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December 2016

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Cite as

Magni C.A. 2016. Capital depreciation and the underdetermination of rate of return: A unifying perspective. *Journal of Mathematical Economics*, 67 (December), 54-79.

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Abstract. This paper shows that the notion of rate of return is best understood through the lens of the average-internal-rate-of-return (AIRR) model, first introduced in Magni (2010a). It is an NPV-consistent approach based on a *coherent* definition of rate of return and on the notion of *Chisini mean*, it is capable of solving the conundrums originated by the rate-of-return notion and represents a unifying theoretical paradigm under which every existing measure of wealth creation can be subsumed. We show that a rate of return is *underdetermined* by the project's cash-flow stream; in particular, a unique return function (not a unique rate of return) exists for every project which maps depreciation classes into rates of return. The various shapes a rate of return can take on (internal rate of return, average accounting rate of return, modified internal rate of return etc.) derive from the (implicit or explicit) selection of different depreciation patterns. To single out the appropriate rate of return for a project, auxiliary assumptions are needed regarding the project's capital depreciation. This involves value judgment. On one side, this finding opens terrain for a capital valuation theory yet to be developed; on the other side, it triggers the creation of a toolkit of domain-specific and purpose-specific metrics that can be used, jointly or in isolation, for analyzing the economic profitability of a given project. We also show that the AIRR perspective has a high explanatory power that enables connecting seemingly unrelated notions and linking various disciplines such as economics, finance, and accounting. Some guidelines for practitioners are also provided.

Keywords. Investment decision, project appraisal, rate of return, depreciation, capital, net present value, AIRR, mean.

1 Introduction

This paper shows that the notion of rate of return is *unique* but the rate of return of an economic transaction is *not unique*. In particular, the paper uses a coherence principle and the notion of *Chisini mean* to establish an underdetermination principle, according to which a rate of return cannot be singled out from a cash-flow stream as such. Rather, any cash-flow stream is associated with a *unique* return function which maps capital values into rates of return, so that infinitely many combinations of capital and rate generate the same net present value (NPV). A rate of return is singled out only if a well-defined statement upon capital depreciation is made by the analyst, implicitly or explicitly. The resulting approach is the so-called average internal rate of return (AIRR), which has been introduced in Magni (2010a) and developed in a vast array of subsequent papers (e.g., Magni 2011a, 2013a,b 2014a,b, 2015a).

The AIRR is a theoretical paradigm based on a mathematically simple and economically meaningful model, which coherently defines a rate of return as a growth rate of capital (and, therefore, as a *ratio*). The notion of Chisini mean enables fulfilling this 'coherence' for multiperiod projects as well as one-period projects, so a rate of return for a multiperiod project can be viewed as a generalized weighted mean of constituent one-period rates. This unique structure makes possible a deeper analysis of economic profitability that the traditional NPV analysis cannot accomplish and is capable of subsuming every single profitability measure (NPV included) into a unifying framework.

The AIRR paradigm completely solves individual problems related to specific metrics, for example, problems related to IRRs (see Magni 2013a for a list of 18 flaws) and problems related to the accounting rates (economic significance and relations with NPV and IRR). But, even more compellingly, it reinvents the notion of rate of return by turning the entire issue on its head: The rate-of-return notion can only make some sense if it is associated with the notion of *capital*, which is in turn connected with the actual economic transactions underlying the project undertaking and the economic milieu where the project is operated. This means that economically meaningful auxiliary assumptions are needed to supply a specific rate of return for the asset under consideration. Judgmental valuation is then necessary. Furthermore, as the traditional measures are indeed encompassed in the AIRR structure, this theoretical framework naturally legitimates the use of *multiple* measures for analyzing an economic transaction, so giving theoretical support to the widely reported empirical finding that practitioners and managers do jointly use different measures for evaluating a project.

This paper also illustrates the explanatory power of the AIRR approach and supplies some hints for investigating various issues such as the non-uniqueness of the cost of capital (COC); the definition of a stronger notion of NPV-consistency; the role of financing periods and investment periods in value creation; the relations between AIRR and Modigliani and Miller's propositions and their generalization; the link between the AIRR notion and Keynes's (1936) notion of user cost; the relations among the weighted average COC (wacc), the cost of equity, and the cost of debt.

The remainder of the paper is structured as follows. Section 2 supplies some notational conventions. Section 3 selectively systematizes the literature on the rate of return. Section 4 presents the AIRR approach, which is based on the notion of coherence of a rate of return and the notion of Chisini mean. The section introduces the iso-value curve, proves the NPV-consistency of the approach and generalizes AIRR to time-variant COCs in both discrete and continuous time. Section 5 presents AROI, a variant of AIRR, and discusses the relations with Keynes's notion of user cost. Section 6 discusses the underdetermination of the rate of return by a cash-flow stream and the notion of depreciation class. Section 7 presents 12 different rates of return related to different depreciation classes, among which are the internal rate of return (IRR) the modified IRR (MIRR), and the average accounting rate, and briefly discusses the economic meaning and the piece of information supplied by each one. Section 8 provides some guidelines for practitioners for singling out the relevant rate of return and presents three numerical applications. Sections 9-11 supply some insights for future research. In particular, section 9 presents a new, more stringent definition of NPV-consistency based on sensitivity analysis and shows that three rates of return, associated with different depreciation classes, possess it in a strict sense; section 10 blends Modigliani and Miller's well-known results (Modigliani and Miller 1958, 1963; Miller and Modigliani 1961) with the AIRR approach and shows how they can be reframed in terms of rate of return, allowing for time-variant rates of return and finite-lived firms; it also shows that the COC, as well as the project rate of return, is not unique, for it depends on capital depreciation; section 11 discusses the conflict existing between the quest for a capital valuation theory which should be capable of supplying an appropriate notion of *capital* value in any circumstance and the finding that different depreciation patterns supply different but not mutually exclusive pieces of information that may enrich the economic analysis. Some concluding remarks end the paper.

2 Notational conventions

Consider an economic agent (investor, manager) and an economic transaction, P , (e.g. project, firm, security) which is described by a sequence of a cash flows $\vec{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \backslash \{\vec{0}\}$. Let r represent the minimum attractive rate of return (if the asset is an investment) or the maximum attractive financing rate (if the asset is a financing). Assume an arbitrage-free, frictionless market (i.e., no transaction costs, no flotation costs, no taxes) exists where securities are traded. Let $\mathrm{N}_k^h = \{h, h+1, \dots, k\},\ h, k \in \mathbb{N}, h < k$ be the set of all natural numbers from h to k. The cash flow x_t, available at time $t \in \mathrm{N}_n^0$ can be interpreted either as a certainty-equivalent or as an expected value of a risky monetary amount. In the former case, r is the equilibrium risk-free rate. In the latter case, r expresses the expected rate of return of an asset traded in the security market which is equivalent in risk to project P, so it is obtained as the sum of the risk-free rate and a risk premium; thus, it represents the so-called (opportunity) *cost of capital* (COC). The economic (or market) value of P at time t is "the price the project would have if it were traded" (Mason and Merton 1985, pp. 38-39); in other terms, it is the price, v_t, of a replicating portfolio generating the same cash flows as the project's from time $t+1$ to time n: $\mathrm{v}_t = \sum_{k=t+1}^n x_k\,(1+r)^{t-k}$ for $t \in \mathrm{N}_n^0$. We will also use the symbol $d_{k,h} = (1+r)^{h-k},\ h, k \in \mathbb{N}, h \leq k$ for denoting the value at time h of one monetary unit available at time k. If the term structure of interest rate is non-flat (or the risk premium is not constant), a vector of forward rates $\vec{r} = (r_1, r_2, \dots, r_n)$ replaces r and the discount factor $d_{k,h}$ is redefined as $d_{k,h} = \prod_{j=h+1}^k \left(1 + r_j\right)^{-1}$. The net present value (NPV) is the difference between the current economic value and the investment cost, that is, $NPV = \mathrm{v}_0 - c_0 = \sum_{k=1}^n x_k\, d_{k,0} - c_0$ where $c_0 = -x_0$ is the capital committed, so that $NPV = \sum_{k=0}^n x_k\, d_{k,0}$. It is widely known that maximization of a project NPV is equivalent to the maximization of the investors' wealth: A firm's managers should then undertake a project if and only if $NPV > 0$ (McMinn 2005, Rubinstein 1973, Brealey et al. 2011, Berk and DeMarzo 2014).

NPV acceptability criterion. A project P is acceptable (creates value) if and only if $NPV > 0$.

An *equilibrium asset* is here defined as an asset whose NPV is zero (i.e., it is value-neutral).

3 The rate-of-return notion in the literature.

The literature on the rate-of-return notion is immense and offers a huge amount of frequently unrelated papers in different fields: Economic theory, engineering economics, finance, mathematics, accounting. In many cases no explicit link has been drawn between one contribution and another and, as soon as the amount of contributions on rate of return grew exponentially in the different disciplines, authors became less and less aware of the work of other authors working in other fields. This section reconstructs the story proposing a selective, systematic recount of events, by conceptually (and formally) chaining contributions that would be otherwise viewed as unrelated.

The modern story of the notion of *rate of return* for multiperiod assets is strictly connected with the related notions of *rate of return over cost* (Fisher 1930), *internal rate of return* (IRR) (Boulding 1935) and *marginal efficiency of capital* (Keynes 1936). It was Boulding who appears to have first called an economic asset's rate of return "internal". Dealing with multi-period projects, he explicitly assumed that "there is some rate of return [...] which is characteristic of the investment as a whole. This is of course a rate of interest, or a rate of discount. But it must be emphasized that it is a rate of interest which the enterprise itself produces [...] That is to say, it is an internal rate" (p. 478). Let $\vec{x}$ be the firm's or project's cash-flow stream and let c_t denote its capital at time t. Boulding (1935) observed that the increase in capital is equal to

$$c_t - c_{t-1} = \sigma \cdot c_{t-1} - x_t \tag{1}$$

where σ here denotes an assumed constant (and exogenously fixed) remuneration rate (see Boulding 1935, eqs. (3)-(5)). Note that c_t is a function of σ and one can rewrite (1) as

$$c_t(\sigma) = c_{t-1}(\sigma)(1+\sigma) - x_t. \tag{2}$$

Solving for n and considering the initial condition $c_0 = -x_0$,[1] one gets $c_n(\sigma) = -\sum_{t=0}^{n} x_t(1+\sigma)^{n-t}$. As the "total amount invested at the date of final liquidation is zero" (Boulding 1935, p. 481), one gets $-\sum_{t=0}^{n} x_t(1+\sigma)^{n-t} = 0$, whence, dividing by σ,

$$\sum_{t=0}^{n} x_t(1+\sigma)^{-t} = 0 \tag{3}$$

(Boulding 1935, eq. (12)). While Boulding assumed σ to be known, he realized that eq. (3) can also be used for 'reverse engineering', that is, used as "an equation from which we can calculate the internal rate of return of the enterprise itself, treating the net revenue series as given and [σ] as an unknown." (p. 481) Considering the NPV function $\varphi(u) = \sum_{t=0}^{n} x_t\,(1+u)^{-t}$, eq. (3) can be written as $\varphi(\sigma) = 0$, which is the standard IRR equation. The resulting IRR decision criterion can then be stated as follows.

IRR acceptability criterion.
Rule (I). An investment is acceptable if and only if $\sigma > r$.
Rule (II). A financing is acceptable if and only if $\sigma < r$.

As r is the equilibrium rate, rule (I) means that an investment is acceptable if the project's rate of return is greater than the rate of return that might be earned by investing the same amount in the security market. Rule (II) means that a financing is acceptable if the project's financing rate is smaller than the financing rate that might be paid by borrowing the same amount in the security market. Unfortunately, while the NPV criterion is applicable with no need of distinguishing investments from financings, the IRR acceptability criterion needs such a distinction, and it is not always easy to understand, at a first sight, whether a project is an investment or a financing.

If the function $\varphi(u)$, $u \in (-1, +\infty)$ is monotonically decreasing or increasing, IRR exists and is unique. Problems of existence and uniqueness of IRR related to nonmonotonicity were soon recognized (Boulding 1936, Wright 1936, Samuelson 1937). However, only two decades later the problem started to attract scholars' attention (e.g., Lorie and Savage 1955, Solomon 1956): Existence and uniqueness of σ in eq. (3) is a necessary condition for the IRR criterion to be employed,[2] which is the reason why a growing amount of contributions started to appear in the literature in order to cope with the issue of multiple solutions or no solution. Many scholars focused on conditions for an IRR to exist and be unique and strived to provide more and more general conditions under which an IRR exists and is unique. For example, Pitchford and Hagger (1958) identified a class of projects with a unique IRR in the interval $(-1, +\infty)$.

Proposition 1 (Pitchford and Hagger 1958) *If the cash-flow stream $\vec{x}$ possesses one and only one change of sign, then the IRR, σ, exists and is unique* in the interval $(-1, +\infty)$.

[1] Boulding (1935) assumed $x_0 = 0$, so his cash-flow stream begins with x_1.
[2] The condition is not sufficient: For example, if $\vec{x} = (-6.4, 16, -10)$, then $\sigma = 25\%$ is unique. However, $\varphi(u) = -6.4 + 16(1+u)^{-1} - 10(1+u)^{-2} \le 0$ for any u and $\partial\varphi(u)/\partial u > 0$ if and only if $u > 0.25$. This makes it unclear whether the project is an investment or a financing, and, therefore, whether the IRR is an investment rate or a financing rate. As a result, the traditional IRR criterion is inapplicable.

Soper (1959) extended the class of projects with unique IRRs, and his results were later generalized by Gronchi (1986) as follows.[3]

Proposition 2 (Soper 1959, Gronchi 1986) *Assume* $x_0 < 0$*. If* σ *is an IRR of* $\vec{x}$*, and* $\sum_{j=0}^{t} x_j (1+\sigma)^{t-j} \leq 0$ *for* $t \in \mathrm{N}_{n-1}^0$*, then an IRR,* σ*, exists and is unique in the interval* $(-1, +\infty)$.

Owing to (2), the Soper-Gronchi condition can be reframed in terms of nonnegative capital: $-\sum_{j=0}^{t} x_j (1+\sigma)^{t-j} = c_t(\sigma) \geq 0$ for all $t \in \mathrm{N}_{n-1}^0$, which implies that the firm injects capital in the project in every period. This in turn implies that, if multiple IRRs exist, $c_\tau(\sigma) < 0$ for some $\tau \in \mathrm{N}_{n-1}^0$, which means that the firm finds itself in a financing position, that is, it borrows from the project.

A further extension of the class of projects with unique IRR in $(-1, +\infty)$ was later accomplished by Kaplan (1965, 1967) by making use of the Sturm's sequence. See also Gronchi (1987, p. 44).

Other scholars focused on the interval $(0, +\infty)$. For example, Jean (1968) proved the following proposition.

Proposition 3 (Jean 1968, Gronchi 1987) *Assume* $x_0 < 0$*. If* $\sum_{t=0}^{n} x_t > 0$ *and the cash-flow stream* $\vec{x}$ *possesses one or two changes of sign, then the IRR,* σ*, exists and is unique in the interval* $(0, +\infty)$*.*

The class of projects described by Jean was then extended by Norstrøm (1972) as follows.

Proposition 4 (Norstrøm 1972) *Let* $X_t = \sum_{j=0}^{t} x_j$ *be the cumulative cash flow. If the sequence* $\vec{X} = (X_0, X_1, \dots, X_n)$ *possesses one and only one change of sign, and* $X_n \neq 0$*, then an IRR,* σ*, exists and is unique in* $(0, +\infty)$*.*

The class of projects described by Norstrøm was in turn extended in the following years by several scholars, including the following result provided by Bernhard (1979, 1980).

Proposition 5 (Bernhard 1979, 1980, Gronchi 1987) *Consider* $m_t = \sum_{j=0}^{t} \binom{n-j}{t-j} x_j$*,* $j \in \mathrm{N}_n^0$*. If the sequence* $\vec{m} = (m_0, m_1, \dots, m_n)$ *possesses one and only one change of sign, then an IRR,* σ*, exists and is unique in the interval* $(0, +\infty)$*.*

(See also Aucamp and Eckardt 1976, De Faro 1978, Pratt and Hammond 1977, 1979).

Notwithstanding the efforts made by many economists to search for wider and wider classes of multiperiod projects having a unique IRR, the picture was not encouraging from a practical perspective. Managers, professionals, and other practitioners need a reliable rate of return for *any* single project, regardless of whether it belongs or not to the above mentioned classes of projects. A first attempt to provide a resolution to this problem was proposed in the 1950s by several authors, including Solomon (1956), Baldwin (1959), Kirshenbaum (1965), Lin (1976) and Athanasopoulos (1978). The underlying idea of their contributions is to assume that the interim cash flows are reinvested (or financed, if negative) at a given rate, y, up to the terminal date n. A modified cash-flow stream $\vec{x}^M = (x_0, 0, \dots, 0, \sum_{t=1}^{n} x_t (1+y)^{n-t})$ is obtained. Assuming $x_0 \cdot \sum_{t=1}^{n} x_t (1+y)^{n-t} < 0$ and applying Proposition 1, a unique IRR exists which is the solution σ^M of

$$x_0 (1+y)^n + \sum_{t=1}^{n} x_t (1+y)^{n-t} = 0 \tag{4}$$

The solution, σ^M, is widely known as *modified internal rate of return* (MIRR).[4] If $x_0 < 0$ (>), the project is interpreted as an investment (financing). Hence,

[3] Prof. Gronchi generalized several results. His monograph (Gronchi 1984/1987) is an excellent recount of the first decades of the economic literature on the IRR notion.

MIRR acceptability criterion. If $x_0 < 0$ ($x_0 > 0$), the project is worth undertaking if and only if $\sigma^M > r$ ($\sigma^M < r$).

The MIRR is included in popular financial spreadsheet packages and is well appreciated by practitioners and academics (e.g., Ryan and Ryan 2002, Kierulff 2008, Lefley 2015); it is mentioned in many textbooks of corporate finance (e.g., Brealey et al., 2011; Ross et al., 2011; Pike et al. 2012, Berk and DeMarzo, 2014) and engineering economics (e.g., Herbst, 2002; Hartman 2007; Newnan et al, 2009; Blank and Tarquin, 2012, 2014). However, the MIRR approach has an important drawback: The MIRR is not the rate of return of the project; rather, it is the IRR of a modified project, which is a *portfolio* of (i) the project and (ii) the reinvestments of the interim cash flows; also, if the reinvestment rate is not equal to the COC, r, the MIRR acceptability criterion is not logically equivalent to the NPV acceptability criterion (see also section 7.2). In addition, it is often highlighted by scholars that there are many ways to modify a cash-flow stream in such a way that its IRR exists and is unique. Ross et al. (2011) illustrate three methods (the discounting approach, the reinvestment approach, the combination approach) and state that "there is no clear reason to say one of our three methods is better than any other" (Ross et al. 2011, p. 250. See also Magni 2015a,b).

Around the same time, another well-known solution was provided by Karmel (1959) and Arrow and Levhari (1969). Karmel (1959) made the assumption that a project can be truncated at any time in order to find a unique IRR for any project. He considered the truncated projects $P_t = (x_0, x_1, \ldots, x_t)$, $t \in \mathrm{N}_n^0$ and considered the IRR of any truncated project P_t. Then, he suggested the "maximum" IRR should be identified in the following way:

(i) exclude all truncated project P_t that have no IRR
(ii) if at least one project has multiple IRRs, then pick the greatest one
(iii) pick the IRRs selected in steps (i)-(ii) and choose the greatest one. Let $\sigma^{\max}$ be such an IRR.

Proposition 6 (Karmel 1959) *Let P_j be the truncated project associated with $\sigma^{\max}$. If the investor truncates the projects at time j, then the truncated project belongs to the class of Soper projects (see Proposition 2) and, therefore, the IRR, $\sigma^{\max}$, is unique.*

With hindsight, it is now evident that the economic content of this proposal is limited. Besides the fact that the three rules (i)-(iii) have no economic rationale, the assumption of truncation is itself economically dubious and essentially irrelevant. However, the result was sufficiently interesting mathematically to convince a substantial number of scholars to use such an assumption in several works (e.g., Hicks 1970, Flemming and Wright 1971, Nuti 1973, Burmeister 1974, Sen 1975, Eatwell 1975; Ross et al. 1980). Within this strand of literature, particularly important is the contribution of Arrow and Levhari (1969), who wrote a well-known paper which used the assumption of truncation in a new way. Consider the truncated projects $P_t = (x_0, x_1, \ldots, x_t)$, $t \in \mathrm{N}_n^0, x_0 < 0$, and consider the present value of the truncated projects, here denoted as $\varphi(P_t, u) := \sum_{j=0}^{t} x_j (1+u)^{-j}$, $t \in \mathrm{N}_n^0$. Let $\varphi(P, u) \coloneqq \max\{\varphi(P_t, u) : t \in \mathrm{N}_n^0\}$ denote the maximum present value.

Proposition 7 (Arrow and Levhari 1969) *The function $\varphi(P, u)$ is such that* $\lim_{u \to -1^+} \varphi(P, u) = +\infty$ *and* $\lim_{u \to +\infty} \varphi(P, u) = x_0 < 0$*; further, it is strictly decreasing. Therefore, the solution of $\varphi(P, \sigma) = 0$ exists and is unique in the interval* $(-1, +\infty)$.

While the authors correctly stated that "we prove that if, with a given constant rate of discount, we choose the truncation period so as to maximize the present value of the project, then the internal rate of return of the truncated project is unique" (Arrow and Levhari 1969, p. 560), this is not what one

[4] It is worth noting that the MIRR approach was anticipated, some two centuries earlier, by the French actuary Duvillard (1755-1832) (see Biondi 2006).

expects from a definition of rate of return of a project, as noted above. Norstrøm (1970) and Gronchi (1987) showed, in different ways, that the IRR associated with Karmel's criterion is equal to the IRR associated with Arrow and Levhari's criterion. Unfortunately, this equivalence, while mathematically important, did not make the two proposals more reliable economically and was finally abandoned (but see Promislow and Spring 1996 , Theorem 5.1).

Notwithstanding the massive contributions focused on (existence and) uniqueness of a rate of return, some scholars began to realize that the attention drawn to the uniqueness of a real root in the traditional IRR equation was excessive, for uniqueness does not guarantee the IRR to be economically meaningful: Even if the project has a unique IRR, its financial nature is not clear whenever the Soper-Gronchi condition does not hold (i.e., $c_t(\sigma) < 0$ for some t). The notion of a *mixed* project was then introduced (Teichroew, Robichek and Montalbano 1965a,b): In a mixed project, the firm invests funds in the project in some periods (lending position) and absorbs funds from the project in some other periods (financing position). For example, consider $\vec{x} = (-100, 160, -115, 106, -64, 22)$. The IRR is unique in the interval $(-1, +\infty)$ and is equal to $\sigma = 0.1$. Despite the fact that $\varphi(u)$ is monotonically decreasing in $(-1, +\infty)$, the vector of the IRR-implied capital amounts is $\vec{c}(0.1) = (100, -50, 60, -40, 20)$, which means that in the first, third and fifth periods the IRR acts as an investment rate, whereas in the second and fourth periods the IRR acts as a financing rate. Therefore, the financial nature of (a project and) an IRR may be ambiguous—and the IRR criterion inapplicable—even in those cases where the IRR is unique and the NPV function is monotonic. So, a problem of meaningfulness adds to the problems of existence and uniqueness: The IRR "is not a proper measure of return on investment. *This is a crucial criticism of the IRR—even though it may be unique, and real in the mathematical sense, this in itself is not a sufficient condition for it to be a correct measure of return on investment*" (Herbst 1978, p. 367, italics in original) and "a unique, real IRR is no assurance that the rate obtained is a proper measure of return on investment (or cost of financing)" (Herbst 1978, p. 369). Teichroew, Robichek and Montalbano (1965a,b) (henceforth, TRM) tried to solve the problem by explicitly recognizing that there may be periods where the entrepreneur injects capital in the project (investment) and some other periods where the entrepreneur subtracts capital from the project (financing). To this end, TRM allowed for a dual evolution of the capital such that, in case of investment ($c_t > 0$), the growth rate for capital is a *project investment* rate (PIR, here denoted as σ_I), whereas, in case of financing ($c_t < 0$), the growth rate for capital is a *project financing* rate (PFR, here denoted as σ_F). Denoting as i_t the growth rate,

$$i_t = \begin{cases} \sigma_F & \text{if } c_{t-1} \leq 0 \\ \sigma_I & \text{if } c_{t-1} > 0. \end{cases} \tag{5}$$

As a result, the capital evolves according to the following recurrence equation:

$$c_t = c_{t-1}(1 + i_t) - x_t = \begin{cases} c_{t-1}(1 + \sigma_F) - x_t & \text{if } c_{t-1} \leq 0 \\ c_{t-1}(1 + \sigma_I) - x_t & \text{if } c_{t-1} > 0 \end{cases} \tag{6}$$

which evidently generalizes eq. (2). The pair $\vec{\sigma} = (\sigma_F, \sigma_I)$ represents a generalization of the constant IRR and the terminal condition $c_n = c_n(\sigma_F, \sigma_I) = 0$ generalizes the condition $c_n(\sigma) = 0$. TRM showed that $c_n(\sigma_F, \sigma_I) = 0$ implicitly defines two strictly increasing functions: An investment-rate function $\sigma_I = f(\sigma_F)$ and a financing-rate function $\sigma_F = g(\sigma_I)$, which is the inverse function of the former: $g(f(u)) = u$. They proved a fundamental result.

Proposition 8 *For any acceptable interest rate (i.e., belonging to the domain of the implicit functions)*

(i) $\varphi(u) > 0$ if and only if $f(u) > u$ or, equivalently,

(ii) $\varphi(u) > 0$ if and only if $u > g(u)$.[5]

(TRM 1965a, Theorem V; TRM 1965b, p.176.). Armed which such a result, they could state two symmetric acceptability criteria, which are equivalent to the NPV criterion.

Proposition 9 (TRM criteria)

(A) If $\sigma_F = r$, then accept the project if and only if $\sigma_I = f(r) > r$.

(B) If $\sigma_I = r$, then accept the project if and only if $\sigma_F = g(r) < r$.

Rule (A) means that, if one assumes that, in the financing periods, the project lends to the firm at a financing rate equal to the market financing rate ($\sigma_F = r$), then the project is acceptable if the PIR earned in the investment periods is greater than the market investment rate holding in the security market ($\sigma_I > r$). Rule (B) means that, if one assumes that, in the investment periods, the firm lends to the project at an investment rate equal to the market rate of return ($\sigma_I = r$),[6] then the project is acceptable if and only if the PFR in the financing periods is smaller than the market financing rate ($\sigma_F < r$). This model has found favor in the literature, especially among engineering economists (e.g., Mao 1967, Oakford et al. 1977, Gronchi 1984/1987, Ward 1994, Herbst 2002, Blank and Tarquin 2012, Chiu and Garza Escalante 2012, Park 2013, Magni 2014a. See also Broverman 2008, p. 274, Kellison 2009, pp. 289-291), sometimes represented with different labels (e.g., Kulakov and Kulakova 2013),[7] used for choosing one IRR among multiple IRRs (Weber 2014) or for measuring performance of financial investments (Becker 2013). However, it is evident that this model is restrictive for several reasons. First, it only works if either $\sigma_F = r$ or $\sigma_I = r$. In all other cases, the inequations $f(r) > r$ or $g(r) < r$, while logically equivalent to $NPV = \varphi(r) > 0$, are economically non-significant, for r is not the actual PFR and, therefore, $f(r)$ is not the actual PIR (analogously, r is not the actual PIR and $g(r)$ is not the actual PFR). Second, both PIR and PFR are assumed to be time-invariant, whereas, in general, a firm's or project's capital (either positive or negative) grows at a time-varying rate. Third, the COC, r, is assumed to be time-invariant as well. This is a reasonable assumption in many situations, but a more general framework should account for time-varying equilibrium rates (i.e. for a non-flat structure $\vec{r}$ of interest rates).[8]

In addition, the TRM model is incomplete because neither σ_I nor σ_F represent the project rate of return. Rather, they represent, respectively, the investment rate in the lending periods and the financing rate in the borrowing periods. In order to provide the overall project's rate of return, the rates σ_F, σ_I need to be somehow aggregated, something that TRM did not address. Notwithstanding these drawbacks, TRM had the merit of showing that the quest for uniqueness was ill-placed and that the IRR concept cannot be considered a reliable one in general.

In recent times, the idea of distinguishing between investment rates and financing rates, which was at the forefront of the TRM approach, was resurrected within the very realm of the IRR approach in an innovative way by Hazen (2003). His main result can be stated in the following way.

[5] That (i) and (ii) are equivalent follows from the fact that $f(\cdot)$ and $g(\cdot)$ are strictly increasing and $f\big(g(u)\big) = u = g(f(u))$.

[6] It may be helpful to recall that r (in this paper as well as in TRM 1965a,b) is assumed to be both a lending rate and a financing rate.

[7] Kulakov and Kulakova's (2013) Generalized Internal Rate of Return is equal to TRM's PIR and the Generalized External Rate of Return is equal to TRM's PFR.

[8] Recently, Chiu and Garza Escalante (2012) extended Rule (A) allowing for time-variant COCs, r_t. In particular, they assumed $\sigma_{t,F} = r_t$ if $c_{t-1} < 0$ and $\sigma_{t,I} = r_t + a_I(1 + r_t)$ if $c_{t-1} \geq 0$. Their result was further generalized by Magni (2014, Theorems 6-7), who assumed: $\sigma_{t,F} = r_t + a_F(1 + r_t)$ if $c_{t-1} < 0$ and $\sigma_{t,I} = r_t + a_I(1 + r_t)$ if $c_{t-1} \geq 0$ (see section 7.3, Remark 7.2, for the more general case of time-varying COCs and unconstrained $\sigma_{t,F}$ and $\sigma_{t,I}$).

Proposition 10 *Suppose at least one IRR exists in* $(-1, +\infty)$ *and let* $\sigma_1, \dots, \sigma_m$ *be the sequence of IRRs of a project. Then,*

$$NPV = \frac{\sigma_j - r}{1 + r} \cdot \sum_{t=1}^{n} c_{t-1}(\sigma_j) \cdot d_{t-1,0} \qquad \forall j \in \mathrm{N}_m^1 \tag{7}$$

(recall that $d_{t-1,0} = (1 + r)^{-(t-1)}$ is the discount factor). This result enabled the author to generalize the distinction between overall investment and financing status of a project, whereas TRM originally only referred to single periods, and to provide mixed projects with a well-defined financial nature: A project is a *net investment* if lending positions exceed financing positions, i.e., $\sum_{t=1}^{n} c_{t-1}(\sigma_j) d_{t,0} > 0$; otherwise, if $\sum_{t=1}^{n} c_{t-1}(\sigma_j) d_{t,0} < 0$, it is a *net financing*. In case of net investment (financing), the IRR is, overall, an investment (financing) rate at which the net capital $\sum_{t=1}^{n} c_{t-1}(\sigma_j) d_{t,0}$ is invested (borrowed). The resulting decision criterion is as follows.

Hazen's (2003) criterion. *Let* σ_j *be any IRR. If a project is a net investment, it should be accepted if and only if* $\sigma_j > r$. *If the project is a net financing, it should be accepted if and only if* $\sigma_j < r$.

(See Hazen 2003, Theorem 4). The author's result reconciled the IRR and the NPV for accept-reject decisions. His finding implies that *any* one IRR can be used for assessing a project's economic profitability and making accept-reject decisions, as long as it is identified as an investment rate or as a financing rate. A project possessing multiple IRRs may be viewed as an investment project or as a financing project at the evaluator's discretion, depending on the choice of the IRR. So, on one hand, Hazen's (2003) proposal technically solved the multiple-IRR problem related to mixed projects;[9] on the other hand, it triggered a new problem: If any IRR captures economic profitability, which is the *true* or, at least, *relevant* rate of return (if there is any)? While Hazen (2003) left the analyst free to make the preferred choice, an answer to this question was proposed, one year later, by Hartman and Schafrick (2004). The authors used the economic distinction between investment and financing to identify the relevant IRR among a bundle of multiple ones. In particular, for any project having multiple IRRs, they identified the financial nature of the project on the basis of the monotonicity of the present-value function, $\varphi(u) = \sum_{t=0}^{n} x_t (1 + u)^{-t}$: The authors partitioned the graph of $\varphi(u)$ into investment and financing regions: An investment region (a, b) is such that $\partial\varphi(u)/\partial u < 0$ for every $u \in (a, b)$; a financing region (a, b) is such that $\partial\varphi(u)/\partial u > 0$ for every $u \in (a, b)$. They locate the region in which the cost of capital, r, resides; the IRR that resides in the same partition as the cost of capital is the 'relevant' IRR. Consequently, the associated criterion for accept-reject decision, recommended by Hartman and Schafrick (2004), is the following one.

Hartman and Schafrick's (2004) criterion. *Let* σ^R *be the relevant IRR. If the partition where* σ^R *lies is an investment region, then the project should be accepted if and only if* $\sigma^R > r$. *If the partition where* σ^R *lies is a financing region, then the project should be accepted if and only if* $\sigma^R < r$.[10]

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[9] Hazen (2003) proposed a method for coping with no real-valued IRR as well (see also Pierru 2010 and Osborne 2010, 2014 on complex-valued rates of return).

[10] In the span of 80 years, a large number of scholars advanced (sometimes conflicting) proposals for choosing the relevant IRR among a bundle of multiple ones. For example, Dorfman (1981) recommended the use of the largest root; Cantor and Lippman (1983, 1995) endorsed the use of the smallest nonnegative IRR; Cannaday et al. (1986) and Colwell (1995), dealing with a dual rate problem, suggested to choose the one which is increased by a marginal increase in a cash flow; Bidard (1999) suggested the use of the maximal IRR; Zhang (2005) recommended a technique which is based on the number of even or odd IRRs which are greater than the COC; Shestopaloff and Shestopaloff (2013) recommended the use of the largest root; Weber (2014) recommended the use of the smallest (or greatest) IRR among the ones which are greater (or smaller) than the COC if $\sigma_I(r) \geq r$ (<). However, these contributions did not challenge the IRR notion itself, which, as we will see later, is the real problem (see also Ramsey 1970, Howe 1991, Bosch et al,. 2007 for solutions not involving the use of IRR). (It should be noted that the economic rationale of the proposed solution is not always clear. For example,

Some decades earlier, other scholars, such as Hirshleifer (1958) and Bailey (1959) had provided a substantive criticism of the IRR, over and beyond the problems posed by the existence/multiplicity issue and by investment/financing conundrum: The IRR is an erroneous concept in itself, even if the IRR exists and is unique, and *even if the project is a pure investment* (i.e., $c_t(\sigma) > 0$ for all $t \in \mathrm{N}^1_{n-1}$). In particular, Bailey (1959) was the first one to acknowledge that the assumption of time-invariant rates of return is unwarranted. In general, a firm's or a project's capital rate of growth i_t, is time-variant, so a sequence $\vec{\imath} = (i_1, i_2, \dots, i_n)$ of time-variant period rates better describes the path of growth rates. The IRR sequence $(\sigma, \sigma, \dots, \sigma)$ is only a particular case, having no special economic significance:

> "recognition of the correct general solution of the investment problem has been hindered by the habit of thinking in terms of a single, long-term rate of interest" (Bailey 1959, p. 477); "This is an example of the 'paradox' that has attracted so much attention in connection with investment decision criteria. It should be evident, however, that this paradox is merely an accident of the simplifying device of dealing with a single long-term rate of interest, and that it has no special importance." (Bailey 1959, pp. 478–479)

The equilibrium forward rates are time-variant as well, a case that the IRR cannot cope with:

> "Martin J. Bailey has emphasized this to me that it is precisely when this occurs (when there exists a known pattern of future variations of [r]) that the internal-rate-of-return rule fails most fundamentally…With a known pattern of varying future [r], shifts in the relative desirability of income in different periods are brought about. In the usual formulation the internal rate of return can take no account of this. In such a case, one might have an investment for which [the IRR] was well defined and unique and still not be able to determine the desirability of the investment opportunity" (Hirshleifer 1958, p. 350).

In other words, two fundamental problems arise even if the project has a unique IRR and is not a mixed project:

(1) the assumption of constant growth, $i_t = \sigma$, for all $t \in \mathrm{N}^1_n$ is very limiting (and generally unrealistic), and so the IRR-implied capital $c_t(\sigma)$ is fictitious, being automatically generated by the very mathematical procedure used for defining the IRR (see eq. (1)-(3))
(2) the IRR cannot capture an asset's economic profitability if the term structure of interest rate is non-flat, for a rate cannot be compared with a sequence of time-variant COCs .

Therefore, these two authors considered the problem of existence and uniqueness only as the signs of a general inadequacy of the IRR to capture the notion of rate of return: "the fact that the use of [the IRR] leads to the correct decision in a particular case or a particular class of cases does not mean that it is correct in principle … These instances of failure of the multi-period internal-rate-of return rule … are, of course, merely the symptom of an underlying erroneous conception" (Hirshleifer 1958, p. 346-7).

Hirshleifer (1958) stated that the conundrum of the rate-of-return notion should be somehow solved reminding that a rate of return must be coherent with the piece of information it is supposed to supply, that is, it must be a ratio, for it represents "a rate of growth of capital funds invested in a project [and] the idea of a rate of growth involves a *ratio*" (Hirshleifer, 1958, p. 346-7, italics added). And Bailey (1959) made it clear that a comparison method should be developed, which enables a whole sequence of time-variant period rates $\vec{\imath} = (i_1, i_2, \dots, i_n)$ to be compared with a whole sequence of time-variant COCs $\vec{r} = (r_1, r_2, \dots, r_n)$. As a possible route, he suggested that "the criterion for

the choice of the relevant IRR in Weber 2014, named *selective* IRR, is associated with a mathematical, ratcheting, procedure which is not supported by any substantive economic argument.)

multiperiod investments can center on a short-term rate when all other short-term rates are assumed to be equal to the equilibrium rates" (p. 479). (See also section 7 below).

More recently, other unsuspected flaws of the IRR were unearthed (see Magni 2013a for a compendium of 18 flaws) and a complete solution to the conceptual and mathematical problems of the rate-of-return notion has been advanced in Magni (2010a) and then developed in a subsequent series of papers (e.g., Magni, 2013a,b, 2014a,b, 2015a). The author turns the issue on its head by getting rid of the IRR equation and of the assumption that the growth rate for capital is constant. A new theoretical perspective is introduced, named Average Internal Rate of Return (AIRR), a unifying framework which encompasses all the traditional capital budgeting metrics and, at the same time, is capable of systematically taking account of the insights by (i) Hirshleifer (1958) (a return rate is a ratio), (ii) Bailey (1959) (capital growth is time-variant), (iii) TRM (1965a,b) (a project can be a financing in some periods), and (iv) Hazen (2003) (a distinction is needed between investment projects and financing projects). In addition, it is free from the flaws and paradoxes that mar the IRR (see Magni 2013 a).

The starting point of the AIRR approach is that the analysis should not be based on a *relevant* IRR that automatically generates a capital stream $\vec{c}(\sigma)$ which is unrelated with the actual economic transactions, but, rather, on a *relevant* capital stream $\vec{c} = (c_0, c_1, \dots, c_{n-1})$, representing the underlying actual economic transactions, that generates a rate of return which fulfills points (i)-(iv) above.

In the following section we show that the notion of Chisini mean originates a coherent rate of return that changes the way the rate-of-return notion is usually cognized.

4 The AIRR paradigm

This section summarizes the AIRR approach, introduced in Magni (2010a) and developed in many subsequent papers (see references), and shows that it can be viewed as a most general framework stemming from the notion of coherent rate of return and the notion of Chisini mean.

4.1 The AIRR notion

We first consider a basic notion of *coherent* rate of return, based on a given capital C and a given return $\mathcal{I}$. Then, we introduce the notion of Chisini mean and apply it to a portfolio of n assets with given capital amounts $C_1, C_2, \dots, C_n$ and returns $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n$, respectively, showing that the Chisini mean of the constituent assets' rates of return is a coherent return rate for the portfolio. Finally, we apply this result to an n-period project with cash-flow stream $\vec{x} = (x_0, x_1, \dots, x_n)$ associated with a vector of beginning-of-period capital values $\vec{c} = (c_0, c_1, \dots, c_{n-1})$, using the fact that any project can always be interpreted as portfolio of n (one-period) assets and making use of a fundamental economic relation linking capital, income/return and cash flow.

Definition 1 (Coherence) *A rate of return $\imath$ is coherent if it can be expressed as an amount of return $\mathcal{I}$ per unit of invested capital $\mathcal{C}$, that is, $\imath = \mathcal{I}/\mathcal{C}$. In other words, a coherent rate of return is a rate of growth for capital enjoying the following product structure:*

$$\imath \cdot \mathcal{C} = \mathcal{I}. \tag{8}$$

Definition 1 simply formalizes Hirshleifer's (1958) idea that a rate of return should be a ratio if it is to be coherent with its purpose.

Proposition 11 *Let I be the income earned by investing capital C at the coherent rate i and let R be the return that could be earned by investing the same amount at the coherent rate r. Then*

$$i = i(C) = r + \frac{\Delta I}{C} \tag{9}$$

where $\Delta I = I - R$.

Proof. By Definition 1, (8) holds with $\mathcal{C} = C$ and $\imath = i$ or $\imath = r$, respectively. Therefore, $I = i \cdot C$, $R = r \cdot C$, so that $\Delta i = i - r$ is a coherent rate of return as well, for

$$\Delta i \cdot C = \Delta I \tag{10}$$

which is a special case of (8). Hence, the thesis follows.■

The proposition says that, given a coherent COC, and an excess return, ΔI, a rate of return depends on capital; more precisely, it is a homographic function of the capital put into play (see Figure 2). Note that, given any combination of (C, i), the product (10) is invariant under changes in C (the area of the rectangle with base $|C|$ and height $|\Delta i|$ does not change) and that, if no capital C is fixed, the rate of return is not determined.

Definition 2 (Chisini mean) *Given a real-valued function f of n variables $u_1, u_2, \dots, u_n$, the Chisini mean of n values $u_1, u_2, \dots, u_n$ associated with f is the number $\overline{u}$ (if it exists) such that*

$$f(\overline{u}, \overline{u}, \dots, \overline{u}) = f(u_1, u_2, \dots, u_n). \tag{11}$$

Essentially, the Chisini mean is the number $\overline{u}$ that, substituted to $u_1, u_2, \dots, u_n$, leaves invariant the value of f. For this reason, eq. (11) may be interpreted as an *invariance requirement*. The choice of the function f depends on the object of the analysis: Different choices of f lead to different kinds of mean. Eq. (11) may have no solution or multiple solutions (see Chisini 1929, de Finetti 1931. See also Graziani and Veronese 2009 for details and applications). Definition 2 says that if one replaces the various data $u_1, u_2, \dots, u_n$ with a unique value $\bar{u}$, then the value taken on by the function does not change. So, $\bar{u}$ indeed represents an overall measure (a mean) that gathers information from the individual data $u_1, u_2, \dots, u_n$.

Consider now a portfolio of n assets and let I_t, C_t be, respectively, the return and the capital of the t-th asset. The following result holds.

Proposition 12 (Portfolio rate of return) *The rate of return of a portfolio is a weighted arithmetic mean of the holding period rates:*

$$\bar{\imath} = w_1 i_1 + w_2 i_2 + \dots + w_n i_n \tag{12}$$

where $w_t = C_t / \sum_{t=1}^n C_t$ and $i_t = I_t / C_t$.

Proof. The sum of the assets' returns, $I = \sum_{t=1}^n I_t$, expresses the overall return generated by the portfolio and can be framed as a function of the return rates: $I = I(i_1, i_2, \dots, i_n) = \sum_{t=1}^n i_t \cdot C_t$. Imposing the Chisini's invariance requirement on I one finds $I(\bar{\imath}, \bar{\imath}, \dots, \bar{\imath}) = I(i_1, i_2, \dots, i_n)$. This equation has a unique solution: $\bar{\imath} = \sum_{t=1}^n C_t i_t \,/ \sum_{t=1}^n C_t$. ■

Consider a multiperiod project P whose cash-flow stream is $\vec{x} = (x_0, x_1, \dots, x_n)$ and let $\vec{c} = (c_0, c_1, \dots, c_{n-1})$ be a vector of values expressing the beginning-of-period capital amounts invested in P. We call $\vec{c}$ the *capital stream* or *investment stream*; let i_t be the growth rate of the capital in the interval $[t-1, t]$. The capital invested in the project grows at the rate i_t and jumps down by x_t (or jumps up by x_t if the latter is negative) at the end of the period. Therefore, at the beginning of the interval $[t, t+1]$, the capital is $c_t = c_{t-1}(1 + i_t) - x_t$. More generally, the following relation holds:

$$c_t = c_{t-1} + I_t - x_t \tag{13}$$

where I_t is redefined as the project income (return) at time t. The above recursive equation is a fundamental relation in economics, finance, and accounting (see Magni 2009a,b, 2010b) which holds for $c_{t-1} = 0$ as well (if $c_{t-1} \neq 0$, then $i_t = I_t / c_{t-1}$). The boundary conditions are $c_0 = -x_0$ and $c_n = 0$. The terminal condition implies $\sum_{t=0}^n x_t \prod_{k=1}^t (1 + i_k)^{-1} = 0$; the vector $\vec{\imath} = (i_1, \dots, i_n)$ can then be referred to as an *internal* vector (Weingartner 1966). We now show that, by interpreting the n-

period project as a portfolio of n one-period assets, we may apply Proposition 12 and get the project's rate of return. In this framework, the invested capital C_t of the t-th constituent asset is equal to the discounted value of the project's capital c_{t-1}.

Proposition 13 (AIRR) *Let $\vec{x} = (x_0, x_1, \dots, x_n)$ be the cash-flow stream of a multi-period project P and let $\vec{c} = (c_0, c_1, \dots, c_{n-1})$ be the associated capital stream. Then, $i = \bar{\imath}$, with $C_t = c_{t-1}d_{t,0}$ and $C = \sum_{t=1}^{n} C_t$. More precisely,*

$$\bar{\imath} = \frac{\sum_{t=1}^{n} c_{t-1} i_t \, d_{t,0}}{\sum_{t=1}^{n} c_{t-1} d_{t,0}} \tag{14}$$

Proof. An n-period project P can be viewed as a portfolio of n one-period projects $P^{(t)}$, whose cash-flow stream is $\vec{x}_{(t)} = -c_{t-1}\vec{1}_t + (c_t + x_t)\vec{1}_{t+1}$ where c_t is the capital invested at time t and $\vec{1}_t \in \mathbb{R}^{n+1}, t \in \mathrm{N}_n^1$ is a vector of zeros except the t-th component which is equal to 1. Therefore, $\vec{x} = \sum_{t=1}^{n} \vec{x}_{(t)}$. The return of each one-period project is $I_t = c_t + f_t - c_{t-1}$ and $i_t = (c_t + x_t - c_{t-1})/c_{t-1}$ is the associated rate of return. The (present value of the) excess return is $\Delta I_t = c_{t-1}(i_t - r)d_{t,0}$, which implies, $i_t c_{t-1} = rc_{t-1} + \Delta I_t / d_{t,0}$. The (present value of the) overall project return is $\sum_{t=1}^{n} i_t c_{t-1} d_{t,0} = \sum_{t=1}^{n} \big(rc_{t-1}d_{t,0} \; + \Delta I_t\big)$, whence $\sum_{t=1}^{n} i_t C_t = rC + \Delta I$ where $\Delta I = \sum_{t=1}^{n} \Delta I_t$ is the (discounted value of) the portfolio's excess return. Dividing by C, $\bar{\imath} = \sum_{t=1}^{n} i_t C_t \,/C = r + \Delta I / C = i$. ■

Remark 4.1 Note that $\bar{\imath} = I/C = \sum_{t=1}^{n} c_{t-1} i_t \, d_{t,0} / \sum_{t=1}^{n} c_{t-1} d_{t,0}$ can be rewritten as

$$\bar{\imath} = \frac{\sum_{t=1}^{n} c_{t-1} i_t \, d_{t-1,0}}{\sum_{t=1}^{n} c_{t-1} d_{t-1,0}} \tag{15}$$

so that the denominator is the *present* value of the capital amounts. In general, $\bar{\imath} = \sum_{t=1}^{n} i_t c_{t-1} \, (d_{t,0}/d_{k,0}) / \sum_{t=1}^{n} c_{t-1}(d_{t,0}/d_{k,0})$ for any $k \in \mathbb{N}$. In any case, the value of interest and the value of capital differ by one period.

The rate $i = \bar{\imath}$ is called Average Internal Rate of Return (AIRR), being a weighted mean of the one-period (internal) rates i_t.[11] Note that there is no need to assume $c_t > 0$. The case $c_t < 0$ signals a financing position in a period, and the AIRR is well-defined as long as $C \neq 0$, so one may interpret it as a *generalized* weighted arithmetic mean. If $C = 0$ the AIRR is not defined, as should be expected: A *coherent* rate is a quantity, amount, or degree of something measured per unit of something else, so it is a *ratio*. Thus, it is not defined if the base is zero. For example, suppose an investor borrows \$100 at 10% and invests it at 15%. The net invested capital is $C = 0$, the return is $I = 5$. In this example, the investor gets something for nothing. In principle, one might define the project rate of return as being *infinite*, as is done by some economists and practitioners (e.g. Krugman 1979, p. 320, Shemin 2004, p. 15, Nosal and Wang 2004, pp. 2-3, Smithers 2013, p. 66). In particular, $i = +\infty$ $(-\infty)$ if $I > 0$ $(<)$. However, this is not necessary, not useful, and not even relevant, as it will become apparent later.

It is worth noting that Proposition 13 does not depend on the choice of the interim capital values: It holds for any vector $\vec{c} = (c_0, c_1, \dots, c_{n-1})$ such that $c_0 = -x_0$. As $\bar{\imath}$ does depend on the choice of $\vec{c}$, the proposition identifies a *class* of rates of return associated with a cash-flow stream. This class of metrics exists and is unique for any project and each metric i is a coherent rate of return associated with a given capital stream $\vec{c}$. The AIRR notion naturally triggers a new definition of investment and financing, which generalizes Hazen's (2003) definition.

[11] We will henceforth use the symbols i and $\bar{\imath}$ interchangeably.

Definition 3 (investment/financing) *If* $C > 0$, *the project is a (net) investment; if* $C < 0$, *the project is a (net) financing.*

Whether or not a project is a net investment or a net financing depends on whether, overall, the lending positions ($C_t > 0$) exceed the financing positions ($C_t < 0$). In the former case, $\bar{\imath}$ represents an investment rate, in the latter case it represents a financing rate. For example, if an investor invests \$100 at 5% and then borrows \$60 at 4%, one might say that the investor invests a net \$40 and the overall investment rate is 6.5%. Symmetrically, if an investor invests \$60 at 4% and then borrows \$100 at 5%, one might say that the investor borrows a net \$40 and the overall financing rate is6. 5%.

4.2 NPV-consistency and value additivity

For a relative measure of worth to be economically acceptable, it must be *NPV-consistent.* This important notion is well-established in the literature: Essentially, a metric/criterion is *NPV-consistent* or *NPV-compatible* if, applying the criterion, the decision maker makes the same decisions that he or she would make if he applied the NPV criterion (i.e., accepting positive-NPV projects). NPV-consistent metrics are then "numbers that lead to the same investment decisions as the npv-rule." (Pfeiffer 2004, p. 915). Accounting scholars use the expression "goal congruence" as a synonymous expression: "Goal congruence requires that the managers have an incentive to accept all positive NPV projects" (Gow and Reichelstein 2007, p. 115).[12] This standard notion is unanimously accepted in the (accounting, financial, engineering, managerial) literature and can be formally summarized as follows.

Definition 4 (NPV-consistency) *Consider a metric* $\phi \in \mathbb{R}$ *expressing a relative measure of worth and a market-dependent benchmark* $\psi = \psi(r) \in \mathbb{R}$:

i. *let P be an investment: then,* ϕ *is said to be NPV-consistent if* $\phi > \psi$ *whenever* $NPV > 0$;

ii. *let P be a financing: then,* ϕ *is said to be NPV-consistent if* $\phi < \psi$ *whenever* $NPV > 0$.

Note that, to fulfill the standard notion of NPV-consistency, it is necessary to distinguish investment projects from financing projects, given that, in case of an investment project, NPV is positive if the return rate is greater than the cost of capital whereas, in a financing project, the NPV is positive if the borrowing rate is smaller than the COC. Definition 3 just accounts for such a distinction.[13]

AIRR acceptability criterion. *Let P be an investment (financing). Then, the project is economically profitable if and only if* $i > r$ ($i < r$).

(See Table 1).

Proposition 14 *The AIRR approach is NPV-consistent.*

Proof. As $x_t = I_t - (c_t - c_{t-1})$ for every $t \in \mathrm{N}_n^1$ and for any $\vec{c}$, one can write $NPV = \sum_{t=0}^{n} x_t d_{t,0} = -c_0 + \sum_{t=1}^{n}(I_t - c_t + c_{t-1})d_{t,0}$, whence

$$NPV = \sum_{t=1}^{n}(I_t - rc_{t-1})d_{t,0} = \sum_{t=1}^{n}(i_t C_t - rC_t) = I - R = \Delta I.$$

Dividing by C, $i = r + \Delta I/C = r + NPV/C$, which implies $NPV = C \cdot (i - r)$. Then, Definition 4 is fulfilled, with $\phi = i$ and $\psi = r$. ■ [14]

[12] See also Pasqual et al. (2013), Magni (2009a) and references therein.

[13] It is evident, from (7), that Hazen's (2003) approach is NPV-consistent, by picking $\phi = \sigma_j$ for some j and $C = \sum_{t=1}^{n} c_{t-1}(\sigma_j) d_t$. Likewise, Hartman and Schafrick's (2004) criterion is, by construction, NPV-consistent. From Propositions 8-9, it is also evident that TRM approach is NPV-consistent as well.

[14] It is worth warning the reader that the Appendix of Weber's (2014) paper misrepresents the AIRR approach and, in addition, the author makes the erroneous (and very naïve) claim that any existing approach in the

Table 1. Financial nature of i and r

	i	r	AIRR acceptability criterion
Net investment ($C > 0$)	Rate of return	Minimum attractive rate of return	$i > r$
Net financing ($C < 0$)	Financing rate	Maximum attractive financing rate	$i < r$

Let $P = P^1 + P^2 + \cdots + P^m$ be a portfolio of m multiperiod assets. Magni (2013b, p. 752) defines the capital of a portfolio of assets as the sum of the capital amounts of the constituent assets: $c_t(P) = \sum_{j=1}^{m} c_t(P^j)$ which implies $\sum_{j=1}^{m} C(P^j) = C(\sum_{j=1}^{m} P^j)$. Therefore, the AIRR preserves value additivity. This also implies that the AIRR of a portfolio is the mean of the AIRRs of the constituent assets weighted by their capital amounts: $i(P, C(P)) = \sum_{j=1}^{m} C(P^j) i(P^j)/C(P)$ where $i(P^j)$ and $C(P^j)$ denote, respectively, the AIRR and the committed capital of project j. From the proof of Proposition 14, the computational shortcut $i = r + NPV/C$ is derived. Applying it to a portfolio of assets, one finds that the Hotelling AIRR can be written as $i(P, C(P)) = r + NPV/C(P)$.

4.3 Time-variant COCs, continuous-time AIRR and NPV

Bailey's (1959) quest for a measure capable of taking into account a vector $\vec{\imath}$ of time-variant rates is fulfilled. There still remains the issue of how to cope with a vector $\vec{r}$ of time-variant COCs. This long-standing problem is easily solved with a suitable AIRR of the equilibrium rates.

Proposition 15 (AIRR generalized) *Let $\vec{r} = (r_1, r_2, \dots, r_n)$ be the vector of COCs (equilibrium rates). Then, for any capital stream $\vec{c}$, the project AIRR and the project COC are generalized as*

$$\bar{\imath} = \frac{i_1 c_0 d_{1,0} + \cdots + i_n c_{n-1} d_{n,0}}{c_0 d_{1,0} + c_1 d_{2,0} + \cdots + c_{n-1} d_{n,0}} \qquad \bar{r} = \frac{r_1 c_0 d_{1,0} + \cdots + r_n c_{n-1} d_{n,0}}{c_0 d_{1,0} + c_2 d_{1,0} + \cdots + c_{n-1} d_{n,0}}. \tag{16}$$

The product structure is then generalized as $NPV = C(\bar{\imath} - \bar{r})$, so the AIRR acceptability criterion is unvaried, with $\phi = \bar{\imath}, \psi = \bar{r}$.

Proof. The proof is straightforward, considering that the market return can be written as $R = R(r_1, r_2, \dots, r_n) = \sum_{t=1}^{n} r_t \cdot C_t$. Imposing Chisini's invariance requirement on R one finds $R(\bar{r}, \bar{r}, \dots, \bar{r}) = R(r_1, r_2, \dots, r_n)$, whence (16). Therefore, $NPV = I - R = C(\bar{\imath} - \bar{r})$ whence the acceptability criterion. ■

It is worth noting that eq. (16) may be conveniently replaced by

$$\bar{\imath} = \frac{i'_1 c_0 + i'_2 c_1 d_{1,0} + \cdots i'_n c_{n-1} d_{n-1,0}}{c_0 + c_1 d_{1,0} + \cdots c_{n-1} d_{n-1,0}} \qquad \bar{r} = \frac{r'_1 c_0 + r'_2 c_1 d_{1,0} + \cdots r'_n c_{n-1} d_{n-1,0}}{c_0 + c_1 d_{1,0} + \cdots c_{n-1} d_{n-1,0}} \tag{17}$$

where $i'_t = i_t/(1 + r_t)$ is the beginning-of-period value of i_t.[15] In such a way, the AIRR is the mean of time-adjusted holding period rates and both numerator and denominator refer to time 0. In essence, $\bar{\imath} = I/C$ is the ratio of overall return to overall capital C where C is redefined as $C = c_0 + c_1 d_{1,0} +$

literature barring the author's is NPV-inconsistent (the author seems to be uninformed of all the scholarly research conducted on NPV-consistency of relative or absolute metrics in the economic, accounting, financial, and managerial literature).

[15] Eq. (17) can also be viewed as a linear combination of the holding period rates: $\bar{\imath} = \alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n$ where $\alpha_t = \frac{c_{t-1} d_{t,0}}{c_0 + c_1 d_{1,0} + \cdots c_{n-1} d_{n-1,0}}$ (analogously for $\bar{r}$).

$\cdots + c_{n-1}d_{n-1,0}$, which implies that $\bar{\imath}$ and $\bar{r}$ are interpreted as *instantaneous* rates. While the results hold no matter which one is used, eq. (17) might be considered more economically significant and more appealing than (16) for both scholars and practitioners, for the capital base C (the denominator) in (17) represents the *present* value of the overall capital committed. Owing to NPV-consistency, one can also write $\bar{\imath} = \bar{r} + NPV/C$.

Remark 4.2 Note that both $\bar{\imath}$ and $\bar{r}$ depend on $\vec{c}$, which means that the COC itself is not uniquely associated with a given cash-flow stream $\vec{x}$. The comparison of $\bar{\imath}$ and $\bar{r}$ is a comparison between the mean rates of return of two assets: $\bar{\imath}$ is the mean rate of return of a disequilibrium project P with cash-flow stream $\vec{x} = (x_0, x_1, \dots, x_n)$ and capital stream $\vec{c}$, while $\bar{r}$ is the mean rate of return of an equilibrium asset P° with cash-flow stream $\vec{x}^\circ = (x_0, x_1^\circ, \dots, x_n^\circ)$, where $x_t^\circ = c_{t-1}^\circ(1 + r_t) - c_t^\circ$, and capital stream $\vec{c}^\circ = \vec{c}$ (i.e., P° replicates the interim values of P).

Remark 4.3 It is also worth noting that asset P^{v_0} with cash-flow stream $\vec{x}^{\mathrm{v}_0} = (-\mathrm{v}_0, x_1, x_2, \dots, x_n)$ is itself an equilibrium asset, just like P° (they share the same internal vector $\vec{r}$), but their mean rates of return are different. Respectively,

$$\bar{r} = \frac{r_1'\, c_0 + r_2' c_1 d_{1,0} \dots + r_n' c_{n-1} d_{n-1,0}}{c_0 + c_1 d_{1,0} + \cdots + c_{n-1} d_{n-1,0}} \qquad \bar{\rho} = \frac{r_1' \mathrm{v}_0 + r_2' \mathrm{v}_1 d_{1,0} + \cdots + r_n' \mathrm{v}_{n-1} d_{n-1,0}}{\mathrm{v}_0 + \mathrm{v}_1 d_{1,0} + \cdots + \mathrm{v}_{n-1} d_{n-1,0}} \tag{18}$$

The rate $\bar{r}$ is the project COC, whereas $\bar{\rho}$ is a generalization of Modigliani and Miller's weighted average cost of capital (wacc) (see section 10).

A fuzzy approach of the AIRR paradigm has also been advanced in Guerra et al. (2012, 2014) in discrete time, and a generalization of the AIRR approach to a continuous-time setting has been sketched in Magni (2013b, Remark 3.2). For the latter, it suffices to consider a continuous stream of cash flows $x(t)$ and a continuous investment stream $c(t)$, $t \in \mathbb{R}$. Letting $c'(t)$ be the derivative of $c(t)$, the instantaneous return is $I(t) = x(t) - c'(t)$ while $i(t) = I(t)/c(t)$ is the instantaneous rate of return. Therefore, the overall return is expressed as $\int_0^n i(t)c(t)\, e^{-\int_0^t r(s)\mathrm{d}s} \mathrm{d}t$, where $r(s)$ denotes the instantaneous forward rate. Hence, the continuous AIRR and the continuous COC are defined as

$$\bar{\imath} = \frac{\int_0^n i(t)c(t)\, e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}t}{\int_0^n c(t)\, e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}t}, \quad \bar{r} = \frac{\int_0^n r(t)c(t)\, e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}t}{\int_0^n c(t)\, e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}t} \tag{19}$$

which generalizes (17). Likewise, $NPV = I - R = \int_0^n i(t)c(t)\, e^{-\int_0^t r(s)\mathrm{d}s} dt - \int_0^n r(t)c(t)\, e^{-\int_0^t r(s)\mathrm{d}s} dt$ so that $NPV = C(\bar{\imath} - \bar{r})$ where $C = \int_0^n c(t)\, e^{-\int_0^t r(s)\mathrm{d}s} \mathrm{d}t$ is the overall capital invested (or borrowed).

The notion of *residual income* is widely known in finance and accounting as a measure of value creation and fulfills the product structure period by period: $c_{t-1}(i_t - r_t)$. (see also Peasnell 1981, 1982, Peccati 1989, Brief and Peasnell 1996, O'Hanlon and Peasnell 2002; see Magni 2009a, 2010b on the relations among NPV, accounting magnitudes and the notion of residual income). The AIRR approach preserves the product structure of residual incomes in overall terms for any investment stream $\vec{c}$, which means that NPV can be viewed as an overall residual income: $NPV = I - R = C(\bar{\imath} - \bar{r})$. In other words, NPV is incorporated in the AIRR approach as a size-scaled excess AIRR. The size-scaling turns a relative measure of worth into an absolute one. However, the AIRR approach captures additional information that the traditional formula $NPV = \sum_{t=0}^n x_t d_{t,0}$ cannot provide. In particular, the product structure enables decomposing the economic value created into investment scale (C) and economic efficiency ($\bar{\imath} - \bar{r}$). The former gathers information on the net amount of capital committed, the latter measures the incremental return rate. The same NPV can be generated by different combinations of investment scale and economic efficiency, so the AIRR product structure supplies information on whether and how much of the value created depends on the investment size or the project's economic efficiency. Also, as opposed to the traditional NPV approach, the AIRR approach clarifies whether value has been created because funds are invested at an average rate of return which is greater than the average lending rate available in the market ($C > 0, \bar{\imath} > \bar{r}$) or, vice versa, because funds are borrowed at an average financing rate which is smaller than the average financing rate available in the market ($C < 0, \bar{\imath} < \bar{r}$). Moreover, it quantifies the contribution of the investment periods, where the project creates value owing to efficient investments, and the contribution of the financing periods, where the project creates value owing to efficient financings (see eq. (30) in section 7.3). Finally, it makes it possible to isolate the NPV generated by the equityholders and the NPV generated by the debtholders (see Magni 2015a for details).

5. AIRR, AROI and Keynes's notion of user cost

One might ask whether using undiscounted values in the mean leads to some economically significant rate of return for any given r. The answer is positive. Consider

$$\bar{\imath}(0) = \frac{i_1 c_0 + i_2 c_1 + \cdots + i_n c_{n-1}}{c_0 + c_1 + \cdots + c_{n-1}}. \tag{20}$$

This is a variant of the AIRR notion, which is called Aggregate Return on Investment (AROI) (see Magni 2009b, 2011c, 2015b, 2016). It can be justified economically as follows. Consider the alternative course of action consisting of a replicating portfolio P^* whose contributions and distributions exactly match the project's cash flows from time 0 to time $n-1$. At time n, the terminal distribution includes the last cash flow and the liquidation value c_n^* . The cash-flow stream is then $\vec{x}^* = (x_0, x_1, \dots, x_{n-1}, x_n + c_n^*)$. The market value of the replicating asset is $c_t^* = c_{t-1}^*(1+r) - x_t, c_0^* = c_0$, so that $c_n^* = -\sum_{t=0}^{n} x_t\,(1+r)^{n-t} = -NPV(1+r)^n$. While the period return from the project is $I_t = i_t c_{t-1}$, the period return from the replicating portfolio is rc_{t-1}^*. The difference, $\xi_t(\vec{c}) = i_t c_{t-1} - rc_{t-1}^*$, is an excess return informing about the period above-normal profit. Summing through 1 to n, one gets $\sum_{t=1}^{n} \xi_t(\vec{c}) = \sum_{t=1}^{n}(x_t - c_{t-1} + c_t) - \sum_{t=1}^{n}(x_t - c_{t-1}^* - c_t^*) = -c_n^* = NPV(1+r)^n$ (see also Magni 2009a, 2010b, 2012). Let $g(i_1, i_2, \dots, i_n) = \sum_{t=1}^{n}(i_t c_{t-1} - rc_{t-1}^*)$. Imposing Chisini's invariance requirement, $g(i_1, i_2, \dots, i_n) = g(\bar{\jmath}, \bar{\jmath}, \dots, \bar{\jmath})$, one gets

$$NPV(1+r)^n = \mathrm{K} \cdot (\bar{\jmath} - \bar{\theta}) \tag{21}$$

where $\mathrm{K} = \sum_{t=1}^{n} c_{t-1}$, $\bar{\jmath} = \bar{\imath}(0)$ and $\bar{\theta} = r \cdot \sum_{t=1}^{n} c_{t-1} / \sum_{t=1}^{n} c_{t-1}^*$. The latter is a modified COC that takes account of the different scale of the two courses of action. Note that, owing to the product structure in (21), the AROI, $\bar{\jmath}$, and the modified COC, $\bar{\theta}$, are coherent rates of return which, multiplied by the overall (undiscounted) capital K, supply, respectively, the project (undiscounted) return and the replicating portfolio's (undiscounted) return. It is worth stressing the fact that the equality $\sum_{t=1}^{n} \xi_t\,(\vec{c}) = NPV(1+r)^n$ holds for any sequence $\vec{c}$ of capital values. Therefore, it follows from (20) that AROI is a class of (infinitely many) NPV-consistent rates $\bar{\jmath}$, the appropriate COC being $\psi = \bar{\theta}$. The class of AROIs can be used for both capital budgeting purposes and investment performance measurement (Magni 2011c, 2015b, 2016, Altshuler and Magni 2015, Jiang 2017) and has the nice property that one can reframe it using cash flows: $\bar{\jmath} = \sum_{t=0}^{n} x_t \,/\, \sum_{t=0}^{n-1} c_t$ (Magni 2011c, Proposition 4.1).

The AROI notion is strictly connected with the notion of *user cost*, introduced in Keynes' (1936, 1967) *General Theory of Employment, Interest and Money*, also called *prime depreciation* by Lerner (1943). (See also Torr 1992). We show that this notion triggers an interesting rate of return whose associated capital stream is the same as that associated with the *economic AIRR*, presented later in section 7. Referring to an entrepreneur, Keynes defines user cost as the difference between "the value of his capital equipment at the end of the period . . . and . . . the value it might have had at the end of the period if he had refrained from using it" (Keynes, 1967, p. 66) or, equivalently, as the difference between "the value of the entrepreneurial stock and equipment had they not been used and… their value after use" (Scott, 1953, p. 370); thus, it compares two different choices: "The choice between . . . using a machine for a purpose and using it for another" (Coase, 1968, p. 123); it then represents the "opportunity cost of putting goods and resources to a certain use" (Scott, 1953, p. 369), an economic measure of "the opportunity lost when another decision is carried through" (Scott, 1953, p. 375). Coase (1968) relabeled it *depreciation through use* as opposed to *depreciation through time*. In our context, the entrepreneur makes the choice of either putting its capital resources in project P or investing them at the rate r. In other words, the alternative course of action consists of P^*. As seen, the capital evolves according to $c_t^* = c_{t-1}^*(1+r) - x_t, t \in \mathrm{N}_{n-1}^1, c_0^* = c_0$. The replicating asset's cash-flow vector is $\vec{x}^* = (x_0, x_1, \dots, x_n + c_n^*)$ where, in general, $c_n^* \neq 0$. The user cost, U_t, at a given date t, is formally computed as "the discounted value of the additional prospective yield which would be obtained at some later date" (Keynes, 1967, p. 70). It is then the difference between the market value of P and the market value of P^*: $U_t = \mathrm{v}_t - c_t^*$, where $\mathrm{v_t} = \sum_{k=t+1}^n x_k d_{k,t}$, $c_t^* = \sum_{k=t+1}^n x_k d_{k,t} + c_n^* d_{n,0}$, $t \in \mathrm{N}_{n-1}^1$. The depreciation through time of user cost is

$$U_{t-1} - U_t = (\mathrm{v}_{t-1} - c_{t-1}^*) - (\mathrm{v_t} - c_t^*).$$

Note that $(\mathrm{v}_{t-1} - \mathrm{v_t}) - (c_{t-1}^* - c_t^*) = r\mathrm{v}_{t-1} - rc_{t-1}^* = \xi_t(\vec{\mathrm{v}})$ where $\vec{\mathrm{v}} = (\mathrm{v}_0, \mathrm{v}_1, \dots, \mathrm{v}_{n-1})$ is the vector of P's economic values. (The depreciation of) user cost is then a special case of $\xi_t(\vec{c})$ generated by picking $c_t = \mathrm{v}_t$ for all t. Summing 1 through n, one gets $\sum_{t=1}^n (U_{t-1} - U_t) = \mathrm{v}_0 - c_0 + c_n^* = NPV - \sum_{t=0}^n x_t(1+r)^{n-t} = NPV - NPV(1+r)^n$. Now, consider the capital stream $\vec{\mathrm{v}}^{c_0} = (-c_0, \mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_{n-1})$. Picking $\vec{c} = \vec{\mathrm{v}}^{c_0}$, the associated excess return is $\xi_t(\vec{\mathrm{v}}^{c_0}) = j_t c_{t-1} - rc_{t-1}^*$ where $j_1 = (\mathrm{v}_1 + x_1 - c_0)/c_0$ and $j_t = r$ for $t > 1$, with $c_t = \mathrm{v}_t$ for $t > 1$. As $\sum_{t=1}^n \xi_t(\vec{c}) = NPV(1+r)^n$ holds for any $\vec{c}$ such that $c_0 = -x_0$, it holds for $\vec{\mathrm{v}}^{c_0}$ as well. Therefore, recalling that $rc_{t-1}^* = x_t - (c_{t-1}^* - c_t^*)$, one finds

$$\sum_{t=1}^n (U_{t-1} - U_t) = NPV - \sum_{t=1}^n \xi_t(\vec{\mathrm{v}}^{c_0}) = \mathrm{v}_0 - c_0 + \sum_{t=1}^n \left(x_t - (c_{t-1}^* - c_t^*)\right) - \left(j_1\, c_0 + \sum_{t=2}^n r\mathrm{v}_{t-1}\right).$$

We can view the above expression as a function of the rates j_1 and r appearing in the last summand: $\sum_{t=1}^n (U_{t-1} - U_t) = g(j_1, r)$. Imposing Chisini's invariance requirement $g(j_1, r) = g(k, k)$, one gets the project rate of return associated with $\vec{\mathrm{v}}^{c_0}$:

$$k = \frac{j_1 c_0 + r\mathrm{v}_1 + r\mathrm{v}_2 + \cdots r\mathrm{v}_{n-1}}{c_0 + \mathrm{v}_1 + \mathrm{v}_2 + \cdots + \mathrm{v}_{n-1}}. \tag{22}$$

This equation is a special case of (20) with $\vec{\imath} = (j_1, r, r, \dots, r)$ and $\vec{c} = \vec{c}^{\mathrm{v}_0}$, that is, $k = \bar{\jmath}(\vec{\mathrm{v}}^{c_0})$. We call k the *Keynesian rate of return*. It is worth noting that k is an AROI derived from imposing a Chisini's invariance requirement upon the sum of $U_{t-1} - U_t$; but the latter expresses the *depreciation through time* of user cost, which is in turn a *depreciation through use*. So, while it was believed that "economists cannot afford to lump together, as 'depreciation', changes in present value caused by the passage of time, and by use" (Scott, 1953, p. 371), the AROI, a variant of the AIRR, does enable one to lump together depreciation through time and depreciation through use and convert it into an economically significant rate of return (see also Magni 2009a, 2010b on user cost), such that $(c_0 + \sum_{t=1}^{n-1} \mathrm{v}_t) \cdot (k - \bar{\theta}) = NPV$ with $\bar{\theta} = r \cdot (c_0 + \sum_{t=1}^{n-1} \mathrm{v}_t) / \sum_{t=1}^n c_{t-1}^*$ (see Magni 2011c, pp. 163-164, for the AROI with time-variant COCs, r_t).[16]

6. Underdetermination of the rate of return by cash flows

[16] As a simple numerical example, consider $\vec{x} = (-100, 60, 50, -10, 40), r = 40\%$. The market values are $\mathrm{v}_1 = 45.19, \mathrm{v}_2 = 13.26, \mathrm{v}_3 = 28.57$ and $j_1 = (45.2 + 60 - 100)/100 = 5.19\%$. Therefore, k is a weighted average of 5.19% (weighted by 100) and 40% (weighted by 87), which results in $k = 21.39\%$. Alternatively, using cash flows: $k = (-100 + 60 + 50 - 10 + 40)/(100 + 87) = 21.39\%$.

Assuming an exogenously given COC, r, a cash-flow stream $\vec{x}=(x_0,x_1,x_2,\dots,x_n)$ unambiguously determines the NPV and the initial capital ($c_0=-x_0$); it does not unambiguously determine the interim values $c_1,\dots,c_{n-1}$, and, therefore, cannot unambiguously determine the rate of return.

The so-called *underdetermination of theory by data* is a well-known issue in science (see Duhem, 1914; Schlick, 1931; Quine, 1951). Underdetermination is a relation between evidence and theory. Evidence underdetermines theory in the sense that the evidence cannot determine, i.e., prove, the theory. Given a sequence of empirically observed data, and a graph which plots those data in an xy-plane, there are many infinite functions that exactly pass through those points (Schlick, 1931). In our context, this means that we may only observe two data points: At time 0, the capital invested is c_0 and, at time n, after liquidation, the capital is $c_n=0$. Given two points $(0,c_0)$ and (n,c_n) on the (t,c_t)-plane, any sequence of interim values represents a depreciation function that passes through those two points. In other words, different investment streams generate different rates of return: $i\colon \vec{c}\mapsto i(\vec{c})$. Therefore, a cash-flow stream underdetermines the investment stream and, hence, the rate of return: The link between cash flows and rate of return is not deductive.

Figure 1 illustrates five different capital streams for a 5-period project whose cash-flow stream is $\vec{x}=(-100,\ 40,\ 50,\ 20,\ -10,\ 30)$. In principle, all of the five capital streams are feasible. Any sequence of capital values generates, in general, a different rate of return i (Table 2).

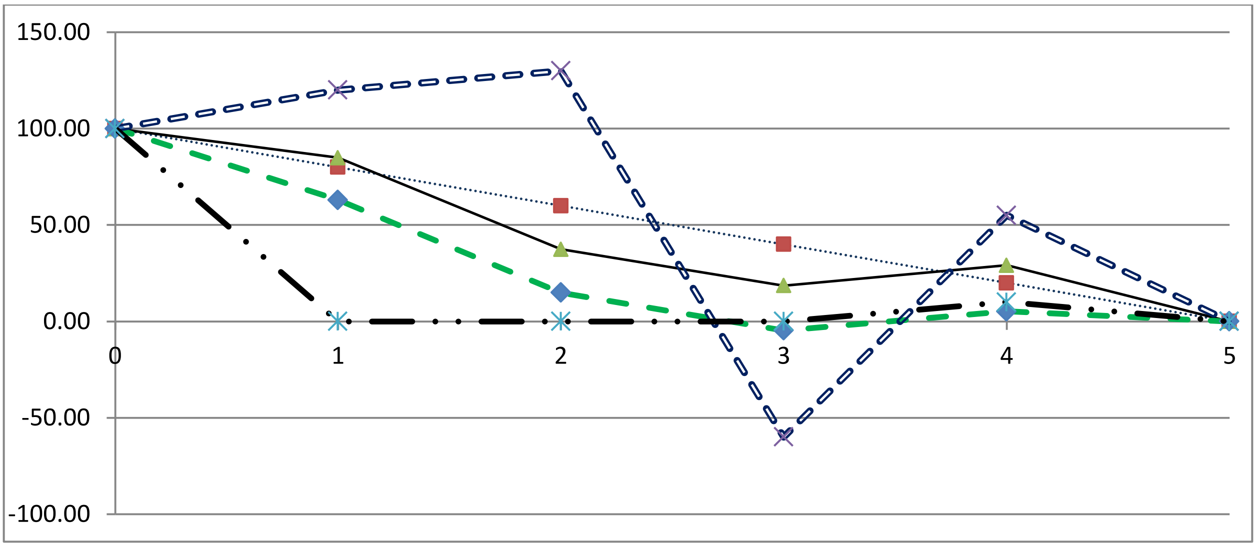


Figure 1. Different investment streams for the cash-flow stream $\vec{x}=(-100,\ 40,\ 50,\ 20,-10,\ 30)$

Table 2. Different investment streams and different rates of return for the cash-flow stream $\vec{x}=(-100,\ 40,\ 50,\ 20,-10,\ 30)$ ($r=3\%$).

time	x	$c(1)$	$c(2)$	$c(3)$	$c(4)$	$c(5)$
0	-100.00	100.00	100.00	100.00	100.00	100.00
1	40.00	80.00	84.90	63.00	120.00	0.00
2	50.00	60.00	37.45	14.89	130.00	0.00
3	20.00	40.00	18.57	-4.66	-60.00	0.00
4	-10.00	20.00	29.13	5.20	55.00	10.00
5	30.00	0.00	0.00	0.00	0.00	0.00
NPV	21.26					
C		280.20	253.00	170.44	323.30	105.71
AIRR		10.59%	11.40%	15.47%	9.58%	23.11%

The notion of *equivalence class* or *depreciation class* in the context of rates of return and NPV has been introduced and developed in Magni (2009b, 2010a,c, 2011b,c). An equivalence class of investment streams is a set of investment streams that share the same overall invested (or borrowed) capital. More precisely, let $S_{\vec{c}} = \{\vec{c} \in \mathbb{R}^n : \vec{c} = (c_0, c_1, \dots, c_{n-1}), c_0 = -x_0\}$ be the set of all possible depreciation patterns for project P and consider the subset of all vectors $\vec{c} = (c_0, c_1, \dots, c_{n-1})$ such that $c_0 = -x_0$ and $C = \sum_{t=1}^{n} c_{t-1} d_{t,0}$. For two vectors $\vec{a}, \vec{b} \in S_{\vec{c}}$, define $\vec{a} \sim \vec{b}$ if $A = B$. Such a relation induces a partition of $S_{\vec{c}}$ in equivalence classes of capital streams. For a given $\vec{x}$, an equivalence class $[C]$ unambiguously determines the overall capital C and the rate $\bar{\imath}$; that is, $i: \vec{c} \mapsto C(\vec{c}) \mapsto i(C)$. Therefore, a rate of return depends on the depreciation class $[C]$. This result is but a corollary of Propositions 11-13.

Corollary 1 *Given a cash-flow stream $\vec{x}$, there is a biunivocal relation between a depreciation class $[C] \subset S_{\vec{c}}$ and a rate of return $\bar{\imath}$. Therefore, for any project, a rate of return is singled out if and only if a depreciation class $[C]$ is selected, implicitly or explicitly.*

In other words, the AIRR perspective shows that a cash-flow stream is uniquely associated (not with a return *rate* but) with a return *function*, $i(C) = r + NPV/C$, which is graphically represented by an iso-value curve consisting of the set of pairs (C, i) that lead to the same NPV (see Figure 2).

Multiple (indeed, infinite) rates exist for any cash-flow stream, each of which is associated with its own depreciation class $[C]$. The traditional quest for *uniqueness* of a project rate of return, long sought for, was then ill-formulated: Multiple rates of return exist conveying different pieces of information.

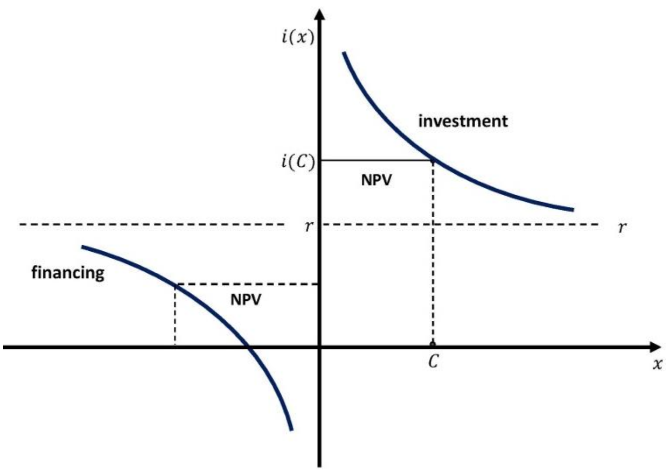
Note that, if the COC is time-varying, the degree of underdetermination is even higher, since $\bar{\imath} = \bar{\imath}(\vec{c})$ is a function of $\vec{c}$, not C, so, in general, different capital streams $\vec{c}$ supply different AIRRs, even if the overall capital C (and, therefore, the depreciation class) is the same.

7 Depreciation classes and rates of return

This section presents some economically significant metrics that can be generated from the AIRR approach by changing the depreciation pattern $\vec{c}$. This sheds light on the analogies and differences that known (and unknown) measures bear to one another: They are all AIRRs associated with different depreciation patterns. We assume that the COC is constant, unless otherwise stated, so these metrics are all AIRRs associated with different depreciation *classes* (Table 3 collects the metrics presented in the section).

Table 3. Different capital depreciation classes lead to different points on the iso-value curve

$(C, i(C))$	**AIRR**	**depreciation**
(C^{σ}, σ),	IRR	Hotelling
$(C^{a}, i(C^{a}))$	DA-AIRR	mark-up
(C^{M}, σ^{M})	MIRR	modified Hotelling
$(C^{I,F}, i(C^{I,F}))$	dual AIRR	dual Hotelling
$(C_1, i(C_1))$	IC-AIRR=SRR	cash-flow
$(X^{-}, i(X^{-}))$	TC-AIRR	cash-flow
$(V^{c_0}, i(V^{c_0}))$	EAIRR	economic
$(C^{*}, i(C^{*}))$	RP-AIRR	replicating (discrete)
$(C^{TS}, i(C^{TS}))$	TS-AIRR	replicating



		(continuous)
$(B, i(B))$	AROA	book value
$(C^{SL}, i(C^{SL}))$	SL-AIRR	straight-line
$(M, i(M))$	MV-AIRR	market

Figure 2. The iso-value curve of a value-creating project ($NPV > 0$), assuming time-invariant COC. Different pairs (C, i) generate the same NPV, which is the area of the rectangle with base $|C|$ and height $|i - r|$ (excess AIRR). If $C > 0$, the project is interpreted as a net investment, if $C < 0$ the project is interpreted as a net financing. Each pair $(C, i(C))$ is a point lying on the iso-value curve.

7.1 The Hotelling class, the IRR, and the Direct Alpha.

Let $\vec{c}(\sigma) = (c_0(\sigma), c_1(\sigma), \dots, c_{n-1}(\sigma)), c_0(\sigma) = c_0$, be the capital stream under (Boulding's) assumption of constant rate of return: $c_t(\sigma) = c_{t-1}(\sigma)(1+\sigma) - x_t$. We henceforth refer to it as the project's *Hotelling value* (Hotelling 1925). Let $C^\sigma = \sum_{t=1}^n C_t(\sigma)$. Consider the *Hotelling class*, denoted as $[C^\sigma] = \{\vec{c} \in S_{\vec{c}} \text{ such that } \vec{c} \simeq \vec{c}(\sigma)\}$. Thus, C^σ is the total capital associated with the Hotelling class and $i(C^\sigma) = r + NPV/C^\sigma$ is the *Hotelling* AIRR. However, from (2) and (3), we know that σ is a project IRR. This implies that an IRR is a Hotelling AIRR: $i(C^\sigma) = r + (\sigma C^\sigma - rC^\sigma)/C^\sigma = \sigma$. This in turn implies that the IRR does not unambiguously identify the capital stream: The equation $\sum_{t=1}^n c_{t-1} d_{t,0} = C^\sigma$ has infinitely many solutions of the type $(c_0, c_1, \dots, c_{n-1})$ with $c_0 = -x_0$. In other words, the Hotelling class contains infinitely many investment streams with time-variant period rates i_t that generate C^σ as the invested capital; picking one of those streams, the weighted average of the corresponding period rates is just the IRR. Therefore, unlike the usual interpretation, an IRR is uniquely associated with a class $[C^\sigma]$ of depreciation schedules, not with the Hotelling stream $\vec{c}(\sigma)$, which is only one of infinitely many streams belonging to $[C^\sigma]$ (see Magni 2010a, Theorem 3): For any $\vec{c} \in [C^\sigma]$, $i(C^\sigma) = \sigma$. As a result, "a (real-valued) IRR is a particular case of AIRR generated by a Hotelling class of investment streams. The class contains infinite elements, so there exist infinite investment streams which give rise to that IRR as the AIRR of the class." (Magni 2010a, p. 163).

Proposition 16 *A project IRR is the AIRR associated with the Hotelling class. Therefore, an IRR is a generalized arithmetic mean of (generally time-variant) holding period rates* i_t.

Graphically, the IRR and its associated capital determine a point $(C^\sigma, \sigma) = (C^\sigma, i(C^\sigma))$ on the iso-value curve. Multiple IRRs, $\sigma_j, j \in \mathrm{N}_m^1$, imply multiple Hotelling classes, $[C^{\sigma_j}]$ and multiple points on the iso-value curve. As a result, Proposition 10 is a particular case of Proposition 15, so Hazen's criterion can be viewed as a particular case of the AIRR criterion.

Considering that $c_t(\sigma)$ can be framed as $c_t(\sigma) = \sum_{k=t+1}^n x_k (1+\sigma)^{t-k}$, one can introduce a new alternative definition of IRR.

Definition 5 (IRR) *Given an exogenously fixed COC, r, an IRR is a solution of the following equation:*

$$\sigma = r + \frac{\sum_{t=0}^n x_t (1+r)^{-t}}{\sum_{t=0}^n \sum_{k=t+1}^n x_k (1+\sigma)^{t-k} (1+r)^{-t}}.$$

(The relation is circular, but a simple spreadsheet can easily find the solutions, if any exist).

Remark 7.1 The IRR belongs to the AIRR class, but its depreciation class is implicitly derived from the IRR equation (see Boulding's derivation above). This "automatism" prevents IRR from preserving value additivity. Let $x_t(P^j)$ be the time-t cash flow of a project $P^j, j \in \mathrm{N}_p^1$. The Hotelling value of portfolio $P = \sum_{j=1}^p P^j$ differs in general from the sum of the Hotelling values of the constituent assets P^j: $c_t\left(\sum_{j=1}^p P^j\right) \neq \sum_{j=1}^p c_t(P^j)$. As an example, consider a portfolio of three assets whose cash-flow streams are $\vec{x}^1 = (-150, 30.5, 0, 120, 15)$, $\vec{x}^2 = (-50, 0, 120, 0, 17)$, $\vec{x}^3 = (90, -15, -20, -25, -30)$ so that P's cash-flow stream is $\vec{x} = \vec{x}^1 + \vec{x}^2 + \vec{x}^3 = (-110, 15.5, 99, 5, 95, 2)$. The (unique) IRRs of the assets are $\sigma^1 = 3.69\%, \sigma^2 = 59.19\%, \sigma^3 = 0.2\%$. The Hotelling streams are $\vec{c}(P^1) = (150, 125, 129.7, 14.5)$, $\vec{c}(P^2) = (50, 79.6, 6.7, 10.7)$, $\vec{c}(P^3) = (-90, -75, -55, -30)$. Conversely, the portfolio's IRR is $\sigma^{1+2+3} = 32.24\%$, and its Hotelling stream is $\vec{c}(P) = (110, 130.3, 72.8, 1.5) \neq (110, 129.6, 81.4, -4.9) =, \vec{c}(P^1) + \vec{c}(P^2) + \vec{c}(P^3)$. Therefore, IRR does not preserve value additivity, which implies that a portfolio's IRR is not equal to the Hotelling AIRR of the portfolio: $\sigma^{1+2+3} \neq i\big(C^{\sigma^{1+2+3}}\big) = i(C^{\sigma_1} + C^{\sigma_2} + C^{\sigma_3})$, the latter being equal to the weighted average of the individual IRRs, $i\big(C^{\sigma^{1+2+3}}\big) = (\sum_{j=1}^{3} C^{\sigma_j} \cdot i(C^{\sigma_j})) / \sum_{j=1}^{3} C^{\sigma_j}$ as shown in section 4.2 (see also Magni 2013a, Danielson 2016, Jiang 2017).

Griffiths (2009) and Gredil et al. (2014) endorsed the use of the *Direct Alpha* method for private equity investments (PEI). The alpha the authors refer to is incorporated in a continuous-time model according to which a PEI's (continuous) rate of return, $i(\tau), \tau \in \mathbb{R}$, is the sum of a benchmark return, expressed as a force of interest, $r(\tau)$, and an (assumed constant) excess term, α: $i(\tau) = r(\tau) + \alpha$. In our context, this implies, using the continuous-time AIRR setting,

$$NPV = \int_0^n i(\tau)c(\tau)\, e^{-\int_0^\tau r(s)\mathrm{d}s}\mathrm{d}\tau - \int_0^n r(\tau)c(\tau)\, e^{-\int_0^\tau r(s)\mathrm{d}s}\mathrm{d}\tau = \alpha C^\alpha \tag{23}$$

where $c(\tau) = -\int_0^\tau x(u)\, e^{\int_u^\tau r(s)\mathrm{d}s} \cdot e^{\alpha(\tau-u)}\, du$ is the capital value at τ and $C^\alpha = \int_0^n c(\tau)\, e^{-\int_0^\tau r(s)\mathrm{d}s} d\tau$ is the aggregate capital (see section 4.3). Discretizing time, the forward rate r_t is such that $(1 + r_t) = e^{\int_{t-1}^t r(s)\mathrm{d}s}, t \in \mathrm{N}_n^1$ and the (discrete) direct alpha is $a = e^\alpha - 1$. Therefore, the capital is recursively obtained by marking up the equilibrium rate: $c_t = c_t(a) = c_{t-1}(a)(1 + r_t)(1 + a) - x_t$. Recalling that $c_n = 0$, the (discrete) direct alpha can be found as a solution of $\sum_{t=0}^n x_t d_{t,0}(1 + a)^{-t} = 0$ and, hence $c_t(a) = -\sum_{k=0}^t (x_k/d_{t,k}) \cdot (1 + a)^{t-k}, t \in \mathrm{N}_{n-1}^1$. The corresponding *direct-alpha AIRR* is computed as in (17) with $c_t = c_t(a)$. Equivalently, $\bar{\imath} = \bar{r} + \frac{NPV}{C^a}$ where $C^a = \sum_{t=1}^n c_{t-1}(a) d_{t-1,0}$. Therefore, $C^a(\bar{\imath} - \bar{r}) = NPV$ and, using (23), $\alpha = (\bar{\imath} - \bar{r}) \cdot \left(\frac{C^a}{C^\alpha}\right)$, which tells us that the (discrete) direct alpha is a modified excess AIRR.

If the force of interest is constant, the term structure of interest rate is flat: $r_t = r$ for all t, and $r(t) = \delta = \ln(1 + r)$ for all t. Then, $e^{\int_0^t \delta \cdot \mathrm{d}s} \cdot e^{\alpha t} = e^{(\delta+\alpha)t} = (1 + q)^t$ for all t, where $q = r + a(1 + r)$. Hence, $c_n = -\sum_{t=0}^n x_t \cdot (1 + q)^{n-t} = 0$ which implies $q = \sigma = \bar{\imath}$ and

$$\alpha = \ln(1 + \sigma) - \ln(1 + r).$$

The latter relation means that, if, multiple IRRs arise, multiple direct alphas arise as well. However, if the force of interest $r(\tau)$ (or, equivalently, the equilibrium rate r_t) is not constant, multiple direct alphas may arise even if the IRR is unique. For example, consider $\vec{x} = (-100, 585, -1056.3, 540.54)$ and let $\vec{r} = (0.5, 0.4, 0.3, 0.2)$. The unique IRR is 8.82%, but $\sum_{t=0}^n x_t d_{t,0}(1 + a)^{-t} = 0$ has three solutions: $a_1 = 0.1$, $a_2 = 0.3$, $a_3 = 0.5$ (and, correspondingly, $\alpha_1 = 0.0953$, $\alpha_2 = 0.2624, \alpha_3 = 0.4055$). Each direct alpha is associated with its own corresponding depreciation pattern.

7.2 The modified Hotelling class and the MIRR

As seen in section 3, the MIRR is itself an IRR of the cash-flow vector $\vec{x}^M = (x_0, 0, \ldots, 0, \sum_{t=1}^n x_t\, (1 + y)^{n-t})$.

Proposition 17 *The MIRR is the AIRR associated with the modified project's Hotelling class* $[C^{\sigma^M}]$:

$$\sigma^M = r + \frac{NPV}{C^{\sigma^M}}. \tag{24}$$

In particular, if a project IRR exists, then, for any IRR, σ_j, the MIRR is a weighted average of σ_j, and the reinvestment/financing rate y:

$$\sigma^M = \lambda\sigma_j + (1-\lambda)y \tag{25}$$

where λ expresses the capital invested in P as a proportion of total capital C^{σ^M}.

Proof. Consider the cash-flow vector $\vec{x}^{(t)} = -x_t \cdot \vec{1}_t + x_t\,(1+r)^{n-t} \cdot \vec{1}_n$ where $\vec{1}_t$ is a vector of zeroes except the t-th component, which is equal to 1, $t \in \mathrm{N}_{n-1}^1$. The modified project $\vec{x}^M$ can then be viewed as a portfolio composed of P and the additional projects $P^{(t)}$ whose cash flow streams are $\vec{x}^{(t)}, t \in \mathrm{N}_{n-1}^1$: $\vec{x}^M = \vec{x} + \vec{x}^{(1)} + \vec{x}^{(2)} + \cdots + \vec{x}^{(n-1)}$. As $P^{(t)}$ has no interim cash flows, its capital is $c_k^{(t)} = c_{k-1}^{(t)}(1+y)$. This implies $c_k^{(t)} = 0$ for $k < t$ and $c_k^{(t)} = x_t(1+y)^{k-t}$ for $k \geq t$. Hence, the overall (Hotelling) capital is $C^{(t)}(r) = \sum_{k=1}^n c_{k-1}^{(t)} d_{k,0}$. The overall capital put in place by the interim reinvestment/financing operations is then $\sum_{t=1}^{n-1} C^{(t)}(r)$. The net return from project P is $\sigma_j \cdot C^{\sigma_j}$ and the net return from the reinvestments/financings is $y \cdot \sum_{t=1}^{n-1} C^{(t)}(r)$. Therefore, exploiting the associativity property of the Chisini mean:[17]

$$\sigma^M = \frac{\sigma_j C^{\sigma_j} + y\sum_{t=1}^{n-1} C^{(t)}(r)}{C^{\sigma^M}}. \tag{26}$$

where $C^{\sigma^M} = C^{\sigma_j} + \sum_{t=1}^{n-1} C^{(t)}(r)$. The thesis follows with $\lambda = C^{\sigma_j}/C^{\sigma^M}$.[18] ∎

Note that, if $y = r$, then $\sigma^M = \lambda\,\sigma_j + (1-\lambda)r$ and the additional projects $P^{(t)}$ are value-neutral: $NPV\big(P^{(t)}\big) = 0$ for all $t \in N_{n-1}^1$. Hence, the overall NPV of the modified project is equal to the NPV of P: $NPV = NPV\big(P + \sum_{t=1}^{n-1} P^{(t)}\big) = C^{\sigma^M} \cdot (\sigma^M - r)$; by Definition 4, the MIRR is NPV-consistent (consistency is not guaranteed if $y \neq r$). Note that $C^{\sigma^M}(\sigma^M - r) = \lambda C^{\sigma^M}(\sigma_j - r)$. If $C^{\sigma^M} > 0$ and $\lambda < 0$, the MIRR is an investment rate whereas the project IRR is a financing rate.

7.3 TRM (1965) – The dual Hotelling class and the dual rate of return

As seen in section 3, TRM (1965a,b) did not provide any rate of return for the entire project. In addition, they introduce the restriction that either the PIR or the PFR is equal to the COC, r. The AIRR approach enables filling the gap under more general assumptions. Let σ_I and σ_F be the PIR and PFR, respectively, not necessarily equal to r. Let a *mixed project* be defined (more generally than in TRM) as a project such that there exists some $k, j \in N_{n-1}^0$ such that $c_k \cdot c_j < 0$. In this case, one can split up the project into an investment region and a financing region: The investment region is the set $T^I = \{t\colon t \in \mathrm{N}_{n-1}^0 \wedge c_t \geq 0\}$ of those periods where the firm invests the capital in the project (the investment rate, σ_I, applies); the financing region is the set $T^F = \{t\colon t \in \mathrm{N}_{n-1}^0 \wedge c_t < 0\}$ of periods where the firms subtracts capital from the project (the financing rate, σ_F, applies). Let $C^I = C^I(\sigma_I, \sigma_F) = \sum_{t\in T^I} C_t > 0$ denote the total capital loaned to the project and $C^F = C^F(\sigma_I, \sigma_F) = \sum_{t \in T^F} C_t < 0$ denote the total capital loaned from the project. Consider the *dual Hotelling* class $[C^{I,F}]$ where $C^{I,F} = C^I + C^F$. Propositions 11-13 trigger what we call the *dual AIRR*:

$$i(C^{I,F}) = r + \frac{NPV}{C^{I,F}} \tag{27}$$

or, equivalently, $\bar{\imath} = (C^I\sigma_I + C^F\sigma_F)/(C^I + C^F)$. The point $(C^{I+F}, i(C^{I+F}))$ lies on the iso-value curve. Under TRM's more restrictive assumptions, either $\sigma_F = r$ (so that $\bar{\imath} = (C^I\sigma_I(r) + C^F \cdot r)/(C^I + C^F)$, with $C^j = C^j(\sigma_I(r), r)$, $j = I, F$) or $\sigma_I = r$ (so that $\bar{\imath} = (C^I \cdot r + C^F \cdot \sigma_F(r))/(C^I + C^F)$, with $C^j = C^j(r, \sigma_F(r)), j = I, F$).

[17] See also Magni (2013b, Section 4) on portfolios' capital values and rates of return.

[18] It should be noted that, in case of multiple IRRs, λ depends on σ_j: Different IRRs generate different values of λ.

The AIRR paradigm naturally induces a generalization of eq. (27). Allowing for time-variant investment rates $\sigma_{t,I}$ as well as financing rates $\sigma_{t,F}$, consider the AIRR of the investment rates and the AIRR of the financing rates, respectively:

$$\bar{\sigma}_I = \frac{\sum_{t\in T^I} \sigma_{t,I} \cdot C_t}{\sum_{t\in T^I} C_t}, \quad \bar{\sigma}_F = \frac{\sum_{t\in T^F} \sigma_{t,F} \cdot C_t}{\sum_{t\in T^F} C_t}. \tag{28}$$

The first rate is a PIR and the latter is a PFR. Likewise, we define a market investment AIRR and a market financing AIRR:

$$\bar{r}_I = \frac{\sum_{t\in T^I} r_t C_t}{\sum_{t\in T^I} C_t}, \quad \bar{r}_F = \frac{\sum_{t\in T^F} r_t C_t}{\sum_{t\in T^F} C_t}. \tag{29}$$

The net project return is $I = \bar{\sigma}_I\, C^I + \bar{\sigma}_F\, C^F$ while the net market return is $R = \bar{r}_I\, C^I + \bar{r}_F\, C^F$. As $NPV = I - R$, the NPV can be decomposed into an investment NPV and a financing NPV:

$$NPV = \underbrace{C^I(\bar{\sigma}_I - \bar{r}_I)}_{\text{investment NPV}} + \underbrace{C^F(\bar{\sigma}_F - \bar{r}_F)}_{\text{financing NPV}}. \tag{30}$$

Unlike the traditional NPV approach, the AIRR approach enables analysts to unearth the causes of a given NPV: Value can be created not only by investing funds at a greater return rate than the market investment rate, but also by borrowing funds from the project at a rate which is smaller than the market financing rate. For example, if the net working capital of a project is negative and sufficiently high in absolute value (as opposed to fixed assets), then the capital is negative: It means that customers and suppliers are "coining" money for the firm.[19]

Further, there exist a unique rate of return/cost $\bar{\imath}$ and a unique COC, $\bar{r}$, which are the (generalized) weighted mean of the PIR and PFR and the weighted mean of the investment market AIRR and financing market AIRR, respectively:

$$\bar{\imath} = \frac{C^I\bar{\sigma}^I + C^F\bar{\sigma}^F}{C^I + C^F}, \quad \bar{r} = \frac{C^I\bar{r}^I + C^F\bar{r}^F}{C^I + C^F}. \tag{31}$$

(see also Magni 2014a).

Remark 7.2 As seen above, the direct-alpha setting assumes that the forward rates are marked up by a constant term $a = \ln(1+\alpha)$ so that $i_t = r_t + a(1+r_t)$. One can generalize and assume that the direct alpha is not constant but depends on the sign of the interim value, so that $i_t = r_t + a_I(1+r_t)$ if $c_{t-1} \geq 0$ or $i_t = r_t + a_F(1+r_t)$ if $c_{t-1} < 0$, $a_I \neq a_F$. This is a special case of dual Hotelling depreciation. Therefore, the DA-AIRR, $\bar{\imath} = \bar{r} + NPV/C^a$ is a special case of the dual AIRR presented in (31), obtained assuming $a_I = a_F = a$ (see also Magni 2014a, Theorem 8 and footnote 8 above).

7.4 Rate of return on initial contribution (IC-AIRR) and rate of return on total contributions (TC-AIRR)

Consider the class of those capital streams associated with the initial investment (Magni 2010a, pp. 168-169; Magni 2013a, p. 99). This is the class $[C_1]$, induced by $\vec{c} = (c_0, 0, \dots, 0)$. The related AIRR expresses the rate of return per unit of initial investment:

$$i(C_1) = r + \frac{NPV}{C_1}. \tag{32}$$

We call it the *initial-contribution* AIRR (IC-AIRR). It is worth noting that it is an affine transformation of the well-known profitability index (PI): $PI = \frac{NPV}{c_0} = (i(C_1) - r)/(1+r)$. We will later show that

[19] Such a business strategy has been used by such firms as Wal-Mart, Amazon, and McDonald's. Typical businesses that may have negative working capital are restaurants, grocery stores, online retailers, discount retailer, broadcasting firms (see a detailed example in Magni 2015a).

$i(C_1)$ is equal to j_1, the first-period rate of return incorporated in the Keynesian rate of return, k (section 5).

Consider the class of those capital streams associated with the total cash outflow (e.g., Magni 2010a, pp. 168-169; Magni 2013a, p. 99). Let $x^-_{h_j}$ denote the absolute value of an outflow and let $h_j \in \mathrm{N}^0_n$ be the date at which it occurs, $j \in A \subset \mathrm{N}^0_n$. Consider the class $[X^-]$ induced by $\vec{c} = (x^-_t$ if $t = h_j$; 0 otherwise). We call the associated rate of return the *total-contribution* rate of return (TC-AIRR), since it expresses the rate of return per unit of total outflow:

$$i(X^-) = r + \frac{NPV}{X^-}. \tag{33}$$

The IC-AIRR and the TC-AIRR can be seen as the rates of return that are obtained by choosing a depreciation class where investments are expensed as incurred (cash-flow depreciation).[20]

7.5 Bailey (1959) and efficient markets – Shareholder rate of return (SRR) and Economic AIRR (EAIRR)

What happens if one follows Bailey's (1959) advice of assuming $i_t = r_t$ for all t except one? Bailey (1959) did not specify which rate should be allowed to differ from the cost of capital. This section shows that the choice made by an efficient market is unambiguous: $i_1 \neq r_1, i_t = r_t$ for $t > 1$. More specifically, assume that a firm has the opportunity of investing in a project P whose cash-flow stream is $\vec{x} = (x_0, x_1, \dots, x_n)$ and its shares are traded in the security market (which, as specified earlier, we assume to be frictionless and arbitrage-free). For the sake of simplicity, we maintain the assumption that $r_t = r$ for all t (generalization is straightforward via Proposition 15).

When the decision of undertaking a project is made public by the firm a state of disequilibrium occurs due to new information. The stock price adjusts instantaneously to arbitrage away the disequilibrium (see Ross et al. 2011, pp. 326–327). As Rubinstein (1973, p. 172) puts it, "when a project is undertaken, the firm can be viewed in temporary disequilibrium", and for the project to be worth undertaking, "it is . . . necessary to show that, given the market strives for equilibrium, the market value of the stock will increase by more than the investment outlay" (Bierman and Hass, 1973, p. 122). Shareholders' wealth increase can be interpreted as a 'windfall gain' accruing to shareholders at time 0, generated by the firm's ability of investing in a project that is more profitable than the standard market rate r. After the new equilibrium has been achieved, "there is no tendency for subsequent increases and decreases" (Ross et al., 2011 p. 327). In particular, "once the 'windfall gain' is realized through the increase in value of the owners' stock, income will continue to be realized at a rate of exactly $[r]$ ... for the remainder of the project's life" (Robichek and Myers 1965, p. 11). We now formalize such a disequilibrium-to-equilibrium process and show that a frictionless, arbitrage-free market triggers an unambiguous project rate of return, which we call *Economic AIRR* (EAIRR), where all the holding period rates, except the first one, are equal to the equilibrium rate, r. To this end, we will use the capital stream $\vec{\mathrm{v}}^{c_0} = (c_0, \mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_{n-1})$. Consider the *economic* class $[V^{c_0}] = \{\vec{c} \in S_{\vec{c}}$ such that $\vec{c} \simeq \mathrm{v}^{c_0}\}$ (note that $\vec{\mathrm{v}} \neq \vec{\mathrm{v}}^{c_0}$).[21]

Proposition 18 (Economic AIRR) *In an efficient market, where disequilibrium is quickly arbitraged away, investors' rational behavior induces the economic class* $[V^{c_0}]$. T*he project rate of return is the weighted average of the disequilibrium rate,* i_1*, and the equilibrium rate,* r*:*

$$\bar{\imath} = i_1\mu + r(1-\mu) \tag{34}$$

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[20] The vector $\vec{c}(5)$ in Table 2 refers to the equivalence class of total cash outflow, so 23.11% is the TC-AIRR.

[21] Lindblom and Sjögren (2009) endorsed the use of $\vec{\mathrm{v}}^{c_0}$ (referred to as "strict market-based depreciation schedule") for increasing goal congruence in managerial performance evaluation. They showed that such a choice is superior to ordinary straight-line, annuity-based or IRR-based depreciation schedules (see also Brealey et al. 2011, p. 331.).

where $i_1 = i(C_1)$, $\mu = C_1/V^{c_0}$, and $V^{c_0} = c_0 d_{1,0} + \sum_{t=2}^{n} \mathrm{v}_t\, d_{t,0}$. *Equivalently,*

$$\bar{\imath} = i(V^{c_0}) = r + \frac{NPV}{V^{c_0}}. \tag{35}$$

Proof. The current market value of the firm's equity is $E_0 = s \cdot p$, where s is the number of outstanding shares. Denoting as e_t the firm's prospective equity cash flow for $t \in \mathrm{N}_q^1, q \leq +\infty$, before acceptance of the project, the following equilibrium relation holds: $E_0 = (e_1 + E_1) \cdot (1 + r)^{-1}$ (where E_1 is the end-of period equity market value) which implies

$$sp(1 + r) = e_1 + E_1. \tag{36}$$

After acceptance, the firm's equity cash flows become $e_t' = e_t + x_t$ with $x_t = 0$ for $t > n$. The market reacts to fully reflect new information. With no loss of generality, we assume that the firm issues Δs shares at the new equilibrium price $p' = p + \Delta p$ to finance the project, so that $\Delta s \cdot p' = c_0$, whence $\Delta s = c_0/(p + \Delta p)$. This implies that, after equilibrium is restored, the firm's equity is $E_0' = (s + \Delta \mathrm{s}) \cdot (p + \Delta p)$ and the new equilibrium relation is $E_0' = (e_1 + E_1 + x_1 + \mathrm{v}_1) \cdot (1 + r)^{-1}$, which implies

$$(s + \Delta \mathrm{s})(p + \Delta p)(1 + r) = e_1 + x_1 + E_1 + \mathrm{v}_1 \tag{37}$$

The project's first-period return rate, i_1, generated by the temporary state of disequilibrium is

$$i_1 = (x_1 + \mathrm{v}_1)/c_0 - 1. \tag{38}$$

Subtracting (36) from (37), one finds $x_1 + \mathrm{v}_1 = (s\Delta p + \Delta s(p + \Delta p))(1 + r)$. Hence, exploiting the equality $c_0 = \Delta s(p + \Delta p)$, (38) can be framed as $i_1 = s\Delta p(1 + r)/\Delta s(p + \Delta p) + r$. However, $s\Delta p = E_0' - E_0 - c_0 = (x_1 + \mathrm{v}_1)(1 + r)^{-1} - c_0 = \sum_{t=0}^{n} x_t d_{t,0} = NPV$, so that $i_1 = r + NPV/C_1 = i(C_1)$. Consider now that, after the first period, shareholder income will continue to be realized at the equilibrium rate r for $t \in \mathrm{N}_q^1$, which just means that the project's invested capital will be v_t for every $t \in \mathrm{N}_n^1$. Therefore, the capital stream is $\vec{\mathrm{v}}^{c_0} = (c_0, \mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_{n-1})$ and the internal vector is $\vec{\imath} = (i(C_1), r, r, \dots, r)$. The value-weighted average of such rates is (34). ■

(See also Magni 2013a, 2014a).[22]

Remark 7.3 The above proposition says that the IC-AIRR is the first-period disequilibrium rate of return generated in an efficient market. We call it *shareholder rate of return* (SRR) (see also Fernandez's, 2002, ch. 1, notion of shareholder return). Note that $i_1 = i(C_1) = j_1$, that is, the IC-AIRR is incorporated in the Keynesian rate of return. It is also worth noting that the Keynesian rate of return and the EAIRR share the same internal vector $\vec{\imath} = (j_1, r, r, \dots, r)$ (and, therefore, the same depreciation schedule $\vec{\mathrm{v}}^{c_0}$), but the former uses undiscounted weights, whereas the latter uses discounted weights.

Remark 7.4 The proof of Proposition 18 makes it clear that the shareholders' wealth increase is just the project NPV: $\Delta p \cdot s = \mathrm{v}_0 - c_0$.[23] This is the windfall gain of a shareholder owning stocks before the undertaking of the project who sells the shares after the announcement of project undertaking. It is worth noting that $j_1 = (x_1 + \mathrm{v}_1 - c_0)/c_0$ can be decomposed into two shares: Shareholders earn an instantaneous gain equal to NPV at time 0 and a subsequent return equal to $r\mathrm{v}_0$ at the end of the period so that $j_1 = \frac{NPV}{c_0} + \frac{r\mathrm{v}_0}{c_0}$.

Remark 7.5 We have shown that (i) the EAIRR is the *value*-weighted *arithmetic* mean of j_1 and r, (ii) the Keynesian rate of return is the (*undiscounted*)-*value*-weighted *arithmetic* mean of j_1 and r, (iii) the MIRR is the *value*-weighted *arithmetic* mean of σ and r. Barry and Robison (2014) found the

[22] The vector $\vec{c}(2)$ in Table 2 refers to the economic class, so 11.4% is the EAIRR.

[23] Note that the project is worth undertaking if and only if $\Delta p > 0$. This is equivalent to $EAIRR > r$, as we should expect (because $\Delta p = NPV/s$).

result that MIRR is a *time*-weighted *geometric* mean of j_1 and r. More precisely, $1+\sigma^M = \sqrt[n]{(1+j_1)(1+r)^{n-1}}$.

7.6 Bailey (1959) and Anthony (1975) – Replicating portfolio's rate of return and return on time-scaled contribution

In the previous section, Bailey's (1959) recommendation was fulfilled with the internal vector $\vec{\imath}=(j_1,r,r,\dots,r)$, which means that value creation is recognized in the first period. However, his recommendation might also be fulfilled by picking the internal vector $\vec{\imath}=(r,r,\dots,r,j_n)$, $j_n=x_n/c^*_{n-1}-1$, which means that value creation is recognized only in the last period. In terms of capital depreciation, this means that the capital stream is $\vec{c}^*=(c_0,c^*_1,c^*_2,\dots,c^*_{n-1})$ (see section 5). Apparently, the first one to endorse this pattern was Anthony (1975). He viewed equity as shareholders' credit and recommended to use a depreciation schedule such that equity interest should be recognized at the market rate. In such a way, every period is value-neutral and income (and, therefore, value creation) is recognized only in the last period. In later years, the same depreciation pattern was used by several different authors, in disparate contexts, with different labels and justified in different ways and for various purposes (Long and Nickels 1996, O'Hanlon and Peasnell 2002, Drukarczyk and Schueler 2000, Magni 2000. See Magni 2004, 2005, 2010b, 2012, Ghiselli Ricci and Magni 2014 for discussions and findings related to this depreciation pattern). We now discuss the AIRR associated with such a depreciation plan. Consider a portfolio traded in the market such that the contributions and distributions exactly match the project's cash flows. Assume, with no loss of generality, $r_t=r$; then, the replicating portfolio's value is $c^*_t=c^*_{t-1}(1+r)-x_t$ and the overall capital is $C^*=\sum_{t=1}^n c^*_{t-1}d_{t,0}$. We call the corresponding AIRR the *replicating portfolio AIRR* (RP-AIRR):

$$i(C^*)=r+\frac{NPV}{C^*}. \tag{39}$$

This AIRR (and the related undiscounted variant AROI, $\bar{\jmath}=\sum_{t=0}^n x_t\,/\sum_{t=1}^n c^*_{t-1}$) can be employed in corporate projects, real estate investments, project financing transactions (Altshuler and Magni 2012, Magni 2015b, 2016) and for investment performance measurement, as private equity investments are often matched against (replicating) benchmark portfolios (Long and Nickels 1996, Gredil et al. 2014, Magni 2014b, Altshuler and Magni 2015).[24] As noted above, EAIRR and RP-AIRR are symmetric: EAIRR recognizes value creation in the first period whereas the RP-AIRR recognizes value creation in the last period. Nonnegativity of interim values in (39), often a required feature in Private Equity, can be easily ensured by picking $c_t=\max\,[c^*_{t-1}(1+r_t)-x_t;0]$.

Most recently, Jiang (2017) applied the AIRR approach on a capital base $C^{TS}=\sum_{t=0}^n x_t d_{t,0}(n-t)$, which represents the time-scaled net contribution of the project, so that

$$i(C^{TS})=r+\frac{NPV}{C^{TS}} \tag{40}$$

is an AIRR on *time-scaled contribution* (TS-AIRR). Making use of the above mentioned replicating portfolio, the author measured the capital $c(\tau)$ in continuous time. Denoting as $t_i\in\mathbb{R}$ the date where discrete cash flows x_{t_i} are available, and allowing for a time-variant benchmark rate $r(\tau)$, the capital is $c(\tau)=c(t_{i-1})e^{\int_{t_{i-1}}^{\tau} r(s)\mathrm{d}s}$ for $\tau\in[t_{i-1},t_i]$, $i\in\mathrm{N}_n^1$ and $c(t_i)=c(t_{i-1})e^{\int_{t_{i-1}}^{t_i} r(s)\mathrm{d}s}-x_{t_i}$ for $i\in\mathrm{N}_{n-1}^1$. The author introduced a three-step decomposition aimed at guaranteeing nonnegativity of capital values and showed that the overall capital of the replicating portfolio is equal to the overall project's

[24] The vector $\vec{c}(3)$ in Table 2 refers to the replicating class, so 15.47% is the replicating AIRR.

time-scaled net contribution $C = \int_{t_0}^{t_n} c(\tau)\, e^{\int_{t_0}^{\tau} r(s)\mathrm{d}s}\, \mathrm{d}\tau = \sum_{t=0}^{n} x_t d_{t,0}(n-t) = C^{TS}$. Eq. (40) supplies a relevant piece of information: The rate of return associated with the project's time-scaled net contribution is equal to the rate of return that would obtain by measuring the project return as per unit of the overall continuous-time benchmark capital. The author also mentioned (39) as the discrete version of (40).

7.7 The average return on asset (AROA)

In many financial investments, cash flows are often the results of an investor's decisions (e.g. deposits and withdrawals from a fund). However, the situation in corporate projects (and project financing transactions) is subtly different. Here, cash flows must be inferred from forecasts of sales and costs, that is, from accounting magnitudes. In particular, pro forma financial statements are drawn, which record period-by-period revenues, operating costs, depreciation, interest expenses, taxes, as well as working capital requirements, net property, plant and equipment. We now show that the project rate of return associated with the book value of assets is an average Return On Assets (ROA) which, compared with the COC, correctly captures the economic value created by the project. Let b_{t}^{w} and b_t^f denote the net working capital requirements and the fixed assets, respectively. The book value of the project at time t is $b_t = b_t^w + b_t^f$ and $\mathrm{Dep}_t = b_{t-1}^f - b_t^f$ measures capital depreciation. Total accruals are given by $\Delta b_t^f + \Delta b_t^w = \Delta b_{\mathrm{t}}$ where Δ denote a variation (i.e. $\Delta y_t \coloneqq y_t - y_{t-1}$). Letting π_t be the earnings before interest, taxes, depreciation and amortization, the project's free cash flow, x_t, is then derived from the accounting estimates in the following way:

$$x_t = \underbrace{(\pi_t - \mathrm{Dep}_t)\cdot(1-\omega)}_{NOPAT_t} - \Delta b_t \qquad x_0 = -b_0 \tag{41}$$

where ω is the company's tax rate. The prospective ROA is defined as the ratio of the net operating profit after taxes (NOPAT, also known as 'unlevered net income') to book value of assets: $ROA_t = NOPAT_t / b_{t-1}$. In this context, the cost of capital, r, is given by the project-specific weighted average cost of capital (wacc). It is the expected rate of return of an asset equivalent in risk to the project under consideration. The AIRR associated with the book values (and, in general, with the accounting class $[B]$), is the project ROA, which is an average accounting rate of return obtained from the project ROA's:

$$i(B) = \frac{B_1 \cdot ROA_1 + B_2 \cdot ROA_2 + \cdots + B_n \cdot ROA_n}{B_1 + B_2 + \cdots + B_n} = \overline{ROA} \tag{42}$$

or, equivalently, $i(B) = r + NPV/B$. It is then evident that the ROAs supply important pieces of economic information that can be aggregated by a Chisini mean.

Proposition 19 (AROA) *The AROA is the AIRR associated with the accounting class* $[B]$*. The economic value created can then be viewed as a function of the prospective ROAs:* $NPV = B \cdot (\overline{ROA} - r)$*. The AROA correctly captures a project's economic profitability when compared with the project wacc:* $\overline{ROA} \gtreqless r$*.*

This interesting result is in sharp contrast with the accounting literature on economic rate of return, according to which accounting rates of return do not provide any information about a project's or firm's economic profitability (e.g. Harcourt 1965, Solomon 1966, Livingstone and Salamon 1970, Gordon 1974, Kay 1976, Fisher and McGowan 1983, Luckett 1984, Salamon 1985, Kay and Mayer 1986, Gordon and Stark 1989, Whittington 1988, Feenstra and Wang 2000, Stark 2004. See also Magni 2009a, 2011c, Magni and Peasnell 2012, 2015).

7.8 Straight-line rate of return

Consider the assumption of a constant linear decrease in the project's value and the related depreciation class. In a sense, this assumption is opposite to the assumption of constant exponential growth associated with the IRR. Let a capital sum be invested at a growth rate equal to γ. Then, the capital at time t will be $c_t = c_0(1+\gamma)^t, t \in \mathrm{N}_n^1$. If no interim cash flows occur, then $x_n = c_0(1+\gamma)^n$ and $\gamma = \sigma$ is the project IRR. Conversely, in case of a linear decrease at a rate γ, the capital at time t will be $c_t = c_0(1-\gamma t)$. Using the boundary condition $c_n = 0$, one gets $c_n = c_0(1-\gamma n) = 0$ so that $\gamma = 1/n$. This implies that capital depreciates uniformly through time: $c_{t-1} - c_t = \gamma c_0$. This depreciation path is the well-known straight-line depreciation, such that $c_t = c_0\left(1-\frac{t}{n}\right), t \in \mathrm{N}_n^1$. The straight-line depreciation class $[C^{SL}]$ is such that $C^{SL} = \sum_{t=1}^{n} c_0\left(1-\frac{t-1}{n}\right) d_{t,0}$. The associated AIRR is here called *straight-line* AIRR (SL-AIRR):

$$i(C^{SL}) = r + \frac{NPV}{C^{SL}}. \tag{43}$$

This SL-AIRR always exists and is unique, so it will be available in any circumstance, and the financial nature of the project is unambiguously identified by the sign of x_0. If $x_0 > 0$ (<), then $C^{SL} > 0$ (<) and the project is a net investment (financing). We will see in the next section that the SL-AIRR (as well as the IC-AIRR) enjoys a stronger form of NPV-consistency than the traditional one.[25]

7.9 AIRR on market values (MV-AIRR)

Consider an investment in a financial asset or fund/portfolio and assume the analyst is evaluating its *ex post* performance (so, n denotes current time). In this case, $x_t < 0$ denotes a contribution and $x_t > 0$ denotes a distribution. Let m_t be the fund's ending values as observed in the market (before cash movements). Then, the beginning-of-period market value at time t is $m_t^b = m_t - x_t$ and $i_t = m_t/m_{t-1}^b - 1$ is the fund's holding period rate. Let r be the holding period rate of a *benchmark* portfolio which acts as a cost of capital in period $[t-1, t]$. The mean of these return rates, weighted by the respective market values, is the AIRR on (observed) market values:[26]

$$\bar{\imath} = i(M) = r + \frac{NPV}{M}. \tag{44}$$

where $M_t = m_{t-1}^b d_{t,0}$ is the discounted beginning value and $M = \sum_{t=1}^{n} M_t$ is the overall invested value. Recall that the dependence on the market class $[M]$ holds under the assumption of constant COC. As already noted, if COCs are time-variant, then i is not a function of M but a function of $\vec{m}^b$: Using (15),

$$\bar{\imath} = \frac{\sum_{t=1}^{n} m_{t-1}^b i_t d_{t,0}}{\sum_{t=1}^{n} m_{t-1}^b d_{t-1,0}} \tag{45}$$

where $d_{t-1,0} = \prod_{k=1}^{t-1}(1+i_k)^{-1}$. (The MV-AIRR can be used for *ex ante* valuation as well; in this case, the values are estimated).

Other capital bases are possible, and the analyst may even blend two or more depreciation classes and generate a new AIRR (See the blended EAIRR in Cuthbert and Magni 2016. See also Magni 2013a, section "Relations with IRR", and Magni 2014b, Section 6).

8. The *relevant* rate of return: A guide for practitioners

[25] The vector $\vec{c}(1)$ in Table 2 refers to straight-line depreciation, so 10.59% is the SL-AIRR.
[26] In this context, "market value" does not mean "economic value" (i.e., discounted value of prospective cash flows). It is the price at which the asset can be purchased or sold in the market.

The AIRR theory unveils the fundamental ambiguity of the rate-of-return notion owing to the phenomenon of underdetermination illustrated above. While this calls for a capital valuation theory yet to be developed, it also enriches the economic content of the rate-of-return notion (see section 11). However, practitioners may find it cumbersome to investigate multiple rate/capital combinations; they often need a well-defined criterion to choose the *relevant* capital stream and therefore the relevant rate of return. In this section, we sketch some simple guidelines to help practitioners choose the relevant rate of return.

We divide investments into two categories:

(i) investments whose cash flows are estimated starting from pro-forma financial statements (e.g., corporate projects, project financing transactions, engineering projects) or whose capital amounts are explicitly predetermined (e.g., loans)

(ii) investments whose cash flows do not depend on pro forma financial statements (e.g., real asset investments, financial assets, funds, portfolios).

In the former case, it is compelling to use the capital value recorded in the pro forma balance sheet as the relevant capital. In this case, the cash flows (and, therefore, the project NPV) explicitly depend on accounting constructs and, in particular, on the pro forma book values. More precisely, from (41),

$$NPV = \left[\underbrace{(\pi_t - \text{Dep}_t)\cdot(1-\omega)}_{NOPAT_t} - b_{\text{t}} + b_{\text{t}-1}\right] d_{t,0}.$$

This means that the relevant rate of return is the AROA. To choose a different pattern, such as the Hotelling depreciation (which generates the IRR), would boil down to negating those very capital values that have been used for estimating the project's cash flows (and, therefore, have determined the economic profitability), thereby resulting in a contradictory series of interim values, that is, book values for cash flows and Hotelling values for rate of return. In these cases, the use of IRR for should be discouraged and the use of the AROA is appropriate as the relevant rate of return.

In Tables 4-6 we use HomeNet's example presented in Berk and DeMarzo (2014, pp. 234-244), with slightly modified inputs, to show the easiness and intuitiveness of the computation of AROA for a corporate project (we assume $r = 12\%$). The resulting rate of return is 65.5%. The NPV is 13,724 and is obtained thanks to an investment scale of 28,732 (present value of invested amounts)[27] and a project efficiency of 53.5%=65.5%−12% (excess AROA). Note that the project IRR is $\sigma = 59\%$, quite different from the 65.5% generated by the estimated capital values. The vector of Hotelling values, derived from the IRR assuming that the capital grows at a constant force of interest, is $\vec{c}(0.59) = (12{,}500;\ 11{,}175.5; 8{,}069.5; 4{,}930.8; 440.2)$. This capital stream (as well as any other in the Hotelling class) is inconsistent with the capital stream $\vec{b} = (12{,}500; 9{,}500; 5{,}500; 3{,}300; 1{,}600)$ that has been used to determine (the cash flows and) the NPV. Note that the IRR is a function of the cash-flow stream, $\vec{x}$, but the latter is a function of the pro forma book values, b_t. So, on one hand, the IRR is affected by the pro forma book values; on the other hand, the IRR generates capital values that are different from the pro forma book values, which is contradictory. Tables 6-7 cope with the same project, under the assumption of time-variant COCs, a case which is often considered impossible to cope with in the corporate finance literature (we use eq. (17) for computing $\bar{\imath}$ and $\bar{r}$).

A second example is a (variant of) a real-life investment illustrated in Hartman (2007, pp. 344-345). Sunoco Inc. faces the opportunity of building a coke-making plant with an annual capacity of 0.65 million tons per year in order to supply plants of International Steel Group (ISG) Inc. The investment cost is \$140 (numbers in million), and ISG agrees to purchase the coke (needed for producing steel) for 14 years. Table 9 summarizes the inputs for revenues, materials, labor, energy

[27] We have used (15) with $c_t = b_t$ so the denominator of $\bar{\imath}$ is the *present* value of the invested amounts (or, which is the same, (42) with $B_t = b_{t-1}d_{t-1,0}$).

and the respective growth rates. Overhead is constant at \$15, the COC is 12%. The investment is to be depreciated under the straight-line method and the salvage value is estimated to be zero. Production begins at time 0. The unlevered net income (NOPAT) at time t is

$$\begin{aligned} NOPAT_t &= (\pi_t - \text{Dep}_t)(1-\omega) \\ &= (\text{Annual Production} \cdot \text{Price}_1 \cdot (1.02)^{t-1} - \text{Materials}_1(1.02)^{t-1} - \text{Labor}_1(1.01)^{t-1} \\ &\quad - \text{Energy}_1(1.01)^{t-1} - \text{Overhead} - \text{Dep}_t)(1-\omega). \end{aligned}$$

The AROA is $\overline{\text{RO}A} = 22.7\%$. Note that this is equal to the SL-AIRR, given that depreciation is linear. The project creates value, since $22.7\% > 12\%$ and the economic value created is $NPV = 65.98$ (see Table 9). Note that one does not even have to unravel cash flows to compute the NPV. And again, one can decompose the NPV into investment scale and economic efficiency: $NPV = 65.98 = 688.04 \cdot (0.227 - 0.12)/1.12$. The IRR is 19.6%, inconsistent with the estimated capital values.

Table 4. HomeNet's project: Pro forma accounting and Free Cash Flow

	Year	0	1	2	3	4	5
Incremental Earnings Forecast							
Sales			23,500	23,500	23,500	23,500	
Cost of goods sold			-9,500	-9,500	-9,500	-9,500	
Gross Profit			14,000	14,000	14,000	14,000	
SG&A			-3,000	-3,000	-3,000	-3,000	
Depreciation			-1,500	-1,500	-1,500	-1,500	-1,500
EBIT			9,500	9,500	9,500	9,500	-1,500
Income tax (40%)			-3,800	-3,800	-3,800	-3,800	600
Unlevered Net Income (NOPAT)			5,700	5,700	5,700	5,700	-900
Free Cash Flow							
Plus: Depreciation			1,500	1,500	1,500	1,500	1,500
Less: Capital Expenditures		-7,500					
Less: Increases in NWC		-5,000	1,500	2,500	700	200	100
Free Cash Flow		**-12,500**	**8,700**	**9,700**	**7,900**	**7,400**	**700**

Table 5. HomeNet's project: The Net Present Value

	Year	0	1	2	3	4	5
Net Present Value							
Free Cash Flow		**-12,500**	**8,700**	**9,700**	**7,900**	**7,400**	**700**
Project cost of capital	12%						
Discount factor		1.000	0.893	0.797	0.712	0.636	0.567
PV of Free Cash Flow		-12,500	7,768	7,733	5,623	4,703	397
NPV	**13,724**						

Table 6. HomeNet's project: The AROA as the relevant project's rate of return

	Year	0	1	2	3	4	5

Capital invested							
Net Fixed Assets		7,500	6,000	4,500	3,000	1,500	0
NWC		5,000	3,500	1,000	300	100	0
Invested capital		12,500	9,500	5,500	3,300	1,600	0
PV of capital		12,500	8,482	4,385	2,349	1,017	0
total capital invested	28,732						
$\overline{ROA}$	**65.5%**						
excess AROA	53.5%						
project NPV	13,724						

Table 7. HomeNet's project: The Net Present Value, assuming time-variant COCs

	Year	**0**	**1**	**2**	**3**	**4**	**5**
Net Present Value							
Free Cash Flow		**-12,500**	**8,700**	**9,700**	**7,900**	**7,400**	**700**
Project cost of capital			3%	5%	8%	10%	12%
Discount factor		1.000	0.971	0.925	0.856	0.778	0.695
PV of Free Cash Flow		-12,500	8,447	8,969	6,764	5,760	486
NPV	**17,925**						

Table 8. HomeNet's project: The AROA as the relevant project's rate of return, assuming time-variant COCs

	Year	**0**	**1**	**2**	**3**	**4**	**5**
Capital invested							
Net Fixed Assets		7,500	6,000	4,500	3,000	1,500	0
NWC		5,000	3,500	1,000	300	100	0
Invested capital		12,500	9,500	5,500	3,300	1,600	0
PV of capital		12,500	9,223	5,086	2,825	1,245	0
total capital invested	30,879						
project COC	5.09%						
$\overline{ROA}$	**63.1%**						
excess AROA	58.0%						
project NPV	17,925						

Table 9. Sunoco's investment: Inputs

Investment	$140	M	Energy	$12 M
Annual Prod.	0.65	M tons	*g* (Energy)	1%
Unit Price	$250	per ton	Overhead	$15 M
g (Price)	2%		Salvage Value	$0
Materials	$27.5	M	COC	12%

g (Materials)	2%	M	Dep Method	straight-line
Labor	$75	M	Tax rate	35%
g (Labor)	1%		Periods	14 years

g=growth rate

Table 10. Sunoco's investment: EBITs and ROAs

Time	Dep. charge	Book value	EBIT	NOPAT	ROA
0		$140.00			
1	$10.00	$130.00	$23.00	$14.95	10.7%
2	$10.00	$120.00	$24.83	$16.14	12.4%
3	$10.00	$110.00	$26.71	$17.36	14.5%
4	$10.00	$100.00	$28.63	$18.61	16.9%
5	$10.00	$90.00	$30.60	$19.89	19.9%
6	$10.00	$80.00	$32.61	$21.20	23.6%
7	$10.00	$70.00	$34.68	$22.54	28.2%
8	$10.00	$60.00	$36.80	$23.92	34.2%
9	$10.00	$50.00	$38.97	$25.33	42.2%
10	$10.00	$40.00	$41.19	$26.77	53.5%
11	$10.00	$30.00	$43.46	$28.25	70.6%
12	$10.00	$20.00	$45.79	$29.77	99.2%
13	$10.00	$10.00	$48.18	$31.32	156.6%
14	$10.00	$0.00	$50.62	$32.90	329.0%
$\boldsymbol{NPV}$ = 65.98		$688.04			$\overline{\boldsymbol{ROA}}$ =22.7%

If case (ii) occurs, it is compelling to use market values. For example, in financial portfolios, the investment's values as they are determined by the market are evidently the relevant ones and, as in the previous case, the use of IRR, which is associated with the Hotelling values, would contradict the values that are actually observed in the market. So, the MV-AIRR is a correct rate of return and IRR is inappropriate. Whenever true market values cannot be observed, one can use the values of the replicating portfolio as a proxy for the investment's values, whence the RP-AIRR will be the relevant rate of return. In Table 11 we present an example of an investment (e.g., a private equity investment) with known market prices and the associated MV-AIRR. The investment size is $2,720.3, the rate of return is $\bar{\imath}$ =11.89%, the COC is $\bar{r}$ = 11.01%, so that $NPV = 23.8 = 2{,}720.3 \cdot (11.89\% - 11.01\%)$. The IRR, $\sigma = 14.39\%$, is rather different from the MV-AIRR. The latter is computed by explicitly using true market values, the former incorrectly implies that the value of the investment should increase at a constant force of interest or follow an equivalent pattern of the Hotelling class. In Table 12, we illustrate the same investment under the assumption that interim market values are not known (initial value and terminal value are known and kept at $350 and $1441.1); the RP-AIRR is then the relevant rate of return. Table 13 summarizes the guidelines.

Table 11. Private Equity Investment: Known asset values

period	contributions (+) distributions (-)	portfolio's rates of return	Net Asset Value ending	Net Asset Value beginning	benchmark's rates of return	discount factors
t	$-x_t$	i_t	m_t	m_t^b	r_t	$d_{t,0}$
0	350		0	350.0		1
1	200	30.0%	455.0	655.0	25.0%	0.8000
2	500	-20.0%	524.0	1024.0	20.0%	0.6667
3	-160	-10.0%	921.6	761.6	-11.0%	0.7491
4	-110	60.0%	1218.6	1108.6	40.0%	0.5350
5	-1441.1	30.0%	1441.1	0.0	12.0%	0.4777
NPV	23.8			C =2,720.3		
MV-AIRR ($\bar{\imath}$)	11.89%					
COC ($\bar{r}$)	11.01%					

Table 12. Private Equity Investment: Unknown (interim) asset values

period	contributions (+) distributions (-)	portfolio's rates of return	Net Asset Value ending	Net Asset Value beginning	benchmark's rates of return	discount factors
t	$-x_t$	i_t	m_t	m_t^b	r_t	$d_{t,0}$
0	350		0	350.0		1
1	200	25.0%	437.5	637.5	25.0%	0.8000
2	500	20.0%	765.0	1265.0	20.0%	0.6667
3	-160	-11.0%	1125.9	965.9	-11.0%	0.7491
4	-110	40.0%	1352.2	1242.2	40.0%	0.5350
5	-1441.1	16.0%	1441.1	0	12.0%	0.4777
NPV	23.8			C =3,091.4		
RP-AIRR ($\bar{\imath}$)	11.40%					
COC ($\bar{r}$)	10.63%					

Table 13. Guidelines for practitioners

Cash-flow stream	Type of project		Depreciation	Rate of return
Cash flows depends on pro forma financial statements	Corporate project, project financing transaction, engineering project, loan, etc.		book value	AROA
Cash flows do not depend on pro forma financial statements	Real estate asset, financial portfolio, fund, etc.	Market values are available	market	MV-AIRR
		Market value are not available	replicating	RP-AIRR

9. Insights for future research (I): A stronger definition of NPV-consistency

Definition 4 is a necessary condition for a metric to be considered reliable. Future research might be addressed to restrict the set of NPV-consistent valuation metrics to a subset that should enjoy more stringent properties. In this section, we introduce the new notion of *strong NPV-consistency*.

Consider a model and let $\vec{u} = (u_1, u_2, \ldots, u_m) \in X \subseteq \mathbb{R}^m, m > 1$ be a set of input factors (key drivers), which affect the output of the model. The model is described by an objective function $f: X \to \mathbb{R}$ such that $y = f(u_1, \ldots . u_n)$ is a scalar output. In our context, a relevant piece of information that a valuation metric f should supply is not simply value creation/destruction (and, therefore, acceptance or rejection) but also how sensitive is f (and, therefore, how robust is the decision) to changes in the inputs.[28] Let $\vec{u}^0 = (u_1^0, u_2^0, \dots, u_m^0) \in X$ be the estimated value of the parameters (base-case value) and let $y^0 = f(\vec{u}^0)$ be the corresponding value of f. In particular, important pieces of managerial information are (i) the direction in which the output y^0 would change if the input parameters change, (ii) the magnitude of the changes, (iii) the ranking of the parameters based on the value drivers' impact on the valuation metric. In this way, the robustness of the output can be assessed and invaluable information on the key drivers of the project and the riskiness of the output can be drawn. Sensitivity analysis (SA) is a commonly used methodology for studying these issues in practice. In particular, local SA studies the variation of the output in the neighborhood of $\vec{u}^0$, while global SA considers variation of the inputs within the entire space X of variability of the inputs. Analysis can be made for changes of individual inputs or for simultaneous changes of more (or all) inputs (see Saltelli et al. 2008).

Let $\vec{R} = (R_1, \dots, R_m) \in \mathbb{R}^m$ be the vector expressing the importance measures (relevances) of the model parameters. Consider the binary relation $\succ$ such that $j \succ i$ if and only if $|R_j| > |R_i|, j \neq i$. This means that j ranks higher than i, that is, input j has a greater impact on the output than input i. For example, the vector $\vec{R} = (-0.1, 0.5, 0.01)$ informs that the output increases under changes in inputs 2 and 3, but decreases under changes in input 1, and that input 2 ranks higher than input 1, which in turn ranks higher than input 3. There are several SA techniques that provide importance measures and ranking of the key drivers (see Borgonovo and Plischke 2016, Pianosi et al. 2016). In our context, the natural question arises whether a given rate of return is consistent with the NPV in the sense that it provides the same ranking of a project's value drivers as the one provided by the NPV. We then introduce the following

Definition 6 (strong NPV-consistency) *Given a technique of sensitivity analysis T, a valuation metric ϕ possesses strong NPV-consistency under T if and only if it fulfills Definition 4 and the ranking of the value drivers provided by ϕ is the same as that provided by the NPV. The consistency is said to be strict if the vector $\vec{R}$ of important measures of ϕ is the same as the NPV's. If consistency holds for any technique T, then ϕ is said to possess absolute NPV-consistency.*

An important issue that deserves to be investigated is whether there is some rate of return that enjoys strong NPV-consistency. Some recent contributions have analyzed the relation between IRR and NPV, building upon Borgonovo and Apostolakis' (2001) Differential Importance Measure (DIM), which measures the importance of the input factors around the base-value case $\vec{u}^0$. The DIM is defined as the ratio of the partial differential of f with respect to u_j to the total differential of the function, assuming the function is differentiable and the vector of the first derivatives $\nabla f(\vec{u})$ is not orthogonal to the vector of increments du:

$$DIM_{u_j} = \frac{\left(\frac{\partial f}{\partial u_j}\right) \mathrm{d}u_j}{\sum_{j=1}^m \left(\frac{\partial f}{\partial u_j}\right) \mathrm{d}u_j}.$$

The ratio measures how much of the output change can be attributed to input j. The vector of the importance measures is $\vec{R} = (DIM_{u_1}, DIM_{u_2}, \dots, DIM_{u_m})$. Borgonovo and Peccati (2004, 2006) and Percoco and Borgonovo (2012) showed that IRR and NPV provide not only different DIMs but also different rankings. This implies that IRR does not possess a strong NPV-consistency under the DIM

[28] Typically, the key (or value) drivers in a corporate project are the sequences of the prospective revenues and costs, and the so-called accruals: Accounts receivables, accounts payables, inventory, liquid assets, depreciation of fixed assets, etc. (see also eq. (41)).

technique. Whether the degree of the NPV-inconsistency is high or low is an issue yet to be explored (see Marchioni and Magni 2016 for some results). Conversely, (at least) two AIRRs possess strong NPV-consistency under DIM (in a strict form).

Proposition 20 *Given* a COC, r, *the SL-AIRR and the IC-AIRR are strongly NPV-consistent in a strict form under the DIM method.* [29]

Proof. Let f be the NPV function: $f=\varphi$. For any fixed r, each of the AIRRs can be viewed as an affine function of the project's value drivers: $\phi=r+\varphi(u_1,u_2,\dots,u_n)/C^p$ where $p=SL,IC$. Then, the DIM of ϕ with respect to u_j coincides with the DIM of NPV with respect to u_j:

$$DIM_{u_j}^{\phi}=\frac{\left(\frac{1}{C^p}\cdot\frac{\partial\varphi_{u_j}}{\partial u_j}\right)\mathrm{d}u_j}{\sum_{j=1}^m\left(\frac{1}{C^p}\cdot\frac{\partial\varphi_{u_j}}{C^p}\right)\mathrm{d}u_j}=\frac{\left(\frac{\partial\varphi_{u_j}}{\partial u_j}\right)\mathrm{d}u_j}{\sum_{j=1}^m\left(\frac{\partial\varphi_{u_j}}{\partial u_j}\right)\mathrm{d}u_j}=DIM_{u_j}^{NPV}.\ \blacksquare$$

More compellingly, it can be shown that the two above mentioned AIRRs are strongly NPV-consistent under many other SA techniques, such as the standardized regression coefficient (Saltelli and Marivoet 1990, Bring 1994, Saltelli et al. 2008), the individual and total sensitivity indices in variance-based decomposition models (Sobol' 1993, 2001, Saltelli et al. 2008), Helton's (1993) normalized partial derivative, Borgonovo's (2010a, 2010b) Finite Change Sensitivity Index. (See Marchioni and Magni 2016 for the proof). Indeed, they might even enjoy an absolute NPV-consistency, for they are affine transformations of the NPV and, presumably, an affine transformation of a function f presents the same ranking of the parameters as f for *any* SA method.

10. Insights for future research (II) : The rate of return in a Modigliani and Miller's world

In three famous papers, Modigliani and Miller (MM) set out the foundations of modern finance theory (Modigliani and Miller 1958, 1963; Miller and Modigliani 1961). MM focused on infinite-lived assets (firms) whose shares are traded in a frictionless, arbitrage-free market where no taxes or transactions or flotation costs occur and fair pricing of securities is guaranteed by rational investors, who are assumed to be price takers, and the firm's investment policy is fixed. Under these assumptions, the authors presented three path-breaking results. The first one (MM-I) states that the value of a firm is independent of the capital structure, that is, of the way the firm is financed; the second one (MM-II) shows that the cost of levered equity (required return to equity) increases linearly with an increase in the leverage ratio (ratio of market value of debt to market value of equity); the third one (dividend irrelevance) states that the corporate payout policy is irrelevant for stockholders (i.e., it does not affect the firm's share price).

Let r^U, r^L be the required return to assets for an unlevered (i.e., zero-debt) and levered company, and let r^e and r^d be the required return to equity and to debt, respectively. Let E_0 and D_0 be the current market value of equity and debt. Further, $\mathrm{v}_0^L=E_0+D_0$ denotes the market value of the firm and v_0^U is the unlevered value of the firm. Finally, let x be the expected value of the firm's free cash flow (FCF), distributed to equityholders and debtholders. We can formalize MM's results in the following ways:

[29] The assumption of a fixed COC is not unrealistic, since in practice the COC is often used as a subjectively determined hurdle rate, determined on the basis of several variables such as decision flexibility, future opportunities, rationing of managerial skills, strategic considerations, agency costs, and costs of external financing. Risk is an important factor as well, but the practitioners' notion of risk is not equivalent to that of analytical models used in academia (see Magni 2009c).

(MM-I) The capital structure of a perpetual firm is irrelevant or, equivalently, the firm value is constant under changes in the debt/equity ratio:

$$r^L = \frac{x}{\mathrm{v}_0^L} = \frac{x}{\mathrm{v}_0^U} = r^U = r. \tag{46}$$

(MM II) Given r^U and r^d, the equity cost of capital increases linearly with the debt-to-equity ratio:

$$r^e = r^U + \left(r^U - r^d\right)\frac{D_0}{E_0}. \tag{47}$$

The above equation has also the important interpretation according to which the unlevered cost of capital is a weighted average cost of capital (wacc):

$$r^U = \frac{r^e E_0 + r^d D_0}{E_0 + D_0} \tag{48}$$

(Dividend Irrelevance) If the firm is unlevered, and e_t denotes the equity cash flow, then

$$\mathrm{v}_t^U = \frac{\mathrm{v}_{t+1}^U + e_{t+1}}{1 + r_{t+1}^e} = \frac{\mathrm{v}_{t+1}^U + e_{t+1} + h_{t+1}}{1 + r_{t+1}^e} \tag{49}$$

for all $t \in \mathbb{N} \cup \{+\infty\}$ *and* $h_t > 0$.

The AIRR framework enables rewriting and generalizing all MM results allowing for time-variant rates, r_t, r_t^e, r_t^d and for finite-lived assets. Let $\bar{\rho}_t^U$ be the expected return on unlevered assets; hence, the firm's value at time t is such that $\mathrm{v}_t^U = \mathrm{v}_{t-1}^U(1 + \bar{\rho}_t^U) - x_t$ where x_t is the time-t FCF, $t \in \mathrm{N}_n^1$. At the same date, the equity market value and the market value of debt are such that $E_t = E_{t-1}(1 + r_t^e) - e_t$ and $D_t = D_{t-1}\left(1 + r_t^d\right) - d_t$, where d_t denotes the cash flow to debt. Using the equality $x_t = e_t + d_t$, no-arbitrage pricing implies $\mathrm{v}_t^U = \mathrm{v}_t^L = E_t + D_t$ for every $t \in N_n^0$ and $\mathrm{v}_t^U = \mathrm{v}_t^L = E_{t-1}(1 + r_t^e) - e_t + D_{t-1}\left(1 + r_t^d\right) - d_t$ for every $t \in \mathrm{N}_n^1$. Imposing Chisini's invariance requirement on $\mathrm{v}_t^L = \mathrm{v}_t^L\left(r_t^e, r_t^d\right)$ one finds $\bar{\rho}_t^L = \frac{r_t^e E_{t-1} + r_{t-1}^d D_{t-1}}{E_{t-1} + D_{t-1}}$. This implies $\mathrm{v}_t^L = \mathrm{v}_{t-1}^L(1 + \bar{\rho}_t^L) - x_t$ and $\bar{\rho}_t^L = \bar{\rho}_t^U = \bar{\rho}_t$ is invariant under changes in the capital structure. Now, the market value of the overall shareholder return is $R^e = \sum_{t=1}^n r_t^e E_{t-1} d_{t,0}^e$ where $d_{t,0}^e = \prod_{k=1}^t (1 + r_k^e)^{-1}$, while the market value of the debtholders' interest is $R^d = \sum_{t=1}^n r_t^d D_{t-1} d_{t,0}^d$ where $d_{t,0}^d = \prod_{k=1}^t \left(1 + r_k^d\right)^{-1}$. Imposing Chisini's invariance requirement upon both $R^e = R^e(r_1^e, r_2^e, \dots, r_n^e)$ and $R^d = R^d(r_1^d, r_2^d, \dots, r_n^d)$ one gets the expected return on equity, $\bar{\rho}^e = R^e/E$ where $E = \sum_{t=1}^n E_{t-1} d_{t-1,0}^e$ and the expected return on debt, $\bar{\rho}^d = R^d/D$, where $D = \sum_{t=1}^n D_{t-1} d_{t-1,0}^d$. Analogously, $R = \sum_{t=1}^n \bar{\rho}_t \mathrm{v}_{t-1} d_{t,0}$ is the stakeholders' (equityholders+debtholders) return, where $d_{t,0} = \prod_{t=1}^n (1 + \bar{\rho}_t)^{-1}$. Imposing Chisini's invariance requirement upon $R = R(\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_n)$ one gets the average wacc: $\bar{\rho} = R/\mathrm{v}^U$ where $\mathrm{v}^U = \sum_{t=1}^n \mathrm{v}_{t-1}^U d_{t-1,0}$. However, $\mathrm{v}_t^U = \mathrm{v}_t^L = E_t + D_t$ in every period, and $R = R^e + R^d$, so that $\mathrm{v}^U = E + D$ and

$$\frac{R}{\mathrm{v}^U} = \frac{R}{\mathrm{v}^L} = \bar{\rho} \tag{50}$$

which generalizes (MM-I). Also, $\bar{\rho}\mathrm{v}^U = \bar{\rho}^e E + \bar{\rho}^d D$ whence

$$\bar{\rho} = \frac{\bar{\rho}^e E + \bar{\rho}^d D}{E + D} \qquad \text{or} \qquad \bar{\rho}^e = \bar{\rho} + \left(\bar{\rho} - \bar{\rho}^d\right)\frac{D}{E} \tag{51}$$

which generalizes (MM-II).

As for the dividend irrelevance theorem, consider that MM treat the case where $h_t > 0$ for every t, that is, dividends are greater than FCFs, x_t $(= e_t)$, while a more general definition of dividend irrelevance includes the case where $h_t > 0$, that is, FCFs are not entirely distributed (see De Angelo and De Angelo 2006, 2007). In the former case, financing is needed to raise funds, in the latter case projects must be chosen where the excess cash retained is invested. Therefore, dividend irrelevance means that current stockholders' wealth does not change if and only if the use of extra funds $h_t \in \mathbb{R}$ is value-neutral. Stated equivalently, it says that dividend policy is irrelevant if the full present value of FCF is distributed to shareholders (see also De Angelo and De Angelo 2006, 2007, Magni 2010d). As $\mathrm{v}_0^U = \mathrm{v}_0^U(\vec{h}) = \sum_{t=1}^{\infty}(x_t + h_t)d_{t,0}$, we can reframe the thesis as $\mathrm{v}_0^U(\vec{h}) = \mathrm{v}_0^U(\vec{0})$ for every $\vec{h} \in \mathbb{R}^n$, $n \leq \infty$, which implies $\mathrm{v}_0^U(\vec{h}) - \mathrm{v}_0^U(\vec{0}) = \sum_{t=1}^{n} h_t d_{t,0} = 0$ (Magni 2010d. p. 235). This means that dividend policy irrelevance holds if and only if $\vec{h}$ is an equilibrium asset. We then apply the AIRR approach to $\vec{h}$. Its value, denoted as z_t, is such that $z_t = z_{t-1}(1 + i_t) - h_t$. Owing to the product structure, one gets $\sum_{t=1}^{n} h_t d_{t,0} = Z \cdot \left(\bar{\imath}(Z) - \bar{r}(Z)\right)$. Therefore, dividend irrelevance means

$$\bar{\imath}(Z) = \bar{r}(Z) \text{ for every } \vec{h}. \tag{52}$$

This means that, under the assumptions made, the average expected return on the extra asset $\vec{h}$, $\bar{\imath}(Z)$, is equal to its average COC, $\bar{r}(Z)$. (Note that $i_t = r_t$ is a sufficient but not necessary condition for (52) to hold.)

As seen in section 4, whenever the equilibrium rates are time-variant, the COC changes under changes in the capital stream. While $\bar{\rho} = (\sum_{t=1}^{n} r_t \mathrm{v}_{t-1} d_{t,0})/(\sum_{t=1}^{n} \mathrm{v}_{t-1} d_{t-1,0})$ is the mean return of the equilibrium asset P^{v_0}, with cash-flow stream $(-\mathrm{v}_0, x_1, x_2, \dots, x_n)$, the rate $\bar{r} = (\sum_{t=1}^{n} r_t c_{t-1} d_{t,0})/(\sum_{t=1}^{n} c_{t-1} d_{t-1,0})$ is the mean return of an equilibrium asset P° whose cash-flow stream is $\vec{x} = (x_0, x_1^{\circ}, x_2^{\circ}, \dots, x_n^{\circ})$, which is unambiguously associated with $\vec{c} = (c_0, c_1, \dots, c_{n-1})$ (see Remark 4.2). For capturing economic profitability of a project P the rate of return $\bar{\imath}$ should be compared with $\bar{r}$, not $\bar{\rho}$ (see eq. (18)). Like $\bar{\rho}$ in (51), the rate $\bar{r}$ can be itself decomposed into average cost of equity and average cost of equity, so that $\bar{r} = (\bar{r}^e C^e + \bar{r}^d C^d)/(C^e + C^d)$, with obvious meaning of the symbols (see Magni 2015a for details). Researchers may be interested in investigating the relation between time-variant equilibrium rates, ρ_t, and time-variant disequilibrium rates r_t, and use the AIRR paradigm to provide further insights on the relations among values, rates, and the effect of capital structure on value creation and rates of return, as well as the relations between dividend policy and rates of return.

11. Insights for future research (III): The quest for a capital valuation theory and the empirical perspective

Projects and firms are entities that cannot be formally described by a mere sequence $\vec{x}$ of cash flows. Economically, they are described by a set of actions undertaken by the investors, a set of economic transactions involving several different subjects (managers, shareholders, employees, customers, providers, government etc.) and a collections of tools that are necessary to undertake the actions and make the transactions. Formally, this description is best captured by the relations among capital ($\vec{c}$), rate of return ($\vec{\imath}$), and cash flow ($\vec{x}$), which are connected by the recursive identity $c_t = c_{t-1}(1 + i_t) - x_t$. A new definition of project is naturally derived.

Definition 7 (project) *A project is a triplet* $(\vec{c}, \vec{\imath}, \vec{x}) = (c_0, \dots, c_{n-1}; i_1, i_2, \dots, i_n; x_0, x_1, \dots, x_n)$ *of capital amounts, rates of returns, cash flows (with* $c_0 = -x_0$*).*

It is then no wonder that the notion of rate of return depends on capital valuation: One has to "account" for capital. Penman (2010, p. 111) makes essentially the same point: "[rate of return] can only be observed by doing some accounting; it is a product of the accounting employed to measure it." To compute the appropriate rate of return, one must specify the value of the capital amounts, and there is no single way to determine the capital stream. This insight was anticipated by Vatter (1966) sixty years ago: "unless capital recovery process is specified, there is no single way to measure the annual productivity of the investment" (Vatter, 1966, p. 687). This evidently calls for a theory of capital valuation capable of associating the proper capital stream with the asset under consideration.

The underdetermination of the rate of return has then two sides. On one side, without determining the capital, there cannot be a rate of return. And the determination of of capital cannot be determined automatically from a mathematical (i.e., deductive) procedure: It is an economic issue and a case-specific and purpose-specific one, in the sense that it depends on (i) the investor's economic milieu, (ii) the implied economic transactions, (iii) the purpose of the analysis, (iv) the available information set, (v) the estimation tools, (vi) the subjective preferences, (vii) the piece(s) of information required.

In other words, the choice of capital values (and, therefore, C) involves value judgment. Any reluctance to make value judgments will prevent the achievement of the appropriate/correct economic rate of return. The AIRR approach stimulates the evaluator to make an informed choice of the appropriate depreciation class. On the basis of the above mentioned (and, possibly, other) features, a capital valuation theory, yet-to-be developed, should be capable of helping the analyst select the appropriate depreciation class.

For example, as seen in section 8, book-value depreciation (and, therefore, AROA) seems to be compelling for capital investments, corporate projects, project financing transactions, where pro forma financial statements are available (see Magni 2013a, Bosch-Badia et al. 2014, Magni 2015a, Mørch et al. 2016) or for capturing a firm's economic profitability in a given interval of time (see also Magni and Peasnell 2012, 2015), for which accounting data are often available. In these cases, the use of the IRR is erroneous, for it is associated with a depreciation class $[C^{\sigma}]$ which negates the depreciation class $[B]$ which has been used for deriving the prospective cash flows. So, an IRR paradox is that it does depend on the pro forma book values but they are then denied with the assumption that the capital depreciation is $\vec{c}(\sigma)$ (or an equivalent one in $[C^{\sigma}]$).

The EAIRR is best suited in those situations where a market-implied rate of return is sought and the market is considered efficient. It is also useful when no explicit information is available on the capital stream and only cash flows are available (e.g., Magni 2013a; Magni 2014a; Barry and Robison, 2014, Bosch-Badia et al., 2014, Jiang 2017) or when goal congruence is an issue for managerial performance evaluation (Lindblom and Sjögren, 2009).

In the above situations and in all those situations where market values are relevant but are not available, the RP-AIRR may also be used: The benchmark portfolio's capital can be viewed as a proxy for the project true market value (see Altshuler and Magni 2015). Jiang's (2017) AIRR on time-scaled contribution, along with the proposed decomposition process, provides useful information about return on time-scaled net contribution.

The SL-AIRR is, in depreciation terms, opposite to the IRR, in that it represents a linear depreciation as opposed to an exponential appreciation. The former is sometimes a more acceptable assumption. Further, unlike the IRR, the assumption of linear depreciation makes the SL-AIRR unique (so no problem arises of choosing a 'relevant' SL-AIRR) and enjoys a much more robust NPV-consistency than the IRR, as previously seen.

IC-AIRR and TC-AIRR are useful whenever the analyst aims at measuring the economic profitability of an asset as per unit of initial investment or total contribution, respectively. IC-AIRR enjoys strong NPV-consistency as well as SL-AIRR.

The use of the dual Hotelling depreciation and the dual AIRR might be advantageous in those cases where the capital values change sign, so signaling periods where funds are invested into the project and periods where funds are subtracted from the project. As seen, this makes it possible to decompose the economic value created into investment NPV and financing NPV, and therefore accomplish a more sophisticated analysis than the traditional NPV analysis (see also Magni 2014a, 2015a).

The use of the DA-AIRR is advisable if the internal vector is a multiple of the market vector: $\vec{\imath} = e^{\alpha} \cdot \vec{r} = (1 + a) \cdot \vec{r}$.

As seen in section 8, the MV-AIRR is particularly compelling for financial investments and real estate investments (see Altshuler and Magni 2012) and, in general, for any cash-flow stream that is not derived from pro forma financial statements.

Other AIRRs will entail different assumptions on capital depreciation and it is even possible to blend different depreciation classes for various purposes (e.g. see the blended EAIRR in Cuthbert and Magni 2016, used for Private Financial Initiatives). And if only some values are known, while others are unknown, they can be interpolated in various ways (see Magni 2014b, Jiang 2017). Even the IRR benefits from the AIRR approach in case of time-variant COCs (see Magni 2013a, pp. 105-107, Magni 2014b, section 6. See also Pressacco et al. 2014, Danielson 2016).

On the other side, underdetermination unfolds a cornucopia of measures of worth: The analyst can rest on a powerful toolkit of rates of return, since he is left free to choose one or more values for C and, therefore, one or more rates of return. Any such choice represents a different financial description of the same project. The AIRR approach stimulates the investigation of (i) the economic meaning of each possible choice of C, (ii) the pieces of information that can be derived from each choice, (iii) the corresponding financial interpretations of each choice (lending vs. borrowing), (iv) the informational content of each choice, (v) the domain and range of applicability of each choice.

From this point of view, the AIRR may be viewed as a descriptive theory of rate of return, not a normative one. It does not define (i.e., impose) a rate of return for a cashflow stream, but lets rates of return depend on the choice of the decision maker/analyst. Empirical evidence supports the idea that real-life different decision makers choose and use different metrics/rates of return for analyzing an investment (Remer and Nieto 1995a,b, Graham and Harvey 2001, Sandahl and Sjögren, 2003, Slagmulder et al., 1995; Hahn and Kuhn, 2012) and that various metrics are often used for the same project (Remer et al., 1993; Lefley,1996; Lindblom and Sjögren, 2009). This is consistent with the AIRR approach, which might then be considered the theoretical support for such an empirical evidence.

Evidently, an intrinsic tension and potential conflict may exist in the quest for a capital valuation theory, intrinsically aimed at narrowing the set of rates of return (more specifically, picking up a singleton containing the 'relevant' or 'correct' rate of return) and the fact that multiple rates of return expand the options and provide the analysts with different (but not mutually exclusive) pieces of information that enrich the economic analysis. In section 8 we have argued that, for practitioners, three AIRRs deserve a preferential status: If a project's capital depreciation is explicitly determined (so that the cash-flow stream is drawn from pro forma financial statements), the AROA is relevant rate of return. In all other cases, the relevant rate of return is the MV-AIRR or, if market values are not known, the RP-AIRR (see Table 12).

Concluding remarks

The AIRR approach, introduced in 2010, is a theoretical paradigm that unifies the concept of rate of return while at the same time extending the number of rates of return available for an economic transaction. A rate of return is underdetermined by cash flows, so while a unique return (AIRR) function exists for any project, a rate of return is fleshed out if and only if a depreciation pattern for the capital is selected. More precisely, the return function maps depreciation classes (or capital streams, if COC is time-variant) into rates of return. Consideration of the capital value is then essential for activating a rate of return. Thanks to the notion of Chisini mean, we show that a multiperiod project's rate of return is the capital-weighted average of the various period rates and, as such, it is a *ratio* of total income to total capital, coherently with the function of "rate of growth" it is supposed to serve. We showed that the venerable IRR belongs to the class of AIRRs, which means that IRR is not a constant rate but an average of (generally time-variant) period rates generated by a Hotelling depreciation class that includes infinitely many depreciation schedules. In many cases, the Hotelling class is not appropriate for it contradicts the cash flow estimates. The MIRR is itself a special case of AIRR, as well as the average accounting rate of return, the shareholder rate of return, the project investment rate (project financing rate) of a mixed project and many (indeed, all) rate-of-return metrics.

The AIRR framework is NPV-consistent, and the NPV itself can be viewed as a size-scaled excess rate of return. However, AIRR enables a more refined economic analysis than NPV: Economic profitability depends on both investment size and project efficiency and the AIRR approach enables the evaluator to decompose economic profitability into investment size (total capital invested) and efficiency (excess AIRR). The same NPV can be found with either a small investment size and a high rate of return or a small rate of return and a high investment size. Also, unlike a traditional NPV analysis, the AIRR analysis enables understanding whether value is created because funds are invested at a rate of return which is higher than the COC or, rather, because funds are borrowed at a financing rate which is smaller than the COC. The role of equity and debt in value creation can also be assessed by the AIRR approach.

The AIRR paradigm does not merely solve all the problems incurred by the IRR, but re-defines the notion of rate of return, and sheds lights on the strict link between the notions of capital and rate of return, and, therefore, the link among economics, finance, and accounting. The AIRR model is extremely flexible and can manage disparate instances of projects, either real assets or financial portfolios, either in *ex ante* settings or *ex post* ones. It naturally illustrates and generalizes MM's findings in a rate-of-return perspective. Future researches may be addressed to investigating the link between a rate of return and the capital structure, and the effect of dividend policy on rates of return. Also, a more reliable definition of NPV consistency is introduced in this paper which will hopefully stimulate research on the subset of AIRRs that possess it and the degree of NPV-inconsistency of those which do not possess it. Finally, a capital valuation theory will help analysts to choose among different capital depreciation patterns and pick up an appropriate rate of return for each project, with the understanding that multiple rates of return can be (and are often) used in practice, for they refine the economic analysis by supplying different pieces of economic information.

Acknowledgments. The author wishes to thank Joseph Hartman, Gordon Hazen, Ken Peasnell, Lorenzo Peccati for their feedback and comments on the paper. The usual disclaimer applies. The author also wishes to thank Atsushi Kajii and Roman Slowinski for their professional competence and academic integrity as editors.

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