal{C}_{\mathfrak{B}}(\mathcal{B})$, if $g(b) > 0$ for all $b \in \mathcal{B}$, then $\mathbb{E}_{\mathcal{B}}[g] > 0$. For every credence μ, there is a unique compatible conditional expectation structure $\mathbf{E}$; furthermore, if μ has full support, then $\mathbf{E}$ is strictly monotonic (see Theorem 4.3 from Pivato and Vergopoulos (2017b)).

Example 8. (a) Let $\mathcal{S} = (0, 1)$, let $\mathfrak{Bas}(0, 1)$ be the Boolean algebra of basic open subsets of $(0, 1)$, and let μ be the credence from Example 6(a). Then the unique μ-compatible conditional expectation structure is defined as follows: for any $\mathcal{B} = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N)$ in $\mathfrak{E}$, and any $g \in \mathcal{G}_{\mathfrak{Bas}}(0, 1)$,

$$\mathbb{E}_{\mathcal{B}}[g] \quad := \quad \frac{1}{\mu[\mathcal{B}]} \left(\int_{a_1}^{b_1} g(x) \, \mathrm{d}x \; + \; \int_{a_2}^{b_2} g(x) \, \mathrm{d}x \; + \; \cdots \; + \; \int_{a_N}^{b_N} g(x) \, \mathrm{d}x \right).$$

(b) Let $\mathcal{S}$ be a Baire space, let $\mathfrak{Bai}(\mathcal{S})$ be the Boolean algebra of sets with the Baire property, and let $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0, 1]$ be a residual charge, which induces a credence on $\mathfrak{R}(\mathcal{S})$ as in Example 6(b). The unique μ-compatible conditional expectation structure on $\mathfrak{R}(\mathcal{S})$ is defined by $\mathbb{E}_{\mathcal{R}}[f] := \dfrac{1}{\nu[\mathcal{R}]} \displaystyle\int_{\mathcal{R}} f \, \mathrm{d}\nu$ for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and $f \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$.[19] $\diamondsuit$

SEU representations. We will say that a function $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-*measurable* if $u^{-1}(\mathcal{B}) \in \mathfrak{D}$ for any $\mathcal{B} \in \mathfrak{Bas}(\mathbb{R})$. For example, if $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$, then any open, continuous real-valued function on $\mathcal{X}$ is $\mathfrak{D}$-measurable (Fremlin, 2006, Appendix 4A2B, item (f)(iii), p.453). If $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-measurable, and $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is $(\mathfrak{B}, \mathfrak{D})$-comeasurable, then $u \circ \alpha : \mathcal{S} \longrightarrow \mathbb{R}$ is $\mathfrak{B}$-comeasurable, and thus it can be evaluated by a conditional expectation structure (Pivato and Vergopoulos, 2017b, Proposition 5.4(a)).

Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, and let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be a $\mathfrak{B}$-indexed conditional preference structure on $\mathcal{A}$. Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a $\mathfrak{D}$-measurable, continuous "utility" function. Let μ be a credence on $\mathfrak{B}$. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be the (unique) conditional expectation structure that is compatible with μ. The pair (u, μ) is a *subjective expected utility* (SEU) representation for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ if, for any $\mathcal{B} \in \mathfrak{B}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, we have

$$\Big(\alpha \succeq_{\mathcal{B}} \beta\Big) \iff \Big(\mathbb{E}_{\mathcal{B}}\left[u \circ \alpha\right] \geq \mathbb{E}_{\mathcal{B}}\left[u \circ \beta\right]\Big). \tag{13}$$

Theorem 9 *Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces, with $\mathcal{X}$ connected. Let $\mathfrak{B}$ and $\mathfrak{D}$ be non-trivial Boolean subalgebras of $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$ respectively. Let $\mathcal{A}$ be a collection of bounded continuous and $(\mathfrak{B}, \mathfrak{D})$-comeasurable functions from $\mathcal{S}$ into $\mathcal{X}$. Let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then, it satisfies* (CEq), (C), (M), (Dom), (Sep), (PC), *and* (TC) *if and only if it has an SEU representation* (13) *with full support. Finally, μ is unique, and u is unique up to positive affine transformation.*

The SEU representation obtained in Theorem 9 is somewhat simpler but less informative than the representations obtained so far: the agent's beliefs only assign probabilities to the observable events, and do not describe the way she copes with her informational constraints. But its advantage lies in its greater generality. This SEU representation does not require any specific topological assumption on the state space, and it accommodates situations where the observable measurements only arise from a subalgebra of regular subsets. Thus, Theorem 9 is adapted to decision problems where technological constraints restrict both the acts which are feasible and the information which is available to the agent.

The proof of Theorem 9 is in Appendix A, but we will sketch the main steps here. First, we use condition (Rch) to obtain a preference order on $\mathcal{X}^N$, for each $\mathfrak{B}$-partition

[19] See Proposition 6.1 of Pivato and Vergopoulos (2017b).

$\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{S}$. Then, we invoke (Sep), (C), (PC), (TC), (Dom) and (CEq) and a theorem of Wakker (1988) to construct a continuous SEU representation for the latter order. The decision weights associated to each component of $\mathcal{X}^N$ provide the values of the credence μ on each regular subset of the partition $\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$. Axiom (M) ensures that the continuous utility function u thus constructed is $\mathfrak{D}$-measurable. Finally, we show that the expected utility of any act is equal the utility of any of its certainty equivalents. The SEU representation then easily follows. Appendix B presents another result, Theorem B1, which is a simplified version of Theorem 9 that is adapted to situations where $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$. Theorems 1, 2, 3 and 4 are all corollaries of Theorem B1.

8 Stonean representations

In this section, we will provide an SEU representation which combines the strengths of the results in Sections 5, 6 and 7. As in Sections 5 and 6, this representation uses a Borel measure on a compact space (rather than the credences of Section 7). But unlike Sections 5 and 6, it uses *only* a Borel measure, without a liminal structure. Meanwhile, as in Section 7, this representation does not assume $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$; thus, it is compatible with informational constraints. To obtain this representation, we must replace the original state space $\mathcal{S}$ with new state space $\mathcal{S}^*$, which, as in Section 6, can be interpreted as the "subjective state space" of the agent. Formally, $\mathcal{S}^*$ is the *Stone space* of the Boolean algebra $\mathfrak{B}$.[20]

Let $\mathfrak{B}$ be any Boolean algebra. A ***truth valuation*** on $\mathfrak{B}$ is a Boolean algebra homomorphism $v : \mathfrak{B} \longrightarrow \mathfrak{T}$, where $\mathfrak{T} := \{T, F\}$ is the two-element Boolean algebra, with the usual operations $\vee$, $\wedge$ and $\neg$. If $\mathfrak{B}$ is a set of propositions about the world, each of which may be true or false, then v is a complete, logically consistent assignment of truth values to these propositions. In particular, if $\mathfrak{B}$ is the algebra of all events which are observable to the agent, then v is a *complete subjective description* of the world, as seen by this agent.

Let $\sigma(\mathfrak{B})$ be the set of all truth valuations of $\mathfrak{B}$. For any $\mathcal{B} \in \mathfrak{B}$, let $\mathcal{B}^* := \{v \in \sigma(\mathcal{B});\ v(\mathcal{B}) = T\}$. The collection $\{\mathcal{B}^*;\ \mathcal{B} \in \mathfrak{B}\}$ is a base of clopen sets for a topology on $\sigma(\mathfrak{B})$, making it into a compact, totally disconnected Hausdorff space, called the ***Stone space*** of $\mathfrak{B}$. The Boolean algebra structure of $\mathfrak{B}$ is completely encoded in the topology of $\sigma(\mathfrak{B})$.[21]

Let $\mathcal{S}$ be any locally compact Hausdorff space, and let $\check{\mathcal{S}}$ be its Stone-Čech compactification. Let $\mathfrak{B}$ be a Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$, and define $\check{\mathfrak{B}} := \{\check{\mathcal{B}};\ \mathcal{B} \in \mathfrak{B}\}$, where we define $\check{\mathcal{B}}$ as in statement (SČ3) of Section 6. We say that $\mathfrak{B}$ is ***generative*** if $\check{\mathfrak{B}}$ is a base for the topology of $\check{\mathcal{S}}$. For example, the full Boolean algebra $\mathfrak{R}(\mathcal{S})$ is generative (Pivato and Vergopoulos, 2017b, Lemma 8.2). Let $\mathcal{S}^* := \sigma(\mathfrak{B})$. If $\mathfrak{B}$ is generative, then there is a continuous surjection $p : \mathcal{S}^* \longrightarrow \check{\mathcal{S}}$ (Pivato and Vergopoulos, 2017b, Proposition 8.3).[22]

[20] Stinchcombe (1997, §7) also constructs an SEU representation based on Stone spaces. However, he works in a very different framework, and his results are unrelated to ours.

[21] To be precise, the Stone Representation Theorem says that the map $\mathcal{B} \mapsto \mathcal{B}^*$ is an isomorphism from $\mathfrak{B}$ to the Boolean algebra of clopen subsets of $\sigma(\mathfrak{B})$. Meanwhile, the Stone Duality Theorem says that σ is a functorial isomorphism between the category of Boolean algebras and the category of compact, totally disconnected Hausdorff spaces; see e.g. Johnstone (1986, §4.1) or Fremlin (2004, §311-§312).

[22] If $\mathcal{S}$ is compact, then $\check{\mathcal{S}} = \mathcal{S}$. If also $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$, then $p : \mathcal{S}^* \longrightarrow \mathcal{S}$ is called the ***Gleason cover*** of $\mathcal{S}$; it

Heuristically, p is defined as follows. For any $s \in \mathcal{S}$, let $\mathfrak{B}_s := \{\mathcal{B} \in \mathfrak{B};\ \ s \in \mathcal{B} \text{ or } s \in \neg\mathcal{B}\}$. Then we can define a "partial" truth valuation $v_s : \mathfrak{B}_s \longrightarrow \{T, F\}$ by setting $v_s(\mathcal{B}) = T$ if $s \in \mathcal{B}$ and $v_s(\mathcal{B}) = F$ if $s \in \neg\mathcal{B}$. Typically, v_s is not *logically complete*, because $\mathfrak{B}_s \subsetneq \mathfrak{B}$. The preimage $p^{-1}\{s\}$ is the set of all *logically consistent* ways of extending v_s to a *complete* truth valuation on $\mathfrak{B}$. To be precise, $v_s(\mathcal{B})$ is not defined for any $\mathcal{B} \in \mathfrak{B}$ with $s \in \partial\mathcal{B}$. If $\overline{v} \in p^{-1}\{v_s\}$ then $\overline{v}$ "attaches" s to either $\mathcal{B}$ or $\neg\mathcal{B}$, for every $\mathcal{B} \in \mathfrak{B}$ with $s \in \partial\mathcal{B}$; furthermore, $\overline{v}$ makes these attachments in a logically consistent way.

Let $\mathcal{X}$ be a Hausdorff space. Recall that any function $\alpha \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$ has a unique continuous extension $\check{\alpha} \in \mathcal{C}(\check{\mathcal{S}}, \mathcal{X})$ (see statement (SČ2) from Section 6). If we define $\alpha^* := \check{\alpha} \circ p$, then $\alpha^* \in \mathcal{C}(\mathcal{S}^*, \mathbb{R})$. Now let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be a $\mathfrak{B}$-indexed conditional preference structure on $\mathcal{A}$. A *Stonean SEU representation* for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ is given by a normal Borel probability measure μ^* on $\mathfrak{Bor}(\mathcal{S}^*)$ and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{B} \in \mathfrak{B}$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{B})$,

$$\Big(\alpha \succeq_{\mathcal{B}} \beta\Big) \quad \Longleftrightarrow \quad \left(\int_{\mathcal{B}^*} u \circ \alpha^* \ \mathrm{d}\mu^* \quad \geq \int_{\mathcal{B}^*} u \circ \beta^* \ \mathrm{d}\mu^*\right). \tag{14}$$

Theorem 10 *Let $\mathcal{S}$ be a locally compact Hausdorff space, and let $\mathcal{X}$ be a connected Hausdorff space. Let $\mathfrak{B}$ be a generative Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$, let $\mathfrak{D}$ be a nontrivial Boolean subalgebra of $\mathfrak{R}(\mathcal{X})$, and let $\mathcal{A}$ be a collection of bounded, continuous, $(\mathfrak{B}, \mathfrak{D})$-comeasurable functions from $\mathcal{S}$ into $\mathcal{X}$. Let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then, it satisfies* (CEq), (C), (M), (Dom), (Sep), (PC), *and* (TC) *if and only if it admits a Stonean SEU representation* (14)*, where μ^* has full support on $\mathcal{S}^*$. Finally, μ^* is unique, and u is unique up to positive affine transformation.*

Theorem 10 has a natural and appealing interpretation. The agent is only able to observe the events in the algebra $\mathfrak{B}$. So for her, a complete subjective description of the world is given by a (logically consistent) assignment of truth-values to the events in $\mathfrak{B}$ — that is, an element of $\mathcal{S}^*$. The measure μ^* assigns probabilities to such complete subjective descriptions. Given any act $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$, it is possible to represent α as a function converting each complete subjective description into an outcome —that is, a function $\alpha^* : \mathcal{S}^* \longrightarrow \mathcal{X}$. The agent then ranks each act α according to the μ^*-expected utility of α^*. This may seem peculiar, but in fact it is quite psychologically natural. Perhaps $\mathcal{S}$ describes the world "as it really is". But for the agent, $\mathcal{S}^*$ describes the world *as she experiences it.* Thus, for her, an SEU representation on $\mathcal{S}^*$ might be more natural than one on $\mathcal{S}$ itself.

There is an interesting duality between Theorem 10 and the model of Lipman (1999). In Lipman's model, as in ours, the agent is equipped with a mental vocabulary of propositions, each of which can be either true or false. In Lipman's model, the "true" state space is the set of all *logically consistent* assignments of truth-values to these propositions. But the agent's *subjective* state space also includes some logically *inconsistent* truth-value assignments; these so-called "impossible possible worlds" reflect her lack of logical omniscience. In our model, by contrast, $\mathcal{S}$ is a set of consistent but *logically incomplete* truth-value assignments, whereas $\mathcal{S}^*$ is the set of all consistent and *complete* assignments.

plays an important role in categorical topology (Johnstone, 1986, §3.10).

Which of $\mathcal{S}$ or $\mathcal{S}^*$ is the "correct" state space is a matter of interpretation. According to the Law of the Excluded Middle, every statement is either true or false. Thus, only $\mathcal{S}^*$ could be the correct state space, and $\mathcal{S}$ is at best some "subjective approximation"; the incompleteness of the truth-value assignments in $\mathcal{S}$ must reflect a lack of information or a failure of the imagination. But there are also countervailing views in philosophy, according to which some statements simply do not *have* an objectively correct truth value. In the philosophical analysis of vagueness, this view is called *supervaluationism*; in the philosophy of time, it is called *metaphysical indeterminism* (about the future); and in the philosophy of mathematics, it aligns closely with *intuitionism* (concerning formally undecidable propositions). Finally, it is a plausible attitude towards lacunae in systems of "socially constructed truths" such as legal corpora, codes of etiquette, and the grammars of natural languages. Of course, an agent can "take a position" on these statements if she so desires, but she is misguided if she thinks any of these positions is any more objectively correct than any other. According to this view, the (incomplete) valuations in $\mathcal{S}$ already capture all that can be objectively known about the world, and the extra information putatively encoded in the states of $\mathcal{S}^*$ is mere prejudice or unjustified speculation. Nevertheless, Theorem 10 shows that the agent can assign subjective probabilities to these "prejudices" in an internally coherent way, and use them to guide her decisions.

The previous paragraph assumed that $\mathfrak{B}$ was a complete list of all meaningful statements which could be made about the world. But if $\mathfrak{B}$ is incomplete, then it is possible that neither $\mathcal{S}$ nor $\mathcal{S}^*$ is the correct state space; both of them are merely subjective approximations. Then the relevant question is which one is a more suitable basis for decision-making. Theorems 9 and 10 show that the agent can use either $\mathcal{S}$ or $\mathcal{S}^*$ as the domain of her SEU representation; the choice is a matter of mathematical convenience.

9 Prior literature

Several previous papers have restricted the Savage universal domain of preference by introducing measurability constraints. In fact, Savage (1954) explicitly notes that his axiomatic construction works equally well for preferences over acts that are measurable with respect to a σ-algebra, but only produces finitely additive probabilities. Arrow (1970) enriches the Savage axioms so as to further derive the σ-additivity of probability measures. See also Kopylov (2010). Epstein and Zhang (2001) construct a theory of "probabilistic sophistication" on a collection of acts that are measurable with respect to a "λ-system". Finally, Kopylov (2007) provides both SEU maximization and probabilistic sophistication on the weaker structure of so-called *mosaics*, which include Boolean algebras.

Some papers aim explicitly at deriving continuous utility functions from preferences. For example, Grandmont (1972) obtained continuous utility functions in a von Neuman and Morgenstern (1947) framework. Other papers consider acts from a measurable state space into a topological outcome space, typically assumed to be connected and separable. For example, Wakker (1985, 1988) and Wakker (1989a, Chapter 5) characterized continuous, state-independent SEU representations in this setting. Wakker (1987) characterized continuous and state-*dependent* SEU over a finite state space, while Wakker and Zank (1999) characterized it over any measurable space. Wakker (1989b) characterized continuous Choquet expected utility representations over a finite state space. Finally, Casadesus-Masanell et al. (2000) characterized continuous maximin expected utility representations.

Like us, Zhou (1999) considers the case where both the state space and the outcome space are topological spaces, and acts are continuous functions. But unlike us, he restricts attention to the case where the outcomes are themselves lotteries over some finite set of consequences, so that acts correspond to "two-stage lotteries" of the kind considered by Anscombe and Aumann (1963). In this framework, Zhou proves versions of Anscombe and Aumann's SEU representation theorem, as well as the Choquet expected utility representation theorem of Schmeidler (1989), in both cases obtaining continuous utility functions. Unlike Zhou, we do not assume any special structure on the outcome space; our framework is more like the framework of Savage, rather than that of Anscombe and Aumann.

Finally, as we have noted earlier in the paper, several of our SEU representations have common features with past models in decision theory which have employed Stone spaces (Stinchcombe, 1997), Stone-Čech compactifications (Chichilnisky, 1996, 2000, 2009; Chichilnisky and Heal, 1997), or other "subjective state spaces" (Jaffray and Wakker, 1993; Gilboa and Schmeidler, 1994; Mukerji, 1997; Lipman, 1999; Alon, 2015).

Conclusion

This paper has presented a series of SEU representations for preferences under uncertainty. However, it is now well-established that the SEU model is often not descriptively accurate. In some cases, it may not even be normatively compelling, e.g. when the agent faces *ambiguity*, where she lacks even sufficient information to form probabilistic beliefs about the state of the world. Thus, there has been much recent interest in "non-SEU" models of decision-making under uncertainty. However, there has been little exploration of such non-SEU models in explicitly topological environments like the ones considered in this paper.[23] In that respect, the results in the present paper can be seen as benchmarks, which set the stage for future research into non-SEU representations on topological spaces.

We have assumed that only *continuous* acts are feasible. This may seem unduly restrictive. Of course, Borel-measurable functions can be extremely complex, and it is unlikely that all such functions could be technologically feasible acts. But it seems plausible that *piecewise continuous* acts could be feasible (i.e. functions which are continuous on each cell of some regular partition of the state space). By restricting ourselves to *continuous* acts to obtain our SEU representations, we have actually solved a harder problem. It is straightforward to extend these SEU representations to preferences over piecewise continuous acts (Pivato and Vergopoulos, 2017a).

[23]Wakker (1989b), Casadesus-Masanell et al. (2000), and Zhou (1999) are exceptions. But the first two only put a topology on the outcome space.

Appendices

Appendix A contains the proof of Theorem 9. Appendix B contains the proofs of the SEU representations from Sections 4, 5, 6 and 8. Appendix C develops a variant of the liminal SEU representation from Section 5, but for *normal* Hausdorff spaces rather than compact Hausdorff spaces. Finally Appendix D gives necessary and sufficient conditions for (Rch).

A Proof of Theorem 9

The proof of Theorem 9 has two preliminary stages. First, Proposition A3 uses (Rch) and axioms (Sep), (C), (TC), (PC), (Dom), (CEq) and (M) to construct a credence μ on $\mathfrak{B}$ and a continuous and $\mathfrak{D}$-measurable utility function u using a theorem of Wakker (1988) for continuous additive representations. Second, Proposition A6 shows that the expected utility of any act, with respect to μ and u, equals the utility of any certainty equivalent of this act. The rest of the appendix uses these findings to construct the SEU representations and establishes the necessity of the axioms, as well as the uniqueness of the representation. Throughout this appendix, we maintain the following standing assumptions:

> $\mathcal{S}$ and $\mathcal{X}$ are topological spaces, with $\mathcal{X}$ connected. $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} \subseteq \mathfrak{R}(\mathcal{X})$ are nontrivial Boolean subalgebras, $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, and $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ is a conditional preference structure on $\mathcal{A}$ that satisfies (Rch).

Lemma A1 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *satisfies axioms* (C)*,* (Dom) *and* (CEq)*. Let* $\mathcal{B} \in \mathfrak{B}$*. Define the function* $\mathsf{K}_{\mathcal{B}} : \mathcal{X} \longrightarrow \mathcal{A}(\mathcal{B})$ *by* $\mathsf{K}_{\mathcal{B}}(x) := \kappa^x_{\mathcal{B}}$ *for all* $x \in \mathcal{X}$*. Then* $\mathsf{K}_{\mathcal{B}}$ *is continuous relative to the* $\succeq_{\mathcal{B}}$*-order topology on* $\mathcal{A}(\mathcal{B})$*.*

Proof. For any $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$, let $(\alpha, \gamma)_{\succeq_{\mathcal{B}}} := \{\beta \in \mathcal{A}(\mathcal{B});\ \alpha \prec_{\mathcal{B}} \beta \prec_{\mathcal{B}} \gamma\}$. This collection of sets (for all $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$) forms a base for the $\succeq_{\mathcal{B}}$-order topology on $\mathcal{A}(\mathcal{B})$. So it suffices to show that $\mathsf{K}_{\mathcal{B}}^{-1}\left[(\alpha, \gamma)_{\succeq_{\mathcal{B}}}\right]$ is open in $\mathcal{X}$, for all $\alpha, \beta \in \mathcal{A}(\mathcal{B})$.

Axiom (CEq) says there exist $x, z \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$ and $\gamma \approx_{\mathcal{B}} \kappa^z_{\mathcal{B}}$. Define $(x, z)_{\succeq_{\mathrm{xp}}} := \{y \in \mathcal{X};\ x \prec_{\mathrm{xp}} y \prec_{\mathrm{xp}} z\}$. Axiom (C) says this is an open subset of $\mathcal{X}$. Now, for any $y \in \mathcal{X}$, we have

$$\begin{aligned} \left(y \in \mathsf{K}_{\mathcal{B}}^{-1}\left[(\alpha, \gamma)_{\succeq_{\mathcal{B}}}\right]\right) &\iff \left(\mathsf{K}_{\mathcal{B}}(y) \in (\alpha, \gamma)_{\succeq_{\mathcal{B}}}\right) \iff \left(\alpha \prec_{\mathcal{B}} \kappa^y_{\mathcal{B}} \prec_{\mathcal{B}} \gamma\right) \\ &\underset{(*)}{\iff} \left(\kappa^x_{\mathcal{B}} \prec_{\mathcal{B}} \kappa^y_{\mathcal{B}} \prec_{\mathcal{B}} \kappa^z_{\mathcal{B}}\right) \underset{(\dagger)}{\iff} \left(x \prec_{\mathrm{xp}} y \prec_{\mathrm{xp}} z\right) \\ &\iff \left(y \in (x, z)_{\succeq_{\mathrm{xp}}}\right). \end{aligned}$$

Here, $(*)$ is because $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$ and $\gamma \approx_{\mathcal{B}} \kappa^z_{\mathcal{B}}$, while $(\dagger)$ is by axiom (Dom) and its contrapositive. Thus, we see that $\mathsf{K}_{\mathcal{B}}^{-1}\left[(\alpha, \gamma)_{\succeq_{\mathcal{B}}}\right] = (x, z)_{\succeq_{\mathrm{xp}}}$, which is an open subset of $\mathcal{X}$. Since this holds for all $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$, we conclude that $\mathsf{K}_{\mathcal{B}}$ is continuous. $\square$

It will be convenient to use the following equivalent formulation of axiom (PC).

(PC′) Let $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$ as in axiom (Sep). Let $\mathcal{O} \subseteq \mathcal{A}(\mathcal{B})$ be open in the $\succeq_{\mathcal{B}}$-order topology, and let $\alpha \in \mathcal{O}$. Then there exist sets $\mathcal{O}_{\mathcal{D}} \subseteq \mathcal{A}(\mathcal{D})$ and $\mathcal{O}_{\mathcal{E}} \subseteq \mathcal{A}(\mathcal{E})$ which are open in the $\succeq_{\mathcal{D}}$-order topology and $\succeq_{\mathcal{E}}$-order topology, with $\alpha_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, such that, for any $\beta \in \mathcal{A}(\mathcal{B})$, if $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\beta_{\upharpoonright\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, then $\beta \in \mathcal{O}$.

Lemma A2 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *satisfies axioms* (PC), (C), (Dom) *and* (CEq). *Consider a* $\mathfrak{B}$*-partition* $\mathfrak{P} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ *of* $\mathcal{S}$ *with* $N \geq 2$. *There exists a mapping* $\Phi_{\mathfrak{P}} : \mathcal{X}^N \longrightarrow \mathcal{A}$ *that is continuous with respect to the product topology on* $\mathcal{X}^N$ *and the* $\succeq_{\mathcal{S}}$*-order topology on* $\mathcal{A}$ *and satisfies* $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright\mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ *for any* $n \in [1 \ldots N]$ *and any* $\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$.

Proof. Let $\mathbf{x} := (x_1, \ldots, x_N) \in \mathcal{X}^N$. Define $\alpha^1 := \kappa^{x_1}_{\mathcal{E}_1}$, an element of $\mathcal{A}(\mathcal{E}_1)$. Condition (Rch) yields $\alpha^2 \in \mathcal{A}(\mathcal{E}_1 \vee \mathcal{E}_2)$ such that $\alpha^2_{\upharpoonright\mathcal{E}_1} = \kappa^{x_1}_{\mathcal{E}_1}$ and $\alpha^2_{\upharpoonright\mathcal{E}_2} \approx_{\mathcal{E}_2} \kappa^{x_2}_{\mathcal{E}_2}$. Next, (Rch) yields $\alpha^3 \in \mathcal{A}(\mathcal{E}_1 \vee \mathcal{E}_2 \vee \mathcal{E}_3)$ such that $\alpha^3_{\upharpoonright\mathcal{E}_1\vee\mathcal{E}_2} = \alpha^2$ and $\alpha^3_{\upharpoonright\mathcal{E}_3} \approx_{\mathcal{E}_3} \kappa^{x_3}_{\mathcal{E}_3}$. In particular, this means that $\alpha^3_{\upharpoonright\mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in \{1, 2, 3\}$.

Inductively, let $M \in [4 \ldots N]$, and suppose we have some $\alpha^{M-1} \in \mathcal{A}(\mathcal{E}_1 \vee \cdots \vee \mathcal{E}_{M-1})$ such that $\alpha^{M-1}_{\upharpoonright\mathcal{E}_m} \approx_{\mathcal{E}_m} \kappa^{x_m}_{\mathcal{E}_m}$ for all $m \in [1 \ldots M)$. (Rch) yields $\alpha^M \in \mathcal{A}(\mathcal{E}_1 \vee \cdots \vee \mathcal{E}_M)$ such that $\alpha^M_{\upharpoonright\mathcal{E}_1\vee\cdots\vee\mathcal{E}_{M-1}} = \alpha^{M-1}$ and $\alpha^M_{\upharpoonright\mathcal{E}_M} \approx_{\mathcal{E}_M} \kappa^{x_M}_{\mathcal{E}_M}$. In particular, this means that $\alpha^M_{\upharpoonright\mathcal{E}_m} \approx_{\mathcal{E}_m} \kappa^{x_m}_{\mathcal{E}_m}$ for all $m \in [1 \ldots M]$.

Setting $M = N$ in this construction, we obtain some α^N such that $\alpha^N_{\upharpoonright\mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in [1 \ldots N]$. Now define $\Phi_{\mathfrak{P}}(\mathbf{x}) := \alpha^N$. To prove the continuity of $\Phi_{\mathfrak{P}}$, we need a preliminary result, which extends axiom (PC).

Claim 1: *Let* $\mathcal{O} \subseteq \mathcal{A}$ *be open in the* $\succeq_{\mathcal{S}}$*-order topology, and let* $\alpha \in \mathcal{O}$. *Then for all* $n \in [1 \ldots N]$, *there is a set* $\mathcal{O}_n \subseteq \mathcal{A}(\mathcal{E}_n)$ *which is open in the* $\succeq_{\mathcal{E}_n}$*-order topology, with* $\alpha_{\upharpoonright\mathcal{E}_n} \in \mathcal{O}_n$, *such that, for any* $\beta \in \mathcal{A}$, *if* $\beta_{\upharpoonright\mathcal{E}_n} \in \mathcal{O}_n$ *for all* $n \in [1 \ldots N]$, *then* $\beta \in \mathcal{O}$.

Proof. Let $\mathcal{D}_1 := \mathcal{E}_2 \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{B} := \mathcal{E}_1 \vee \mathcal{D}_1$. Setting $\mathcal{D} := \mathcal{D}_1$ and $\mathcal{E} := \mathcal{E}_1$ in axiom (PC′), we obtain some $\mathcal{O}_1 \subseteq \mathcal{A}(\mathcal{E}_1)$ and $\mathcal{Q}_1 \subseteq \mathcal{A}(\mathcal{D}_1)$ with $\alpha_{\upharpoonright\mathcal{E}_1} \in \mathcal{O}_1$ and $\alpha_{\upharpoonright\mathcal{D}_1} \in \mathcal{Q}_1$, such that, for any $\beta \in \mathcal{A}(\mathcal{B})$, if $\beta_{\upharpoonright\mathcal{E}_1} \in \mathcal{O}_1$ and $\beta_{\upharpoonright\mathcal{D}_1} \in \mathcal{Q}_1$ then $\beta \in \mathcal{O}$.

Now let $\mathcal{D}_2 := \mathcal{E}_3 \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{D}_1 := \mathcal{E}_2 \vee \mathcal{D}_2$. Setting $\mathcal{D} := \mathcal{D}_2$ and $\mathcal{E} := \mathcal{E}_2$ in axiom (PC′), we obtain some $\mathcal{O}_2 \subseteq \mathcal{A}(\mathcal{E}_2)$ and $\mathcal{Q}_2 \subseteq \mathcal{A}(\mathcal{D}_2)$ with $\alpha_{\upharpoonright\mathcal{E}_2} \in \mathcal{O}_2$ and $\alpha_{\upharpoonright\mathcal{D}_2} \in \mathcal{Q}_2$, such that, for any $\beta \in \mathcal{A}(\mathcal{D}_1)$, if $\beta_{\upharpoonright\mathcal{E}_2} \in \mathcal{O}_2$ and $\beta_{\upharpoonright\mathcal{D}_2} \in \mathcal{Q}_2$ then $\beta \in \mathcal{Q}_1$. In particular, this means that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\upharpoonright\mathcal{E}_1} \in \mathcal{O}_1$, $\beta_{\upharpoonright\mathcal{E}_2} \in \mathcal{O}_2$ and $\beta_{\upharpoonright\mathcal{D}_2} \in \mathcal{Q}_2$, then $\beta \in \mathcal{O}$.

Inductively, let $M \in [3 \ldots N)$, let $\mathcal{D}_{M-1} := \mathcal{E}_M \vee \cdots \vee \mathcal{E}_N$, and suppose that, for all $m \in [1 \ldots M)$, we have constructed $\mathcal{O}_m \subseteq \mathcal{A}(\mathcal{E}_m)$ (open in the $\succeq_{\mathcal{E}_m}$-topology) with $\alpha_{\upharpoonright\mathcal{E}_m} \in \mathcal{O}_m$, along with some $\mathcal{Q}_{M-1} \subseteq \mathcal{A}(\mathcal{D}_{M-1})$ (open in the $\succeq_{\mathcal{D}_{M-1}}$-topology) with $\alpha_{\upharpoonright\mathcal{D}_{M-1}} \in \mathcal{Q}_{M-1}$, such that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\upharpoonright\mathcal{E}_m} \in \mathcal{O}_m$ for all $m \in [1 \ldots M)$ and $\beta_{\upharpoonright\mathcal{D}_{M-1}} \in \mathcal{Q}_{M-1}$, then $\beta \in \mathcal{O}$. Now let $\mathcal{D}_M := \mathcal{E}_{M+1} \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{D}_{M-1} := \mathcal{E}_M \vee \mathcal{D}_M$. Setting $\mathcal{D} := \mathcal{D}_M$ and $\mathcal{E} := \mathcal{E}_M$ in axiom (PC′), we obtain some $\mathcal{O}_M \subseteq \mathcal{A}(\mathcal{E}_M)$ and $\mathcal{Q}_M \subseteq \mathcal{A}(\mathcal{D}_M)$ with $\alpha_{\upharpoonright\mathcal{E}_M} \in \mathcal{O}_M$ and $\alpha_{\upharpoonright\mathcal{D}_M} \in \mathcal{Q}_M$, such that, for any $\beta \in \mathcal{A}(\mathcal{D}_{M-1})$, if $\beta_{\upharpoonright\mathcal{E}_M} \in \mathcal{O}_M$ and $\beta_{\upharpoonright\mathcal{D}_M} \in \mathcal{Q}_M$ then $\beta \in \mathcal{Q}_{M-1}$. In particular, this means that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\upharpoonright\mathcal{E}_m} \in \mathcal{O}_m$ for all $m \in [1 \ldots M]$ and $\beta_{\upharpoonright\mathcal{D}_M} \in \mathcal{Q}_M$, then $\beta \in \mathcal{O}$.

Suppose $M = N-1$ in the previous paragraph. Then $\mathcal{D}_M = \mathcal{E}_N$. Thus, if we define $\mathcal{O}_N := \mathcal{Q}_{N-1}$, then we have obtained sets $\mathcal{O}_1, \ldots, \mathcal{O}_N$ satisfying the claim. $\diamond$ Claim 1

It remains to show that $\Phi_{\mathfrak{P}}$ is continuous with respect to the product topology on $\mathcal{X}^N$ and the $\succeq_{\mathcal{S}}$-order topology on $\mathcal{A}$. To see this, let $\mathcal{O} \subseteq \mathcal{A}$ be open in the $\succeq_{\mathcal{S}}$-order topology. It is sufficient to show that $\mathcal{U} := \Phi_{\mathfrak{P}}^{-1}(\mathcal{O})$ is open in the product topology on $\mathcal{X}^N$. To do this, let $\mathbf{x} \in \mathcal{U}$; we will construct an open neighbourhood around $\mathbf{x}$ inside $\mathcal{U}$.

Let $\alpha := \Phi_{\mathfrak{P}}(\mathbf{x}) \in \mathcal{A}$. Then, $\alpha \in \mathcal{O}$. For any $n \in [1 \ldots N]$, let $\mathcal{O}_n \subseteq \mathcal{A}(\mathcal{E}_n)$ be the open subset in the $\succeq_{\mathcal{E}_n}$-order topology obtained in Claim 1, and define $\mathcal{V}_n := \mathsf{K}_{\mathcal{E}_n}^{-1}(\mathcal{O}_n)$. By Lemma A1, each $\mathsf{K}_{\mathcal{E}_n}$ is a continuous function from $\mathcal{X}$ to $\mathcal{A}(\mathcal{E}_n)$. So $\mathcal{V}_n$ is an open subset of $\mathcal{X}$ for any $n \in [1 \ldots N]$. Define $\mathcal{V} := \mathcal{V}_1 \times \ldots \times \mathcal{V}_N$; then $\mathcal{V}$ is an open subset of $\mathcal{X}^N$ in the product topology.

Claim 2: $\mathbf{x} \in \mathcal{V}$.

Proof. Any open set in an order topology is a union of order intervals, and any order interval is a union of indifference classes (because if an order interval contains some element γ, then it also contains all other elements which are indifferent to γ). Thus, any open set is a union of indiffererence classes.

Now fix $n \in [1 \ldots N]$. By Claim 1, we have $\alpha_{\upharpoonright \mathcal{E}_n} \in \mathcal{O}_n$. Moreover, by the definition of α and the construction of $\Phi_{\mathfrak{P}}$, we have $\alpha_{\upharpoonright \mathcal{E}_n} = \Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa_{\mathcal{E}_n}^{x_n}$. By the remark in the previous paragraph, and since $\mathcal{O}_n$ is open in the $\succeq_{\mathcal{E}_n}$-order topology, we obtain $\kappa_{\mathcal{E}_n}^{x_n} \in \mathcal{O}_n$. Then, $x_n \in \mathcal{V}_n$. Since this holds for any $n \in [1 \ldots N]$, we obtain $\mathbf{x} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_N = \mathcal{V}$. $\diamond$ Claim 2

Claim 3: $\mathcal{V} \subseteq \mathcal{U}$.

Proof. Let $\mathbf{y} = (y_1, \ldots, y_N) \in \mathcal{V}$ and define $\beta = \Phi_{\mathfrak{P}}(\mathbf{y}) \in \mathcal{A}$. Fix $n \in [1 \ldots N]$. Then $y_n \in \mathcal{V}_n$, so by definition of $\mathcal{V}_n$, we have $\kappa_{\mathcal{E}_n}^{y_n} \in \mathcal{O}_n$. By the construction of $\Phi_{\mathfrak{P}}$, we have $\beta_{\upharpoonright \mathcal{E}_n} = \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa_{\mathcal{E}_n}^{y_n}$. Since $\mathcal{O}_n$ is open in the $\succeq_{\mathcal{E}_n}$-order topology and $\kappa_{\mathcal{E}_n}^{y_n} \in \mathcal{O}_n$, $\mathcal{O}_n$ must also contain any act that is indifferent to $\kappa_{\mathcal{E}_n}^{y_n}$. Thus, $\beta_{\upharpoonright \mathcal{E}_n} \in \mathcal{O}_n$. This holds for any $n \in [1 \ldots N]$. Then, by Claim 1, $\beta \in \mathcal{O}$. Finally, $y \in \Phi_{\mathfrak{P}}^{-1}(\mathcal{O}) = \mathcal{U}$. $\diamond$ Claim 3

Thus, $\mathcal{V}$ is an open neighbourhood around $\mathbf{x}$ (in the product topology), which is contained in $\mathcal{U}$. We can construct such a neighbourhood around any $\mathbf{x} \in \mathcal{U}$. Thus, $\mathcal{U}$ is open in the product topology. Hence $\Phi_{\mathfrak{P}}$ is continuous, as claimed. $\square$

Consider any regular partition $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ of $\mathcal{S}$ with $N \geq 2$. Let $\Phi_{\mathfrak{P}}$ be the mapping from Lemma A2. We then define a preference order $\succeq_{\mathfrak{P}}$ on $\mathcal{X}^N$ in the following way: For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succeq_{\mathcal{S}} \Phi_{\mathfrak{P}}(\mathbf{y})\Big). \tag{A1}$$

Proposition A3 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *satisfies axioms* (Sep), (C), (TC), (PC), (Dom) *and* (CEq). *Then, there exists a credence* μ *on* $\mathfrak{B}$ *with full support and a continuous function* $u : \mathcal{X}\longrightarrow\mathbb{R}$ *such that, for any* $\mathfrak{B}$*-partition* $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ *of* $\mathcal{S}$ *with* $N \geq 2$*, we have*

$$\left(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\right) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(x_n) \geq \sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(y_n)\right), \tag{A2}$$

where $\succeq_{\mathfrak{P}}$ *is defined by formula* (A1)*. Moreover,* μ *is unique, and* u *is unique up to positive affine transformation. Finally,* u *is an ordinal utility function for* $\succeq_{\text{xp}}$*. If* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *also satisfies axiom* (M)*, then* u *is* $\mathfrak{D}$*-measurable.*

Proof. Fix a $\mathfrak{B}$-partition $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ of $\mathcal{S}$ with $N \geq 2$ (such a partition exists because $\mathfrak{B}$ is nontrivial). Define $\succeq_{\mathfrak{P}}$ on $\mathcal{X}^N$ according to formula (A1).

Claim 1: $\succeq_{\mathfrak{P}}$ *is continuous with respect to the product topology on* $\mathcal{X}^N$*.*

Proof. Fix $\mathbf{y} \in \mathcal{X}^N$ and define $\beta := \Phi_{\mathfrak{P}}(\mathbf{y}) \in \mathcal{A}$. Let $\mathcal{O} = \{\alpha \in \mathcal{A},\ \alpha \succ_{\mathcal{S}} \beta\}$; this is an open set in the $\succeq_{\mathcal{S}}$-order topology on $\mathcal{A}$. Then, by Lemma A2, $\mathcal{U} := \Phi_{\mathfrak{P}}^{-1}(\mathcal{O})$ is an open subset of $\mathcal{X}^N$ in the product topology. Moreover, for any $\mathbf{x} \in \mathcal{X}^N$, we have

$$\left(\mathbf{x} \succ_{\mathfrak{P}} \mathbf{y}\right) \quad \underset{(*)}{\Longleftrightarrow} \quad \left(\Phi_{\mathfrak{P}}(\mathbf{x}) \succ_{\mathcal{S}} \beta\right) \quad \underset{(\dagger)}{\Longleftrightarrow} \quad \left(\mathbf{x} \in \mathcal{U}\right),$$

where $(*)$ is by formula (A1) and $(\dagger)$ is by the definition of $\mathcal{U}$. Thus, the strict upper contour set of $\succeq_{\mathfrak{P}}$ at $\mathbf{y}$ is equal to $\mathcal{U}$ and, therefore, an open set in the product topology on $\mathcal{X}^N$. A similar proof works for strict lower contour sets. $\diamond$ `Claim 1`

Claim 2: $\succeq_{\mathfrak{P}}$ *satisfies* Cardinal Coordinate Independence*: For all* $n, m \in [1 \ldots N]$*, all* $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w} \in \mathcal{X}^N$ *and all* $a, b, c, d \in \mathcal{X}$*, if* $a_n\mathbf{x}_{-n} \preceq_{\mathfrak{P}} b_n\mathbf{y}_{-n}$*,* $c_n\mathbf{x}_{-n} \succeq_{\mathfrak{P}} d_n\mathbf{y}_{-n}$ *and* $a_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} b_m\mathbf{w}_{-m}$*, then* $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$*.*

Proof. Define $(a_{\mathcal{B}_n}\alpha), (b_{\mathcal{B}_n}\beta), (c_{\mathcal{B}_n}\alpha), (d_{\mathcal{B}_n}\beta) \in \mathcal{A}$ by $(a_{\mathcal{B}_n}\alpha) := \Phi_{\mathfrak{P}}(a_n\mathbf{x}_{-n})$, $(b_{\mathcal{B}_n}\beta) := \Phi_{\mathfrak{P}}(b_n\mathbf{y}_{-n})$, $(c_{\mathcal{B}_n}\alpha) := \Phi_{\mathfrak{P}}(c_n\mathbf{x}_{-n})$ and $(d_{\mathcal{B}_n}\beta) := \Phi_{\mathfrak{P}}(d_n\mathbf{y}_{-n})$. Then, by the definition (A1) of $\succeq_{\mathfrak{P}}$, we have $(a_{\mathcal{B}_n}\alpha) \preceq_{\mathcal{S}} (b_{\mathcal{B}_n}\beta)$ and $(c_{\mathcal{B}_n}\alpha) \succeq_{\mathcal{S}} (d_{\mathcal{B}_n}\beta)$. Moreover, by the definition of $\Phi_{\mathfrak{P}}$, we have $(a_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^a_{\mathcal{B}_n}$, $(b_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^b_{\mathcal{B}_n}$, $(c_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^c_{\mathcal{B}_n}$ and $(d_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^d_{\mathcal{B}_n}$. Meanwhile, $(a_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_l} \approx_{\mathcal{B}_l} (c_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_l}$ and $(b_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_l} \approx_{\mathcal{B}_l} (d_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_l}$ for all $l \in [1 \ldots N]$ with $l \neq n$. So if we set $\mathcal{Q} := \neg\mathcal{B}_n$, then by (Sep) we obtain $(a_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} (c_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{Q}}$ and $(b_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} (d_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{Q}}$. This shows that $(a \overset{\mathcal{B}_n}{\rightsquigarrow} b) \succeq (c \overset{\mathcal{B}_n}{\rightsquigarrow} d)$.

Meanwhile, define $(a_{\mathcal{B}_m}\gamma), (b_{\mathcal{B}_m}\delta), (c_{\mathcal{B}_m}\gamma), (d_{\mathcal{B}_m}\delta) \in \mathcal{A}$ by $(a_{\mathcal{B}_m}\gamma) := \Phi_{\mathfrak{P}}(a_m\mathbf{v}_{-m})$, $(b_{\mathcal{B}_m}\delta) := \Phi_{\mathfrak{P}}(b_m\mathbf{w}_{-m})$, $(c_{\mathcal{B}_m}\gamma) := \Phi_{\mathfrak{P}}(c_m\mathbf{v}_{-m})$ and $(d_{\mathcal{B}_m}\delta) := \Phi_{\mathfrak{P}}(d_m\mathbf{w}_{-m})$. Proceeding as above, we obtain $(a_{\mathcal{B}_m}\gamma) \succeq_{\mathcal{S}} (b_{\mathcal{B}_m}\delta)$ with $(a_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^a_{\mathcal{B}_m}$, $(b_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^b_{\mathcal{B}_m}$, $(c_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^c_{\mathcal{B}_m}$ and $(d_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^d_{\mathcal{B}_m}$. Moreover, set $\mathcal{Q}' := \neg\mathcal{B}_m$. Then we also have $(a_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{Q}'} \approx_{\mathcal{Q}'} (c_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{Q}'}$ and $(b_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{Q}'} \approx_{\mathcal{Q}'} (d_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{Q}'}$. Now, if it is *not* the case that $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$, then (A1) implies that it is also not the case that $(c_{\mathcal{B}_m}\gamma) \succeq_{\mathcal{S}} (d_{\mathcal{B}_m}\delta)$. Thus, $(c_{\mathcal{B}_m}\gamma) \prec_{\mathcal{S}} (d_{\mathcal{B}_m}\delta)$ (because $\succeq_{\mathcal{S}}$ is a complete order). But this means that $(a \overset{\mathcal{B}_m}{\rightsquigarrow} b) \prec (c \overset{\mathcal{B}_m}{\rightsquigarrow} d)$, which contradicts (TC). Thus, we must have $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$, as claimed. $\diamond$ `Claim 2`

By Claims 1 and 2, and the connectedness of $\mathcal{X}$, the assumptions of Wakker's (1988) Theorem 6.2 are satisfied. So there exist a continuous function $u_{\mathfrak{P}} : \mathcal{X} \longrightarrow \mathbb{R}$ and a probability vector $(\mu_{\mathfrak{P}}(\mathcal{B}_1), \ldots, \mu_{\mathfrak{P}}(\mathcal{B}_N)) \in \Delta([1 \ldots N])$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot u_{\mathfrak{P}}(x_n) \geq \sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot u_{\mathfrak{P}}(y_n)\right). \tag{A3}$$

Moreover, the probability vector is unique, and the function is unique up to positive affine transformation.

By the nontriviality of $\succeq_{\mathcal{S}}$ and axiom (Dom), there exist $l, o \in \mathcal{X}$ with $l \succ_{\text{xp}} o$. Then, still by (Dom), $\kappa^l_{\mathcal{B}} \succ_{\mathcal{B}} \kappa^o_{\mathcal{B}}$ for any $\mathcal{B} \in \mathfrak{B}$. Fix $n \in [1 \ldots N]$, and let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$ be such that $x_n = l$, $y_n = o$, and $x_m = y_m = o$ for any $m \in [1 \ldots N] \setminus \{n\}$. Then, $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright \mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^o_{\mathcal{B}_m} \approx_{\mathcal{B}_m} \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright \mathcal{B}_m}$ for any $m \in [1 \ldots N] \setminus \{n\}$. Let $\mathcal{Q} = \neg \mathcal{B}_n$. By iterative applications of (Sep), we have $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright \mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright \mathcal{Q}}$. Moreover, we have $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright \mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^l_{\mathcal{B}_n} \succ_{\mathcal{B}_n} \kappa^o_{\mathcal{B}_n} \approx_{\mathcal{B}_n} \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright \mathcal{B}_n}$ Another application of (Sep) yields $\Phi_{\mathfrak{P}}(\mathbf{x}) \succ_{\mathcal{B}_n} \Phi_{\mathfrak{P}}(\mathbf{y})$ and, by formula (A1), $\mathbf{x} \succ_{\mathfrak{P}} \mathbf{y}$. Then, from (A2) and the definition of $\mathbf{x}$ and $\mathbf{y}$, we get

$$\mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot [u_{\mathfrak{P}}(l) - u_{\mathfrak{P}}(o)] \quad > \quad 0.$$

This inequality first shows that $\mu_{\mathfrak{P}}(\mathcal{B}_n) > 0$, and this holds for any $n \in [1 \ldots N]$. It also shows $u_{\mathfrak{P}}(l) > u_{\mathfrak{P}}(o)$. Thus, we obtain a unique function $u_{\mathfrak{P}}$ providing a representation as in formula (A3) and satisfying $u_{\mathfrak{P}}(l) = 1$ and $u_{\mathfrak{P}}(o) = 0$. From now on, we assume that the functions $u_{\mathfrak{P}}$ are normalized in this way.

Claim 3: *For any two $\mathfrak{B}$-partitions $\mathfrak{P}$ and $\mathfrak{Q}$ of $\mathcal{S}$, each with at least two cells, $u_{\mathfrak{P}} = u_{\mathfrak{Q}}$. Moreover, if $\mathcal{B} \in \mathfrak{B}$ is a cell in each of $\mathfrak{P}$ and $\mathfrak{Q}$, then $\mu_{\mathfrak{P}}(\mathcal{B}) = \mu_{\mathfrak{Q}}(\mathcal{B})$.*

Proof. Let $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ with $\mathcal{P}_n \in \mathfrak{B}$ for all $n \in [1 \ldots N]$, and let $\mathfrak{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_M)$ with $\mathcal{Q}_m \in \mathfrak{B}$ for all $m \in [1 \ldots M]$. Consider first the case where $\mathfrak{Q}$ *refines* $\mathfrak{P}$ —that is, for all $m \in [1 \ldots M]$, there is some $n \in [1 \ldots N]$ such that $\mathcal{Q}_m \subseteq \mathcal{P}_n$. For all $n \in [1 \ldots N]$, let $\mathcal{M}_n \subseteq [1 \ldots M]$ be the set of $m \in [1 \ldots M]$ such that $\mathcal{Q}_m \subseteq \mathcal{P}_n$. Then, for all $n \in [1 \ldots N]$, the subcollection $\{\mathcal{Q}_m,\ m \in M_n\}$ is a $\mathfrak{B}$-partition of $\mathcal{P}_n$. For all $n \in [1 \ldots N]$, we define

$$p_n \ := \ \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m). \tag{A4}$$

Then, the collection $(p_1, \ldots, p_N)$ is a probability vector on $[1 \ldots N]$. Moreover, for any $\mathbf{x} \in \mathcal{X}^N$, define $\mathbf{x}' \in \mathcal{X}^M$ by setting

$$x'_m \ = \ x_n, \qquad \text{for all } m \in \mathcal{M}_n \text{ and } n \in [1 \ldots N]. \tag{A5}$$

Note that by (Sep) we have the following indifference for any $\mathbf{x} \in \mathcal{X}^N$:

$$\Phi_{\mathfrak{P}}(\mathbf{x}) \ \approx_{\mathcal{S}} \ \Phi_{\mathfrak{Q}}(\mathbf{x}') \tag{A6}$$

Thus, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\begin{aligned}
\left(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\right) \;\underset{\text{(a)}}{\iff}\; & \left(\Phi_{\mathfrak{P}}(\mathbf{x}) \ \succeq_{\mathcal{S}} \ \Phi_{\mathfrak{P}}(\mathbf{y})\right) \;\underset{\text{(b)}}{\iff}\; \left(\Phi_{\mathfrak{Q}}(\mathbf{x}') \ \succeq_{\mathcal{S}} \ \Phi_{\mathfrak{Q}}(\mathbf{y}')\right) \\
\underset{\text{(a)}}{\iff}\; & \left(\mathbf{x}' \succeq_{\mathfrak{Q}} \mathbf{y}'\right) \;\underset{\text{(c)}}{\iff}\; \left(\sum_{m=1}^{M} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(x'_m) \geq \sum_{m=1}^{M} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(y'_m)\right) \\
\underset{\text{(d)}}{\iff}\; & \left(\sum_{n=1}^{N} \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(x_n) \geq \sum_{n=1}^{N} \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(y_n)\right) \\
\underset{\text{(e)}}{\iff}\; & \left(\sum_{n=1}^{N} p_n \cdot u_{\mathfrak{Q}}(x_n) \geq \sum_{n=1}^{N} p_n \cdot u_{\mathfrak{Q}}(y_n)\right).
\end{aligned}$$

Here, both (a) are by equation (A1), (b) is by equation (A6), (c) is by equation (A3), (d) by equation (A5), and (e) is by equation (A4). Thus, $u_{\mathfrak{Q}}$ and $(p_1, \ldots, p_N)$ provide a representation of $\succeq_{\mathfrak{P}}$ as in equation (A3). By uniqueness, we obtain $u_{\mathfrak{P}} = u_{\mathfrak{Q}}$. Moreover, for all $n \in [1 \ldots N]$,

$$\mu_{\mathfrak{P}}(\mathcal{P}_n) \ = \ \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m). \tag{A7}$$

Now, if $\mathfrak{P}$ and $\mathfrak{Q}$ have a common cell $\mathcal{B} \in \mathfrak{B}$, then $\mathcal{B} = \mathcal{P}_n = \mathcal{Q}_m$ for some $n \in [1 \ldots N]$ and $m \in [1 \ldots M]$ such that $M_n = \{m\}$. Then, equation (A7) yields $\mu_{\mathfrak{P}}(\mathcal{B}) = \mu_{\mathfrak{Q}}(\mathcal{B})$.

Now consider the general case, where neither $\mathfrak{P}$ nor $\mathfrak{Q}$ refines the other. Let $\mathfrak{P} \otimes \mathfrak{Q} := \{\mathcal{P} \cap \mathcal{Q};\ \mathcal{P} \in \mathfrak{P} \text{ and } \mathcal{Q} \in \mathfrak{Q}\}$. Then $\mathfrak{P} \otimes \mathfrak{Q}$ is a $\mathfrak{B}$-partition which refines both $\mathfrak{P}$ and $\mathfrak{Q}$. Now apply to the previous argument to $\mathfrak{P}$ and $\mathfrak{P} \otimes \mathfrak{Q}$ on the one hand, and to $\mathfrak{Q}$ and $\mathfrak{P} \otimes \mathfrak{Q}$ on the other hand to conclude. $\diamond$ `Claim 3`

Now, we define a set function $\mu : \mathfrak{B} \longrightarrow [0, 1]$ by setting $\mu(\mathcal{S}) = 1$, $\mu(\varnothing) = 0$ and, for any $\mathcal{B} \in \mathfrak{B}$, $\mu(\mathcal{B}) = \mu_{\mathfrak{P}}(\mathcal{B})$ where $\mathfrak{P} = \{\mathcal{B}, \neg\mathcal{B}\}$. Note that, for any nonempty $\mathcal{B} \in \mathfrak{B}$, $\mu(\mathcal{B}) > 0$ since we have already proved that $\mu_{\mathfrak{P}}(\mathcal{B}) > 0$.

Claim 4: *μ is a credence on $\mathfrak{B}$ with full support.*

Proof. Consider a collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of pairwise disjoint regular subsets in $\mathfrak{B}$, and let $\mathcal{B}$ be its join. Consider first the case where $\mathcal{B} = \mathcal{S}$, and set $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$. Then, $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{S}$. We have

$$\sum_{n=1}^{N} \mu(\mathcal{P}_n) \ \underset{(*)}{=} \ \sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{P}_n) \ \underset{(\dagger)}{=} \ 1. \tag{A8}$$

Here $(*)$ is by Claim 3, and $(\dagger)$ is because $\mu_{\mathfrak{P}}$ is a probability distribution. Now, if $\mathcal{B} \neq \mathcal{S}$, set $\mathcal{P}_{N+1} := \neg\mathcal{B}$. Consider $\mathfrak{Q} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N, \mathcal{P}_{N+1}\}$ and $\mathfrak{Q}' = \{\mathcal{B}, \mathcal{P}_{N+1}\}$, two $\mathfrak{B}$-partitions of $\mathcal{S}$. We have

$$\sum_{n=1}^{N} \mu(\mathcal{P}_n) \ \underset{\text{(a)}}{=} \ 1 - \mu_{\mathfrak{Q}}(\mathcal{P}_{N+1}) \ \underset{\text{(b)}}{=} \ 1 - \mu_{\mathfrak{Q}'}(\mathcal{P}_{N+1}) \ \underset{\text{(c)}}{=} \ \mu_{\mathfrak{Q}'}(\mathcal{B}) \ \underset{\text{(d)}}{=} \ \mu(\mathcal{B}). \tag{A9}$$

Here, both (a) and (c) are by suitable versions of equation (A8) while both (b) and (d) are by Claim 3. Thus, μ is a credence. Finally, μ is fully supported since it has positive values at any nonempty regular subset in $\mathfrak{B}$. $\diamondsuit$ `Claim 4`

Set $u := u_{\mathfrak{Q}}$ for some $\mathfrak{B}$-partition $\mathfrak{Q}$ of $\mathcal{S}$ with at least two cells. For any $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of $\mathcal{S}$ with $N \geq 2$, Claim 3 yields $u = u_{\mathfrak{P}}$ and $\mu_{\mathfrak{P}}(\mathcal{P}_n) = \mu(\mathcal{P}_n)$ for all $n \in [1 \ldots N]$. This, together with equation (A3), completes the proof of formula (A2).

Claim 5: *u is an ordinal utility function for $\succeq_{\text{xp}}$.*

Proof. Fix $x, y \in \mathcal{X}$. Since $\mathfrak{B}$ is nondegenerate, there exists a $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{S}$ with $N \geq 2$. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$ be defined by $x_n := x$ and $y_n := y$ for any $n \in [1 \ldots N]$. Then, we have

$$\begin{aligned}\Big(x \succeq_{\text{xp}} y\Big) \quad &\underset{\text{(a)}}{\Longleftrightarrow} \quad \Big(\kappa^x \succeq_{\mathcal{S}} \kappa^y\Big) \quad \underset{\text{(b)}}{\Longleftrightarrow} \quad \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succeq_{\mathcal{S}} \Phi_{\mathfrak{P}}(\mathbf{y})\Big) \\ &\underset{\text{(c)}}{\Longleftrightarrow} \quad \left(\sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(x) \ \geq \ \sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(y)\right) \\ &\Longleftrightarrow \quad \Big(u(x) \geq u(y)\Big),\end{aligned}$$

where (a) is by the definition of $\succeq_{\text{xp}}$, (b) is because by inductive applications of (Sep) and because $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright \mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^x_{\mathcal{B}_n}$ and $\Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright \mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^y_{\mathcal{B}_n}$, for all $n \in [1 \ldots N]$. Meanwhile, (c) is by formula (A2). $\diamondsuit$ `Claim 5`

Claim 6: *If $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ satisfies axiom* (M)*, then u is $\mathfrak{D}$-measurable.*

Proof. Fix an open interval $\mathcal{O} \subseteq \mathbb{R}$. We must show that $u^{-1}(\mathcal{O}) \in \mathfrak{D}$.

First, suppose $\mathcal{O} = (q, \infty)$ for some $q \in \mathbb{R}$. If $u(x) < q$ for all $x \in \mathcal{X}$, then $u^{-1}(q, \infty) = \emptyset \in \mathfrak{D}$. On the other hand, if $u(x) > q$ for all $x \in \mathcal{X}$, then $u^{-1}(q, \infty) = \mathcal{X} \in \mathfrak{D}$. If neither of these cases holds, then there must exist $x, z \in \mathcal{X}$ with $u(x) \leq q \leq u(z)$. Since u is continuous and $\mathcal{X}$ is connected, the Intermediate Value Theorem yields some $y \in \mathcal{X}$ such that $u(y) = q$. But then $u^{-1}(q, \infty) = \{z \in \mathcal{X};\ \ z \succ_{\text{xp}} y\}$, because, by Claim 5 , u is an ordinal utility representation for $\succeq_{\text{xp}}$. Thus, $u^{-1}(q, \infty)$ is an open upper contour set of $\succeq_{\text{xp}}$, so $u^{-1}(q, \infty) \in \mathfrak{D}$, by axiom (M).

The same argument works if $\mathcal{O} = (-\infty, r)$ for some $r \in \mathbb{R}$. Finally, if $\mathcal{O} = (q, r)$, then $\mathcal{O} = (-\infty, r) \cap (q, \infty)$, so $u^{-1}(\mathcal{O}) = u^{-1}(-\infty, r) \cap u^{-1}(q, \infty)$ is an intersection of two elements of $\mathfrak{D}$, and thus, an element of $\mathfrak{D}$.

Finally, let $\mathcal{B}$ be an arbitrary basic subset of $\mathbb{R}$. Then $\mathcal{B} := (a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N)$ for some $-\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N \leq \infty$. For all $n \in [1 \ldots N]$, let $\mathcal{D}_n := u^{-1}(a_n, b_n)$; then $\mathcal{D}_n \in \mathfrak{D}$ by the previous paragraph, and $u^{-1}(\mathcal{B}) = \bigsqcup_{n=1}^{N} \mathcal{D}_n$. It remains to show that this union is an element of $\mathfrak{D}$.

Claim 6A: $\bigvee_{n=1}^{N} \mathcal{D}_n \subseteq \bigsqcup_{n=1}^{N} \mathcal{D}_n$.

Proof. (by contradiction) Suppose $x \in \left(\bigvee_{n=1}^N \mathcal{D}_n\right) \setminus \left(\bigsqcup_{n=1}^N \mathcal{D}_n\right)$. Now,

$$\bigvee_{n=1}^N \mathcal{D}_n \quad = \quad \mathrm{int}\left[\mathrm{clos}\left(\bigsqcup_{n=1}^N \mathcal{D}_n\right)\right] \quad = \quad \mathrm{int}\left[\bigcup_{n=1}^N \mathrm{clos}(\mathcal{D}_n)\right].$$

Thus, $x \in \mathrm{int}\left[\bigcup_{n=1}^N \mathrm{clos}(\mathcal{D}_n)\right]$, but $x \notin \mathcal{D}_n = \mathrm{int}[\mathrm{clos}(\mathcal{D}_n)]$ for any $n \in [1 \ldots N]$. Thus, if $\mathcal{U}$ is any open neighbourhood around x, then $\mathcal{U}$ overlaps $\bigcup_{n=1}^N \mathrm{clos}(\mathcal{D}_n)$ but $\mathcal{U} \not\subseteq \mathrm{clos}(\mathcal{D}_n)$ for any $n \in [1 \ldots N]$; hence there must be at least two distinct $n, m \in [1 \ldots N]$ such that $\mathcal{U} \cap \mathrm{clos}(\mathcal{D}_n) \neq \emptyset$ and $\mathcal{U} \cap \mathrm{clos}(\mathcal{D}_m) \neq \emptyset$. Define

$$\epsilon \quad := \quad \frac{1}{4} \min_{n \in [1 \ldots N)} (a_{n+1} - b_n).$$

Then $\epsilon > 0$ because $b_n < a_{n+1}$ for all $n \in [1 \ldots N]$, by hypothesis. Let $r := u(x)$, and let $\mathcal{V} := (r - \epsilon, r + \epsilon)$. Then $\mathcal{V}$ is an open neighbourhood around r. Let $\mathcal{U} := u^{-1}(\mathcal{V})$; then $\mathcal{U}$ is an open neighbourhood around x (because u is continuous), so by the previous paragraph there exist distinct $n < m \in [1 \ldots N]$ such that $\mathcal{U} \cap \mathrm{clos}(\mathcal{D}_n) \neq \emptyset$ and $\mathcal{U} \cap \mathrm{clos}(\mathcal{D}_m) \neq \emptyset$. Now, $u(\mathcal{U}) = \mathcal{V}$ (by definition of $\mathcal{U}$), while $u[\mathrm{clos}(\mathcal{D}_n)] \subseteq \mathrm{clos}(a_n, b_n) = [a_n, b_n]$ and $u[\mathrm{clos}(\mathcal{D}_m)] \subseteq \mathrm{clos}(a_m, b_m) = [a_m, b_m]$ (because u is continuous). Thus, we must have $\mathcal{V} \cap [a_n, b_n] \neq \emptyset$ and $\mathcal{V} \cap [a_m, b_m] \neq \emptyset$. But this is impossible, because $\mathcal{V}$ is an interval of length $2\epsilon \leq (a_m - b_n)/2$, by construction.

To avoid the contradiction, we must have $x \in \mathcal{D}_n$ for some $n \in [1 \ldots N]$. This argument applies to all $x \in \bigvee_{n=1}^N \mathcal{D}_n$. Thus, $\bigvee_{n=1}^N \mathcal{D}_n \subseteq \bigsqcup_{n=1}^N \mathcal{D}_n$, as claimed. $\triangledown$ `Claim 6A`

Clearly, $\bigsqcup_{n=1}^N \mathcal{D}_n \subseteq \bigvee_{n=1}^N \mathcal{D}_n$. Together with Claim 6A, this implies that $\bigsqcup_{n=1}^N \mathcal{D}_n = \bigvee_{n=1}^N \mathcal{D}_n$. Thus, it is an element of $\mathfrak{D}$, as desired. $\diamondsuit$ `Claim 6`

This completes the proof. $\square$

Our SEU representation requires one more technical preliminary.

Proposition A4 (Pivato and Vergopoulos, 2017b, Theorem 4.3) *Let $\mathcal{S}$ be any topological space, let $\mathfrak{B}$ be any Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$, and let μ be a credence with full support on $\mathfrak{B}$. There exists a unique, strictly monotonic conditional expectation system $\mathbf{E}$ that is compatible with μ.*

Lemma A5 *Suppose $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ satisfies axiom* (Sep). *For any $\mathcal{B} \in \mathfrak{B}$, consider a $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{B}$. For any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, if $\alpha_{\upharpoonright \mathcal{B}_n} \succeq_{\mathcal{B}_n} \beta_{\upharpoonright \mathcal{B}_n}$ for any $n \in [1 \ldots N]$, then $\alpha \succeq_{\mathcal{B}} \beta$.*

Proof. We proceed by induction. Consider first a subset $\mathcal{B} \in \mathfrak{B}$ and a two-cell partition $\mathfrak{P} = \{\mathcal{B}_1, \mathcal{B}_2\}$ of $\mathcal{B}$. Let $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ be such that $\alpha_{\upharpoonright \mathcal{B}_1} \succeq_{\mathcal{B}_1} \beta_{\upharpoonright \mathcal{B}_1}$ and $\alpha_{\upharpoonright \mathcal{B}_2} \succeq_{\mathcal{B}_2} \beta_{\upharpoonright \mathcal{B}_2}$. By (Rch), there exists $\gamma \in \mathcal{A}(\mathcal{B})$ such that $\gamma_{\upharpoonright \mathcal{B}_1} \approx_{\mathcal{B}_1} \alpha_{\upharpoonright \mathcal{B}_1}$ and $\gamma_{\upharpoonright \mathcal{B}_2} \approx_{\mathcal{B}_2} \beta_{\upharpoonright \mathcal{B}_2}$. Then, we have $\alpha_{\upharpoonright \mathcal{B}_1} \approx_{\mathcal{B}_1} \gamma_{\upharpoonright \mathcal{B}_1}$ and $\alpha_{\upharpoonright \mathcal{B}_2} \succeq_{\mathcal{B}_2} \gamma_{\upharpoonright \mathcal{B}_2}$. By (Sep), we obtain $\alpha \succeq_{\mathcal{B}} \gamma$. Similarly, we have $\gamma_{\upharpoonright \mathcal{B}_1} \succeq_{\mathcal{B}_1} \beta_{\upharpoonright \mathcal{B}_1}$ and $\gamma_{\upharpoonright \mathcal{B}_2} \approx_{\mathcal{B}_2} \beta_{\upharpoonright \mathcal{B}_2}$. Still by (Sep), we obtain $\gamma \succeq_{\mathcal{B}} \beta$. Since $\alpha \succeq_{\mathcal{B}} \gamma$ and $\gamma \succeq_{\mathcal{B}} \beta$, we finally obtain $\alpha \succeq_{\mathcal{B}} \beta$ as desired.

Consider now a subset $\mathcal{B} \in \mathfrak{B}$ and an N-cell partition $\mathfrak{P} = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{B}$ with $N \geq 2$. Let $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ be such that $\alpha_{\upharpoonright \mathcal{B}_n} \succeq_{\mathcal{B}_n} \beta_{\upharpoonright \mathcal{B}_n}$ for any $n \in [1 \ldots N]$. Let $\mathcal{Q} = \mathcal{B}_1 \vee \ldots \vee \mathcal{B}_{N_1}$. By induction, we have $\alpha_{\upharpoonright \mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\upharpoonright \mathcal{Q}}$. But since $\{\mathcal{Q}, \mathcal{B}_N\}$ is a two-cell partition of $\mathcal{B}$, and since we have $\alpha_{\upharpoonright \mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\upharpoonright \mathcal{Q}}$ and $\alpha_{\upharpoonright \mathcal{B}_N} \succeq_{\mathcal{B}_N} \beta_{\upharpoonright \mathcal{B}_N}$, the previous paragraph yields $\alpha \succeq_{\mathcal{B}} \beta$ as desired. $\square$

Proposition A6 *Suppose $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ satisfies axioms* (Sep), (C), (M), (TC), (PC), (Dom) *and* (CEq). *Let μ be the credence and u be the utility function from Proposition A3. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be the unique μ-compatible conditional expectation system from Proposition A4. Then, for any $\mathcal{B} \in \mathfrak{B}$, $\alpha \in \mathcal{A}(\mathcal{B})$ and $x \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{B}} \kappa_{\mathcal{B}}^x$, we have*

$$\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \quad = \quad u(x). \tag{A10}$$

Proof. For any $\mathcal{B} \in \mathfrak{B}$, and any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, we define $\widetilde{\mathbb{E}}_{\mathcal{B}}[g] := \mu[\mathcal{B}]\, \mathbb{E}_{\mathcal{B}}[g]$. Recall that $\mathbb{E}_{\mathcal{B}}[\mathbf{1}] = 1$. Thus, $\widetilde{\mathbb{E}}_{\mathcal{B}}[\mathbf{1}] = \mu[\mathcal{B}]$. Thus, for any $r \in \mathbb{R}$, the linearity of $\mathbb{E}_{\mathcal{B}}$ implies

$$\widetilde{\mathbb{E}}_{\mathcal{B}}[r\mathbf{1}] \quad = \quad r\, \mu[\mathcal{B}]. \tag{A11}$$

Let $\mathcal{B} \in \mathfrak{B}$ and $\alpha \in \mathcal{A}(\mathcal{B})$. Consider first the case where $u \circ \alpha$ is constant over $\mathcal{B}$. Then there exists $y \in \mathcal{X}$ such that $u \circ \alpha(s) = u(y)$ for any $s \in \mathcal{B}$. Then $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] = u(y)$ by the linearity of $\mathbb{E}_{\mathcal{B}}$. On the other hand, by Proposition A3, u is an ordinal utility function for $\succeq_{\text{xp}}$. Therefore, $\alpha(s) \approx_{\text{xp}} y$ for any $s \in \mathcal{B}$. By (Dom), we obtain $\alpha \approx_{\mathcal{B}} \kappa_{\mathcal{B}}^y$ and thus, $\kappa_{\mathcal{B}}^x \approx_{\mathcal{B}} \kappa_{\mathcal{B}}^y$ by transitivity. Still by (Dom) we have $x \approx_{\text{xp}} y$. Thus, $u(x) = u(y)$, because u is an ordinal utility function for $\succeq_{\text{xp}}$. This shows $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] = u(x)$, as desired.

Consider now the case where $u \circ \alpha$ takes at least two different values over $\mathcal{B}$. Let $\beta \in \mathcal{A}$ be such that $\beta_{\upharpoonright \mathcal{B}} = \alpha$. Let $\mathcal{K} := \text{clos}[\beta(\mathcal{S})]$ (i.e. the closure of the image set $\beta(\mathcal{S})$ in $\mathcal{X}$); then $\mathcal{K}$ is compact, because $\beta \in \mathcal{C}_{\text{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}; \mathfrak{D})$. Recall that $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, by Proposition A3. Thus, u is bounded when restricted to $\mathcal{K}$. If we define $\mathcal{U} := u \circ \beta(\mathcal{S})$, then $\mathcal{U}$ is a bounded subset of $\mathbb{R}$ —say, $\mathcal{U} \subseteq [-M, M]$ for some $M \in \mathbb{N}$.

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be large enough that $\frac{1}{N} < \epsilon$. For all $n \in [-MN \ldots MN)$, let $\mathcal{C}_n := (u \circ \beta)^{-1}[\frac{n}{N}, \frac{n+1}{N}]$. Recall that $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-measurable by Proposition A3, while β is $(\mathfrak{B}, \mathfrak{D})$-comeasurable, by the definition of $\mathcal{A}$. Thus, $u \circ \beta$ is $\mathfrak{B}$-comeasurable, by Proposition 5.4(a) of Pivato and Vergopoulos (2017b). Thus if $\mathcal{B}_n := \text{int}(\mathcal{C}_n) \cap \mathcal{B}$, then $\mathcal{B}_n$ is a (possibly empty) element of $\mathfrak{B}$. Let $\mathcal{P}_{-MN} = \mathcal{B}_{-MN}$ and, for any $m \in (-MN \ldots MN)$, define $\mathcal{P}_m := \mathcal{B}_m \cap (\neg \mathcal{B}_{m-1})$. Then $\mathcal{P}_{-MN}, \ldots, \mathcal{P}_{MN-1}$ are disjoint (possibly empty) elements of $\mathfrak{B}$. Let $\mathcal{N} := \{n \in [-NM \ldots NM);\ \ \mathcal{P}_n \neq \emptyset\}$. Finally, define $\mathfrak{P} := \{\mathcal{P}_n\}_{n\in\mathcal{N}}$; then $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{B}$. Since $u \circ \alpha$ takes at least two different values over $\mathcal{B}$, we can take N to be large enough to make sure that $\mathfrak{P}$ has at least two cells.

Claim 1: *For any $n \in \mathcal{N}$, there exist values $x_n, y_n \in \mathcal{X}$ such that*

$$\frac{n}{N} \quad \leq \quad u(x_n) \quad \leq \quad u\circ\alpha(p) \quad \leq \quad u(y_n) \quad \leq \quad \frac{n+1}{N}, \tag{A12}$$

for all $p \in \mathcal{P}_n$.

Proof. Recall that $\mathcal{K} := \mathrm{clos}[\beta(\mathcal{S})]$ is a compact subset of $\mathcal{X}$. Thus, its image $u(\mathcal{K})$ is a compact subset of $\mathbb{R}$, because u is continuous. Thus, the set $\mathcal{W}_n := u(\mathcal{K}) \cap [\frac{n}{N}, \frac{n+1}{N}]$ is compact. Thus, $\underline{w}_n := \min(\mathcal{W}_n)$ and $\overline{w}_n := \max(\mathcal{W}_n)$ are well-defined. Let $x_n \in u^{-1}\{\underline{w}_n\}$ and let $y_n \in u^{-1}\{\overline{w}_n\}$. Thus, $u(x_n) = \underline{w}_n \geq \frac{n}{N}$, while $u(y_n) = \overline{w}_n \leq \frac{n+1}{N}$,

For any $p \in \mathcal{P}_n$, we have $\alpha(p) = \beta(p)$ and $\beta(p) \in \mathcal{K}$, and also $u\circ\alpha(p) \in [\frac{n}{N}, \frac{n+1}{N}]$ by definition of $\mathcal{P}_n$; thus, $u\circ\alpha(p) \in \mathcal{W}_n$. Thus, $\underline{w}_n \leq u\circ\alpha(p) \leq \overline{w}_n$, i.e. $u(x_n) \leq u\circ\alpha(p) \leq u(y_n)$, as claimed. $\diamondsuit$ `Claim 1`

Now define $\underline{u}, \overline{u} \in \mathbb{R}$ in the following way:

$$\underline{u} \quad = \quad \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n)\cdot u(x_n) \quad \text{and} \quad \overline{u} = \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n)\cdot u(y_n).$$

Claim 2: $\underline{u} \quad \leq \quad \widetilde{\mathbb{E}}_{\mathcal{B}}[u\circ\alpha] \quad \leq \quad \overline{u}$

Proof. Fix $n \in \mathcal{N}$. For all $p \in \mathcal{P}_n$, formula (A12) says $u(x_n) \leq u\circ\alpha(p) \leq u(y_n)$, and thus,

$$\begin{aligned} \mu[\mathcal{P}_n]\cdot u(x_n) \quad &\underset{(*)}{=} \quad \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u(x_n)\,\mathbf{1}] \quad \underset{(\dagger)}{\leq} \quad \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u\circ\alpha] \\ &\underset{(\dagger)}{\leq} \quad \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u(y_n)\,\mathbf{1}] \quad \underset{(*)}{=} \quad \mu[\mathcal{P}_n]\cdot u(y_n). \end{aligned} \tag{A13}$$

Here, both $(*)$ are by equation (A11), and both $(\dagger)$ are by inequality (A12) and the monotonicity of the conditional expectation operator $\mathbb{E}_{\mathcal{B}}$. Summing the versions of inequality (A13) obtained for every $n \in \mathcal{N}$, we obtain

$$\underline{u} \quad = \quad \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n)\cdot u(x_n) \quad \leq \quad \sum_{n\in\mathcal{N}} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u\circ\alpha] \quad \leq \quad \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n)\cdot u(y_n) \quad = \quad \overline{u}.$$

The result then follows by noting that the middle term in this inequality is equal to $\widetilde{\mathbb{E}}_{\mathcal{B}}[u\circ\alpha]$ thanks to equation (12) $\diamondsuit$ `Claim 2`

Claim 3: $\underline{u} \leq \mu(\mathcal{B})\cdot u(x) \leq \overline{u}$.

Proof. Fix $o \in \mathcal{X}$. Define $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathcal{X}^{\mathcal{N}}$ in the following way:

$$\text{For any } n \in \mathcal{N}, \quad a'_n \quad := \quad x_n, \quad b'_n \quad := \quad x \quad \text{and} \quad c'_n \quad := \quad y_n.$$

Consider the $\mathfrak{B}$-partition $\mathfrak{Q} = \{\neg\mathcal{B}\} \cup \{\mathcal{P}_n,\ n \in \mathcal{N}\}$ made of $M := 1 + |\mathcal{N}|$ cells (so $M \geq 2$). Define $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^M$ by setting $\mathbf{a} := (\mathbf{a}', o)$, $\mathbf{b} := (\mathbf{b}', o)$ and $\mathbf{c} := (\mathbf{c}', o)$. Finally, let $\Phi_{\mathfrak{Q}}$ be the mapping constructed in Lemma A2.

Now, u is an ordinal utility function for $\succeq_{\text{xp}}$, by Proposition A3. Thus, for all $n \in \mathcal{N}$ and $p \in \mathcal{P}_n$, formula (A12) implies that $x_n \preceq_{\text{xp}} \alpha_{\upharpoonright\mathcal{P}_n}(p) \preceq_{\text{xp}} y_n$. Thus, axiom (Dom) implies that

$$\kappa^{x_n}_{\mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \alpha_{\upharpoonright\mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \kappa^{y_n}_{\mathcal{P}_n}. \tag{A14}$$

Given the defining properties of the mapping $\Phi_{\mathfrak{Q}}$, formula (A14) then implies that

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright\mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \alpha_{\upharpoonright\mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright\mathcal{P}_n}. \tag{A15}$$

Since formula (A15) holds for every $n \in \mathcal{N}$, Lemma A5 further yields

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright\mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \alpha \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright\mathcal{B}}. \tag{A16}$$

Meanwhile, we have $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright\mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa^x_{\mathcal{P}_n}$ for every $n \in \mathcal{N}$. By iterative applications of (Sep), we obtain $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright\mathcal{B}} \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$. But by assumption $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$. Thus, $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright\mathcal{B}} \approx_{\mathcal{B}} \alpha$, by transitivity. Formula (A16) then gives

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright\mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright\mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright\mathcal{B}}. \tag{A17}$$

Moreover, by construction, we have $\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \kappa^o_{\mathcal{Q}}$ where $\mathcal{Q} = \neg\mathcal{B}$. Given this fact and formula (A17), axiom (Sep) implies

$$\Phi_{\mathfrak{Q}}(\mathbf{a}) \quad \preceq_{\mathcal{S}} \quad \Phi_{\mathfrak{Q}}(\mathbf{b}) \quad \preceq_{\mathcal{S}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c}). \tag{A18}$$

By the definition of $\succeq_{\mathfrak{Q}}$ in formula (A1) and its representation obtained in Proposition A3, formula (A18) implies

$$\begin{aligned}
\sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n) \cdot u(a_n) \quad + \quad \mu(\mathcal{Q}) \cdot u(o) \quad &\leq \quad \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n) \cdot u(b_n) \quad + \quad \mu(\mathcal{Q}) \cdot u(o) \\
&\leq \quad \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n) \cdot u(c_n) \quad + \quad \mu(\mathcal{Q}) \cdot u(o),
\end{aligned}$$

which, given the definition of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, reduces to the following formula

$$\underline{u} \quad = \quad \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n) \cdot u(x_n) \quad \leq \quad \mu(\mathcal{B}) \cdot u(x) \quad \leq \quad \sum_{n\in\mathcal{N}} \mu(\mathcal{P}_n) \cdot u(x_n) \quad = \quad \overline{u}.$$

This completes the proof of the claim. ◇ `Claim 3`

Finally, we obtain

$$\begin{aligned}|\widetilde{\mathbb{E}}_{\mathcal{B}}[u\circ\alpha]-\mu(\mathcal{B})\cdot u(x)| \quad &\underset{\text{(a)}}{\leq} \quad |\overline{u}-\underline{u}| \;\underset{\text{(b)}}{\leq}\; \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot|u(y_n)-u(x_n)| \\ &\underset{\text{(c)}}{\leq} \quad \epsilon\cdot\sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n) \;\underset{\text{(d)}}{\leq}\; \epsilon\cdot\mu(\mathcal{B}).\end{aligned}$$

Here, (a) is by Claims 2 and 3, (b) is by the definition of $\underline{u}$ and $\overline{u}$, (c) is inequality (A12), because $1/N<\epsilon$ by definition, and (d) is because μ is a credence on $\mathfrak{B}$ and $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{B}$. This argument works for all $\epsilon>0$. Letting $\epsilon\to 0$, we conclude that $\widetilde{\mathbb{E}}_{\mathcal{B}}[u\circ\alpha]=\mu(\mathcal{B})\cdot u(x)$. Last, since μ‘ has full support, we obtain $\mathbb{E}_{\mathcal{B}}[u\circ\alpha]=u(x)$. $\square$

Finally, we come to the proof of the main result.

Proof of Theorem 9.

SEU representation. Let $u:\mathcal{X}\longrightarrow\mathbb{R}$ be the normalized *ex post* utility function and let μ be the credence with full support from Proposition A3. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique μ-compatible conditional expectation system from Proposition A4.

For any $\mathcal{B}\in\mathfrak{B}$ and $\alpha,\beta\in\mathcal{A}(\mathcal{B})$, axiom (CEq) yields $x,y\in\mathcal{X}$ such that $\alpha\approx_{\mathcal{B}}\kappa^x_{\mathcal{B}}$ and $\beta\approx_{\mathcal{B}}\kappa^y_{\mathcal{B}}$. Then,

$$\begin{aligned}\left(\alpha\succeq_{\mathcal{B}}\beta\right) \quad &\iff \quad \left(\kappa^x_{\mathcal{B}}\succeq_{\mathcal{B}}\kappa^y_{\mathcal{B}}\right) \underset{(*)}{\iff} \left(x\succeq_{\mathrm{xp}} y\right) \\ &\underset{(\dagger)}{\iff} \quad \left(u(x)\geq u(y)\right) \underset{(\diamond)}{\iff} \left(\mathbb{E}_{\mathcal{B}}[u\circ\alpha]\geq\mathbb{E}_{\mathcal{B}}[u\circ\beta].\right)\end{aligned}$$

Here, $(*)$ is by axiom (Dom), $(\dagger)$ is because u is an ordinal utility function for $\succeq_{\mathrm{xp}}$ by Proposition A3, and $(\diamond)$ is by Proposition A6. This equivalence establishes the SEU representation. It remains to show that the representation is unique and demonstrate the necessity of the axioms.

Uniqueness. Let $u,u':\mathcal{X}\longrightarrow\mathbb{R}$ be two continuous and $\mathfrak{D}$-measurable functions, and let $\mathbf{E}:=\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{R}\in\mathfrak{B}}$ and $\mathbf{E}':=\{\mathbb{E}'_{\mathcal{B}}\}_{\mathcal{R}\in\mathfrak{B}}$ be two conditional expectation systems. Let μ and μ' be two credences on $\mathfrak{B}$ with which $\mathbf{E}$ and $\mathbf{E}'$ are respectively compatible. Suppose that (u,μ) and (u',μ') are both SEU representations for the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$. We must show that $\mu=\mu'$ and u is a positive affine transformation of u'.

Let $\mathfrak{E}=\{\mathcal{E}_1,\ldots,\mathcal{E}_N\}$ be a $\mathfrak{B}$-partition of $\mathcal{S}$, with $N\geq 2$ (such a partition exists because $\mathfrak{B}$ is nontrivial). For any $\mathbf{x}\in\mathcal{X}^N$, Lemma A2 yields an act $\alpha^{\mathbf{x}}\in\mathcal{A}$ such that $\alpha^{\mathbf{x}}_{\restriction\mathcal{B}_n}\approx_{\mathcal{B}_n}\kappa^{x_n}_{\mathcal{E}_n}$ for all $n\in[1\ldots N]$. Then we have

$$\begin{aligned}\mathbb{E}_{\mathcal{S}}[u\circ\alpha^{\mathbf{x}}] \;&\underset{(*)}{=}\; \sum_{n=1}^{N}\mu(\mathcal{E}_n)\,\mathbb{E}_{\mathcal{E}_n}[u\circ\alpha^{\mathbf{x}}] \\ &\underset{(\dagger)}{=}\; \sum_{n=1}^{N}\mu(\mathcal{E}_n)\,\mathbb{E}_{\mathcal{E}_n}[u\circ\kappa^{x_n}_{\mathcal{E}_n}] \;=\; \sum_{n=1}^{N}\mu(\mathcal{E}_n)\;u(x_n).\end{aligned}\tag{A19}$$

Here, $(*)$ is by equation (12), while $(\dagger)$ is by the SEU representation and the fact that $\alpha^{\mathbf{x}}_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^{x_n}_{\mathcal{B}_n}$ for all $n \in [1 \ldots N]$. Define $\succeq_{\mathfrak{E}}$ as in equation (A1). By equation (A19), we have for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$

$$\Big(\mathbf{x} \succeq_{\mathfrak{E}} \mathbf{y}\Big) \quad \iff \quad \left(\sum_{n=1}^{N} \mu(\mathcal{E}_n)\ u(x_n) \geq \sum_{n=1}^{N} \mu(\mathcal{E}_n)\ u(y_n)\right).$$

The SEU representation (u', μ') provides similarly the following representation: for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$

$$\Big(\mathbf{x} \succeq_{\mathfrak{E}} \mathbf{y}\Big) \quad \iff \quad \left(\sum_{n=1}^{N} \mu'(\mathcal{E}_n)\ u'(x_n) \geq \sum_{n=1}^{N} \mu'(\mathcal{E}_n)\ u'(y_n)\right).$$

Now, by the uniqueness part of Proposition A3, we obtain that μ and μ' are equal to each other, and that u and u' are positive affine transformation of each other.

Necessity of the axioms. Assume that $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies condition (Rch) and has an SEU representation in the sense of Theorem 9 with respect to a continuous and $\mathfrak{D}$-measurable utility function u and a credence μ with full support. Let $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique, strictly monotonic conditional expectation system defined by μ via Proposition A4. Axiom (Dom) is a simple consequence of the strict monotonicity of each expectation functional in $\mathbf{E}$. Axiom (Sep) follows from the fact that $\mathbf{E}$ satisfies Equation (12). The proofs of the other axioms are somewhat more involved.

Axiom (TC): Fix two disjoint subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, and let $\mathcal{Q}_1 = \neg\mathcal{B}_1$ and $\mathcal{Q}_2 = \neg\mathcal{B}_2$. Fix $x, y, v, w \in \mathcal{X}$. By contradiction, assume that $(x \overset{\mathcal{B}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}_1}{\rightsquigarrow} w)$ but $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$. Then, since $(x \overset{\mathcal{B}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}_1}{\rightsquigarrow} w)$, there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q}_1)$, an (x, α)-bet $(x_{\mathcal{B}_1}\alpha) \in \mathcal{A}$, a (y, β)-bet $(y_{\mathcal{B}_1}\beta) \in \mathcal{A}$, a (v, α)-bet $(v_{\mathcal{B}_1}\alpha) \in \mathcal{A}$ and a (w, β)-bet $(w_{\mathcal{B}_1}\beta) \in \mathcal{A}$ such that $(x_{\mathcal{B}_1}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}_1}\beta)$ while $(v_{\mathcal{B}_1}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}_1}\beta)$. We now show that $u(x) - u(y) \leq u(v) - u(w)$. Indeed, first we have

$$\begin{aligned}
&\mu(\mathcal{B}_1)\Big(u(x) - u(y)\Big) \\
&\underset{(*)}{=} \ \mathbb{E}_{\mathcal{S}}[u \circ (x_{\mathcal{B}_1}\alpha)] - \mu(\mathcal{Q}_1)\mathbb{E}_{\mathcal{Q}_1}[u \circ (x_{\mathcal{B}_1}\alpha)] - \mathbb{E}_{\mathcal{S}}[u \circ (y_{\mathcal{B}_1}\beta)] + \mu(\mathcal{Q}_1)\mathbb{E}_{\mathcal{Q}_1}[u \circ (y_{\mathcal{B}_1}\beta)] \\
&\underset{(\dagger)}{\leq} \ \mu(\mathcal{Q}_1)\Big(\mathbb{E}_{\mathcal{Q}_1}[u \circ (y_{\mathcal{B}_1}\beta)] - \mathbb{E}_{\mathcal{Q}_1}[u \circ (x_{\mathcal{B}_1}\alpha)]\Big).
\end{aligned} \tag{A20}$$

Here, $(*)$ is by formula (B1) in the definition of bets, equation (12) and the SEU representation. Meanwhile, $(\dagger)$ is because $\mathbb{E}_{\mathcal{S}}[u \circ (x_{\mathcal{B}_1}\alpha)] \leq \mathbb{E}_{\mathcal{S}}[u \circ (y_{\mathcal{B}_1}\beta)]$ because $(x_{\mathcal{B}_1}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}_1}\beta)$. Proceeding similarly for $(v_{\mathcal{B}_1}\alpha)$ and $(w_{\mathcal{B}_1}\beta)$, we obtain

$$\mu(\mathcal{B}_1)\Big(u(v) - u(w)\Big) \quad \geq \quad \mu(\mathcal{Q}_1)\Big(\mathbb{E}_{\mathcal{Q}_1}[u \circ (w_{\mathcal{B}_1}\beta)] - \mathbb{E}_{\mathcal{Q}_1}[u \circ (v_{\mathcal{B}_1}\alpha)]\Big). \tag{A21}$$

Meanwhile, by formula (B2) in the definition of bets and the SEU representation, we have $\mathbb{E}_{\mathcal{Q}_1}[u \circ (x_{\mathcal{B}_1}\alpha)] = \mathbb{E}_{\mathcal{Q}_1}[u \circ (v_{\mathcal{B}_1}\alpha)]$ and $\mathbb{E}_{\mathcal{Q}_1}[u \circ (y_{\mathcal{B}_1}\beta)] = \mathbb{E}_{\mathcal{Q}_1}[u \circ (w_{\mathcal{B}_1}\beta)]$. Combining inequalities (A20) and (A21) and using the fact that μ has full support, we obtain

$$u(x) - u(y) \quad \leq \quad u(v) - u(w). \tag{A22}$$

Now, since $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$, there exist $\gamma, \delta \in \mathcal{A}(\mathcal{Q}_2)$, an (x,γ)-bet $(x_{\mathcal{B}_2}\gamma) \in \mathcal{A}$, a (y,δ)-bet $(y_{\mathcal{B}_2}\delta) \in \mathcal{A}$, a (v,γ)-bet $(v_{\mathcal{B}_2}\gamma) \in \mathcal{A}$ and a (w,δ)-bet $(w_{\mathcal{B}_2}\delta) \in \mathcal{A}$ such that $(x_{\mathcal{B}_2}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{B}_2}\delta)$ while $(v_{\mathcal{B}_2}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{B}_2}\delta)$. Thus,

$$\begin{aligned} \mu(\mathcal{B}_2)\Big(u(x) - u(y)\Big) \quad & \underset{(*)}{\geq} \quad \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u \circ (y_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u \circ (x_{\mathcal{B}_2}\gamma)]\Big) \\ & \underset{(\dagger)}{=} \quad \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u \circ (w_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u \circ (v_{\mathcal{B}_2}\gamma)]\Big), \end{aligned} \qquad \text{(A23)}$$

where $(*)$ is obtained like inequality (A20), while $(\dagger)$ is by formula (B2) in the definition of bets and the SEU representation. Combining inequalities (A22) and (A23), we get

$$\mu(\mathcal{B}_2)\Big(u(v) - u(w)\Big) \quad \geq \quad \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u \circ (w_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u \circ (v_{\mathcal{B}_2}\gamma)]\Big). \qquad \text{(A24)}$$

Finally, applying equation (12) and the SEU representation to inequality (A24), we obtain $(v_{\mathcal{B}_2}\gamma) \succeq_{\mathcal{S}} (w_{\mathcal{B}_2}\delta)$. But this contradicts the fact that $(v_{\mathcal{B}_2}\gamma) \prec_S (w_{\mathcal{B}_2}\delta)$.

Axioms (C) and (M): Let $x \in \mathcal{X}$. Let $(x, \rightarrow)_{\succeq_{\rm xp}} := \{y \in \mathcal{X};\ \ y \succ_{\rm xp} x\}$ and let $(\leftarrow, x)_{\succeq_{\rm xp}} := \{y \in \mathcal{X};\ \ y \prec_{\rm xp} x\}$. To verify axiom (C), we must show that these sets are open in $\mathcal{X}$. To verify axiom (M), we must show that they are elements of $\mathfrak{D}$. To verify both, let $r := u(x)$, and observe that $(x, \rightarrow)_{\succeq_{\rm xp}} = u^{-1}(r, \infty)$ and $(\leftarrow, x)_{\succeq_{\rm xp}} = u^{-1}(-\infty, r)$, because u is an ordinal utility representation for $\succeq_{\rm xp}$. Since u is continuous, these preimage sets are open in $\mathcal{X}$. Since u is $\mathfrak{D}$-measurable, these preimage sets are elements of $\mathfrak{D}$.

Axiom (CEq): Let $\mathcal{B} \in \mathfrak{B}$ and let $\alpha \in \mathcal{A}(\mathcal{B})$. Then $\alpha = \alpha'_{\upharpoonright\mathcal{B}}$ for some $\alpha' \in \mathcal{A}$.

Claim 1: clos$[\alpha'(\mathcal{S})]$ *has a* $\succeq_{\rm xp}$*-maximal element and a* $\succeq_{\rm xp}$*-minimal element.*[24]

Proof. (By contradiction) Suppose clos$[\alpha'(\mathcal{S})]$ had no $\succeq_{\rm xp}$-maximal element. Thus, for any $x \in$ clos$[\alpha'(\mathcal{S})]$, there exists some $y \in$ clos$[\alpha'(\mathcal{S})]$ with $y \succ_{\rm xp} x$. In other words, $x \in (\leftarrow, y)_{\succeq_{\rm xp}}$. Thus, the collection $\{(\leftarrow, y)_{\succeq_{\rm xp}};\ \ y \in \text{clos}[\alpha'(\mathcal{S})]\}$ is an open cover for clos$[\alpha'(\mathcal{S})]$.

However, $\alpha' \in \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$, so its image $\alpha'(\mathcal{S})$ is relatively compact; hence clos$[\alpha'(\mathcal{S})]$ is compact. Thus, this open cover has a finite subcover; in other words, there exists some $y_1, \ldots, y_N \in$ clos$[\alpha'(\mathcal{S})]$ such that clos$[\alpha'(\mathcal{S})]$ is covered by the collection $\{(\leftarrow, y_n)_{\succeq_{\rm xp}}\}_{n=1}^N$. Now, let $\overline{y} := \max_{\succeq_{\rm xp}}\{y_1, \ldots, y_N\}$ (this maximum exists because the set is finite). Then $\overline{y} \in$ clos$[\alpha'(\mathcal{S})]$, and $(\leftarrow, y_n)_{\succeq_{\rm xp}} \subseteq (\leftarrow, \overline{y})_{\succeq_{\rm xp}}$ for all $n \in [1 \ldots N]$. Thus, clos$[\alpha'(\mathcal{S})] \subseteq (\leftarrow, \overline{y})_{\succeq_{\rm xp}}$. But clearly, $\overline{y} \notin (\leftarrow, \overline{y})_{\succeq_{\rm xp}}$, whereas $\overline{y} \in$ clos$[\alpha'(\mathcal{S})]$. Contradiction.

To avoid the contradiction, clos$[\alpha'(\mathcal{S})]$ must have a $\succeq_{\rm xp}$-maximal element. The proof for $\succeq_{\rm xp}$-minimal elements is analogous. ◇ `Claim 1`

[24] Actually, we only need to obtain an upper and lower bound for clos$[\alpha'(\mathcal{S})]$ in $\mathcal{X}$. But constructing a maximum and minimum is no more difficult.

Let x be a $\succeq_{\text{xp}}$-minimal element of $\text{clos}[\alpha'(\mathcal{S})]$, and let z be a $\succeq_{\text{xp}}$-maximal element of $\text{clos}[\alpha'(\mathcal{S})]$; these exist by Claim 1. Then $x \preceq_{\text{xp}} \alpha(b) \preceq_{\text{xp}} z$ for all $b \in \mathcal{B}$. Thus, axiom (Dom) implies that $\kappa^x_{\mathcal{B}} \preceq_{\mathcal{B}} \alpha \preceq_{\mathcal{B}} \kappa^z_{\mathcal{B}}$. Thus,

$$u(x) \quad = \quad \mathbb{E}_{\mathcal{B}}[u \circ \kappa^x_{\mathcal{B}}] \quad \underset{(*)}{\leq} \quad \mathbb{E}_{\mathcal{B}}[u \circ \alpha] \quad \underset{(*)}{\leq} \quad \mathbb{E}_{\mathcal{B}}[u \circ \kappa^z_{\mathcal{B}}] \quad = \quad u(z),$$

where both $(*)$ are because of the assumed SEU representation. However, $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, and $\mathcal{X}$ is connected. Thus, the Intermediate Value Theorem yields some $y \in \mathcal{X}$ such that $u(y) = \mathbb{E}_{\mathcal{B}}[u \circ \alpha]$. Thus, $\mathbb{E}_{\mathcal{B}}[u \circ \kappa^y_{\mathcal{B}}] = \mathbb{E}_{\mathcal{B}}[u \circ \alpha]$. But then the assumed SEU representation yields $\kappa^y_{\mathcal{B}} \approx_{\mathcal{B}} \alpha$, as desired.

Axiom (PC): Let $\mathcal{D} \in \mathfrak{B}$ and $\mathcal{E} \in \mathfrak{B}$ be disjoint, and let $\mathcal{G} := \mathcal{D} \vee \mathcal{E}$. Let $\mathcal{O} \subseteq \mathcal{A}(\mathcal{G})$ be open in the $\succeq_{\mathcal{G}}$-order topology, and let $\beta \in \mathcal{O}$. Thus, there exist some $\alpha, \gamma \in \mathcal{A}(\mathcal{G})$ such that $\alpha \prec_{\mathcal{G}} \beta \prec_{\mathcal{G}} \gamma$, and $\mathcal{O}$ contains the order-interval $(\alpha, \gamma)_{\succeq_{\mathcal{G}}}$. Let $a := \mathbb{E}_{\mathcal{G}}[u \circ \alpha]$, $b := \mathbb{E}_{\mathcal{G}}[u \circ \beta]$, and $c := \mathbb{E}_{\mathcal{G}}[u \circ \gamma]$; then $a < b < c$. Let $\epsilon := \min\{b - a, c - b\}$. Then $\epsilon > 0$.

Claim 2: *There exist a subset $\mathcal{O}_{\mathcal{D}} \subseteq \mathcal{A}(\mathcal{D})$, open in the $\succeq_{\mathcal{D}}$-order topology, such that $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$, and such that $|\mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] - \mathbb{E}_{\mathcal{D}}[u \circ \beta_{\upharpoonright\mathcal{D}}]| < \epsilon$ for all $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$.*

Proof. (*Case 1*) First, suppose that $\beta_{\upharpoonright\mathcal{D}}$ is neither $\succeq_{\mathcal{D}}$-maximal nor $\succeq_{\mathcal{D}}$-minimal in $\mathcal{A}(\mathcal{D})$. Then there exists some $\phi_{\mathcal{D}}, \psi_{\mathcal{D}} \in \mathcal{A}(\mathcal{D})$ such that $\phi_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \psi_{\mathcal{D}}$. Now, $\phi_{\mathcal{D}} := \phi'_{\upharpoonright\mathcal{D}}$ and $\psi_{\mathcal{D}} := \psi'_{\upharpoonright\mathcal{D}}$ for some $\phi, \psi \in \mathcal{A}$. Let w be a $\succeq_{\text{xp}}$-minimal element of $\text{clos}[\phi'(\mathcal{S})]$, and let z be a $\succeq_{\text{xp}}$-maximal element of $\text{clos}[\psi'(\mathcal{S})]$; these exist by Claim 1. Then $w \preceq_{\text{xp}} \phi_{\mathcal{D}}(d)$ and $\psi_{\mathcal{D}}(d) \preceq_{\text{xp}} z$ for all $d \in \mathcal{D}$. Thus,

$$\kappa^w_{\mathcal{D}} \quad \preceq_{\mathcal{D}} \quad \phi_{\mathcal{D}} \quad \prec_{\mathcal{D}} \quad \beta_{\mathcal{D}} \quad \prec_{\mathcal{D}} \quad \psi_{\mathcal{D}} \quad \preceq_{\mathcal{D}} \quad \kappa^z_{\mathcal{D}},$$

where the "$\preceq_{\mathcal{D}}$" comparisons are by axiom (Dom), and the "$\prec_{\mathcal{D}}$" comparisons are by the definitions of $\phi_{\mathcal{D}}$ and $\psi_{\mathcal{D}}$. Thus,

$$u(w) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa^w_{\mathcal{D}}] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa^z_{\mathcal{D}}] \quad = \quad u(z),$$

where both $(*)$ are because of the assumed SEU representation. Thus, $u(w) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] < u(z)$. Now, u is continuous, and $\mathcal{X}$ is connected. Thus, the Intermediate Value Theorem yields $x, y \in \mathcal{X}$ such that $\mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon < u(x) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] < u(y) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon$. (It is even possible that w and z themselves already satisfy these inequalities). Thus,

$$\mathbb{E}_{\mathcal{D}}[u \circ \kappa^x_{\mathcal{D}}] \quad = \quad u(x) \quad < \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] \quad < \quad u(y) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa^y_{\mathcal{D}}],$$

so $\kappa^x_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \kappa^y_{\mathcal{D}}$, by the assumed SEU representation. Thus, if we define $\mathcal{O}_{\mathcal{D}} := (\kappa^x_{\mathcal{D}}, \kappa^y_{\mathcal{D}})_{\succeq_{\mathcal{D}}}$, then $\mathcal{O}_{\mathcal{D}}$ is open in the $\succeq_{\mathcal{D}}$-order topology, and $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$. Furthermore, for any $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$, we have $\kappa^x_{\mathcal{D}} \prec_{\mathcal{D}} \omega_{\mathcal{D}} \prec_{\mathcal{D}} \kappa^y_{\mathcal{D}}$, and thus,

$$\begin{aligned} \mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon \quad \underset{(\diamond)}{<} \quad u(x) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa^x_{\mathcal{D}}] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] \\ \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa^y_{\mathcal{D}}] \quad = \quad u(y) \quad \underset{(\diamond)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon, \end{aligned}$$

so that $|\mathbb{E}_\mathcal{D}[u \circ \omega_\mathcal{D}] - \mathbb{E}_\mathcal{D}[u \circ \beta_{\upharpoonright\mathcal{D}}]| < \epsilon$, as desired. Here, the $(*)$ inequalities are by the assumed SEU representation, and $(\diamond)$ inequalities are by the definitions of x and y.

(*Case 2*) Suppose $\beta_{\upharpoonright\mathcal{D}}$ is $\succeq_\mathcal{D}$-maximal in $\mathcal{A}(\mathcal{D})$, but not $\succeq_\mathcal{D}$-minimal. The logic is similar to *Case 1*, so we will be more cursory. There exists some $\phi_\mathcal{D} \in \mathcal{A}(\mathcal{D})$ such that $\phi_\mathcal{D} \prec_\mathcal{D} \beta_{\upharpoonright\mathcal{D}}$. As in Case 1, use Claim 1 to obtain some $w \in \mathcal{X}$ such that $w \preceq_{\text{xp}} \phi_\mathcal{D}(d)$ for all $d \in \mathcal{D}$. Thus, $\kappa^w_\mathcal{D} \preceq_\mathcal{D} \phi_\mathcal{D} \prec_\mathcal{D} \beta_{\upharpoonright\mathcal{D}}$, and thus, $u(w) = \mathbb{E}_\mathcal{D}[u \circ \kappa^w_\mathcal{D}] < \mathbb{E}_\mathcal{D}[u \circ \beta]$.

Now, u is continuous, and $\mathcal{X}$ is connected, so the Intermediate Value Theorem yields $x \in \mathcal{X}$ such that $\mathbb{E}_\mathcal{D}[u \circ \beta] - \epsilon < u(x) < \mathbb{E}_\mathcal{D}[u \circ \beta]$. Thus, $\mathbb{E}_\mathcal{D}[u \circ \kappa^x_\mathcal{D}] = u(x) < \mathbb{E}_\mathcal{D}[u \circ \beta]$, so $\kappa^x_\mathcal{D} \prec_\mathcal{D} \beta_{\upharpoonright\mathcal{D}}$. Thus, if we define $\mathcal{O}_\mathcal{D} := (\kappa^x_\mathcal{D}, \rightarrow)_{\succeq_\mathcal{D}}$, then $\mathcal{O}_\mathcal{D}$ is open in the $\succeq_\mathcal{D}$-order topology, and $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_\mathcal{D}$; in fact, $\beta_{\upharpoonright\mathcal{D}}$ is a $\succeq_\mathcal{D}$-maximal element of $\mathcal{O}_\mathcal{D}$. Thus, for any $\omega_\mathcal{D} \in \mathcal{O}_\mathcal{D}$, we have $\kappa^x_\mathcal{D} \prec_\mathcal{D} \omega_\mathcal{D} \preceq_\mathcal{D} \beta_{\upharpoonright\mathcal{D}}$, and thus,

$$\mathbb{E}_\mathcal{D}[u \circ \beta] - \epsilon \underset{(\diamond)}{<} u(x) = \mathbb{E}_\mathcal{D}[u \circ \kappa^x_\mathcal{D}] \underset{(*)}{<} \mathbb{E}_\mathcal{D}[u \circ \omega_\mathcal{D}] \underset{(*)}{\leq} \mathbb{E}_\mathcal{D}[u \circ \beta] < \mathbb{E}_\mathcal{D}[u \circ \beta] + \epsilon,$$

as desired. Here, the $(*)$ inequalities are by the assumed SEU representation, and the $(\diamond)$ inequality is by the definition of x.

(*Case 3*) Suppose $\beta_\mathcal{D}$ is $\succeq_\mathcal{D}$-minimal in $\mathcal{A}(\mathcal{D})$, but not $\succeq_\mathcal{D}$-maximal. The logic is exactly the same as *Case 2*, but with all the preferences and inequalities reversed.

(*Case 4*) Suppose $\beta_\mathcal{D}$ is both $\succeq_\mathcal{D}$-minimal and $\succeq_\mathcal{D}$-maximal in $\mathcal{A}(\mathcal{D})$. In this case, $\mathbb{E}_\mathcal{D}[u \circ \omega] = \mathbb{E}_\mathcal{D}[u \circ \beta]$ for all $\omega \in \mathcal{A}(\mathcal{D})$. Thus, if we define $\mathcal{O}_\mathcal{D} := \mathcal{A}(\mathcal{D})$, then the claim is trivially satisfied. $\diamond$ `Claim 2`

Claim 3: *There exist a subset $\mathcal{O}_\mathcal{E} \subseteq \mathcal{A}(\mathcal{E})$, open in the $\succeq_\mathcal{E}$-order topology, such that $\beta_\mathcal{E} \in \mathcal{O}_\mathcal{E}$, and such that $|\mathbb{E}_\mathcal{E}[u \circ \omega_\mathcal{E}] - \mathbb{E}_\mathcal{E}[u \circ \beta_\mathcal{E}]| < \epsilon$ for all $\omega_\mathcal{E} \in \mathcal{O}_\mathcal{E}$.*

Proof. The argument is identical to Claim 2. $\diamond$ `Claim 3`

Now let $\omega \in \mathcal{A}(\mathcal{G})$, and suppose $\omega_{\upharpoonright\mathcal{D}} \in \mathcal{O}_\mathcal{D}$ and $\omega_{\upharpoonright\mathcal{E}} \in \mathcal{O}_\mathcal{E}$. Then

$$\begin{aligned} \mu[\mathcal{G}]\,\mathbb{E}_\mathcal{G}[u \circ \omega] &\underset{(*)}{=} \mu[\mathcal{D}]\,\mathbb{E}_\mathcal{D}[u \circ \omega] + \mu[\mathcal{E}]\,\mathbb{E}_\mathcal{E}[u \circ \omega] \\ &\underset{(\dagger)}{<} \mu[\mathcal{D}]\left(\mathbb{E}_\mathcal{D}[u \circ \beta] + \epsilon\right) + \mu[\mathcal{E}]\left(\mathbb{E}_\mathcal{E}[u \circ \beta] + \epsilon\right) \\ &\underset{(*)}{=} \mu[\mathcal{G}]\ (\mathbb{E}_\mathcal{G}[u \circ \beta] + \epsilon) \underset{(\diamond)}{\leq} \mu[\mathcal{G}]\,\mathbb{E}_\mathcal{G}[u \circ \gamma]. \end{aligned}$$

Here, both $(*)$ are by equation (12), $(\dagger)$ is by the inequalities in Claims 2 and 3, and the full support of μ while $(\diamond)$ is by the definition of ϵ. Thus, $\mathbb{E}_\mathcal{G}[u \circ \omega] < \mathbb{E}_\mathcal{G}[u \circ \gamma]$. Thus, by the presumed SEU representation $\omega \prec_\mathcal{G} \gamma$. By an identical argument, $\mathbb{E}_\mathcal{G}[u \circ \omega] > \mathbb{E}_\mathcal{G}[u \circ \alpha]$, and thus, $\omega \succ_\mathcal{G} \alpha$. Thus, $\omega \in (\alpha, \gamma)_{\succeq_\mathcal{G}}$, and thus, $\omega \in \mathcal{O}$, as desired. $\square$

B Proofs of results from Sections 4, 5, 6 and 8

The proofs in this appendix draw heavily on results from a companion paper, which studies credences and their representations by classical probability measures (Pivato and Vergopoulos, 2017b). We will refer to results in the companion paper with the prefix "PV". Thus, "Theorem PV-4.3" should be read as, "Theorem 4.3 from Pivato and Vergopoulos (2017b)."

Theorems 1, 2, 3 and 4 are all obtained as corollaries of the following alternate version of Theorem 9 in the particular case where $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$.

Theorem B1 *Let $\mathcal{S}$ be a nondegenerate topological space, let $\mathcal{X}$ be a connected topological space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then, it further satisfies Axioms* (CEq), (C), (Dom), (Sep), (PC)*, and* (TC) *if and only if it has an SEU representation (u, μ), where u is a continuous function and μ is a credence on $\mathfrak{R}(\mathcal{S})$ with full support. Finally, μ is unique, and u is unique up to positive affine transformation.*

Proof of Theorem B1. In Theorem 9, the $\mathfrak{D}$-measurability of the utility function u only serves to obtain axiom (M), and not the other axioms. To prove the necessity of the axioms, we can therefore proceed here exactly as in Theorem 9. Conversely, in Theorem 9, axiom (M) only serves to obtain the $\mathfrak{D}$-measurability of u. (See Claim 6 in the proof of Proposition A3.) This is only used to make sure that $u \circ \alpha$ is $\mathfrak{B}$-comeasurable for any $\alpha \in \mathcal{A}$. Here, since u and α are continuous, $u \circ \alpha$ is also continuous, and therefore automatically $\mathfrak{R}(\mathcal{S})$-measurable for any $\alpha \in \mathcal{A}$, by Proposition PV-5.4(a). Thus, we do not need axiom (M) in showing the sufficiency of the axioms for the representation. Finally, the uniqueness of the representation can be obtained exactly as in Theorem 9 since the argument invoked there uses neither axiom (M) nor $\mathfrak{D}$-measurability. □

Proof of Theorem 1. "$\Longleftarrow$" Let ν be a residual charge on $\mathcal{S}$ with full support, and $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a continuous function that together provide an SEU representation of $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ as in equation (2). Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained by restricting ν to $\mathfrak{R}(\mathcal{S})$ as in Example 6(b). As explained in Example 8(b), $\mathbb{E}^{\mu}_{\mathcal{B}}[u \circ \alpha] = \int_{\mathcal{B}} u \circ \alpha \, \mathrm{d}\nu$; thus, (u, μ) provides an SEU representation as in Theorem B1. Moreover, since ν has full support, μ also has full support. Thus, $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ must satisfy all of axioms (CEq), (C), (Dom), (Sep), (PC) and (TC) by Theorem B1.

"$\Longrightarrow$" If $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms (CEq), (C), (Dom), (Sep), (PC) and (TC), then Theorem B1 says it has an SEU representation given by a credence μ on $\mathfrak{R}(\mathcal{S})$ with full support, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Since $\mathcal{S}$ is a Baire space, Proposition PV-6.1 yields a residual charge on $\mathcal{S}$ representing μ and $\mathbf{E}$ as in Examples 6(b) and 8(b). Combining this and the credence SEU representation from Theorem B1, we obtain an SEU representation as in formula (2). Since μ has full support, ν has also full support.

Finally, suppose that both (u, ν) and (u', ν') provide a representation of $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}}$. Let μ and μ' be the credences obtained by restricting respectively ν and ν' to $\mathfrak{R}(\mathcal{S})$. Then, by the assumed representations and Example 8(b), (u, μ) and (u', μ') both provide SEU representations as in formula (13). By uniqueness in Theorem B1, u and u' are positive affine transformations of each other. Moreover, μ and μ' are equal to each other. Thus, Proposition PV-2.7 implies that $\nu = \nu'$. □

The proofs of Theorems C1, 2, 3 and 4 are very similar to the proof of Theorem 1, and we only briefly sketch them. They require the following lemma.

Lemma B2 **(a)** *Any nonsingleton Hausdorff space is nondegenerate.*

(b) *If $\mathcal{S}$ is a nondegenerate space, then $\mathfrak{R}(\mathcal{S})$ is nontrivial.*

(c) *Suppose $\mathcal{S}$ is either locally compact or normal Hausdorff. For any nonempty open $\mathcal{O} \subseteq \mathcal{S}$, there is a nonempty $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ with $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{O}$.*

Proof. (a) Since $|\mathcal{S}| \geq 2$, there exist $s_1, s_2 \in \mathcal{S}$ with $s_1 \neq s_2$. Since $\mathcal{S}$ is Hausdorff, there are disjoint open neighbourhoods $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{S}$ around s_1 and s_2. Thus, $\mathcal{O}_1$ and $\mathcal{O}_2$ are both nonempty open subsets which are not dense in $\mathcal{S}$.

(b) Let $\mathcal{O} \subset \mathcal{S}$ be a nonempty, non-dense open subset. Let $\mathcal{C} := \mathcal{S} \setminus \mathcal{O}$. Then $\mathcal{C}$ is a proper closed subset of $\mathcal{S}$ with a nonempty interior (because $\mathcal{O}$ is not dense). Let $\mathcal{R} := \mathrm{int}(\mathcal{C})$. Then $\emptyset \neq \mathcal{R} \neq \mathcal{S}$, and $\mathcal{R}$ is regular, because it is the interior of a closed set. Thus, $\mathfrak{R}(\mathcal{S})$ is nontrivial.

(c) First suppose $\mathcal{S}$ is locally compact. Let $s \in \mathcal{O}$. By local compactness, $\mathcal{O}$ contains a compact subset $\mathcal{K}$ which is also a neighbourhood of s. Let $\mathcal{R} := \mathrm{int}(\mathcal{K})$. Then $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{K} \subseteq \mathcal{O}$, and $\mathcal{R} \neq \emptyset$, because $s \in \mathcal{R}$. Finally, $\mathcal{R}$ is regular, because it is the interior of the closed set $\mathcal{K}$.

Now suppose $\mathcal{S}$ is normal. If $\mathcal{O} = \mathcal{S}$, the statement is trivial. So assume $\mathcal{O} \neq \mathcal{S}$. Let $s \in \mathcal{O}$ and let $\mathcal{C} := \mathcal{S} \setminus \mathcal{O}$. Then $\mathcal{C}$ is closed, and $\{s\}$ is also closed, because $\mathcal{S}$ is Hausdorff. By normality, there exist disjoint open sets $\mathcal{U}_1, \mathcal{U}_2$ containing $\{s\}$ and $\mathcal{C}$. Let $\mathcal{K} = \mathrm{clos}(\mathcal{U}_1)$; the $\mathcal{K}$ is closed and disjoint from $\mathcal{U}_2$. Thus, $\mathcal{K}$ is disjoint from $\mathcal{C}$, so $\mathcal{K} \subseteq \mathcal{O}$. Let $\mathcal{R} := \mathrm{int}(\mathcal{K})$. Then $\mathcal{R}$ is regular (being the interior of a closed set), $\mathcal{R}$ is nonempty (it contains $\mathcal{U}_1$) and $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{K} \subseteq \mathcal{O}$. □

Proof of Theorem 2. By Lemma B2(a,b), $\mathfrak{R}(\mathcal{S})$ is nontrivial. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 1, with equations (PV-6T) and (PV-6U) and Corollary PV-6.7 playing respectively the same role as Example 6(b), Example 8(b) and Proposition PV-6.1. Finally, there is a minor difference in the proof of the full support property of the probability measure in the sufficiency of the axioms. Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained by applying Theorem

B1. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Let ν be the Borel probability measure obtained by applying Corollary PV-6.7 to $\mathbf{E}$. Now, suppose $\nu[\mathcal{O}] = 0$ for some nonempty open subset $\mathcal{O}$ of $\mathcal{S}$. By Lemma B2(c), there exists a nonempty $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ such that $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{O}$. Thus,

$$\begin{aligned} \mu(\mathcal{R}) \underset{\text{(a)}}{=} \nu(\mathcal{R}) + \int_{\partial\mathcal{R}} \phi_{\mathcal{R}} \, \mathrm{d}\nu \underset{\text{(b)}}{\leq} \nu(\mathcal{R}) + \nu(\partial\mathcal{R}) \\ = \nu(\mathrm{clos}(\mathcal{R})) \underset{\text{(c)}}{\leq} \nu(\mathcal{O}) = 0, \end{aligned}$$

where (a) is by Corollary PV-6.7, (b) is by equation (3) and (c) is because $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{O}$. But this contradicts the fact that μ has full support. Thus, $\nu[\mathcal{O}] > 0$ for every nonempty open subset $\mathcal{O}$ of $\mathcal{S}$, so ν has full support. □

Proof of Theorem 3. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 2, with equations (PV-7A) and (PV-7b), Theorem PV-7.2, and Example PV-7.3(a) playing respectively the same role as equations (PV-6T) and (PV-6U) and Corollary PV-6.7. However, in the sufficiency of the axioms, the proof that the Borel probability measure $\dot{\nu}$ on $\dot{\mathcal{S}}$ has full support is slightly different. Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained from Theorem B1. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Let $\dot{\nu}$ be the Borel probability measure on $\dot{\mathcal{S}}$ obtained by applying Theorem PV-7.2 to $\mathbf{E}$. Now, suppose $\dot{\nu}[\dot{\mathcal{O}}] = 0$ for some nonempty open subset $\dot{\mathcal{O}}$ of $\dot{\mathcal{S}}$. Since $\dot{\mathcal{S}}$ is compact Hausdorff, Lemma B2(c) gives a nonempty regular subset $\dot{\mathcal{R}} \in \mathfrak{R}(\dot{\mathcal{S}})$ such that $\mathrm{clos}_{\dot{\mathcal{S}}}(\dot{\mathcal{R}}) \subseteq \dot{\mathcal{O}}$. Define $\mathcal{R} := \dot{\mathcal{R}} \cap \mathcal{S}$. By Lemma PV-7.4(a), $\mathcal{R}$ is a regular subset of $\mathcal{S}$. By equation (PV-7A), we have $\mu(\mathcal{R}) = \dot{\nu}(\dot{\mathcal{R}}) + \int_{\partial\dot{\mathcal{R}}} \dot{\phi}_{\dot{\mathcal{R}}} \, \mathrm{d}\dot{\nu}_{\dot{\mathcal{R}}}$. At this point, we can proceed as in Theorem C1 to obtain $\mu(\mathcal{R}) = 0$, which contradicts the fact that μ has full support. Hence the full support of $\dot{\nu}$. □

Proof of Theorem 4. The proof is very similar to Theorem 3, but using Example PV-7.3(c) instead of Example PV-7.3(a). □

Proof of Theorem 10. Let $\mathcal{S}^*$ be the Stone space of the Boolean algebra $\mathfrak{B}$ —that is, the set of all Boolean algebra homomorphisms from $\mathfrak{B}$ into $\{T, F\}$. Let $\mathfrak{Clp}(\mathcal{S}^*)$ be the set of all clopen subsets of $\mathcal{S}^*$; this is a Boolean algebra under the standard set-theoretic operations of union, intersection, and complementation. The Stone Representation Theorem says there is a Boolean algebra isomorphism $\Phi : \mathfrak{B} \longrightarrow \mathfrak{Clp}(\mathcal{S}^*)$ given by $\Phi(\mathcal{B}) = \mathcal{B}^*$ for all $\mathcal{B} \in \mathfrak{B}$, where $\mathcal{B}^* := \{s^* \in \mathcal{S}^*; \ s^*(\mathcal{B}) = T\}$.

"$\Longleftarrow$" Suppose μ^* is a Borel probability measure on $\mathcal{S}^*$ with full support, and $u : \mathcal{X} \longrightarrow \mathbb{R}$ is a continuous function that together provide a Stonean SEU representation of $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ as in formula (14). For all $\mathcal{B} \in \mathfrak{B}$, define $\mu[\mathcal{B}] := \mu^*[\mathcal{B}^*]$. In other words,

$\mu := \mu^* \circ \Phi$. Then μ is a credence on $\mathfrak{B}$, because Φ is a Boolean algebra isomorphism from $\mathfrak{B}$ to $\mathfrak{Clp}(\mathcal{S}^*)$, and μ^* is a finitely additive probability measure when restricted to $\mathfrak{Clp}(\mathcal{S}^*)$. Furthermore, Theorem PV-8.4 says that $\mathbb{E}^{\mu}_{\mathcal{B}}[u \circ \alpha] = \int_{\mathcal{B}^*} u \circ \alpha^* \, \mathrm{d}\mu^*$ for all $\alpha \in \mathcal{A}$; thus, (u, μ) provides an SEU representation as in formula (13). Meanwhile, μ^* has full support, so $\mu^*[\mathcal{B}^*] > 0$ for all $\mathcal{B}^* \in \mathfrak{Clp}(\mathcal{S}^*)$, and hence, $\mu[\mathcal{B}] > 0$ for all $\mathcal{B} \in \mathfrak{B}$; thus μ also has full support. Thus, Theorem 9 says that $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies the axioms (CEq), (C), (Dom), (Sep), (PC), (M) and (TC).

"$\Longrightarrow$" If $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies Axioms (CEq), (C), (Dom), (Sep), (PC), (M), and (TC), then Theorem 9 says it has an SEU representation (13) given by a credence μ on $\mathfrak{B}$ with full support, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Define the function $\mu^* : \mathfrak{Clp}(\mathcal{S}^*) \longrightarrow [0, 1]$ by setting $\mu^*(\mathcal{B}^*) := \mu[\mathcal{B}]$ for all $\mathcal{B} \in \mathfrak{B}$ —in other words, $\mu^* := \mu \circ \Phi^{-1}$. This is a finitely additive probability measure on $\mathfrak{Clp}(\mathcal{S}^*)$ because Φ^{-1} is a Boolean algebra isomorphism from $\mathfrak{Clp}(\mathcal{S}^*)$ to $\mathfrak{B}$. Theorem PV-8.4 says that μ^* extends to a unique Borel probability measure μ^* on $\mathcal{S}^*$ such that, for any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ and $\mathcal{B} \in \mathfrak{B}$, we have $\mathbb{E}^{\mu}_{\mathcal{B}}[g] = \int_{\mathcal{B}^*} g^* \, \mathrm{d}\mu^*$. In particular, for any $\alpha \in \mathcal{A}$, we have $\mathbb{E}^{\mu}_{\mathcal{B}}[u \circ \alpha] = \int_{\mathcal{B}^*} u \circ \alpha^* \, \mathrm{d}\mu^*$ (because $(u \circ \alpha)^* = u \circ \alpha^*$). Applying this identity to the credence SEU representation (13), we obtain a Stonean SEU representation as in formula (14).

Full support. $\mu[\mathcal{B}] > 0$ for every nonempty $\mathcal{B} \in \mathfrak{B}$. Thus, $\mu^*[\mathcal{B}^*] > 0$ for every nonempty $\mathcal{B}^* \in \mathfrak{Clp}(\mathcal{S}^*)$. But $\mathcal{S}^*$ is totally disconnected, so $\mathfrak{Clp}(\mathcal{S}^*)$ is a base for the topology of $\mathcal{S}^*$. Thus, we deduce that $\mu^*[\mathcal{O}^*] > 0$ for every nonempty open subset $\mathcal{O}^* \subseteq \mathcal{S}^*$.

Uniqueness. Suppose that both (u_1, μ_1^*) and (u_2, μ_2^*) provide Stonean SEU representation for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$. Let $\mu_1 := \mu_1^* \circ \Phi$ and $\mu_2 := \mu_2^* \circ \Phi$; these are credences on $\mathfrak{B}$, and by Theorem PV-8.4, they both provide SEU representations as in formula (13). By uniqueness in Theorem 9, u_1 and u_2 are positive affine transformations of each other, while $\mu_1 = \mu_2$. Thus, the "uniqueness" part of Theorem PV-8.4 says that $\mu_1^* = \mu_2^*$. □

The next lemma is not needed to prove any of our results. But it justifies a claim made at the end of Section 6, where we provided an interpretation of compactification SEU representations. There, we claimed that, if $\mathcal{S} \in \mathfrak{R}(\mathcal{T})$, then $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{R}(\mathcal{T})$. In fact, we will prove a slightly stronger result.

Lemma B3 *Let $\mathcal{T}$ be a topological space, and let $\mathcal{S} \subseteq \mathcal{T}$. Let $\mathfrak{R}(\mathcal{S})$ denote the set of regular subsets of $\mathcal{S}$ with respect to the subspace topology on $\mathcal{S}$. If $\mathcal{S} \in \mathfrak{R}(\mathcal{T})$, then $\mathfrak{R}(\mathcal{S}) = \{\mathcal{R} \in \mathfrak{R}(\mathcal{T});\ \mathcal{R} \subseteq \mathcal{S}\}$.*

Proof. Let $\mathcal{R} \subseteq \mathcal{S}$. We must show that $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ if and only if $\mathcal{R} \in \mathfrak{R}(\mathcal{T})$. First note that $\mathrm{clos}_{\mathcal{T}}(\mathcal{R}) \subseteq \mathrm{clos}_{\mathcal{T}}(\mathcal{S})$, and thus,

$$\mathrm{int}_{\mathcal{T}}[\mathrm{clos}_{\mathcal{T}}(\mathcal{R})] \quad \subseteq \quad \mathrm{int}_{\mathcal{T}}[\mathrm{clos}_{\mathcal{T}}(\mathcal{S})] \quad = \quad \mathcal{S}, \tag{B1}$$

where the last equality is because $\mathcal{S} \in \mathfrak{R}(\mathcal{T})$. Let $\mathfrak{O}(\mathcal{T})$ be the family of open subsets of $\mathcal{T}$, and let $\mathfrak{O}(\mathcal{S})$ be the family of relatively opens subsets of $\mathcal{S}$. Then $\mathfrak{O}(\mathcal{S}) = \{\mathcal{O} \in \mathfrak{O}(\mathcal{T});\ \mathcal{O} \subseteq \mathcal{S}\}$, because $\mathcal{S}$ itself is open in $\mathcal{T}$.

Claim 1: *Let $\mathcal{O} \subseteq \mathcal{T}$. Then*

$$\Big(\mathcal{O} \in \mathfrak{O}(\mathcal{T}) \text{ and } \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{T}}(\mathcal{R})\Big) \iff \Big(\mathcal{O} \in \mathfrak{O}(\mathcal{S}) \text{ and } \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R})\Big).$$

Proof. "$\Longleftarrow$" is because $\mathfrak{O}(\mathcal{S}) \subseteq \mathfrak{O}(\mathcal{T})$ and $\operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \subseteq \operatorname{clos}_{\mathcal{T}}(\mathcal{R})$.

To see "$\Longrightarrow$", suppose $\mathcal{O} \in \mathfrak{O}(\mathcal{T})$ and $\mathcal{O} \subseteq \operatorname{clos}_{\mathcal{T}}(\mathcal{R})$. Then $\mathcal{O} \subseteq \operatorname{int}_{\mathcal{T}}[\operatorname{clos}_{\mathcal{T}}(\mathcal{R})]$, and thus, formula (B1) implies that $\mathcal{O} \subseteq \mathcal{S}$. Thus, $\mathcal{O} \in \mathfrak{O}(\mathcal{S})$ and $\mathcal{O} \subseteq \mathcal{S} \cap \operatorname{clos}_{\mathcal{T}}(\mathcal{R}) = \operatorname{clos}_{\mathcal{S}}(\mathcal{R})$. $\diamondsuit$ `Claim 1`

We now have

$$\begin{aligned} \operatorname{int}_{\mathcal{T}}[\operatorname{clos}_{\mathcal{T}}(\mathcal{R})] &= \bigcup \{\mathcal{O} \in \mathfrak{O}(\mathcal{T})\ ;\ \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{T}}(\mathcal{R})\} \\ &\underset{(*)}{=} \bigcup \{\mathcal{O} \in \mathfrak{O}(\mathcal{S})\ ;\ \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R})\} = \operatorname{int}_{\mathcal{S}}[\operatorname{clos}_{\mathcal{S}}(\mathcal{R})], \end{aligned} \quad \text{(B2)}$$

where $(*)$ is by Claim 1. Thus,

$$\Big(\mathcal{R} \in \mathfrak{R}(\mathcal{T})\Big) \iff \Big(\mathcal{R} = \operatorname{int}_{\mathcal{T}}[\operatorname{clos}_{\mathcal{T}}(\mathcal{R})]\Big) \underset{(\dagger)}{\iff} \Big(\mathcal{R} = \operatorname{int}_{\mathcal{S}}[\operatorname{clos}_{\mathcal{S}}(\mathcal{R})]\Big) \iff \Big(\mathcal{R} \in \mathfrak{R}(\mathcal{S})\Big),$$

where $(\dagger)$ is by formula (B2). □

C SEU representations on normal state spaces

Theorem 2 assumed that the state space $\mathcal{S}$ was a compact Hausdorff space. Theorem 4 extended this to *locally* compact Hausdorff spaces, but the resulting SEU representation involved the Stone-Čech representation, which can be rather unwieldy. This appendix presents a variant of Theorem 2 which works for any *normal* Hausdorff space, and requires neither compactness nor local compactness.

Recall that a topological space $\mathcal{S}$ is ***Hausdorff*** if any pair of points in $\mathcal{S}$ can be placed in two disjoint open neighbourhoods. A Hausdorff space $\mathcal{S}$ is ***normal*** (or "T_4") if, for any disjoint closed subsets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S}$, there exist disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{S}$ with $\mathcal{C}_1 \subseteq \mathcal{O}_1$ and $\mathcal{C}_2 \subseteq \mathcal{O}_2$. For example, any subset of $\mathbb{R}^N$ is normal. More generally, every metrizable space is normal. In particular, every topological manifold is normal. Also, every compact Hausdorff space is normal. Finally, the order topology on any strictly ordered set is both Hausdorff and normal. (However, if $\mathcal{I}$ is an uncountably infinite set, then the topology of pointwise convergence on $\mathbb{R}^{\mathcal{I}}$ is *not* normal.) Thus, almost all topological spaces which would arise naturally in economic applications are normal.

Charges and liminal charge structures. Let $\mathfrak{A}(\mathcal{S})$ be the Boolean algebra generated by the open subsets of $\mathcal{S}$. Thus, $\mathfrak{A}(\mathcal{S})$ contains all open subsets, all closed subsets, and all finite unions and intersections of such sets. A function $\nu : \mathfrak{A}(\mathcal{S}) \longrightarrow [0, 1]$ is a *charge* if it is finitely additive —i.e. $\nu[\mathcal{A} \sqcup \mathcal{B}] = \nu[\mathcal{A}] + \nu[\mathcal{B}]$ for any disjoint $\mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{S})$. We say ν is a *probability* charge if, furthermore, $\nu(\mathcal{S}) = 1$. Another charge ρ is *absolutely continuous* relative to ν if $\rho[\mathcal{B}] = 0$ whenever $\nu[\mathcal{B}] = 0$.

Let ν be a probability charge on $\mathfrak{A}(\mathcal{S})$. For any $\mathcal{B} \in \mathfrak{A}(\mathcal{S})$, let $\nu_{\mathcal{B}}$ be the restriction of ν to a charge on the elements of $\mathfrak{A}(\mathcal{B})$. A *liminal charge structure subordinate to* ν is a collection $\{\rho_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$, where, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{R})$, $\rho_{\mathcal{R}}$ is a charge on $\mathfrak{A}(\partial\mathcal{R})$ which is absolutely continuous with respect to ν, such that, for any regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, we have

$$\rho_{\mathcal{R}_1} + \cdots + \rho_{\mathcal{R}_N} \quad = \quad \nu_{\partial\mathcal{R}_1 \cup \cdots \cup \partial\mathcal{R}_N}. \tag{C1}$$

As with the liminal density structures introduced in Section 5, the liminal charge structure describes how the agent copes with her informational limitations; once she has "observed" the event $\mathcal{R}$, the charge $\rho_{\mathcal{R}}$ describes how much probability she conditionally assigns to $\partial\mathcal{R}$, and how this probability is distributed. To be precise, for any $\mathcal{U} \in \mathfrak{A}(\mathcal{S})$, the conditional probability she assigns to $\mathcal{U}$, given that she has "observed" $\mathcal{R}$, is the following ratio:

$$\frac{\nu(\mathcal{U} \cap \mathcal{R}) + \rho_{\mathcal{R}}(\mathcal{U} \cap \partial\mathcal{R})}{\nu(\mathcal{R}) + \rho_{\mathcal{R}}(\partial\mathcal{R})}. \tag{C2}$$

A charge ν is *normal* if, for every $\mathcal{B} \in \mathfrak{A}(\mathcal{S})$, we have $\nu[\mathcal{B}] = \sup\{\nu[\mathcal{C}];\ \mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{C}$ closed in $\mathcal{S}\}$ and $\nu[\mathcal{B}] = \inf\{\nu[\mathcal{O}];\ \mathcal{B} \subseteq \mathcal{O} \subseteq \mathcal{S}$ and $\mathcal{O}$ open in $\mathcal{S}\}$. A liminal charge structure $\{\rho_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ is *normal* if $\rho_{\mathcal{R}}$ is a normal charge on $\partial\mathcal{R}$ for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. Finally, a charge ν is said to have *full support* if $\nu(\mathcal{O}) > 0$ for any open set $\mathcal{O}$ in $\mathcal{S}$.

Liminal charge SEU representation. Let $\mathcal{X}$ be another topological space, let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A *liminal charge SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ is given by a charge ν on $\mathfrak{A}(\mathcal{S})$, a liminal charge structure $\{\rho_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ subordinate to ν, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \left(\int_{\mathcal{R}} u \circ \alpha \, \mathrm{d}\nu + \int_{\partial\mathcal{R}} u \circ \alpha \, \mathrm{d}\rho_{\mathcal{R}} \geq \int_{\mathcal{R}} u \circ \beta \, \mathrm{d}\nu + \int_{\partial\mathcal{R}} u \circ \beta \, \mathrm{d}\rho_{\mathcal{R}}\right). \tag{C3}$$

In this representation, the value of an act conditional on a regular event $\mathcal{R}$ has two components, which correspond to the two ways in which $\mathcal{R}$ could be the outcome of an observation, as explained above. If $\mathcal{S}$ is a normal Hausdorff space, then we have the following variant of Theorem 2.

Theorem C1 *Let $\mathcal{S}$ be a (nonsingleton) normal Hausdorff space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq), (C), (Dom), (Sep), (PC) *and* (TC) *if and only if it admits a liminal charge SEU representation* (C3), *where ν is a normal probability charge on $\mathfrak{A}(\mathcal{S})$ with full support, and $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is a normal liminal charge structure. Furthermore, ν and $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ are unique, and u is unique up to positive affine transformation.*

Proof of Theorem C1. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 2, with equations (PV-6D) and (PV-6E) and Proposition PV-6.4 playing respectively the same role as equations (PV-6T) and (PV-6U) and Corollary PV-6.7. The proof of full support is as in Theorem 2. □

Theorem C1 is similar to Theorem 1 in that both represent the agent's beliefs using *finitely* additive charges. But Theorem C1 has two advantages over Theorem 1: the charge ν is now *normal*, and the liminal structure explicitly encodes how the agent deals with the limits of her observational technology.

D Necessary and sufficient conditions for (Rch)

In this appendix, we will show that condition (Rch) places an important restriction on the utility function u. We say that a function $u : \mathcal{X}\longrightarrow\mathbb{R}$ is *nonsatiating* if, for any $x \in \mathcal{X}$, there exist $\underline{x}, \overline{x} \in \mathcal{X}$ such that $u(\underline{x}) < u(x) < u(\overline{x})$. Equivalently: u never obtains a global maximum or global minimum on $\mathcal{X}$. (For example, this is the case if $\sup_{x\in\mathcal{X}} u(x) = \infty$ and $\inf_{x\in\mathcal{X}} u(x) = -\infty$.) In particular, if u is continuous and nonsatiating, then $\mathcal{X}$ *cannot* be compact. Our next result says that, if $\mathcal{S}$ and $\mathcal{X}$ satisfy certain conditions, then condition (Rch) is more or less equivalent to a nonsatiating utility function. To state this result, we need two topological preliminaries.

A topological space $\mathcal{X}$ is *contractible* if there is a function $f : [0,1]\times\mathcal{X}\longrightarrow\mathcal{X}$ and a point $z \in \mathcal{X}$ such that $f(0, x) = x$ for all $x \in \mathcal{X}$, whereas $f(1, x) = z$ for all $x \in \mathcal{X}$. (In other words: the identity map on $\mathcal{X}$ is *null-homotopic*.) Heuristically, this mean that $\mathcal{X}$ is not only path-connected, but it has no "holes" of any dimension.[25] For example, any convex subset of $\mathbb{R}^N$ is contractible. More generally, any star-shaped subset of any topological vector space is contractible.[26] However, there exist other, less obvious contractible spaces (even contractible manifolds).

A topological space is *extremally disconnected* if the closure of every open set is open. Clearly, these are rather exotic spaces: any extremally disconnected Hausdorff space is totally disconnected, and any extremally disconnected metrizable space is discrete. It is safe to say that most of the topological spaces which would arise in economic applications are *not* extremally disconnected.

A credence μ is *nonatomic* if, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, if $0 < \epsilon < \mu[\mathcal{R}]$, then there exists some $\mathcal{R}' \in \mathfrak{R}(\mathcal{S})$ with $\mathcal{R}' \subset \mathcal{R}$ and $\mu[\mathcal{R}'] = \epsilon$.

Proposition D1 *Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces, let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ with an SEU representation given by a continuous utility function $u : \mathcal{X}\longrightarrow\mathbb{R}$ and a credence μ with full support.*

[25] To be precise: all the homotopy groups of $\mathcal{X}$ are trivial.

[26] A subset $\mathcal{X}$ of a topological vector space is *star-shaped* if $r\,x \in \mathcal{X}$ for all $x \in \mathcal{X}$ and all $r \in [0, 1]$.

(a) *Suppose $\mathcal{S}$ is a normal Hausdorff space, $\mathcal{X}$ is contractible, and $\mathcal{A} = \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$. If u is nonsatiating and μ is nonatomic, then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies axiom* (Rch).

(b) *If $\mathcal{S}$ is* not *extremally disconnected, and $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies axiom* (Rch), *then u is nonsatiating.*

(c) *If $\mathcal{S}$ is extremally disconnected and $\mathcal{A} = \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$, then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ always satisfies axiom* (Rch).

A first consequence of Proposition D1 is the following restriction on the primitive topologies: if the state space is not extremally disconnected, then the outcome space cannot be compact. (For most applications, this is only a mild loss of generality: let $\mathcal{X}'$ be some large but non-compact subset of $\mathcal{X}$ where $\succeq_{\rm xp}$ has neither a maximal nor a minimal element, and let $\mathcal{A}'$ be the set of elements in $\mathcal{A}$ which range over $\mathcal{X}'$; then we can construct an SEU representation of the agent's preferences on $\mathcal{A}'$ instead.) Proposition D1 also provides a characterization of condition (Rch). For example, if $\mathcal{S}$ is any non-discrete metric space (e.g. a non-discrete subset of $\mathbb{R}^N$) and $\mathcal{X}$ is a convex subset of $\mathbb{R}^M$, while $\mathcal{A} = \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$ and μ is nonatomic, then Proposition D1(a,b) implies that $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies (Rch) if and only if u is nonsatiating.

At the opposite extreme, the Stone space of complete Boolean algebra is extremally disconnected (Fremlin, 2004, Theorem 314S). In particular, $\mathfrak{R}(\mathcal{S})$ is a complete Boolean algebra (Fremlin, 2004, Theorem 314P). Thus, if we apply the machinery from Section 8 in the case $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$, then the resulting space $\mathcal{S}^*$ is extremally disconnected, so Proposition D1(c) says that *any* conditional preference structure on $\mathcal{C}(\mathcal{S}^*, \mathcal{X})$ satisfies (Rch).

The rest of this appendix is the the proof of Proposition D1. First, we need three topological lemmas.

Lemma D2 *Let $\mathcal{S}$ be any topological space. The interior of any closed subset of $\mathcal{S}$ is a regular set.*

Proof. Suppose $\mathcal{C} \subseteq \mathcal{S}$ is closed. Let $\mathcal{D} := {\rm int}(\mathcal{C})$. We want to show that $\mathcal{D}$ is regular. Let $\mathcal{C}' := {\rm clos}(\mathcal{D})$. Then $\mathcal{C}' \subseteq \mathcal{C}$. But $\mathcal{D}$ is (by definition) the largest open subset of $\mathcal{C}$, and since $\mathcal{D} \subseteq \mathcal{C}' \subseteq \mathcal{C}$, that means that $\mathcal{D}$ is also the largest open subset of $\mathcal{C}'$ —hence $\mathcal{D} = {\rm int}(\mathcal{C}')$, as desired. □

Lemma D3 ($\mathcal{S}$ *is extremally disconnected*) $\Longleftrightarrow$ (*Every regular subset of $\mathcal{S}$ is clopen*).

Proof. "$\Longrightarrow$" Let $\mathcal{R}$ be regular. Then $\mathcal{R} = {\rm int}[{\rm clos}(\mathcal{R})]$. But ${\rm clos}(\mathcal{R})$ is open (because $\mathcal{S}$ is extremally disconnected), so ${\rm int}[{\rm clos}(\mathcal{R})] = {\rm clos}(\mathcal{R})$. Thus, $\mathcal{R} = {\rm clos}(\mathcal{R})$, so $\mathcal{R}$ is closed —hence clopen.

"$\Longleftarrow$" Let $\mathcal{O} \subseteq \mathcal{S}$ be open. Let $\mathcal{R} := {\rm int}[{\rm clos}(\mathcal{O})]$; then $\mathcal{R}$ is a regular set (by Lemma D2) which contains $\mathcal{O}$ (because it contains any open subset of ${\rm clos}(\mathcal{O})$). But by hypothesis, $\mathcal{R}$ is clopen. Thus, ${\rm clos}(\mathcal{R}) = \mathcal{R}$. Thus, $\mathcal{R}$ is a closed set containing $\mathcal{O}$, so $\mathcal{R} \supseteq {\rm clos}(\mathcal{O})$. But $\mathcal{R} = {\rm int}[{\rm clos}(\mathcal{O})]$, so $\mathcal{R} \subseteq {\rm clos}(\mathcal{O})$. Thus, $\mathcal{R} = {\rm clos}(\mathcal{O})$. So the closure of $\mathcal{O}$ is open. □

Lemma D4 *Suppose $\mathcal{S}$ is a normal Hausdorff space, and $\mathcal{X}$ is contractible. Let $\mathcal{A} := \mathcal{C}_{\rm b}(\mathcal{S},\mathcal{X})$. Let $\mathcal{R}$ and $\mathcal{Q}$ be regular open subsets of $\mathcal{S}$, such that* $\mathrm{clos}(\mathcal{R})$ *and* $\mathrm{clos}(\mathcal{Q})$ *are disjoint. For any acts α, ξ^0, and ξ^1 in $\mathcal{A}$, there is a bounded continuous function $\Gamma : [0,1]\times\mathcal{S}\longrightarrow\mathcal{X}$ with the following properties:*

(a) *For all $t \in [0,1]$, we get a function $\gamma^t \in \mathcal{A}$ by setting $\gamma^t(s) := \Gamma(t,s)$ for all $s \in \mathcal{S}$.*

(b) *For all $t \in [0,1]$, $\gamma^t_{\restriction\mathcal{R}} = \alpha_{\restriction\mathcal{R}}$.*

(c) *$\gamma^0_{\restriction\mathcal{Q}} = \xi^0_{\restriction\mathcal{Q}}$ and $\gamma^1_{\restriction\mathcal{Q}} = \xi^1_{\restriction\mathcal{Q}}$.*

(d) *There is a compact subset $\mathcal{K} \subseteq \mathcal{X}$ such that $\Gamma(t,s) \in \mathcal{K}$ for all $t \in [0,1]$ and $s \in \mathcal{S}$. Furthermore, the set $\mathcal{K}$ depends only on α, ξ^0, and ξ^1, and is independent of the choice of $\mathcal{Q}$ and $\mathcal{R}$.*

(e) *For any continuous function $u : \mathcal{X}\longrightarrow\mathbb{R}$, and any $T \in [0,1]$, we have*

$$\lim_{t\to T} \left\| u\circ\gamma^t - u\circ\gamma^T \right\|_\infty \quad = \quad 0.$$

Given two acts α and ξ^0, Lemma D4 says that we can construct an act γ^0 which "interpolates" between them, in the sense that it agrees with α on $\mathcal{R}$ and agrees with ξ^0 on $\mathcal{Q}$. Likewise, we can construct an act γ^1 which agrees with α on $\mathcal{R}$ and agrees with ξ^1 on $\mathcal{Q}$. Finally, we can *continuously deform* γ^0 into γ^1 via a continuously varying family of maps $\{\gamma^t\}_{t\in[0,1]}$, such that *every one* of the functions γ^t agrees with α on $\mathcal{R}$. In the jargon of algebraic topology, the function Γ is a *homotopy* from γ^0 to γ^1 *relative to* $\mathcal{R}$. Furthermore, this relative homotopy can be constructed such that it never leaves some compact region $\mathcal{K}$, and when composed with any real-valued function u, the functions $\{u\circ\gamma^t\}_{t\in[0,1]}$ trace a continuous "path" through the space $\mathcal{C}_{\rm b}(\mathcal{S},\mathbb{R})$ in the uniform norm topology.

Proof of Lemma D4. The space $\mathcal{S}$ is normal and Hausdorff, so it is Tychonoff (Willard, 2004, Corollary 15.7). So, let $\overline{\mathcal{S}}$ be the Stone-Čech compactification of $\mathcal{S}$, and let $\iota : \mathcal{S} \hookrightarrow \overline{\mathcal{S}}$ be the canonical embedding. Let $\mathcal{K}_\alpha := \mathrm{clos}[\alpha(\mathcal{S})]$, $\mathcal{K}_0 := \mathrm{clos}[\xi^0(\mathcal{S})]$, and $\mathcal{K}_1 := \mathrm{clos}[\xi^1(\mathcal{S})]$. These are compact subsets of $\mathcal{X}$, because α, ξ^0, and ξ^1 are bounded continuous functions. Let $\mathcal{K}_* := \mathcal{K}_\alpha \cup \mathcal{K}_0 \cup \mathcal{K}_1$; then $\mathcal{K}_*$ is also compact, and α, ξ^0, and ξ^1 are continuous functions from $\mathcal{S}$ into $\mathcal{K}_*$. Thus, by the universal property of the Stone-Čech compactification, we can extend them to continuous functions $\overline{\alpha}$, $\overline{\xi}^0$, and $\overline{\xi}^1$ from $\overline{\mathcal{S}}$ into $\mathcal{K}_*$, such that $\overline{\alpha}\circ\iota = \alpha$, $\overline{\xi}^0\circ\iota = \xi^0$, and $\overline{\xi}^1\circ\iota = \xi^1$ (Willard, 2004, Thm. 19.5; Aliprantis & Border, 2006, Thm. 2.79).

The space $\mathcal{X}$ is contractible, so there is a continuous function $\Xi : [-1,1]\times\overline{\mathcal{S}}\longrightarrow\mathcal{X}$ such that $\Xi(-1,s) = \overline{\alpha}(s)$, $\Xi(0,s) = \overline{\xi}^0(s)$ and $\Xi(1,s) = \overline{\xi}^1(s)$ for all $s \in \overline{\mathcal{S}}$ (Willard, 2004, Theorem 32.7). Let $\mathcal{K} := \Xi\left([-1,1]\times\overline{\mathcal{S}}\right)$. Then $\mathcal{K}$ is a compact subset of $\mathcal{X}$, because $[-1,1]\times\overline{\mathcal{S}}$ is compact and Ξ is continuous. Note that the construction of $\mathcal{K}$ depends only on α, ξ^0 and ξ^1; we have not yet refered to $\mathcal{R}$ and $\mathcal{Q}$.

Let $\overline{\mathcal{Q}} := \text{clos}(\mathcal{Q})$ and $\overline{\mathcal{R}} := \text{clos}(\mathcal{R})$; by hypotheses, these sets are disjoint. Since $\mathcal{S}$ is normal, Urysohn's Lemma yields a function $\phi : \mathcal{S} \longrightarrow [0,1]$ such that $\phi(\overline{\mathcal{R}}) = \{0\}$ and $\phi(\overline{\mathcal{Q}}) = \{1\}$ (Willard, 2004, Thm. 15.6; Aliprantis & Border, 2006, Thm. 2.46; Royden, 1988, Thm 8.7). Now define $\Gamma : [0,1] \times \mathcal{S} \longrightarrow \mathcal{K}$ by setting $\Gamma(t,s) := \Xi[\phi(s) \cdot (t+1) - 1,\ \iota(s)]$, for all $t \in [0,1]$ and all $s \in \mathcal{S}$. Thus, Γ is a continuous function (because it is a composition of continuous functions). Furthermore:

- For all $t \in [0,1]$, if we define the function $\gamma^t : \mathcal{S} \longrightarrow \mathcal{X}$ by setting $\gamma^t(s) := \Gamma(t,s)$ for all $s \in \mathcal{S}$, then γ^t is continuous (because Γ is continuous) and bounded (because its image is contained in the compact set $\mathcal{K}$); thus, $\gamma^t \in \mathcal{A}$.
- For all $t \in [0,1]$, and all $r \in \mathcal{R}$, we have $\gamma^t(r) = \Gamma(t,r) = \Xi[\phi(r) \cdot (t+1) - 1, \iota(r)] = \Xi[0 \cdot (t+1) - 1, \iota(r)] = \Xi[-1, \iota(r)] = \overline{\alpha}[\iota(r)] = \alpha(r)$. Thus, $\gamma^t_{\upharpoonright\mathcal{R}} = \alpha_{\upharpoonright\mathcal{R}}$.
- For all $q \in \mathcal{Q}$, we have $\gamma^0(q) = \Gamma(0,q) = \Xi[\phi(q) \cdot (0+1) - 1, \iota(q)] = \Xi[1 \cdot 1 - 1, \iota(q)] = \Xi[0, \iota(q)] = \overline{\xi}^0[\iota(q)] = \xi^0(q)$. Thus, $\gamma^0_{\upharpoonright\mathcal{Q}} = \xi^0_{\upharpoonright\mathcal{Q}}$
- For all $q \in \mathcal{Q}$, we have $\gamma^1(q) = \Gamma(1,q) = \Xi[\phi(q) \cdot (1+1) - 1, \iota(q)] = \Xi[1 \cdot 2 - 1, \iota(q)] = \Xi[1, \iota(q)] = \overline{\xi}^1[\iota(q)] = \xi^1(t)$. Thus, $\gamma^1_{\upharpoonright\mathcal{Q}} = \xi^1_{\upharpoonright\mathcal{Q}}$.

Thus, Γ verifies all of properties (a)-(d).

To check property (e), let $\overline{\phi} : \overline{\mathcal{S}} \longrightarrow [0,1]$ be the Stone-Čech extension of ϕ, and define the function $\overline{\Gamma} : [0,1] \times \overline{\mathcal{S}} \longrightarrow \mathcal{K}$ by setting $\overline{\Gamma}(t,\overline{s}) := \Xi[\phi(\overline{s}) \cdot (t+1) - 1,\ \overline{s}]$, for all $t \in [0,1]$ and all $\overline{s} \in \overline{\mathcal{S}}$. For all $t \in [0,1]$, define the function $\overline{\gamma}^t : \overline{\mathcal{S}} \longrightarrow \mathcal{K}$ by setting $\overline{\gamma}^t(\overline{s}) := \overline{\Gamma}(t,\overline{s})$ for all $\overline{s} \in \overline{\mathcal{S}}$. Clearly, for all $t \in [0,1]$ and $s \in \mathcal{S}$, we have $\Gamma(t,s) = \overline{\Gamma}[t, \iota(s)]$ and hence $u \circ \Gamma(t,s) = u \circ \overline{\Gamma}[t, \iota(s)]$; in other words, $u \circ \gamma^t(s) = u \circ \overline{\gamma}^t[\iota(s)]$. Thus, for any $t, T \in [0,1]$, $\left\| u \circ \gamma^t - u \circ \gamma^T \right\|_\infty \leq \left\| u \circ \overline{\gamma}^t - u \circ \overline{\gamma}^T \right\|_\infty$. Thus, it suffices to show that $\lim\limits_{t \to T} \left\| u \circ \overline{\gamma}^t - u \circ \overline{\gamma}^T \right\|_\infty = 0$.

To see this, let $\epsilon > 0$. Since $u \circ \overline{\Gamma}$ is continuous, for every $\overline{s} \in \overline{\mathcal{S}}$, there is some $\delta_{\overline{s}} > 0$ and some open neighbourhood $\mathcal{Z}_{\overline{s}}$ around $\overline{s}$ in $\overline{\mathcal{S}}$ such that, if we define $\mathcal{O}_{\overline{s}} := (T - \delta_{\overline{s}}, T + \delta_{\overline{s}}) \times \mathcal{Z}_{\overline{s}}$ (an open neighbourhood of $(T, \overline{s})$), then $|u \circ \overline{\Gamma}(t,\overline{z}) - u \circ \overline{\Gamma}(T,\overline{s})| < \epsilon/2$ for all $(t, \overline{z}) \in \mathcal{O}_{\overline{s}}$. The collection $\{\mathcal{O}_{\overline{s}}\}_{\overline{s} \in \overline{\mathcal{S}}}$ is an open cover of the set $\{T\} \times \overline{\mathcal{S}}$. But $\{T\} \times \overline{\mathcal{S}}$ is compact (because $\{T\}$ and $\overline{\mathcal{S}}$ are compact). Thus, this collection has a finite subcover, say, $\{\mathcal{O}_{\overline{s}_n}\}_{n=1}^N$ for some finite collection $\{\overline{s}_n\}_{n=1}^N$ of points in $\overline{\mathcal{S}}$. Let $\delta_\epsilon := \min\{\delta_{\overline{s}_n}\}_{n=1}^N$.

Claim 1: *Suppose $|t - T| < \delta_\epsilon$. Then $|u \circ \overline{\Gamma}(t,\overline{z}) - u \circ \overline{\Gamma}(T,\overline{z})| < \epsilon$, for all $\overline{z} \in \overline{\mathcal{S}}$.*

Proof. Let $\overline{z} \in \overline{\mathcal{S}}$. Since $\{\mathcal{O}_{\overline{s}_n}\}_{n=1}^N$ is an open cover of $\{T\} \times \overline{\mathcal{S}}$, there is some $n \in [1 \ldots N]$ such that $(T, \overline{z}) \in \mathcal{O}_{\overline{s}_n}$, which means that $\overline{z} \in \mathcal{Z}_{\overline{s}_n}$. Since $|t - T| < \delta_\epsilon \leq \delta_{\overline{s}_n}$, it follows that $(t, \overline{z}) \in \mathcal{O}_{\overline{s}_n}$ also. Thus,

$$
\begin{aligned}
|u \circ \overline{\Gamma}(t,\overline{z}) - u \circ \overline{\Gamma}(T,\overline{z})| \quad &\underset{(\triangle)}{\leq} \quad |u \circ \overline{\Gamma}(t,\overline{z}) - u \circ \overline{\Gamma}(T,\overline{s}_n)| + |u \circ \overline{\Gamma}(T,\overline{s}_n) - u \circ \overline{\Gamma}(T,\overline{z})| \\
&\underset{(\dagger)}{\leq} \quad \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad = \quad \epsilon,
\end{aligned}
$$

as desired. Here ($\triangle$) is the triangle inequality, and ($\dagger$) is two applications of the defining property of $\mathcal{O}_{\bar{s}_n}$. $\diamondsuit$ `Claim 1`

Since Claim 1 holds for all $y \in \overline{\mathcal{S}}$, we conclude that $\left\| u \circ \overline{\gamma}^t - u \circ \overline{\gamma}^T \right\|_\infty < \epsilon$. We can construct such a δ_ϵ for any $\epsilon > 0$. We conclude that $\lim_{t \to T} \left\| u \circ \overline{\gamma}^t - u \circ \overline{\gamma}^T \right\|_\infty = 0$. □

Proof of Proposition D1. (a) Suppose u is nonsatiating. Let $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathcal{S}$, let $\alpha_1 \in \mathcal{A}(\mathcal{R}_1)$, and let $\alpha_2 \in \mathcal{A}(\mathcal{R}_2)$. We must find $\beta \in \mathcal{A}(\mathcal{R}_2)$ compatible with α_1 such that $\beta \approx_{\mathcal{R}_2} \alpha_2$.

Claim 1: *Let $E := \mathbb{E}_{\mathcal{R}_2}[u \circ \alpha_2]$. There exist some some $x_0, x_1 \in \mathcal{X}$ such that $u(x_0) < E < u(x_1)$.*

Proof. (by contradiction) Suppose there was no $x_0 \in \mathcal{X}$ such that $u(x_0) < E$. Thus, $u(x) \geq E$ for all $x \in \mathcal{X}$. In particular, this means that $u \circ \alpha(r) \geq E$ for all $r \in \mathcal{R}$. There are now two cases.

Case 1. If $u \circ \alpha(r) = E$ for some $r \in \mathcal{R}$, then there exists $x \in \mathcal{X}$ such that $u(x) = E$, but by hypothesis, there does not exist any $x \in \mathcal{X}$ such that $u(x) < E$. Thus, u obtains a minimal value of E at x, which contradicts the hypothesis that u is nonsatiating.

Case 2. Proposition A4 says that $\mathbb{E}_{\mathcal{R}}$ is strictly monotonic, because μ has full support. Thus, if $u \circ \alpha(r) > E$ for all $r \in \mathcal{R}$, then $\mathbb{E}_{\mathcal{R}}[u \circ \alpha] > E$. But this contradicts the definition of E.

Either way, we have a contradiction. To avoid this, there must be some $x_0 \in \mathcal{X}$ with $u(x_0) < E$. An identical argument yields some $x_1 \in \mathcal{X}$ with $u(x_1) > E$. $\diamondsuit$ `Claim 1`

Let $\overline{\mathcal{Q}}$ be a closed subset of $\mathcal{R}_2$ with nonempty interior. Since it is a subset of the open set $\mathcal{R}_2$, it is disjoint from the closed set $\text{clos}(\mathcal{R}_1)$. Let $\mathcal{Q} := \text{int}(\overline{\mathcal{Q}})$. This is a nonempty regular subset of $\mathcal{R}_2$.

By definition of $\mathcal{A}(\mathcal{R}_1)$, there is some function $\overline{\alpha}^1 \in \mathcal{A}$ such that $\alpha_1 := \overline{\alpha}^1_{\upharpoonright \mathcal{R}_1}$. Let x_0 and x_1 be the values identified by Claim 1, and let $\xi^0, \xi^1 : \mathcal{S} \longrightarrow \mathcal{X}$ be the constant functions with value x^0 and x^1, respectively. Applying Lemma D4 to $\overline{\alpha}^1$, ξ^0 and ξ^1, we obtain a continuous function $\Gamma : [0,1] \times \mathcal{S} \longrightarrow \mathcal{X}$ with the following properties:

- For all $t \in [0,1]$, we get a function $\gamma^t \in \mathcal{A}$ by setting $\gamma^t(s) := \Gamma(t,s)$ for all $s \in \mathcal{S}$.
- For all $t \in [0,1]$, we have $\gamma^t_{\upharpoonright \mathcal{R}_1} = \alpha_1$; thus, $\gamma^t_{\upharpoonright \mathcal{R}_2}$ is compatible with α_1.
- $\gamma^0(q) = x_0$ and $\gamma^1(q) = x_1$ for all $q \in \mathcal{Q}$.

Furthermore, there is a compact subset $\mathcal{K} \subseteq \mathcal{X}$ such that $\Gamma(t,s) \in \mathcal{K}$ for all $t \in [0,1]$ and $s \in \mathcal{S}$. This set depends only on $\overline{\alpha}_1$, x_0, and x_1, and is independent of the choice of $\mathcal{Q}$. Let $M := \max\{|u(k)|;\ k \in \mathcal{K}\}$; this maximum is well-defined because the function $|u|$ is continuous and $\mathcal{K}$ is compact. Let $\epsilon := \frac{1}{2}\min\{|E - u(x_0)|, |E - u(x_1)|\}$; then $\epsilon > 0$ by Claim 1. Let $\mathcal{Q}^c := \mathcal{R}_2 \cap (\neg\mathcal{Q})$. Since μ is nonatomic, we can make $\mathcal{Q}$ big enough that $\mu[\mathcal{Q}^c] \leq \epsilon\, \mu[\mathcal{R}_2]/M$.

Finally, define $v(t) := \mathbb{E}_{\mathcal{R}_2}[u\circ\gamma^t_{\upharpoonright\mathcal{R}_2}]$ for all $t \in [0,1]$; this yields a function $v : [0,1]\longrightarrow\mathbb{R}$.

Claim 2: $v(0) < E < v(1)$.

Proof.
$$\begin{aligned}\mu[\mathcal{R}_2]\cdot\mathbb{E}_{\mathcal{R}_2}[u\circ\gamma^0_{\upharpoonright\mathcal{R}_2}] &\underset{(*)}{=} \mu[\mathcal{Q}]\cdot\mathbb{E}_{\mathcal{Q}}[u\circ\gamma^0_{\upharpoonright\mathcal{Q}}]+\mu[\mathcal{Q}^c]\cdot\mathbb{E}_{\mathcal{Q}^c}[u\circ\gamma^0_{\upharpoonright\mathcal{Q}^c}] \\ &\underset{(\dagger)}{\leq} \mu[\mathcal{Q}]\cdot u(x_0)+\mu[\mathcal{Q}^c]\cdot M \\ &\leq \mu[\mathcal{R}_2]\cdot u(x_0)+M\cdot\epsilon\,\mu[\mathcal{R}_2]/M \\ &= \mu[\mathcal{R}_2]\cdot u(x_0)+\epsilon\cdot\mu[\mathcal{R}_2] \quad = \quad \mu[\mathcal{R}_2]\cdot(u(x_0)+\epsilon).\end{aligned}$$

Here, $(*)$ is by equation (12), because $\mathcal{R}_2 = \mathcal{Q}\vee\mathcal{Q}^c$. Next, $(\dagger)$ is because $\gamma^0(q) = x_0$ for all $q \in \mathcal{Q}$, while $u(x) \leq M$ for all $x \in \mathcal{X}$.

Dividing both sides by $\mu[\mathcal{R}_2]$, we get $v(0) = \mathbb{E}_{\mathcal{R}_2}[u\circ\gamma^0_{\upharpoonright\mathcal{R}_2}] \leq u(x_0)+\epsilon < E$. Through an identical proof, we establish that $v(1) = \mathbb{E}_{\mathcal{R}_2}[u\circ\gamma^1_{\upharpoonright\mathcal{R}_2}] \geq u(x_1)-\epsilon > E$. $\diamondsuit$ `Claim 2`

Claim 3: *$v : [0,1]\longrightarrow\mathbb{R}$ is a continuous function.*

Proof. Fix $T \in [0,1]$. Lemma D4(e) says that $\lim_{t\to T}\left\|u\circ\gamma^t - u\circ\gamma^T\right\|_\infty = 0$. *A fortiori*, $\lim_{t\to T}\left\|u\circ\gamma^t_{\upharpoonright\mathcal{R}_2} - u\circ\gamma^T_{\upharpoonright\mathcal{R}_2}\right\|_\infty = 0$. Thus, $\lim_{t\to T}\left|\mathbb{E}_{\mathcal{R}_2}[u\circ\gamma^t_{\upharpoonright\mathcal{R}_2}] - \mathbb{E}_{\mathcal{R}_2}[u\circ\gamma^T_{\upharpoonright\mathcal{R}_2}]\right| = 0$ (because $\left\|\mathbb{E}_{\mathcal{R}_2}\right\|_\infty = 1$). In other words, $\lim_{t\to T}|v(t)-v(T)| = 0$, meaning that v is continuous at T. This argument works for all $T \in [0,1]$. $\diamondsuit$ `Claim 3`

Combining the Intermediate Value Theorem with Claims 2 and 3, we obtain some $t \in [0,1]$ such that $\mathbb{E}_{\mathcal{R}_2}[u\circ\gamma^t_{\upharpoonright\mathcal{R}_2}] = E$. But then since (u,μ) is an SEU representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$, we conclude that $\gamma^t_{\upharpoonright\mathcal{R}_2} \approx_{\mathcal{R}_2} \alpha_2$. We have already observed that $\gamma^t_{\upharpoonright\mathcal{R}_2}$ is compatible with α_1. Thus, set $\beta := \gamma^t_{\upharpoonright\mathcal{R}_2}$ to prove the theorem.

(b) By contradiction, suppose u was satiating. In particular, suppose u had a global maximizer $x \in \mathcal{X}$. (The argument for a global minimizer is analogous).

Since $\mathcal{S}$ is *not* extremally disconnected, Lemma D3 says there is some $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ which is *not* a clopen set. (It follows that $\emptyset \neq \mathcal{R} \neq \mathcal{S}$.) Let $\mathcal{Q} = \neg\mathcal{R}$; then $\mathcal{Q} \in \mathfrak{R}(\mathcal{S})$ also. Let $o \in \mathcal{X}$ be such that $u(o) < u(x)$. Without loss of generality, suppose $u(o) = 0$ and $u(x) = 1$. Let κ^x be the constant x-valued act, and let κ^o be the constant o-valued act; by hypotheses, both of these are elements of $\mathcal{A}$. Structural condition (Rch) says that we can find some $\alpha^x_{\mathcal{R}} \in \mathcal{A}$ such that $(\alpha^x_{\mathcal{R}})_{\upharpoonright\mathcal{R}} \approx_{\mathcal{R}} \kappa^x_{\mathcal{R}}$, while $(\alpha^x_{\mathcal{R}})_{\upharpoonright\mathcal{Q}} = \kappa^o_{\upharpoonright\mathcal{Q}}$.

Claim 4: *$\mathcal{O}_1 := (u\circ\alpha^x_{\mathcal{R}})^{-1}(0,\frac{1}{2})$ is a nonempty open subset of $\mathcal{R}$.*

Proof. $\mathcal{O}_1$ is the preimage of the open set $(0,\frac{1}{2})$ under the continuous function $u\circ\alpha^x_{\mathcal{R}}$, so $\mathcal{O}_1$ is open. Furthermore, $u\circ\alpha^x_{\mathcal{R}}(q) = u(o) = 0$ for all $q \in \mathcal{Q}$. Thus, by continuity, $u\circ\alpha^x_{\mathcal{R}}(s) = 0$ for all $s \in \mathrm{clos}(\mathcal{Q}) = \mathcal{S}\setminus\mathcal{R}$. Thus, if $\mathcal{O}_1$ is nonempty, it must be a subset of $\mathcal{R}$. It remains to show that $\mathcal{O}_1$ is nonempty.

Now, $\mathcal{R}$ is *not* closed (by construction), so there exists a net $\{r_\lambda\}_{\lambda\in\Lambda}$ (for some directed set Λ) which converges to some point $s \in \mathcal{S}\setminus\mathcal{R}$. Thus, since $u\circ\alpha^x_{\mathcal{R}}$ is continuous, we have $\lim_{\lambda\in\Lambda} u\circ\alpha^x_{\mathcal{R}}(r_\lambda) = u(s) = 0$. In particular, this means that $r_\lambda \in \mathcal{O}_1$ for some $\lambda \in \Lambda$. Thus, $\mathcal{O}_1 \neq \emptyset$. $\diamondsuit$ `Claim 4`

Now, let $\mathcal{P}_1 := \text{int}[\text{clos}(\mathcal{O}_1)]$; then $\mathcal{P}_1$ is a regular subset of $\mathcal{R}$ (by Lemma D2) which is nonempty by Claim 4 (because it contains $\mathcal{O}_1$). Furthermore, $0 \leq u \circ \alpha^x_{\mathcal{R}}(p) \leq 1/2$ for all $p \in \mathcal{P}_1$. Let $\mathcal{P}_2 := \mathcal{R} \cap (\neg\mathcal{P}_1)$, then $\{\mathcal{P}_1, \mathcal{P}_2\}$ is a regular partition of $\mathcal{R}$, so that

$$\begin{aligned}
\mu[\mathcal{R}]\,\mathbb{E}_{\mathcal{R}}[u \circ \alpha^x_{\mathcal{R}}] \quad &\underset{(*)}{=} \quad \mu[\mathcal{P}_1]\,\mathbb{E}_{\mathcal{P}_1}[u \circ \alpha^x_{\mathcal{R}}] + \mu[\mathcal{P}_2]\,\mathbb{E}_{\mathcal{P}_2}[u \circ \alpha^x_{\mathcal{R}}] \quad \underset{(\dagger)}{\leq} \quad \frac{1}{2} \cdot \mu[\mathcal{P}_1] + 1 \cdot \mu[\mathcal{P}_2] \\
&\underset{(\diamond)}{<} \quad \mu[\mathcal{P}_1] + \mu[\mathcal{P}_2] \quad = \quad \mu[\mathcal{R}] \quad \underset{(\ddagger)}{=} \quad \mu[\mathcal{R}]\,\mathbb{E}_{\mathcal{R}}[u \circ \kappa^x]. \qquad (D1)
\end{aligned}$$

Here, $(*)$ is by equation (12), while $(\dagger)$ is by the monotonicity of the expectation operator $\mathbb{E}_{\mathcal{R}}$, because $(u \circ \alpha^x_{\mathcal{R}})(p) \leq \frac{1}{2}$ for all $p \in \mathcal{P}_1$, while $(u \circ \alpha^x_{\mathcal{R}})(p) \leq u(x) = 1$ for all $p \in \mathcal{P}_2$. Next $(\diamond)$ is because $\mu[\mathcal{P}_1] > 0$, because μ has full support. Finally, $(\ddagger)$ is because for all $r \in \mathcal{R}$, we have $\kappa^x(r) = x$, and thus, $u \circ \kappa^x(r) = u(x) = 1$.

Dividing both sides of inequality (D1) by $\mu[\mathcal{R}]$, we obtain:

$$\mathbb{E}_{\mathcal{R}}[u \circ \alpha^x_{\mathcal{R}}] \quad < \quad \mathbb{E}_{\mathcal{R}}[u \circ \kappa^x].$$

Thus, by the assumed SEU representation, we get $\alpha^x_{\mathcal{R}} \prec_{\mathcal{R}} \kappa^x_{\mathcal{R}}$. But this contradicts the definition of $\alpha^x_{\mathcal{R}}$. To avoid the contradiction, x *cannot* be a global maximizer for u.

(c) Fix two disjoint regular subsets $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}(\mathcal{S})$, and two acts $\alpha_1 \in \mathcal{A}(\mathcal{R}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{R}_2)$. There exist $\tilde{\alpha}_1 \in \mathcal{A}$ and $\tilde{\alpha}_2 \in \mathcal{A}$ such that $\alpha_1 = \tilde{\alpha}_{1\upharpoonright\mathcal{R}_1}$ and $\alpha_2 = \tilde{\alpha}_{2\upharpoonright\mathcal{R}_2}$. Define $\beta \in \mathcal{A}$ by $\beta(s) = \tilde{\alpha}_2(s)$ if $s \in \mathcal{R}_2$ and $\beta(s) = \tilde{\alpha}_1(s)$ otherwise.

Let us show that $\beta \in \mathcal{A}$. For any open subset $\mathcal{V}$ of $\mathcal{X}$, we have by construction that $\beta^{-1}(\mathcal{V}) = (\mathcal{R}_2 \cap \tilde{\alpha}_2^{-1}(\mathcal{V})) \cup (\mathcal{R}_2^{\complement} \cap \tilde{\alpha}_1^{-1}(\mathcal{V}))$. Now, $\mathcal{S}$ is extremally disconnected, so Lemma D3 says that $\mathcal{R}_2$ is clopen. Thus, $\mathcal{R}_2^{\complement}$ is open, so $\mathcal{R}_2^{\complement} = \neg\mathcal{R}_2$. Thus, $\beta^{-1}(\mathcal{V}) = (\mathcal{R}_2 \cap \tilde{\alpha}_2^{-1}(\mathcal{V})) \cup (\neg\mathcal{R}_2 \cap \tilde{\alpha}_1^{-1}(\mathcal{V}))$. Since $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are both continuous, and $\mathcal{R}_2$ and $\neg\mathcal{R}_2$ are both open, we finally obtain that $\beta^{-1}(\mathcal{V})$ is open in $\mathcal{S}$. Hence the continuity of β. To see that β is also bounded, note that $\beta(\mathcal{S}) = (\mathcal{R}_2 \cap \tilde{\alpha}_2(\mathcal{S})) \cup (\mathcal{R}_2^{\complement} \cap \tilde{\alpha}_1(\mathcal{S}))$. Therefore, $\beta(\mathcal{S}) \subseteq \text{clos}(\tilde{\alpha}_1(\mathcal{S})) \cup \text{clos}(\tilde{\alpha}_2(\mathcal{S}))$ and thus $\text{clos}(\beta(\mathcal{S})) \subseteq \text{clos}(\tilde{\alpha}_1(\mathcal{S})) \cup \text{clos}(\tilde{\alpha}_2(\mathcal{S}))$. By assumption, each of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ is bounded, and both $\text{clos}(\tilde{\alpha}_1(\mathcal{S}))$ and $\text{clos}(\tilde{\alpha}_2(\mathcal{S}))$ are compact. Then, $\text{clos}(\tilde{\alpha}_1(\mathcal{S})) \cup \text{clos}(\tilde{\alpha}_2(\mathcal{S}))$ is also compact. Finally, $\text{clos}(\beta(\mathcal{S}))$ is closed in a compact set and, therefore, compact. Hence boundedness. So $\beta \in \mathcal{C}_{\text{b}}(\mathcal{S}, \mathcal{X}) = \mathcal{A}$.

Now, since $\beta_{\upharpoonright\mathcal{R}_1} = \tilde{\alpha}_{1\upharpoonright\mathcal{R}_1} = \alpha_1$ and $\beta_{\upharpoonright\mathcal{R}_2} = \tilde{\alpha}_{1\upharpoonright\mathcal{R}_2} = \alpha_2$, we have that α_1 and α_2 are compatible. Last, we naturally have $\alpha_2 \approx_{\mathcal{R}_2} \alpha_2$, which completes the proof of (Rch). □

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