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March 2017

Online at https://mpra.ub.uni-muenchen.de/77519/
MPRA Paper No. 77519, posted 16 Mar 2017 11:27 UTC

Public debt, corruption and tax evasion: Nash and Stackelberg equilibria

by

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Abstract
Public debt accumulation results to disutility with the problem addressed as whether time path of the public debt is sustainable. In this study the infinite time differential game modeling is used as appropriate tool for the economic analysis that follows. The dynamic game is simple and assumes that the starting point of the public debt model is the well known accounting identity interrelating public debt, interest rate and real government surplus exclusive of interest payments on public debt. In the setting, we consider as stock the public debt and the stress of the regulator is to raise nation's primary surplus. Any surplus increase is not only dependent on government measures, but is also dependent on the known "culture of corruption" and on tax evasion. Thus the process of surpluses' augmentation should be a function of these two factors. Nash and Stackelberg differential game approaches are used to explore strategic interactions. In the Nash equilibrium establishment of cyclical strategies during the game between the group of people involved in illegal activities of corruption and tax evasion in one hand and the government in the other, requires that the discount rate of the group of people involved in illegal actions must be greater than government's discount rate. That is the group of corrupt officials and evaders must be more impatient than government. In the case of hierarchical setting analytical expressions of the strategies and the steady state value of public debt stock are provided. Furthermore a number of propositions are stated.

Keywords: Public debt; Tax evasion**;** Dynamic games; Nash equilibrium; Stackelberg equilibrium

JEL Classifications: C72; H26; H62, H63.

1. Introduction

In this paper we deal with the dynamics of accumulation of a nation's public debt which harms prosperity of the economic agents of a country or a nation. As it is already known public debt accumulation produces disutility, therefore is detrimental to nations' households, because it reduces their consumption in order to meet their future tax burden (Greiner and Fincke, 2009)

Game theory may be used as an appropriate methodology to design efficient actions against accumulation of public debt, as the regulator has to take into consideration the response of victims e.g. the honest taxpayers and so on. As it happens in most cases, every socially undesirable stock is an irreversible fact and therefore one main concern of social planers should be the discovery of effective ways to reduce the sources responsible for the unwished stock accumulation. We use both Nash and Stackelberg differential game approaches to study intertemporal strategic interactions between the group of corrupt officials and the tax evaders on one hand and the government on the other.

The major problem of the public debt accumulation requires finding ways to effectively reduce this unwished stock, maintaining at the same time the standards of the economic process within a country. In the macroeconomic literature the same problem is addressed as whether the time path of public debt is sustainable. Nowadays modern models about sustainability of public debt do not involve the central bank of an economy, as central banks are independent and therefore the nation's government should not rely on central banks when deciding about real public debt reduction. As it is well known the main goal of the government is to achieve sustainable debt policies. In order to show how public debt and (primary) surpluses are (crucially) connected, we resort in the recent literature.

In macroeconomics literature and in the case the interest rate exceeds the growth rate of the gross domestic product (GDP), a given debt policy is said sustainable if the primary surplus relative to GDP is a positive and linearly increasing function of the debt to GDP ratio. If a government pursues a sustainable policy the debt ratio remains constant in the long-run or it converges to zero. As it is pointed out by Greiner and Fincke (2009), a stationary debt to GDP ratio tends to be sustainable if the government raises its own primary surpluses as the public debt rises. Moreover, it is known that there exist a statistically significant positive correlation between government debt levels and a measure of corruption (see also Kaufmann, 2010).

In the present model we introduce the possibility that households will engage in illegal activities such as corruption and tax evasion. Households employed in the public sector consider diverting public funds, earmarked to finance investment projects, for their own private use. Households employed in the private sector consider hiding income from the government to avoid taxation. Following the standards, we model all households as having some aversion to illegal activity and the aversion varies inversely with the average level of corruption by government officials. That is the so called “culture-of-corruption” effect—the average level of corruption among government officials reduces guilt associated with the illegal activity of individual households.

According to Ivanya *et al.* (2015) corruption and tax evasion interact with each other, but the combined presence of both causes net tax rate to be significantly higher than in the benchmark model with no corruption and evasion. Here extending the model of Ivanya *et al.* (2015), we include the interaction between corruption - tax evasion on one side and public debt on the other.

Returning to the model solution problems, one of the main concerns should be the irregularity of multiple equilibrium points. Finding multiple equilibrium points in economic models is not an attractive solution for policy makers. But the recognition of multiple optimal stable equilibrium points may be crucial in order to locate the thresholds separating the basins of attraction surrounding these different equilibria. Starting at a threshold, a rational economic agent is indifferent between moving toward one or the other equilibrium, but a small movement away from the threshold can "destroy" this indifference, leading in a unique optimal course of action.

The introductory one sector, with a convex - concave production function, the optimal growth model of Skiba (1978) was the start of a fast growing literature towards the cyclical solution strategies generated in intertemporal dynamic economic models. Wirl (1995), in resources stock literature, reconsidering a model of Clark *et al* (1979), concludes that equilibrium that falls below the maximum sustainable yield but that exceeds the intertemporal harvest rule due to the positive spillovers allows for optimal, long run, cyclical harvest strategies.

As it is already made clear, the purpose of the present paper is to uncover principles underpinning efficient design of countermeasures against the sources of the undesired public debt accumulation. In particular, we model the optimal balance of competing parties and we intend to find the implications of misspecification at the level for success or failure. An important aim of the first part of our research is the identification of mechanisms generating oscillations of both responsible (for the public debt) agents' illegal activities and periodic countermeasures taken by the government. The discussion of a threshold occurrence is not only limited in the well known (S, s) policies in inventory management, but there are however, other nonlinearities implying oscillatory behavior.

We intend to study this issue by using the methodology of stable limit cycles. Limit cycles, has the intuitive explanation which states that if a trajectory of a continuous dynamical system stays in a bounded region forever, it has to approach a point or a cycle. Cycles gives rise to cyclical policies in economic models, e.g. if a policy trajectory say a higher primary surplus policy, which is restricted in a bounded planar space then this policy sooner or later will retrace its previous steps. Moreover, in higher than the two dimensional systems, sufficient conditions for the existence of limit cycles of nonlinear dynamical systems are provident. Arithmetically the sufficient conditions require that a pair of purely imaginary eigenvalues exists, for a particular value of the bifurcation parameter, and the real part of this pair of eigenvalues changes smoothly its sign as the parameter is altered from below its actual value to above.

The stability of limit cycles is of great importance for the long run behavior of a dynamical system. Economic mechanisms that may be a source of limit cycles, as mentioned by Dockner and Feichtinger (1995) are:

(i) complementarity over time,

(ii) dominated cross effects with respect to capital stocks, and

(iii) positive growth of equilibrium.

The contribution of the paper, in the public economics field, is that it considers the accumulation of public debt as a conflict between two rivals. One is the government policy imposing measures in order to augment primary surplus, which in turn ensures sustainability of public debt, while the other is the group of the corrupted people together with the people that evade both involved in illegal actions. As mentioned, the problem is modeled first as a Nash differential game, for which we explore at equilibrium the possibility of limit cycles and second as a Stackelberg differential game for which we calculate the equilibrium strategies. Such stock accumulation and regulation control models can be found, among others, in Forster

(1980) concerning optimal energy use model and in Xepapadeas (1992) regarding environmental policy design and non-point source pollution.

The remainder of the paper is organized as follows. Section 2 comments on cyclical policies in control of the undesirable public debt, while Section 3 introduces the Nash differential game and gives a necessary condition for cyclical strategies. Section 4 investigates the Stackelberg differential game between the regulator and the corrupted agents and calculates the equilibrium strategies and the players' value functions. The last section concludes the paper.

2. Cyclical policies in public debt accumulation and impatience

An intuitive explanation of cyclical policies in the following public debt model between economic agents involved in illegal activities (corrupt officials and tax evaders) and the government may be the subsequent. The group of people involved in illegal actions derive utility from the higher intensity of its illegal activity, such as corruption and tax evasion, while the other side (i.e. the government) derive utility from the measures taken against the illegal mechanisms (e.g. counter tax evasion). Let us start with rather low and declining stock of public debt. A farsighted regulator, which only gains benefits from the reduction of public debt, will curb its measures against, since further reduction of the public debt's stock would only be possible at high costs. As a consequence the stock of public debt begins to grow again.

Now the corrupt officials together with tax evaders has to react by increasing the intensity of their illegal mechanisms but only moderately, since government measures would not be still very efficient with higher costs and moderate benefits for the government. Moreover this would stabilize the stock of public debt and the dynamical system would approach a stable steady state. If the group of corrupt officials and tax evaders has a high discount rate, that is a realistic assumption, they behave myopically reacting strongly, i.e. they intensify their illegal activities. This provokes measures on the government's side which in turn lead to an increasing reduction of public debt. To avoid public debt's reduction the group of illegally acting economic agents have to reduce the intensity of their mechanisms, so the cycle would close.

3. The Nash Differential Game

Let us denote by $x(t)$ the instantaneous public debt of a country at time t. Without any government's measures taken and also without any other actions on the side of the group of people involved in illegal actions, the stock of resources grows according to the function $g(x)$, which is considered as growth function, obviously dependent on the interest rate, satisfying the conditions $g(0)=0$, $g(x)>0$ for all $x\in(0,K)$, $g(x)<0$ for all $x\in(K,\infty)$, $g''(x)\leq 0$. Starting up the mechanisms, which are responsible for the accumulation of the public debt, is costly for the group of people involved in illegal actions (corrupt officials and tax evaders), e.g. compliance costs. These costs reduce their capital available to their illegal activities.

The reduction of the growth of the public debt, however, does not only depend on the intensity of the illegal activities $\nu(t)$, but is also influenced by the measures $u(t)$ against illegality, undertaken by the government. We set as instrument variables for both sides the intensity of illegal activities $\nu(t)$ and the government's actions $u(t)$ undertaken against, which are assumed non-negatives $\nu(t)\geq 0$, $u(t)\geq 0$.

According to the recent proposed models, one way for the public debt accumulation reduction is the function of primary surplus. In our case the surplus function is denoted by $\phi(u,\nu)$ and is a function of the two control variables, instead of a single time function. Combining the growth of public debt $g(x)$ with the surplus function $\phi(u,\nu)$ the state dynamics can be written as

$$\dot{x} = g(x) - \phi(u,\nu), \qquad x(0) = x_0 > 0 \tag{1}$$

Along a trajectory the non negativity constraint is imposed, that is

$$x(t) \geq 0 \quad \forall t \geq 0 \tag{2}$$

With the assumption of the compliance costs and the damages incurred in the group of illegally acting people, a higher intensity of actions of those people and also the government measures, cause a stronger reduction of their capital resources and therefore we assume the partial derivatives of the surplus' function $\phi(u,\nu)$ to be positive, i.e. $\phi_u > 0,\ \phi_\nu > 0$. Moreover the law of diminishing returns is applied only for the government actions undertaken, that is $\phi_{uu} < 0$ and for simplicity we assume $\phi_{\nu\nu} = 0$.

The utility functions the two players need to maximize defined as follows:

Player 1, the government, derive instantaneous utility, on one hand from the primary surplus $\phi(u,\nu)$, and on the other hand from their measures effort $u(t)$ which gives rise to increasing and convex costs $a(u)$. Moreover the disutility derived by the high level of public debt is described by the increasing function $\delta(x)$. Summing up, the present value of player's 1 utility is described by the following functional

$$J_1 = \int_0^\infty e^{-\rho_1 t} \left[\phi(u,\nu) - \delta(x) - a(u)\right] dt \tag{3}$$

Player 2, the group of economic agents which act illegal (corrupt officials and tax evaders), enjoy immediate utility $\upsilon(x)$ from the level of public debt $x(t)$, and utility, as well, from the intensity ν of their illegal methods used, which is described by the function $\beta(\nu)$. For the utilities $\upsilon(x)$ and $\beta(\nu)$ we assume that are monotonically increasing functions with decreasing marginal returns, therefore for the first derivatives we have $\upsilon'(x)>0,\ \beta'(\nu)>0$ and for the second $\upsilon''(x)<0,\ \beta''(\nu)<0$. So, player's 2 utility function is defined, in its additively separable form, as:

$$J_2=\int_0^{\infty} e^{-\rho_2 t}\left[\upsilon(x)+\beta(\nu)\right]dt \tag{4}$$

3.1. Equilibrium analysis

We begin analysis with the concept of open loop Nash equilibrium, which is based on the fact that every player's strategy is the best reply to the opponent's exogenously given strategy. Obviously, equilibrium holds if both strategies are simultaneously best replies.

The current value Hamiltonians for both players, are defined as follows

$$H_1=\phi(u,\nu)-\delta(x)-a(u)+\lambda\big(g(x)-\phi(u,\nu)\big)$$

$$H_2=\upsilon(x)+\beta(\nu)+\mu\big(g(x)-\phi(u,\nu)\big)$$

The first order conditions, for the maximization problem, are the following system of differential equations for both players:

First, the maximized Hamiltonians are

$$\frac{\partial H_1}{\partial u}=(1-\lambda)\phi_u(u,\nu)-a'(u)=0 \tag{5}$$

$$\frac{\partial H_2}{\partial \nu}=\beta'(\nu)-\mu\phi_\nu(u,\nu)=0 \tag{6}$$

and second the costate variables are defined by the equations

$$\dot{\lambda} = \rho_1 \lambda - \frac{\partial H_1}{\partial x} = \lambda\left[\rho_1 - g'(x)\right] + \delta'(x) \qquad (7)$$

$$\dot{\mu} = \rho_2 \mu - \frac{\partial H_2}{\partial x} = \mu\left[\rho_2 - g'(x)\right] - \upsilon'(x) \qquad (8)$$

3.2. Stability of equilibrium

An interior steady state $\left(x^*, \lambda^*, \mu^*\right)$ with the optimal controls $\left(u^*, \nu^*\right)$ is a solution of the following system (taking steady states):

$$\begin{aligned} & g(x) = \phi(u,\nu) \\ & \lambda\left(\rho_1 - g'(x)\right) = -\delta'(x) \\ & \mu\left(\rho_2 - g'(x)\right) = \upsilon'(x) \\ & (1-\lambda)\phi_u(u,\nu) = a'(u) \\ & \mu\phi_\nu(u,\nu) = \beta'(\nu) \end{aligned}$$

and the Jacobian matrix, evaluated at the steady state, is

$$J = \begin{pmatrix} \frac{\partial}{\partial x}\left[g(x) - \phi(u,\nu)\right] & \frac{\partial}{\partial \lambda}\left[g(x) - \phi(u,\nu)\right] & \frac{\partial}{\partial \mu}\left[g(x) - \phi(u,\nu)\right] \\ \frac{\partial}{\partial x}\left[\lambda\left(\rho_1 - g'(x)\right) + \delta'(x)\right] & \frac{\partial}{\partial \lambda}\left[\lambda\left(\rho_1 - g'(x)\right) + \delta'(x)\right] & \frac{\partial}{\partial \mu}\left[\lambda\left(\rho_1 - g'(x)\right) + \delta'(x)\right] \\ \frac{\partial}{\partial x}\left[\mu\left(\rho_2 - g'(x)\right) - \upsilon'(x)\right] & \frac{\partial}{\partial \lambda}\left[\mu\left(\rho_2 - g'(x)\right) - \upsilon'(x)\right] & \frac{\partial}{\partial \mu}\left[\mu\left(\rho_2 - g'(x)\right) - \upsilon'(x)\right] \end{pmatrix}$$

which after the simple calculations, takes the following final form:

$$J = \begin{pmatrix} g'(x) & -\partial\phi(u,\nu)/\partial\lambda & -\partial\phi(u,\nu)/\partial\mu \\ -\lambda g''(x) + \delta''(x) & \rho_1 - g'(x) & 0 \\ -\mu g''(x) - \upsilon''(x) & 0 & \rho_2 - g'(x) \end{pmatrix} \qquad (9)$$

The main stability analysis is focused in periodic solutions, and therefore we make use of the Hopf bifurcations. Thus, computing determinants and trace of the Jacobian matrix (9) we have

$$\text{tr}J = \rho_1 + \rho_2 - g'(x)$$

$$\det J = g'\left[\rho_1 - g'(x)\right]\left[\rho_2 - g'(x)\right] - \left[\lambda g''(x) - \delta''(x)\right]\left[\rho_2 - g'(x)\right]\frac{\partial \phi(u,\nu)}{\partial \lambda} - \\ -\left[\mu g''(x) + \upsilon''(x)\right]\left[\rho_1 - g'(x)\right]\frac{\partial \phi(u,\nu)}{\partial \mu}$$

The Jacobian (9) possesses two purely imaginary eigenvalues $\pm i\sqrt{\omega}$ if the condition

$$\frac{\det J}{\mathrm{tr}J} = \omega > 0 \text{ holds.}$$

In the following we compute the value of ω as:

$$\omega = \rho_1\rho_2 - \left[g'(x)\right]^2 - \left[\lambda g''(x) - \delta''(x)\right]\frac{\partial \phi(u,\nu)}{\partial \lambda} - \left[\mu g''(x) + \upsilon''(x)\right]\frac{\partial \phi(u,\nu)}{\partial \mu}$$

A Hopf bifurcation can thus only occur if the conditions $\omega > 0$ and the following

$$\rho_1\left[\lambda g''(x) - \delta''(x)\right]\frac{\partial \phi(u,\nu)}{\partial \lambda} + \rho_2\left[\mu g''(x) + \upsilon''(x)\right]\frac{\partial \phi(u,\nu)}{\partial \mu} = \\ = \rho_1\rho_2\left[\rho_1 + \rho_2 - 2g'(x)\right] \qquad (10)$$

has to hold.

In what follows we give specific forms in the functions of the model in order to extract some useful conclusions for periodic solutions.

3.3. Specifications of the model

We specify the functions involved as

Growth function of public debt : $g(x) = Rx(1-x)$ (11.1)

The primary surplus function as a Cobb – Douglas type

$$\phi(u,\nu) = u^{\pi}\nu \qquad (11.2)$$

The government's cost function as a linear function

$$a(u) = au \qquad (11.3)$$

The government's damage function $\delta(x)$ and the agents' utility derived from the resource stock $\upsilon(x)$ in linear forms, respectively

$$\delta(x) = \delta x \qquad (11.4)$$

$$\upsilon(x) = \sigma x \qquad (11.5)$$

Finally, the utility function derived from the illegal mechanisms' intensive usage (on behalf the group of people acting illegally), we assume to be in the form

$$\beta(\nu) = \gamma - \frac{\nu^{\eta-1}}{1-\eta} \qquad (11.6)$$

with $r,\ a,\ \delta,\ \sigma,\ \gamma,\ b > 0,\ \ \pi,\ \eta \in (0,1)$

For the above specifications, the necessary condition for cyclical strategies is given from the next proposition.

Proposition 1

Given the specifications (11.1) - (11.6) *for the functions of the model, a necessary condition for cyclical strategies is that the government is more farsighted than the illegal acting people (corrupt officials and evaders) which are responsible for the unwished public debt, therefore the condition* $\rho_2 > \rho_1$ *has to hold.*

Proof

In appendix A

4. The Stackelberg Setting

In this section we analyze the case in which the two players of the game move hierarchically and the rate of measures taken against illegal activities is chosen by the government before the group of illegal acting agents decides on the rate of their methods, thus the government is the leader.

4.1. The group of illegal acting people as follower

We first consider the optimization problem for the follower, i.e. the group of corrupt officials and tax evaders, which takes the action of the leader as given. The above group face the following objective which is maximized, that is

$$\max_{\nu} \int_{0}^{\infty} e^{-\rho_2 t} \left(\upsilon(x) + \beta(\nu) \right) dt$$

Note that the maximization takes place with respect to the intensity of the mechanisms utilization, which means higher intensive use. The state variable evolves according to (1), for which the growth function is simplified in linear form, i.e. $g(x) = rx$. Thus, the resources equation of motion becomes $\dot{x} = rx - \phi(u, \nu)$. Moreover we assume separability of the model through the separable utility function of the group of illegal acting people. Therefore we assume that the utility enjoyed by the public debt would be in the form, $\upsilon(x) = \sigma x$ and the utility derived from the intensive use of their mechanisms would be in the linear form $\beta(\nu) = \beta\nu$, as well.

The responsible agents' Hamiltonian current value, after the above simplifications is

$$H_2 = \sigma x + \beta\nu + \mu\left[rx - \phi(u, \nu) \right]$$

The first order conditions for an interior solution w.r.t. the control ν is therefore,

$$\frac{\partial H_2}{\partial \nu} = \beta - \mu\phi_{\nu}(u, \nu) = 0$$

and the specification for the primary surplus function the following: $\phi(u, \nu) = u^{\pi}\nu^{\varepsilon}$, $\varepsilon > 1$.

Next we get the optimal control function for the group who act illegal, as follows

$$\phi_{\nu} = \varepsilon u^{\pi} \nu^{\varepsilon-1} = \frac{\beta}{\mu} \quad \Rightarrow \quad \nu^{*}(u) = \left[\frac{\beta}{\varepsilon\mu}\right]^{\frac{1}{\varepsilon-1}} u^{\frac{\pi}{1-\varepsilon}} \tag{12}$$

Now, the adjoint variable μ has to follow the differential equation

$$\dot{\mu} = \rho_2 \mu - \frac{\partial H_2}{\partial x} = \mu[\rho_2 - r] - \sigma \tag{13}$$

Substituting the follower's optimal control function (12) into the primary surplus function, we take the analytical form of the surplus function, as:

$$\phi(u, \nu^{*}(u)) = u^{\pi}(\nu^{*}(u))^{\varepsilon} = u^{\frac{\pi}{1-\varepsilon}} \left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}} \tag{14}$$

The readable expressions of (12) and (14) leads to the conclusion which says that, since $\varepsilon > 1$ and therefore $\frac{\pi}{1-\varepsilon} < 0$, an increase to the measures against the illegal activities on behalf the government, u, results in a more cautious control on behalf the followers', $\nu^{*}(u)$. Similarly, (14) leads to a lower reduction of the public debt, i.e. $\phi(u, \nu^{*}(u))$.

4.2. The Government as regulator

Following Dockner *et al.* (2000) (especially Chapter 5) we formulate the government's problem, for which the leader has to take into account the dynamics of the optimal decisions of the follower, expressed by the adjoint equation (13). Equation (13) now becomes the second state's evolution, so the leader's problem now is treated as an optimal control problem with two state variables. Moreover, combining with the early calculated analytical form (14), the leader's objective functional becomes (we assume the damage function $\psi(\nu)$, due to intensive use of mechanisms, in the form $\psi(\nu)=\psi\nu$)

$$\max_u \int_0^\infty e^{-\rho_1 t}\left(u^{\frac{\pi}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}}-au-\delta x-\psi\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{1}{1-\varepsilon}}u^{\frac{\pi}{1-\varepsilon}}\right)dt \qquad (15)$$

which is subject to both state dynamics, the original resource's dynamics plus the intensity's shadow price dynamics which stems from the follower's maximization problem, i.e. the following dynamics

$$\dot{x}=rx-u^{\frac{\pi}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}} \qquad (16)$$

$$\dot{\mu}=\mu[\rho_2-r]-\sigma \qquad (17)$$

The Hamiltonian current value of the above system (15) - (17) becomes

$$H_1=u^{\frac{\pi}{1-\varepsilon}}\left\{\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}}(1-\lambda)-\psi\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{1}{1-\varepsilon}}\right\}-au-\delta x+\lambda rx+\xi\left[\mu(\rho_2-r)-\sigma\right]$$

with $\lambda,\ \xi$ to denote the adjoint variables of the states $x,\ \mu$ respectively. We note again that the shadow price μ for the group of illegal acting people now becomes the new state variable for the government's problem.

Taking first order conditions we are able to express analytically the leader's optimal control u^* as a function of the adjoints $\lambda,\ \xi$, that is,

$$\frac{\partial H_1}{\partial u}=\frac{\pi}{1-\varepsilon}u^{\frac{\pi+\varepsilon-1}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}}\left(1-\lambda-\frac{\psi\varepsilon\mu}{\beta}\right)-a=0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad u^*(\lambda,\xi)=\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \qquad (18)$$

Moreover, the adjoints follow the differential equations

$$\dot{\lambda} = \rho_1 \lambda - \frac{\partial H_1}{\partial x} = \lambda(\rho_1 - r) + \delta \qquad (19)$$

$$\dot{\xi} = \rho_1 \xi - \frac{\partial H_1}{\partial \mu} = \xi(\rho_1 - \rho_2 + r) - \frac{au}{\pi \mu} \qquad (20)$$

Note that, thanks to state separability, the government's adjoint variable ξ, with respect to the follower's adjoint variable μ, has no influence on the leader's optimization problem.

The findings in the Stackelberg game are summarized in the following proposition.

Proposition 2.

In the Stackelberg game with the government as leader and the group of illegal acting people as follower, a feasible solution exists, iff ψ *is sufficient large, i.e. iff*

$$\psi > \frac{\beta(1-\lambda)}{\varepsilon\mu} = \frac{\beta(\rho_1 - r + \delta)(\rho_2 - r)}{\varepsilon\sigma(\rho_1 - r)} \quad \text{or} \quad \varepsilon > \frac{\beta(1-\lambda)}{\psi\mu} \qquad (21)$$

The optimal strategies are then given by

$$u_S^* = \left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \qquad (22)$$

$$\nu_S^* = \left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{1-\pi}{\pi+\varepsilon-1}} \qquad (23)$$

the optimal primary surplus function is given by

$$\phi_S^*(u,\nu) = \left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \qquad (24)$$

The steady state value of the public debt is given by

$$x_S^{\infty}(u,v)=\frac{1}{r}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{25}$$

Proof

The values $(22)-(25)$ follow, from further substitutions of (18), from the maximization condition $\frac{\partial H_1}{\partial \nu}=0$ and from the steady state condition $\dot{x}=0 \;\Leftrightarrow\; rx-h_S^*(u,\nu)=0$.

Proposition 3.

In the Stackelberg game the analytic forms of the objective functionals are given by[1]

$$J_S^1=\frac{a(1-\pi-\varepsilon)}{\pi\rho_1}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}}-\frac{\delta x_0}{\rho_1-r} \tag{26}$$

$$J_S^2=\frac{\beta(\varepsilon-1)}{\varepsilon\rho_2}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{1-\pi}{\pi+\varepsilon-1}}+\frac{\sigma x_0}{\rho_2-r} \tag{27}$$

Proposition 4.

For the values of ψ such that

$$\frac{\beta(1-\lambda)}{\varepsilon\mu}<\psi<\frac{\beta(1-\lambda)}{\mu}$$

the regulator as leader act more cautiously and the follower more aggressively compared to the Nash case[2]*. This leads to a higher function of the primary surplus and a higher profit of the Stackelberg follower compared to the Nash case. For values*

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[1] Proof available on request.

[2] For the Nash differential game exposition, see Halkos and Papageorgiou (2011a, b).

of ψ *larger than* $\frac{\beta(1-\lambda)}{\mu}$, *i.e.* $\left(\psi > \frac{\beta(1-\lambda)}{\mu}\right)$, *the government acts more aggressively and the group of corrupt officials and tax evaders more cautiously compared to the Nash case, leading to a lower primary surplus function and a lower objective value for the follower compared to the Nash case.*[3]

Since ψ is the crucial variable which measures damages due to the intensive use of illegal mechanisms, it is obvious (from proposition 4) that for large values of ψ the regulator follows more truculent policy, but for small values of ψ the leader's policy is holding back.

4. Conclusions

The purpose of this paper was to investigate the dynamics of the public debt accumulation together with the actions undertaken for counter accumulation. For this purpose we setup a very simple model of accumulation. We model the public debt as an accumulated stock first in a simultaneous (Nash) game. The Nash game takes place between the government which uses as control its counter–accumulation policy and the group of corrupt officials and tax evaders using the intensity of their illegal actions as their control variable. The economic analysis that follows in the game's solution, focused on cyclical policies, reveals the possibility of limit cycles. As a result we found the sufficient condition for the cyclical policies existence. According to that result it suffices, assuming different discount rates, the group of illegal acting agents' discount rate to be greater than the government's discount rate.

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[3] Proof available on request.

In the second setting, we extend the simultaneous move in a hierarchical (Stackelberg) differential game, for which the government undertakes the role of leader, while the answerable agents undertake the follower's role. In the above Stackelberg game, we first calculate the analytical expressions of the player's strategies and the analytical expression of the reducing resources function.

We also found the steady state of the agents' resources stock. The analytical expressions of the value functions are finally calculated. The last proposition of the paper concerns about the behavior of the primary surplus function. To be more precise, we found the interval between one crucial parameter of the model lies. If this parameter lies between certain values the reducing function takes higher values, leading therefore to higher profits for the follower, compared with the Nash case.

On the other hand, if the parameter takes a higher value than the threshold, the government acts more aggressively and the polluters more cautiously, leading to a lower reducing function and therefore to a lower objective value for the follower, in comparison to the Nash case.

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Appendix A

Proof of proposition 1.

With the specifications, given in subsection entitled "**3.3. Specifications of the model**", one can compute

$g'(x)=R(1-2x)$, $g''(x)=-2R$, $\phi_u(u,\nu)=\gamma u^{\gamma-1}$, $\phi_\nu(u,\nu)=u^\pi$, $a'(u)=a$, $\beta'(\nu)=\nu^{\eta-2}$, $\delta'(x)=\delta$, $\upsilon'(x)=\upsilon$

$$\frac{\partial H_1}{\partial u}=0 \Leftrightarrow (1-\lambda)\phi_u(u,\nu)=a'(u) \Leftrightarrow (1-\lambda)\gamma u^{\pi-1}\nu=a \qquad (A.1)$$

$$\frac{\partial H_2}{\partial \nu}=0 \Leftrightarrow \beta'(\nu)=\mu\phi_\nu(u,\nu) \Leftrightarrow \mu u^\pi=\nu^{\eta-2} \qquad (A.2)$$

Combining $(A.1)$ and $(A.2)$ the optimal strategies take the following forms

$$u^*=\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{(\eta-2)/[1+(1-\eta)(1-\pi)]} \qquad (A.3),$$

$$\nu^*=\mu^{(\pi-1)/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi/[1+(1-\pi)(1-\eta)]} \qquad (A.4)$$

and the optimal reducing function becomes

$$\phi(u^*,\nu^*)=\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]} \qquad (A.5)$$

with the following partial derivatives

$$\frac{\partial\phi}{\partial\lambda}=\frac{\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]}}{(1-\lambda)}\frac{\pi(\eta-1)}{1+(1-\eta)(1-\pi)}= \qquad (A.6)$$

$$=\frac{\phi(u^*,\nu^*)}{(1-\lambda)}\frac{\pi(\mu-1)}{1+(1-\eta)(1-\pi)}$$

$$\frac{\partial\phi}{\partial\mu}=\frac{\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]}}{\lambda_2}\frac{-1}{1+(1-\eta)(1-\pi)}= \qquad (A.7)$$

$$=\frac{\phi(u^*,\nu^*)}{\mu}\frac{-1}{1+(1-\eta)(1-\pi)}$$

Both derivatives (A.6), (A.7) are negatives due to the assumptions on the parameters $\pi,\ \eta\in(0,1)$ and on the signs of the functions derivates, that is $\phi_u>0,\ \phi_\nu>0,\ \upsilon'(x)>0,\ \delta'(x)>0$, which ensures the positive sign of the adjoints $\lambda,\ \mu$.

Bifurcation condition $\omega\ =\dfrac{\det(J)}{\operatorname{tr}(J)}$ now becomes

$\rho_1\rho_2\left[\rho_1+\rho_2-2g'(x)\right]=\lambda\rho_1 g''(x)\dfrac{\partial\phi}{\partial\lambda}+\mu\rho_2 g''(x)\dfrac{\partial\phi}{\partial\mu}$, which after substituting the values from (A.6), (A.7) and making the rest of algebraic manipulations, finally yields (at the steady states)

$$\frac{\phi(u_\infty,\nu_\infty)g''(x)}{1+(1-\eta)(1-\pi)}\left[\rho_1\pi(1-\eta)\frac{\delta}{\delta+g'(x)-\rho_1}-\rho_2\right]-\rho_1\rho_2\left[\rho_1+\rho_2-2g'(x)\right]=0\ (\text{A.8})$$

Where we have set $\dfrac{\lambda}{1-\lambda}=\dfrac{\delta}{\rho_1-g'(x)-\delta}$ stemming from the adjoint equation $\dot\lambda=\lambda(\rho_1-g'(x))-\delta'(x)$, which at the steady states reduces into $\lambda=\delta'(x)/(\rho_1-g'(x))$.

Condition $w>0$ after substitution the values from (A.6), (A.7) becomes

$$w=\rho_1\rho_2-\left[g'(x)\right]^2+\frac{\phi(u,\nu)g''(x)}{1+(1-\eta)(1-\pi)}\left[\pi(1-\eta)\frac{-\delta}{g'(x)+\delta-\rho_1}+1\right]>0\qquad(\text{A.9})$$

The division of (A.8) by ρ_1 yields

$$\frac{\phi(u_\infty,\nu_\infty)g''(x)}{1+(1-\eta)(1-\pi)}\left[\pi(1-\eta)\frac{\delta}{\delta+g'(x)-\rho_1}-\frac{\rho_2}{\rho_1}\right]-\rho_2\left[\rho_1+\rho_2-2g'(x)\right]=0\ (\text{A.10})$$

The sum $(\text{A.9})+(\text{A.10})$ must be positive, thus after simplifications and taking into account that (at the steady state) $\phi(u_\infty,\nu_\infty)=g(x)$, we have:

$g(x)g''(x)\dfrac{\rho_1-\rho_2}{\rho_1\left[1+(1-\eta)(1-\pi)\right]}>\left[\rho_2-g'(x)\right]^2$ and the result $\rho_2>\rho_1$ follows from the strict concavity of the growth function $g''<0$, since it is supposed $0<\eta<1$ and $0<\pi<1$, as well.