Candidates' Uncertainty and Error Distribution Models in Electoral Competitions.

Bagh, Adib

University of Kentucky

2014

Candidates' Uncertainty and Error Distribution Models in Electoral Competitions

Adib Bagh

June 30, 2014

Departments of Economics and Mathematics
University of Kentucky.

Abstract. Error distribution models provide a simple and convenient approach for introducing candidates' uncertainty in voting models. In such models, given a profile of announced strategies by the players, each candidate can compute the fraction of voters that will vote for him but only up to a random error. We show that the standard practice of assuming that the random error term enters the model additively and that it is independent of the announced policies actually leads to logical inconsistencies. Specifically, we list three assumptions that are frequently imposed when the error distribution approach is used. We then show that, under such assumptions, the error distribution models imply that some candidates believe that certain logically impossible events can take place with a strictly positive probability. We propose a modification of error distribution models that circumvents this problem. Moreover, for electoral competition between two candidates over a unidimensional policy space, our modified model allows us to investigate the pure strategy strategy Nash equilibria of voting games that incorporate voter bias as well as incorporating disagreement between the candidates regarding the preferences of the voters.

Key words. Probabilistic voting models, error distribution models, voter bias, non-policy preferences, Nash Equilibrium.

JEL classification:D72, C72

1. Introduction

In probabilistic voting models, the uncertainty facing the candidates is typically expressed in terms of functions that assign to each candidate his subjective probability of winning the election conditional on the announced policies by all candidates. In models with a continuum of voters, these functions are often specified as primitives of the model Wittman (1983), Calvert (1985). However, it is also possible to derive these functions from more basic assumptions of uncertainty regarding the voters. This is normally done using two different approaches. The first approach postulates that the candidates are uncertain about the distribution of the types of voters (the state-space approach in Roemer (2001)), which implies uncertainty regarding the type of the median voter Ball (1999), Alesina (1988), Hansson & Stuart (1984). The second approach introduces uncertainty to the model by postulating that each candidate is uncertain about the fraction of voters that will vote for him (the error distribution approach in Roemer (2001)). More specifically, once all the candidates announce their policies, each candidate can accurately compute the fraction of voters that prefer one policy over another. However, the candidates are uncertain about how many voters will actually vote for them on election day. More precisely, when the policies of all the candidates are announced, candidate 1 (without loss of generality) can compute FR, which is the fraction of voters who prefer the policy of this candidate to his rival's, regardless to who proposed these policies. This is done by using the information the candidate has about the distribution of the preferences (types) of the voters. A fraction $\widehat{FR} = FR + \xi$ of the voters will then vote for candidate 1 on election day. The variable ξ is a random error term that is usually assumed to be independent of the policies announced by the candidates (Saporiti (2008), Roemer (2001), Roemer (2003)). Given FR, the distribution of ξ, and a specific rule to win the election (for example a simple majority rule), candidate 1 can compute his probability of winning the election. Error distribution models have the advantage of clearly distinguishing between the probability of victory of some candidate and his expected share of votes. Winning probabilities can be more important in winner-takes-all competitions whereas votes shares are more important when candidates maximize their share of voters. The facts that ξ enters these models additively and that it is independent of the announced policies considerably simplify the calculations of the winning probabilities of the candidates. This in turn leads to more tractable analysis of the equilibria of electoral competitions (e.g Chapters 3 and 4 in Roemer (2001), and

Saporiti (2008)). In this paper, we show that the practice of assuming that the error term ξ is both additive and independent of the announced policies can lead to logical inconsistencies. Specifically, we list three assumptions that are frequently imposed when the error distribution approach is used. We then show that, under such assumptions, the " standard" error distribution models imply that some candidates believe that certain logically impossible events can take place with a strictly positive probability. The starting point for analyzing probabilistic voting models is often establishing the existence of Nash equilibrium for the underlying game. However, asserting the existence of pure Nash equilibria in a game where players assign strictly positive probabilities to impossible events invariably undermines any argument that can be used to motivate interest in such equilibria. We, therefore, introduce a modification of error distribution models that circumvents this problem by abandoning the additivity of the error term while maintaining its independence of the announced policies.

Most current models of electoral competitions assume that the candidates have identical views regarding voters' policy preferences. In other words, the candidates agree on how the ideal policies of the voters are distributed over the policy space (in the state-space approach, the candidates agree on the distribution of the ideal policy of the median voter). However, pre-election debates between candidates often reflect a disagreement over "what the voters really want". This indicates some heterogeneity in the beliefs that candidates have about the policy preferences of the voters. Furthermore, candidates' uncertainty regarding the outcome of the election can arise even if voters' policy preferences are common knowledge with certainty (see Erikson & Roomero (1990), Coughlin (1990), Burden (1997), and Eguia (2007)). The most common reasons for such uncertainty are i) voters' non-policy preferences such as preferences over the personal attributes of the candidates, ii) random events that impact actual behavior of the voters on election day such as events that affect voter turnout and mistakes by voters. All these factors contain a large unobservable component, and therefore it is convenient to model their role as a random error that impacts the the calculations of the probability of winning the election by each candidate. The modified model error distribution model we introduce in this paper allows us to model candidates' uncertainty in a logically consistent, robust, and tractable manner. As Section 3 will demonstrate, for electoral competitions between between two candidates over a uni-dimensional policy space, our model is robust in the sense that it does not require the candidates to hold identical beliefs about the policy preferences of the voters, and it only imposes very general assumptions on the source of uncertainty facing the candidates. It is tractable in the sense that, in voting games based on our model, one can readily establish the existence of pure strategy equilibria and then investigate the impact of the parameters of the model on these equilibria.

2. The need to modify the standard error distribution model

Consider a competition between two candidates that announce polices in a set $T \subseteq I\!R^n$. We list three assumptions that often accompany error distribution models.

Assumption A: voters are represented by a probability space (μ, Θ) where each type is represented by a point in Θ. The types are weighted by the measure μ in the sense that for any measurable set $B \subseteq \Theta$, $\mu(B)$ is the fraction of the voters that belong to the set B.

The second assumption requires some definitions. Let t_1 and t_2 in T be respectively the policies announced by candidate 1 and candidate 2. Define

$$S(t_1, t_2) = \{\theta \in \Theta | \text{ voter } \theta \text{ strictly prefers policy } t_1\}$$

and

$$I(t_1, t_2) = \{\theta \in \Theta | \text{ voter } \theta \text{ is indifferent between } t_1 \text{ and } t_2\}$$

We assume that the sets I and S are μ measurable. The fraction of voters that intend to vote for t_1 is given by

$$FR(t_1, t_2) = \mu(S(t_1, t_2)) + \frac{1}{2}\mu(I(t_1, t_2)),$$

and $FR(t, t) = \frac{1}{2}$.

Assumption B: for any $0 < \delta < 1$, either

(a) there exist t_1 and t_2 in $[0, 1]$ such that $1 - \delta < FR(t_1, t_2) < 1$,

or

(b) there exist t_1 and t_2 in $[0, 1]$ such that $0 < FR(t_1, t_2) < \delta$.

Assumption B is satisfied in most spatial voting models in the literature. Specifically, consider any model where

i) $\Theta = T = A \subset I\!R^n$ (i.e., types are identified with their ideal policies, and μ is continuous with respect to the Lebesgue measure on $I\!R^n$)

(ii) for any θ and t in A, the utility of the voter of type θ from policy t is given by the function

$$v(\theta, t) = r(||\theta - t||),$$

where $r : I\!R_+ \longrightarrow I\!R$ is a strictly decreasing function and $|| \cdot ||$ is the norm in $I\!R^n$.

(iii) there exist $\hat{t} \in A$ and a sequence $t^n \longrightarrow \hat{t}$ in A such that

$$\lim_{n \longrightarrow +\infty} \mu\{\theta \in A| \ ||\theta - \hat{t}|| > ||\theta - t^n||\} = 0$$

Then, any such model must satisfy Assumption B. The function r is typically given by $r = -|| \cdot ||$ or $r = -|| \cdot ||^2$. Note that given (i) and (ii), (iii) depends only on the geometry of A as the following examples illustrate.

Example 1. Assume that $\Theta = T = [0, 1]$, and (i) and (ii) above hold. Assume further that $r = -|| \cdot ||$. Then, for any t_1 and t_2,

$$FR(t_1, t_2) = \begin{cases} \frac{t_1 + t_2}{2} & \text{if } t_1 < t_2 \\ \frac{1}{2} & \text{if } t_1 = t_2 \\ 1 - \frac{t_1 + t_2}{2} & \text{if } t_2 < t_1 \end{cases}$$

Therefore,

$$FR(0, t_2) = \mu[0, \frac{t_2}{2}].$$

The continuity of μ implies that $FR(0, t_2)$ can be made arbitrarily close to 0 be taking t_2 that is close enough to 0, and hence (b) of Assumption B is satisfied.

Example 2. Consider another model where $\Theta = T$ is the unit ball in $I\!R^n$. Again assume $r = -|| \cdot ||$. Let t_1 be a point on the boundary of the unit ball. Let t_2^n be a sequence of points inside the ball that approach t_1 along the ray emanating from the origin and heading towards t_1. Then, $\lim_{t_2^n \longrightarrow t_1} \mu(FR(t_1, t_2^n)) = 0$, and Assumption B is satisfied.

Similarly, if $\Theta = T = A \subset I\!R^n$ is a compact convex polygon with non-empty interior, then the model will satisfy Assumption B. In fact, for models satisfying conditions (i) and (ii), a sufficient condition guaranteeing that (iii) holds can be formulated in terms of the compactness of A and the existence of a point on the boundary of A where A is locally convex (e.g. 0 in $[0,1]$ and every point on the boundary of the unit ball are examples of such a point). However, formally deriving such a condition is beyond the scope of this note.

Our last assumption is the defining feature of the standard error distribution model.

Assumption C: candidate 1 believes that the fraction of voters that will vote for him on election day is given by

$$\widehat{FR} = FR(t_1, t_2) + \xi$$

where ξ is random variable over $[-\beta, \beta]$ with cdf G that $G(0) = \gamma > 0$ and G is continuous at 0.

Normally, G is assumed to be continuous on all of $[-\beta, \beta]$ and $G(0) = \frac{1}{2}$.

Proposition 1. Any model satisfying assumptions A through C is internally inconsistent.

Proof. Let E be the event that candidate believes that the fraction of voters voting for him is either strictly larger than 1 or strictly less than 0. By the definition of E, clearly we should have $Pr(E) = 0$. However,

$$Pr(E) = Pr(FR(t_1, t_2) + \xi > 1) + Pr(FR(t_1, t_2) + \xi < 0)$$

Hence,

$$Pr(E) = Pr(\xi > 1 - FR(t_1, t_2)) + Pr(\xi < -FR(t_1, t_2))$$

and

$$Pr(E) = 1 - G(1 - FR(t_1, t_2)) + G(-FR(t_1, t_2)).$$

Assumption B implies that there exists t_1 and t_2 in $[0,1]$ such that either $FR(t_1, t_2)$ is arbitrarily close to 0 or $FR(t_1, t_2)$ is arbitrarily close to 1. Hence, Assumption C implies that either $G(1 - FR(\mu(t_1, t_2))$ or $G(-FR(t_1, t_2))$ can be made arbitrarily close to γ. This implies that $P(E)$ is strictly positive, which is not possible. $\square$

Note that the assumptions of Proposition 1 impose no restrictions on the objective of the candidates, and these assumptions do not specify the conditions for winning the election (e.g a simple majority rule vs more complicated rules). To illustrate some of the complications that arise from error distribution models in a setting that satisfies assumptions A through C, consider the following example.

Example 3. The error distribution models used in Saporiti (2008) satisfy Assumptions A, B, and C. Take for example the model of Section 3.2 in Saporiti (2008) where $T = \Theta = [0, 1]$, the types are distributed on $[0, 1]$ with a continuous cdf F, $v(\theta, t) = -||\theta - t||$, and ξ is uniformly distributed on $[-\beta, \beta]$. Assume a simple majority rule. The results in Saporiti (2008) show that in this model, when candidates are policy and office motivated, the set of equilibria of an electoral competition between the candidates depend on the relative values of β and the "intrinsic values" that candidates attribute to winning. In particular, for certain values of these parameters, the model possesses pure strategy equilibria. However, according to Proposition 1, the model is logically inconsistent for <u>any value</u> of β. To see this, note that this model satisfies Assumptions A and C. As in Example 1, and for any $\beta > 0$, the model also satisfies Assumption B. Hence, such a model is logically inconsistent.

The inconsistency highlighted in Example 3 above also appears in examples 3.1 and 3.2 in Roemer (2001) as well as in some of the results derived in sections 3.2, 3.4, and 3.5 in Roemer (2001). Of course, these cases, which also satisfy conditions A through C, were used to illustrate and simplify the derivation of well-known results. After all, the results in sections 3.2, 3.4, and 3.5 of Roemer (2001) had been previously obtained using the state-space approach. Using error distribution models in a setting that satisfies assumptions A, B, and C in order to introduce and prove new results is harder to justify.

Possible modifications of the error distribution model

Alternative model 1: At first glance, it might appear that the problem discussed above can be avoided by simply limiting the support of ξ to insure that $FR + \xi$ remains inside the interval $[0, 1]$. However, such approach will lead to a type of candidate uncertainty that would -in effect- trivialize the model. More specifically, for any t_1 and t_2 in T, let $\underline{\delta} = -FR(t_1, t_2)$ and $\overline{\delta} = 1 - FR(t_1, t_2)$. Let ξ_{t_1,t_2} be a random variable distributed on $[\underline{\delta}, \overline{\delta}]$ with a continuous cdf $H_{t_1,t_2}(\xi)$. Assuming a simple majority rule, let π denote the subjective probability of winning for player 1. Thus,

$$\pi(t_1,t_2) = Pr(\widehat{FR}(t_1,t_2) > \frac{1}{2})$$

and

$$\pi(t_1,t_2) = 1 - H_{t_1,t_2}(\frac{1}{2} - FR(t_1,t_2))$$

If ξ_{t_1,t_2} is symmetric on $[\underline{\delta},\overline{\delta}]$ (e.g. uniform on $[\underline{\delta},\overline{\delta}]$), then for any t_1 and t_2 in T

$$\pi(t_1,t_2) = 1 - H_{t_1,t_2}(\frac{1}{2} - FR(t_1,t_2)) = \frac{1}{2}.$$

and $\pi(t_1,t_2)$ does not enter the computations of the best reply function of candidate 1. In a Downsian competition, this implies that any platform (t_1,t_2) is an equilibrium, and in the Wittman model, this implies that the only equilibrium is (t_1^*,t_2^*), the most preferred platforms for candidates 1 and 2 respectively.[1] The same phenomenon will occur, if instead of requiring $FR+\varepsilon$ to remain in the same interval, we assume that for every (t_1,t_2), ξ_{t_1,t_2} is a horizontal translation of some fixed ξ on $[-\frac{1}{2},\frac{1}{2}]$ i.e., if $h_{t_1,t_2}(\xi) = h(\xi - (\frac{1}{2} - FR(t_t,t_2)))$ where h_{t_1,t_2} and h are respectively the pdfs of ξ_{t_1,t_2} and ξ.

Alternative model 2: A second obvious approach would be to model the uncertainty in the following way. Define $\widehat{FR}$ by

$$\widehat{FR}(t_1,t_2) = \xi \times FR(t_1,t_2)) \tag{1}$$

where ξ be a random variable defined on $[0,1]$. However, this model does not allow (unobservable) factors that can make $\widehat{FR}(t_1,t_2)$ larger than $FR(t_1,t_2)$ (what if there could be a shock to the economy or if there is a weather condition on election day that lowers the participation of voters who prefer t_2 more than it lowers the participation of those who prefer t_1?). Note also that $t_1=t_2=t$ in this model implies $\pi(t,t) = Pr(\xi \times FR(t,t) > \frac{1}{2}) = Pr(\xi \times \frac{1}{2} > \frac{1}{2}) = 0$. This means that candidate 1 believes that the voters are extremely biased against him, which would rule out any convergence results in a Downsian setting.[2]

[1] In a Downsian model, candidates only care about winning the elections. In a Wittman model, candidates only care about the policy implemented by the winner regardless to who wins the election.

[2] In a Downsian model, candidates maximize their probabilities of winning the elections. A standard result for such models when Assumptions A,B and C hold is that in equilibrium both candidates announce the same policy (Theorem 3.1 in Roemer (2001)).

3. A viable modification of the standard error distribution model

As we have just demonstrated in the previous section, the "obvious" modifications of the error distribution model are not satisfactory. We therefore propose the following modification.

Alternative model 3: The most promising approach for fixing the problem with error distribution models is an approach that is extensively used in the econometric literature on discrete choice models. For a given (t_1, t_2), define the ratio

$$\Gamma(t_1, t_2) = \frac{\widehat{FR}(t_1, t_2)}{1 - \widehat{FR}(t_1, t_2)}, \tag{2}$$

as the relative weight of voters that will vote for candidate 1 on election day. The candidate uses the distribution of types to compute $FR(t_1, t_2)$. He then estimates Γ using an equation of the form

$$\Gamma(t_1, t_2) = \varphi(FR(t_1, t_2), \xi), \tag{3}$$

where the exact functional form on φ is specified by the modeler, and the random variable ξ represents an estimation error. The expression Γ in (2) can take any value in $[0, +\infty)$. Therefore, as long as φ is such that $\varphi(FR(t_1, t_2), \xi) \in [0, +\infty)$ for any values of ξ and FR, we can avoid all the inconsistencies caused by the classic error distribution model. The candidate finally computes the probability of winning the election by computing

$$\pi(t_1, t_2) = Pr(\Gamma(t_1, t_2) > 1).$$

In order to apply this model in various examples voting games, we need to choose a specific and convenient functional form for φ. Henceforth, let

$$\varphi(FR, \xi) = \frac{1}{\xi} FR.$$

This means that for any (t_1, t_2), we have

$$\widehat{FR}(t_1, t_2) = \frac{FR(t_1, t_2)}{FR(t_1, t_2) + \xi} \tag{4}$$

where ξ is random variable distributed on $[0, +\infty)$ with a continuous cdf H. Assuming a simple majority rule, the probability of a victory for candidate 1 is given by

$$\pi(t_1,t_2) = Pr(\frac{FR(t_1,t_2)}{FR(t_1,t_2)+\xi} > \frac{1}{2}) = H(FR(t_1,t_2)). \tag{5}$$

Unlike equation (1), equation (4) implies that we can have $\widehat{FR} > FR$ or $\widehat{FR} < FR$ depending on the realization of ξ. Furthermore, the support of ξ can be a proper subset of $[0,+\infty]$. Hence, unlike the standard error distribution model, equation (4) allows the candidate to believe that the fraction of voters that will vote for him on election day is bounded $0 < a < \widehat{FR}(t_1,t_2) < b < 1$ no matter what the platforms t_1 and t_2 are (e.g. consider the famous "47%" comment by Mitt Romney in the last U.S. presidential elections).[3]

We now incorporate model 3 in a voting game between two candidates over a uni-dimensional policy space. In all what follows, we assume $T = \Theta = [0,1]$ (i.e we identify each voter with his/her ideal policy in $[0,1]$). We initially assume that the candidates agree on the location of the median voter but might disagree on the exact distribution of voters over $[0,1]$. In other words, for $i \in \{1,2\}$, candidate i believes that the distribution of ideal policies of the voters on $[0,1]$ is given by the pdf (cdf) f_i (F_i) where f_1 and f_2 share the same median at t^m with $0 < t^m < 1$.

Assumption D: Candidate 1 believes that, for any (t_1,t_2), the fraction of voters that prefer t_1 over t_2 is given by

$$FR_1(t_1,t_2) = \begin{cases} F_1(\frac{t_1+t_2}{2}) & \text{if } t_1 < t_2 \\ \frac{1}{2} & \text{if } t_1 = t_2 \\ 1 - F_1(\frac{t_1+t_2)}{2}) & \text{if } t_2 < t_1 \end{cases} \tag{6}$$

Similarly, candidate 2 believes that the fraction of voters that prefer t_1 over t_2 is given by

$$FR_2(t_1,t_2) = \begin{cases} 1 - F_2(\frac{t_1+t_2}{2}) & \text{if } t_1 < t_2 \\ \frac{1}{2} & \text{if } t_1 = t_2 \\ F_2(\frac{t_1+t_2)}{2}) & \text{if } t_2 < t_1 \end{cases} \tag{7}$$

Moreover, we assume F_1 and F_2 are continuous and $F_1(t^m) = F_2(t^m) = 1/2$ for some $0 < t^m < 1$.

The requirement that both F_1 and F_2 have the same median will be eventually relaxed (see As-

[3] A month before the 2012 US presidential election, presidential candidate Mitt Romeny, speaking at private fund raising event, said that he believes that 47 percent of the voters will vote for the other candidate (president Obama) "no matter what".

sumption D' at the end of this Section). Equations (6) and (7) are consistent with assuming that voters have concave and single-peaked preferences whose peaks are distributed on $[0, 1]$ according to F_i and that the winner is chosen using a simple majority rule. We also allow the candidates to perceive bias from the voters. As in equation (4), for any (t_1, t_2), let

$$\widehat{FR}_1(t_1, t_2) = \frac{FR_1(t_1, t_2)}{FR_1(t_1, t_2) + \xi_1} \tag{8}$$

where ξ_1 is random variable distributed on $[0, +\infty)$ with a continuous cdf H_1. The probability of a victory for candidate 1 is given by

$$\pi_1(t_1, t_2) = Pr(\frac{FR_1(t_1, t_2)}{FR_1(t_1, t_2) + \xi_1} > \frac{1}{2}) = H_1(FR_1(t_1, t_2)). \tag{9}$$

Similarly, player 2 perceives that fraction

$$\widehat{FR}_2(t_1, t_2) = \frac{FR_2(t_1, t_2)}{FR_2(t_1, t_2) + \xi_2} \tag{10}$$

will actually vote for him on election day where ξ_2 is random variable with a cdf H_2 on $[0, +\infty)$. Then, the probability of a victory for candidate 2 is given by

$$\pi_2(t_1, t_2) = Pr(\frac{FR_2(t_1, t_2)}{FR_2(t_1, t_2) + \xi_2} > \frac{1}{2}) = H_2(FR_2(t_1, t_2)). \tag{11}$$

Remark 1: Assumption D and equations (9) and (11) imply he following properties of π_i:

i) $\pi_1(\cdot, t^m)$ and $\pi_2(t^m, \cdot)$ are continuous on $[0, 1]$

ii) For every t_2, $\pi_1(\cdot, t_2)$ is weakly increasing (i.e. non decreasing) on $[0, t_2)$ and weakly decreasing on $(t_2, 1]$. For $t_2 \geq t^m$, $\pi_1(\cdot, t_2)$ is actually weakly decreasing on $[t_2, 1]$

iii) For every t_1, $\pi_2(t_1, \cdot)$ is weakly increasing on $[0, t_1)$ and weakly decreasing on $(t_1, 1]$. For $t_1 \leq t^m$, $\pi_2(t_1, \cdot)$ is actually weakly increasing on $[0, t_1]$.

The second half of (ii) is a result of the following two observations: First, if $t_2 = t^m$, then $\pi_1(\cdot, t^m)$ is continuous on $[0, 1]$, and as t_1 increases over the interval $[t^m, 1]$, $FR(t_1, t^m)$ weakly decreases and so does $\pi_i(t_1, t^m)$. Second, if $t_2 > t_m$, then $\pi_1(\cdot, t_2)$ is no longer continuous at $t_1 = t_2$. However, $1 - F_1(t_2) < 1/2$, and therefore

$$\lim_{t \searrow t_2} \pi_1(t, t_2) = H_1(1 - F_1(t_2)) < H_1(1/2) = \pi_1(t_2, t_2).$$

The second half of (iii) is obtained using a similar argument. In particular, note that if $t_1 = t^m$, $\pi_2(t^m, t_2)$ is continuous on $[0,1]$, and as t_2 increases over the interval $[0, t^m]$, $\pi_2(t_1, t^m)$ weakly increases. If $t_1 < t^m$, then $F_2(t_1) < 1/2$ and

$$\lim_{t \nearrow t_1} \pi_2(t_1, t) = H_2(F_2(t_1)) < H_2(1/2) = \pi_2(t_1, t_1).$$

The notation $\lim_{t \searrow t_2}$ denotes "t approaching t_2 from above" and $\lim_{t \nearrow t_1}$ denotes "t approaching t_1 from below".[4]

Most probabilistic voting models (see Calvert (1985),Hansson & Stuart (1984), Alesina (1988), Ball (1999), Roemer (1997), Saporiti (2008), Droubvelis & Vriend (2014)) impose the following assumptions on the winning probabilities:

Agreement Assumption: $\pi_1(t_1, t_2) = 1 - \pi_2(t_1, t_2)$

Unbiasedness Assumption: $\pi_1(t_1, t_2) = \pi_2(t_2, t_1)$

Further discussion on these assumptions can be found in Wittman (1983) and Calvert (1985). The Agreement and Unbiasedness assumptions are very convenient because they simplify the computation of the payoff functions of the candidates, and they imply that the sum of the payoffs is "well-behaved". In particular, in a Downsian model, the Agreement and Unbiasedness assumptions imply that the game is zero-sum. More generally, these two assumptions imply that the payoffs of the candidates are reciprocally upper semi-continuous, a property that is often needed to establish the existence of equilibria in a discontinuous game (see Reny (1999) and Saporiti (2008)). However, expressions $\pi_1(t_1, t_2)$ and $\pi_2(t_1, t_2)$ represents subjective probabilities. Therefore, whether or not the above assumptions are reasonable depends on the source of uncertainty facing the candidates. In particular, these assumption may not hold when the uncertainty aries from non-policy considerations by voters. Consider a scenario where both candidates believe that voters are biased against them for different reasons. For example, one candidate might believe that his race causes net voter bias against him while the other candidate might believe that his religious belief causes net voter bias against him. In this case, both the Agreement and the Unbiasedness assumptions will, most likely, be violated. Moreover, the bias when $t_1 = t_2$ may be more significant than when t_1 and t_2 are

[4] Alternatively, one can use $\lim_{t \longrightarrow t_2^+}$ and $\lim_{t_1 \longrightarrow t_1^-}$.

very different. In this context, it is no longer reasonable to model the bias -as in Wittman (1983)- as a simple vertical shift of the winning probabilities. The Agreement and the Unbiasedness assumptions may also fail when the uncertainty in the model results from a possible difference between the intentions of the voters and their behavior on election day. This includes, but it is not limited to, the possibility of voting errors. For example, ξ_1 and ξ_2 can represent candidates uncertainty about how voters react to polls. Suppose after the candidates simultaneously announce (t_1, t_2), polls are conducted and the results are communicated to the voters. These results may in fact impact the turnout on election day; some voters might become complacent about the victory of their candidate, and therefore decide not to vote.[5] For more on the impact of polls and public information in general on election outcomes, please see Goeree & Großer (2007) and Taylor & Yildirim (2009).

Remark 2: Even when $f_1 = f_2 = f$ and f is uniform on $[0, 1]$ and $H_1 = H_2 = H$, we can still have $\pi_1(t_1, t_2) \neq 1 - \pi_2(t_1, t_2)$ (see Example 8 in Section 4). Suppose we, in fact, have $f_1 = f_2 = f$. If $H_1 = H_2 = H$, then equations (9), (11), and Assumption D imply $\pi_1(t_1, t_2) = \pi_2(t_2, t_1)$. If, in addition, $H(1/2) = 1/2$, then we also have $\pi_1(t, t) = \pi_2(t, t) = 1/2$. Note that when $H(1/2) \neq 1/2$, our model allows for a type of a weak bias that only appears in the case the two candidates announce identical platforms. Finally, if $H \equiv 1$ on $[0, 1]$, then $\pi_1(t_1, t_2) = 1 - \pi_2(t_1, t_2)$, and the Agreement assumption is satisfied.

Despite the fact the model 3 does not, in general, satisfy the Agreement and the Unbiasedness assumptions, we can still obtain establish the existence of pure strategy equilibria in voting models with various types of candidates' motivation. As already noted by several authors, any such existence results will invariably require *some* quasi-concavity assumptions on U_i, which in turn require concavity assumptions on the winning probabilities π_i. We, therefore, require the following.

Assumption E: for any $t_2 \geq t^m$, $\pi_1(\cdot, t_2)$ is differentiable and concave in t_1 on $[0, t_2)$, and for any $t_1 \leq t^m$, $\pi_2(t_1, \cdot)$ is differentiable and concave in t_2 on $(t_1, 1]$.

Remark 3: The concavity requirements in the above assumption can be replaced by log-concavity. Please note that we are not requiring $\pi_1(\cdot, t_2)$ and $\pi_2(t_1, \cdot)$ to be concave (or log-concave) on all

[5] Many political analysts asserted that a low turnout played a significant role in the loss of House Majority Leader Eric Cantor in a GOP primary to a tea party challenger in June of 2014. Leading to the election, Cantor was projected to win by a large margin. Only 12 percent of eligible voters actually participated in the election, and Cantor lost the election.

of $[0,1]$. Such assumption would be -for all practical purposes- impossible to hold. To see this, note that, for any $t^m < t_2 < 1$, $\pi_1(\cdot, t_2)$ is, in general, discontinuous at t_2. Therefore, it cannot be concave since every concave function has to be continuous on the interior of its domain. Moreover, Conditions E can be easily reduced to conditions on H_i and F_i. For $t_1 < t_2$, we have

$$\frac{\partial \pi_1}{\partial t_1} = \frac{1}{2} h_1(F_1(\frac{t_1+t_2}{2})) f_1(\frac{t_1+t_2}{2}) \tag{12}$$

and

$$\frac{\partial \pi_2}{\partial t_2} = \frac{-1}{2} h_2(1 - F_2(\frac{t_1+t_2}{2})) f_2(\frac{t_1+t_2}{2}). \tag{13}$$

When $t_2 \geq t^m$, $\frac{t_1+t_2}{2}$ cannot be less than $\frac{t^m}{2}$, and when $t_1 \leq t^m$, $\frac{t_1+t_2}{2}$ cannot be more than $\frac{1+t^m}{2}$. Therefore, taking the derivatives of (12) and (13) and re-arranging the resulting expressions we can show that Assumption E holds, if

$$\frac{h_1'(F_1(z))}{h_1(F_1(z))} \leq \frac{-f_1'(z)}{f_1^2(z)} \tag{14}$$

for z in $[\frac{t^m}{2}, 1]$, and

$$\frac{h_2'(1-F_2(z))}{h_2(1-F_2(z))} \leq \frac{f_2'(z)}{f_2^2(z)} \tag{15}$$

for z in $[0, \frac{1+t^m}{2}]$. In particular, condition E holds if both candidates agree that the voters are uniformly distributed on $[0,1]$ and H_1 and H_2 are concave (e.g. ξ_1 and ξ_2 are exponentially distributed).

<u>Candidates with mixed motivation:</u>

We consider a competition between two candidates who care about winning the election as well as about the policy implemented by the winner. We follow Saporiti (2008) and model the payoffs of the candidates as

$$U_i(t_1, t_2) = \pi_i(t_1, t_2)[\psi_i(t_1, t_2) + K_i], \tag{16}$$

where $\psi_1(t_1, t_2) = v_1(t_1) - v_1(t_2)$ and $\psi_2(t_1, t_2) = v_2(t_2) - v_2(t_1)$.

We assume that $t = 0$ and $t = 1$ are respectively the preferred of candidates one and two. More specifically, we posit

Assumption F: Assume v_1 is decreasing and concave on $[0,1]$, differentiable on $(0,1]$, and $v_1(0) = 0$. Similarly, v_2 is increasing and concave on $[0,1]$, differentiable on $(0,1]$, and $v_2(1) = 0$.

Unlike the results in Ball (1999) and Saporiti (2008), Assumption D does not specify functional forms for v_1 and v_2. Henceforth, we will use $\mathcal{G} = (X, U)$ to denote the two-player game with $X = X_1 \times X_2$, $X_1 = X_2 = [0, 1]$, payoffs given by (16), and π_1 and π_2 respectively given by (9) and (11).

The partial derivatives $\frac{\partial U_i}{\partial t_i}$ may fail to exist at the point (t^m, t^m). However, we can define "directional" partials.

$$\frac{\partial U_1}{\partial t_1}^{-}(t^m, t^m) = \lim_{t_1 \nearrow t^m} \frac{\partial U_1}{\partial t_1}(t_1, t^m)$$

and

$$\frac{\partial U_2}{\partial t_2}^{+}(t^m, t^m) = \lim_{t_2 \searrow t^m} \frac{\partial U_2}{\partial t_2}(t^m, t_2).$$

We also define the constants

$$\alpha_1 = \sup_{t_2 \in (t^m, 1]} \frac{\partial U_1}{\partial t_1}(t^m, t_2)$$

and

$$\alpha_2 = \inf_{t_1 \in [0, t^m)} \frac{\partial U_2}{\partial t_2}(t_1, t^m)$$

The following proposition illustrates how alternative model 3 can be used to generate existence results under very weak assumptions.

Proposition 2. Consider a game $\mathcal{G}$ that satisfies Assumptions E, D, and F. Assume that the candidates satisfy one of the following two conditions:

(a) $\frac{\partial U_1}{\partial t_1}^{-}(t^m, t^m) \geq 0$ and $\frac{\partial U_2}{\partial t_2}^{+}(t^m, t^m) \leq 0$

(b) $\alpha_1 \leq 0$ and $\alpha_2 \geq 0$

Then, the game $\mathcal{G}$ has a pure strategy equilibrium.

The proof of Proposition 2 is in the appendix. Our proposition is very similar in spirit to Theorem 1 in Saporiti (2008). However, Proposition 2 and Theorem 1 in Saporiti (2008) differ in a number of important ways. First, we do not impose any Agreement or Unbiasedness assumptions. Second, we do not assume that $K_1 = K_2$ nor do we assume a specific functional form for v_i. Third, Theorem 1 in Saporiti (2008) is based on the problematic classic error distribution model (Assumption 2 in Saporiti (2008)) while our proposition is valid for any error model with winning probabilities satisfying Assumption E and the properties listed in Remark 1. This includes, among others, the error model specified in equations (8) and (10). Finally, our concavity assumptions are imposed on π_i over specific subintervals of $[0, 1]$ and not over the entire interval $[0, 1]$ (see the beginning of Remark 3).

Condition (a) in Proposition 2 insures that t^m is in the best response set of every player when the other player also plays t^m, and therefore (t^m, t^m) is an equilibrium. This condition is satisfied when both K_1 and K_2 are above some threshold. This should not be surprising since as K_1 and K_2 become "very large", the candidates become essentially office motivated, and in such case a standard argument can show that (t^m, t^m) is an equilibrium. Condition (a) in Proposition 2 can then inform us on how large K_1 and K_2 must be before we can claim that (t^m, t^m) is an equilibrium as the next example demonstrates.

Example 4. Assume $\xi_i = \xi_2 = \xi$ and $\xi \equiv 1$ on $[0, 1]$, and hence $\widehat{FR}_i = FR_i$. Assume further that $f_1 \equiv 1$ on $[0, 1]$ but f_2 is given by the following expression

$$f_2(z) = \begin{cases} \frac{16z}{3} & \text{if } \ z \le 1/4 \\ \frac{4}{3} & \text{if } \ 1/4 \le z \le 3/4 \\ -\frac{16z}{3} + \frac{16}{3} & \text{if } \ 3/4 < z \le 1 \end{cases} \tag{17}$$

Note that Agreement assumption is violated despite the fact that the median for both f_1 and f_2 is 1/2. For any v_1 and v_2 that satisfy Assumption F, simple calculations show that Assumption E is satisfied, and that condition (a) of Proposition 2 is equivalent to

$$f_1(1/2)K_1 + v_1'(1/2) \ge 0$$

and

$$-f_2(1/2)K_2 + v_2'(1/2) \le 0.$$

In other words, assumption (a) of Proposition 2 holds, if and only if,

$$K_1 \ge \frac{-v_1'(1/2)}{f_1(1/2)} \quad \text{and} \quad K_2 \ge \frac{v_2'(1/2)}{f_2(1/2)}. \tag{18}$$

Intuitively, (18) says that if both candidates believes that there is mass of voters around 1/2 that is large relative to $v_i'(1/2)$, then (t^m, t^m) will be an equilibrium even when K_1 and K_2 are small.

Take for example $v_1 = -t$ and $v_2 = -(1-t)$. In this case, if $K_1 \geq 1$ and $K_2 \geq 3/4$, then $(1/2, 1/2)$ is an equilibrium. If we take let $v_1 = -t^2$ and $v_2 = -(1-t)^2$, then we still have $v_1'(1/2) = -1$ and $v_2'(1/2) = 1$, and our conclusion remains the same; for $K_1 \geq 1$ and $K_2 \geq 3/4$, $(1/2, 1/2)$ is an equilibrium.

As the proof of Proposition 2 will demonstrate, $\alpha_1 \leq 0$ implies the following: for any $t_2 \geq t^m$, the best response of candidate 1 to this t_2 will be in $[0, t^m]$. Similarly, $\alpha_2 \geq 0$ implies that, for any $t_1 \leq t^m$, the best response of candidate 2 to this t_1 will be in $[t^m, 1]$. This means that the game $\mathcal{G}$ can be replaced by a simpler game with $X_1 = [0, t^m]$ and $X_2 = [t^m, 1]$. In general, condition (b) will be satisfied when both K_1 and K_2 are below some threshold. Consider the following simple example where the candidates agree on the distribution of voters but not on the distribution of ξ_i.

Example 5. Assume the candidates agree that voters are uniformly distributed over $[0, 1]$ (i.e $f_1 = f_2 = f$ with $f \equiv 1$ on $[0, 1]$). For $i \in \{1, 2\}$, assume h_i is equal to constant β_i on $[0, 1]$ and $0 \leq \beta_i \leq 1$.[6] Assume $\beta_1 \neq \beta_2$, $v_1(t) = -|t|$, and $v_2(t) = -|1-t|$. Then, for any $K_i \geq 0$ conditions D, E, and F hold. Note that

$$\pi_1(t, t) = \beta_1 \neq \beta_2 = \pi_2(t, t),$$

and hence, despite its simplicity, this example does not satisfy the Agreement and Unbiasedness assumptions. Moreover,

$$\frac{\partial U_1}{\partial t_1}^{-}(1/2, 1/2) = \alpha_1 = \frac{1}{2}\beta_1[K_1 - 1], \tag{19}$$

and

$$\frac{\partial U_2}{\partial t_2}^{+}(1/2, 1/2) = \alpha_2 = \frac{1}{2}\beta_2[1 - K_2]. \tag{20}$$

When $0 \leq K_i \leq 1$ for $i \in \{1, 2\}$, (19) and (20) imply that condition (b) of Proposition 2 holds. When $K_i \geq 1$ for $i \in \{1, 2\}$, (19) and (20) imply that condition (a) of Proposition 2 holds. Therefore, for any $K_i \geq 0$, if both $K_i \leq 1$ or if both $K_i \leq 1$, then the game has a pure strategy equilibrium. Assuming that the candidates are located at the extreme points of the interval $[0, 1]$ is not essential to the conclusion of this example. Let $v_1(z) = -|a - z|$ and $v_2 = -|b - z|$ with $0 < a < 1/2 < b < 1$. Consider the game played between the two candidates over the interval $[a, b]$. For $t^m \leq t_2 \leq b$, $\pi_1(\cdot, t_2)$ is differentiable on (a, t_2) and concave on $[a, t_2)$. For any $a \leq t_1 \leq t^m$, $\pi_2(t_1, \cdot)$ is

[6] When the support of ξ_i is larger than $[0, 1]$, then β_i is strictly less than 1.

differentiable on (t_1, b) and concave on $(t_1, b]$. Clearly v_1 and v_2 satisfy Assumption F on $[a, b]$. Moreover, (19) and (20) still hold. Therefore, Proposition 2 implies that the game on $[a, b]$ has an equilibrium (t_1^*, t_2^*) with $a \le t_1^* \le t^m \le t_2^* \le 1$. For any $0 \le t_1 < a$,

$$U_1(t_1, t_2^*) \le U_1(a, t_2^*) \le U(t_1^*, t_2^*)$$

and for any $b \le t_1 \le 1$

$$U_1(t_1, t_2^*) \le U_1(b, t_2^*) \le U(t_1^*, t_2^*).$$

Therefore, for $0 \le t_1 \le 1$,

$$U_1(t_1, t_2^*) \le U(t_1^*, t_2^*).$$

Similarly, we can show that for $0 \le t_2 \le 1$,

$$U_1(t_1^*, t_2^*) \le U(t_1^*, t_2^*).$$

Therefore, (t_1^*, t_2^*) is also an equilibrium for game played on $[0, 1]$.

Remark 4: In examples 4 and 5 and in our examples in Section 4, the payoff of each player can fail to be quasi-concave in its own strategy over $[0, 1]$. Therefore, the standard results for the existence of pure strategy equilibria, including Proposition 3.1 in Reny (1999), cannot be applied directly. We now relax the assumption that both candidates agree on the location of the median voter,

Assumption D': Same as Assumption D except $F_1(t_1^m) = F_2(t_2^m) = 1/2$ for $0 < t_1^m < t_2^m < 1$.

For large K_1 and K_2, there is little hope to have a pure strategy equilibrium. In fact, for very large K_i, each candidate essentially will want to maximize his probability of winning. This will imply that $BR_1(t_2)$ is undefined except when $t_2 = t_1^m$. Yet (t_1^m, t_1^m) is not an equilibrium since candidate two will do better by slightly changing his platform. One other hand, when K_1 and K_2 are both low, we have the following proposition

Assumption E': For any $t_2 \ge t_1^m$, $\pi_1(\cdot, t_2)$ is differentiable and concave in t_1 on $[0, t_2)$, and for any $t_1 \le t_2^m$, $\pi_2(t_1, \cdot)$ is differentiable and concave in t_2 on $(t_1, 1]$.

For example, equations (12) and (13) imply that E' holds when, for $i \in \{1, 2\}$, H_i is concave on $[0, 1]$, F_1 is concave, and F_2 is convex. Note that the concavity of F_1 means candidate one believes that the voters are skewed toward $t = 0$ while the convexity of F_2 implies that candidate two believes that the voters are skewed toward $t = 1$.

Proposition 3. Assume the game $\mathcal{G}$ satisfies Assumptions D', E' and F. Assume further that $\alpha_1 \leq 0$ and $\alpha_2 \geq 0$. Then, the game $\mathcal{G}$ has a pure strategy equilibrium.

The proof is in the Appendix. When $K_1 = K_2 = 0$, Proposition 3 can be used to establish the existence of a pure strategy equilibrium for Wittman models with $f_1 \neq f_2$.

Example 6. Consider a game $\mathcal{G}$ with $K_1 = K_2 = 0$. Assume F_1 is concave, F_2 is convex, and $0 < t_1^m < t_2^m < 1$. Assume further that $H_1 = H_2 = H$ with $H \equiv 1$ on $[0, 1]$. Let $v_1(t) = -t$ and $v_2(t) = -(1 - t)$. Inequalities (14) and (15) imply that $U_1(\cdot, t_2)$ is concave on $[0, t_2)$, $U_2(t_1, \cdot)$ is convex on $(t_1, 1]$, and hence Assumption E' holds.

To show that $\alpha_1 \leq 0$, we compute

$$\frac{\partial U_1}{\partial t_1}(t_1, t_2) = 1/2 f_1(\frac{t_1 + t_2}{2})[v_1(t_1) - v_1(t_2)] + v_1'(t_1) F_1(\frac{t_1 + t_2}{2}).$$

Therefore,

$$\frac{\partial U_1}{\partial t_1}^{[} - (t_1^m, t_1^m) = v_1'(t_1^m) F_1(t_1^m) = -F_1(t_1^m) \leq 0.$$

For any $t_2 > t_1$ and a concave F_1, we have

$$\begin{aligned}\frac{\partial^2 U_1}{\partial t_2 \partial t_1}(t_1, t_2) &= 1/4 f'(\frac{t_1 + t_2}{2})[v_1(t_1) - v_1(t_2)] - 1/2 f(\frac{t_1 + t_2}{2}) v_1'(t_2) + 1/2 v_1'(t_1) f(\frac{t_1 + t_2}{2}) \\ &= 1/4 f_2'(\frac{t_1 + t_2}{2})[v_1(t_1) - v_1(t_2)] \leq 0.\end{aligned}$$

Hence, for any $t_2 > t_1^m$,

$$\frac{\partial U_1}{\partial t_1}(t_1^m, t_2) \leq \frac{\partial U_1}{\partial t_1}(t_1^m, t_1^m) \leq 0,$$

and $\alpha_1 \leq 0$.

Similarly, we have

$$\frac{\partial U_2}{\partial t_2}^{+}(t_2^m, t_2^m) = 1 - F_2(t_2^m) \geq 0.$$

For $t_1 < t_2$ and a convex F_2, we have

$$\frac{\partial^2 U_2}{\partial t_1 \partial t_2}(t_1, t_2) = -1/4 f_2'(\frac{t_1 + t_2}{2})[v_1(t_1) - v_1(t_2)] \leq 0.$$

Hence, for any $t_1 < t_2^m$,

$$0 \leq \frac{\partial U_2}{\partial t_2}(t_2^m, t_2^m) \leq \frac{\partial U_2}{\partial t_2}(t_1, t_2^m),$$

and $\alpha_2 \geq 0$. By Proposition 3, the game has a pure strategy equilibrium.

So far, we have assumed that candidates' uncertainty is expressed by equations 8 and 10. More generally, this uncertainty can be expressed using equations 2 and 3. For $i \in \{1, 2\}$, define

$$\Gamma_i(t_1, t_2) = \frac{\widehat{FR}_i(t_1, t_2)}{1 - \widehat{FR}_i(t_1, t_2)},$$

and let

$$\Gamma_i(t_1, t_2) = \varphi_i(FR_i(t_1, t_2), \xi_i),$$

To obtain a results similar to Proposition 2 and 3, we only need to impose conditions on φ_i and ξ_i such that the resulting π_i satisfy Assumption E and the properties listed in Remark 1.

4. Further examples

The typical examples of non-existence of pure equilibria in probabilistic spatial voting models violate both of conditions (a) and (b) in Proposition 2. Therefore, such examples do not contradict our results.

Example 7. [Example 6 in Ball (1999)] Assume the candidates agree that the voters are uniformly distributed over $[0, 1]$. For $i \in \{1, 2\}$, assume $\xi_i = \xi_2 = \xi$ and ξ is uniform on $[0, 1]$ (no bias and $t^m = 1/2$). Assume further $v_1(t) = -0.5t^2$ and $v_2(t) = -0.5(1 - t)^2$. When $K_1 = 3$ and $K_2 = 0.05$, this game is shown not to have a pure strategy equilibrium in Ball (1999). Note that for $K_1 = 3$, player one fails to satisfy condition (b) in Proposition 2, and for $K_2 = 0.05$, player two fails to satisfy condition (a) in the same proposition. Therefore, this example does not contradict Proposition 2. In fact, if we modify the example so that both K_i are larger or equal to 3, or so that both K_i are less or equal to 0.05, then the game will have a pure strategy equilibrium by Proposition 2.

Example 4, 5, 6, and 7 suggest that when the heterogeneity in office motivation is not very large (either both K_i are large or both are small) then equilibrium in pure strategy exists even when the payoffs of the players are not quasi-concave in their own actions. Proposition 4 in Saporiti (2008) suggests a " converse" of the previous statement; as the difference between K_1 and K_2 increases above certain point, the game will fail to have a pure strategy equilibrium.

Our next example shows how Proposition 2 can be applied when ξ is not constant on $[0,1]$.

Example 8. Assume the candidates agree the the voters are uniformly distributed over $[0,1]$. Assume $\xi_1 = \xi_2 = \xi$, where ξ_i is a random variable with values in $[0,+\infty)$ and a pdf

$$h(\xi) = \frac{1}{(\xi+1)^2}$$

Then, $H(\xi) = \frac{\xi}{\xi+1}$, and using (9) and (11), we obtain

$$\pi_1(t_1,t_2) = \begin{cases} \frac{t_1+t_2}{2+t_1+t_2} & \text{if} \quad t_1 < t_2 \\ \frac{1}{3} & \text{if } t_1 = t_2 \\ \frac{2-(t_1+t_2)}{4-(t_1+t_2)} & \text{if} \quad t_2 < t_1 \end{cases}$$

Similarly,

$$\pi_2(t_1,t_2) = \begin{cases} \frac{2-(t_1+t_2)}{4-(t_1+t_2)} & \text{if} \quad t_1 < t_2 \\ \frac{1}{3} & \text{if} \quad t_1 = t_2 \\ \frac{t_1+t_2}{2+t_1+t_2} & \text{if} \quad t_2 < t_1 \end{cases}$$

Our particular choice of h implies that both candidates believe that the voters are biased against them; given identical platforms each candidate believes that his chance of winning the election is only $\frac{1}{3}$ and $\pi_1 \neq 1 - \pi_2$,

Now assume $v_1(t) = -t$, $v_2(t) = -(1-t)$. The facts that $f_1 = f_2 = f$, f is constant, and H is concave imply, via Remark 3, that Assumption E is satisfied. We apply Proposition 2 to obtain the following results:

If $K_1 = K_2 = 0$, then $(t_1^*, t_2^*) = (0,1)$ is the only pure strategy equilibrium; Note first that if (t_1^*, t_2^*) is an equilibrium, then $t_1^* < t_2^*$. Furthermore, if $0 < t_1 < t_2 \leq 1$, then t_1 cannot be a best response of candidate 1 to t_2. Finally $t_1 = 0$ is a best response of candidate 1 to $t_2 = 1$ and $t_2 = 1$ is a best response of candidate 2 to $t_1 = 0$ since $U_1(\cdot, 1)$ is decreasing and $U_2(0, \cdot)$ is increasing on $[0,1]$. This is the classic divergence result.

If $0 < K_i < 3/2$ for $i \in \{1,2\}$, then condition (b) of Proposition 2 holds. In particular, routine calculations show that $\alpha_1 \leq 0$, if, for $1/2 < t_2 \leq 1$, we have

$$2K_1 \leq 9/4 + t_2 + t_2^2.$$

The above inequality holds for any $1/2 \leq t_2 \leq 1$ as long as $K_1 \leq \frac{3}{2}$. Hence, $K_1 \leq \frac{3}{2}$ implies that $\alpha_1 \leq 0$. Similarly, we can show that $K_2 \leq \frac{3}{2}$ also implies that $\alpha_2 \geq 0$.

Finally, if $K_i > 3/2$ for $i \in \{1, 2\}$, then it is straightforward to show that condition (a) of Proposition 2 holds, and $(1/2, 1/2)$ is a pure strategy equilibrium for the game. Moreover, if $K_i \geq 4$, then for any $t_2 > 1/2$, $\frac{\partial U_1}{\partial t_1}(\cdot, t_2) \geq 0$ and for any $t_2 > t_1$, $\frac{\partial U_2}{\partial t_2}(t_1, \cdot) \leq 0$. Therefore, $(1/2, 1/2)$ is the only pure strategy equilibrium of this game, and we have the standard convergence to the median voter.

5. Conclusion

Despite their intuitive appeal, error distribution models where the error term ξ is both additive and independent of the announced policies can lead to logical inconsistencies. These inconsistencies arise whether or not the policy space is uni-dimensional and regardless to what assumptions we impose on the objectives of the candidates. Therefore, we introduce an alternative formulation of error distribution models that does not require the additivity of the error term but maintains its independence of the announced policies. For voting games between two candidates over a uni-dimensional policy space -in addition to being logically consistent- this alternative formulation provides a tractable approach to modeling candidates' uncertainty regarding the actual behavior of the voters on election day. Most of the classic results in probabilistic voting models can be reproduced as special cases of this new formulation. The formulation of the model, as expressed by equations 2 through 5, remains valid when the policy space is multi-dimensional. For the existence results we derived from such model, however, the uni-dimensionality of the policy space is essential.

Our model distinguishes between two different types of heterogeneity among the candidates. The first type is the heterogeneity in the objectives of the candidates, which is reflected in the difference in office motivation for the two candidates (i.e the difference between K_1 and K_2 in equation (16)). Proposition 2 in our paper, Proposition 1 in Saporiti (2008), and Example 6 in Ball (1999) suggest that when this heterogeneity is low, the voting game will have a pure strategy equilibrium. As this heterogeneity increases, the existence of a pure strategy equilibrium becomes less likely. The second type is a heterogeneity in the beliefs the candidates have about voters' preferences and their actual behavior on election day. On one hand, this heterogeneity allows for a richer representation of the uncertainty facing the candidates. On the other hand, the presence of such heterogeneity rules out the standard assumptions of Agreement and Unbiasedness used in most of the current models, and the proofs of the existence of pure equilibria have to be modified accordingly.

Appendix

In order to prove Proposition 2, we will need the following four technical lemmas. The first two translate the properties of winning probabilities into properties for the payoff functions of the candidates. The last two detail the roles of assumptions (a) and (b) in Proposition 2.

Lemma A1 Consider a game $\mathcal{G}$ that satisfies Assumption E, D, and F. The following must hold:

(i) the functions $U_1(\cdot, t_m)$ and $U_2(t_m, \cdot)$ are continuous on $[0,1]$

(ii) for any t_2, $U_1(\cdot, t_2)$ is weakly decreasing over $(t_2, 1]$. If, in addition, $t_2 \geq t^m$, then $U_1(\cdot, t_2)$ is weakly decreasing over $[t_2, 1]$

(iii) for any t_1, $U_2(t_1, \cdot)$ is weakly increasing over $[0, t_1)$. If, in addition, $t_1 \leq t^m$, then $U_2(t_1, \cdot)$ is weakly increasing over $[0, t_1]$

Proof. The above claims follow immediately from Remark 1 and the facts that v_1 is continuous and decreasing on $[0,1]$ and v_2 is continuous and increasing on $[0,1]$.

Lemma A2 Consider a game $\mathcal{G}$ that satisfies Assumption E, D, and F. For every $t_2 \geq t^m$, $U_1(\cdot, t_2)$ is concave on $[0, t_2)$, and for every $t_1 \leq t^m$, $U_2(t_1, \cdot)$ is concave on $(t_1, 1]$.

Proof. Fix some $t_2 \geq t^m$. $U_1'' = \pi_1''\psi_1 + 2\pi_1'\psi_1' + \psi_1''\pi_1$, where primes denote partials with respect to t_1. Therefore, the concavity of $\pi_1(\cdot, t_2)$ (Assumption E), the concavity of v_1 (Assumption F), and the fact that $\pi_1(\cdot, t_2)$ is increasing and v_1 is decreasing on $[0, t_2)$ (Remark 1, Assumption D, and Assumption F) imply that $U_i'' \leq 0$ on $[0, t_2)$. Similarly, for a fixed $t_1 < t^m$, we have $U_2'' = \pi_2''\psi_2 + 2\pi_2'\psi_2' + \psi_2''\pi_2$, where now primes denote partials with respect to t_2. Now Therefore, the concavity of $\pi_2(t_2, \cdot)$, the concavity of v_2, and the fact that $\pi_2(t_1, \cdot)$ is weakly decreasing and v_2 is increasing on $(t_1, 1]$ imply that $U_2'' \leq 0$ on $(t_1, 1]$. $\square$

Lemma A3 Consider a game $\mathcal{G}$ that satisfies Assumption E, D, and F. If $\frac{\partial U_1}{\partial t_1}^-(t^m, t^m) \geq 0$, then

$$t^m \in Argmax_{t_1 \in [0,1]} U_1(t_1, t^m).$$

Similarly, if $\frac{\partial U_2}{\partial t_2}^+(t^m, t^m) \leq 0$, then

$$t^m \in Argmax_{t_2 \in [0,1]} U_2(t^m, t_2)$$

.

Proof. The assumption $\lim_{t_1 \nearrow t^m} \frac{\partial U_1}{\partial t_1}(t_1, t^m) \geq 0$ and the concavity of U_1 on $[0, t_2)$ -obtained via Lemma A2- imply that $U_1(\cdot, t^m)$ is weakly increasing on $[0, t^m)$. Lemma 1 implies that $U_1(\cdot, t^m)$ is weakly decreasing on $[t^m, 1]$. Hence, $t^m \in Argmax_{t_1 \in [0,1]} U_1(t_1, t^m)$. The proof of our claim regarding U_2 is obtained in the same manner. □

Lemma A4 Consider a game $\mathcal{G}$ that satisfies Assumption E, D, and F. If $\alpha_1 \leq 0$, then for any $t_2 \geq t^m$ we have

$$Argmax_{t_1 \in [0,t^m]} U_1(t_1, t_2) = Argmax_{t_1 \in [0,1]} U_1(t_1, t_2) \tag{A1}$$

Similarly, if $\alpha_2 \geq 0$, then for any $t_1 \leq t^m$ we have

$$Argmax_{t_2 \in [t^m,1]} U_2(t_1, t_2) = Argmax_{t_2 \in [0,1]} U_2(t_1, t_2) \tag{A2}$$

Proof. When $t_2 = t^m$, $(A1)$ follows from parts (i) and (ii) of Lemma A1. When $t_2 > t^m$, the fact that $\frac{\partial U_1}{\partial t_1}(t^m, t_2) \leq 0$, and the concavity of $U_1(\cdot, t_2)$ on $[0, t_2)$ (via Lemma A2) imply that $\frac{\partial U_1}{\partial t_1}(\cdot, t_2) \leq 0$ on $[t^m, t_2)$, and $U_1(\cdot, t_2)$ is weakly decreasing on $[t^m, t_2)$. Combining this with (i) of Lemma A1 implies that $U_1(\cdot, t_2)$ is weakly decreasing on $[t^m, 1]$. Hence, equation $A1$ holds. Using a similar argument, we show, for any any $t_1 \leq t^m$, $\frac{\partial U_2}{\partial t_2}(t_1, t^m) \geq 0$ on $[0, t^m]$ and equation $(A2)$ holds.
□

We are now ready to prove Proposition 2.

Proof of Proposition 2. First, if (a) holds, then $(t_1^*, t_2^*) = (t^m, t^m)$ is an equilibrium by Lemma A3. Now assume (b) holds. Lemma A4 then implies, for any $t_2 \geq t^m$,

$$Argmax_{t_1 \in [0,1]} U_1(t_1, t_2) = Argmax_{t_1 \in [0,t^m]} U_1(t_1, t_2). \tag{A3}$$

Similarly, Lemma A4 also implies that, for any $t_1 \leq t^m$,

$$Argmax_{t_2 \in [0,1]} U_2(t_1, t_2) = Argmax_{t_2 \in [t^m,1]} U_2(t_1, t_2). \tag{A4}$$

Let $\hat{\mathcal{G}}$ be the game that results from $\mathcal{G}$ by restricting the strategy sets of candidates 1 and 2 to $\hat{S}_1 = [0, t^m]$ and $\hat{S}_2 = [t^m, 1]$ respectively. The payoffs of every candidate is now continuous and concave in his own strategy. Therefore, $\hat{\mathcal{G}}$ has an equilibrium (t_1^*, t_2^*) in pure strategies by Proposition 3.1 in Reny (1999). Hence,

$$U_1(t_1^*, t_2^*) \geq U_1(t_1, t_2^*), \quad \text{for all} \quad t_1 \in [0, t^m], \tag{$A5$}$$

and

$$U_2(t_1^*, t_2^*) \geq U_2(t_1^*, t_2), \quad \text{for all} \quad t_2 \in [t^m, 1]. \tag{$A6$}$$

Since $t_2^* \geq t^m$, $(A3)$ and $(A5)$ imply

$$U_1(t_1^*, t_2^*) \geq U_1(t_1, t_2^*), \quad \text{for all} \quad t_1 \in [0, 1]. \tag{$A7$}$$

Similarly, Since $t_1^* \leq t^m$, $(A4)$ and $(A6)$ implies

$$U_2(t_1^*, t_2^*) \geq U_2(t_1^*, t_2), \quad \text{for all} \quad t_2 \in [0, 1]. \tag{$A8$}$$

Clearly, A7 and A8 imply that (t_1^*, t_2^*) is an equilibrium for the original game $\mathcal{G}$. $\square$

To prove Proposition 3, we modify Remark 1 and Lemma A4

Remark $1'$**:** Assumption D' and equations (9) and (11) imply he following properties of π_i:

i) $\pi_1(\cdot, t_1^m)$ and $\pi_2(t_2^m, \cdot)$ are continuous on $[0, 1]$

ii) For every t_2, $\pi_1(\cdot, t_2)$ is weakly increasing (i.e. non decreasing) on $[0, t_2)$ and weakly decreasing on $(t_2, 1]$. For $t_2 \geq t_1^m$, $\pi_1(\cdot, t_2)$ is weakly decreasing on $[t_2, 1]$

Similarly,

iii) For every t_1, $\pi_2(t_1, \cdot)$ is weakly increasing on $[0, t_1)$ and weakly decreasing on $(t_1, 1]$. For $t_1 \leq t_2^m$, $\pi_2(t_1, \cdot)$ is actually weakly increasing on $[0, t_1]$.

Lemma $4'$**.** If $\alpha_1 \leq 0$, then for any $t_2 \geq t_1^m$ we have

$$Argmax_{t_1 \in [0,1]} U_1(t_1, t_2) = Argmax_{t_1 \in [0, t_1^m]} U_1(t_1, t_2) \tag{$A'1$}$$

Similarly, if $\alpha_2 \geq 0$, then for any $t_1 \leq t_2^m$ we have

$$Argmax_{t_2 \in [0,1]} U_2(t_1, t_2) = Argmax_{t_2 \in [t_2^m, 1]} U_2(t_1, t_2). \tag{$A'2$}$$

The proofs of the above claims are essentially the same as the proofs of Remark 1 and Lemma A4.

Proof of Proposition 3. Consider the game let $\tilde{\mathcal{G}}$ be the game that results from $\mathcal{G}$ by restricting the strategy sets of candidates 1 and 2 to $\tilde{S}_1 = [0, t_1^m]$ and $\tilde{S}_2 = [t_2^m, 1]$ respectively. The game $\tilde{G}$ has an equilibrium (t_1^*, t_2^*) (again by Proposition 3.1 in Reny (1999)). Hence,

$$U_1(t_1^*, t_2^*) \geq U_1(t_1, t_2^*), \quad \text{for all} \quad t_1 \in [0, t_1^m].$$

Lemma $4'$ implies that, for any $t_2 \geq t_1^m$,

$$Argmax_{t_1 \in [0, t_1^m]} U_1(t_1, t_2) = Argmax_{t_1 \in [0,1]} U_1(t_1, t_2).$$

Since $t_2^* > t_1^m$, we now have

$$U_1(t_1^*, t_2^*) \geq U_1(t_1, t_2^*), \quad \text{for all} \quad t_1 \in [0, 1]. \tag{A9}$$

Similarly, Lemma $4'$ implies that, for any $t_1 \leq t_2^m$,

$$Argmax_{t_2 \in [t_2^m, 1]} U_2(t_1, t_2) = Argmax_{t_2 \in [0,1]} U_2(t_1, t_2). \tag{A11}$$

Since $t_1^* < t_2^m$, $(A11)$ and the fact that (t_1^*, t_2^*) is an equilibrium of $\tilde{\mathcal{G}}$ imply that

$$U_2(t_1^*, t_2^*) \geq U_1(t_1^*, t_2) \quad \text{for all} \quad t_2 \in [0, 1]. \tag{A12}$$

Clearly, $(A9)$ and $(A12)$ imply that (t_1^*, t_2^*) is also an equilibrium for $\mathcal{G}$. $\square$

References

Alesina, A. 1988. Credibility and Policy Convergence in a Two-Party System with Rational Voters. *AER*, **78**, 796–805.

Ball, R. 1999. Discontinuity and non-existence of equilibirum in the probabilistic spatial voting model. *Social Choice and and Welfare*, **16**, 533–555.

Burden, B. C. 1997. Deterministic and Probabilistic Voting Models. *American Journal of Political Science*, **41**, 1150–1169.

Calvert, R. L. 1985. Robsutness of the multidimensional voting model:Candidate motivation, uncertainty, and convergence. *Amercan Jounral of Political Science*, **3**, 69–95.

Coughlin, Peter J. 1990. *Candidate Uncertainty and Electoral Equilibria in Advances in the Spatial Theory of Voting,ed. J. M Enelow and M. Hinich.* Cambridge University Press.

Droubvelis, M. Saporiti, A, & Vriend, N J. 2014. Political motivations and electoral competition: Equilibrium analysis and experimental evidence. *Games and Economic Behavior*, **183**, 86–115.

Eguia, J. X. 2007. Citizen Candidates Under Uncertainty. *Social Choice and Welfare*, **29**, 317–331.

Erikson, R. S., & Roomero, D. W. 1990. Candidate Equilibrium and the Behavioral Model of the Vote. *American Journal of Political Science Review*, **84**, 1103–11126.

Goeree, J. K., & Großer, J. 2007. Welfare reducing polls. *Economic Theory*, **31**, 51–68.

Hansson, I., & Stuart, C. 1984. Voting competitions with interested politicians: Paltforms do not converge to the prefrences of the median voter. *Public choice*, **44**, 431–441.

Reny, P. 1999. On the Existence of Equilibrium in Discontinuous Games. *Econometrica*, **167**, 1029–1056.

Roemer, J. E. 1997. Political-Economic equilibrium when parties represent constituents:The undimensional case. *Social Choice and Welfare*, **14**, 479–502.

Roemer, J. E. 2001. *Political Competition.* Harvard University Press.

Roemer, J. E. 2003. Indeterminacy of Citizen-Candidate Equilibrium. *Cowles Foundation Discussion Paper No. 1410.*

Saporiti, A. 2008. Existence and Uniquness of Nash Equilibrium in Electoral Competition Game: The Hybrid Case. *Journal of Public Economic Theory*, **10**, 827–857.

Taylor, C. R., & Yildirim, H. 2009. Public Information and electoral bias. *Mimeo.*

Wittman, D. 1983. Candidate motivation: A synthesis of alternatives the prefrences of the median voter. *American Political Science Review*, **77**, 142–157.