Centralizing Disconnected Markets? An Irrelevance Result

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Abstract

This article compares centralized with disconnected markets in which $n > 2$ agents trade two perfectly divisible goods. In a multi-goods uniform-price double auction (centralized market) traders can make their demand for one good contingent on the price of the other good. Interlinking demands across goods is - by design - not possible when each good is traded in separate, single-good uniform-price double auctions (disconnected market). Here, agents are constrained in the way they can submit their joint preferences. I show for a class of models that equilibrium allocations and efficiency of centralized and disconnected markets nevertheless coincide when the total supply of the goods is known or perfectly correlated. This suggests that disconnected markets perform as well as centralized markets when the underlying uncertainty that governs the goods' market prices is perfectly correlated.

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JEL classification: *D44, D47, D82, G14*

Modern economies consist of markets with different structures. Some markets are centralized. They offer multiple goods within the same platform. Some others are disconnected in that only one good is sold or traded per platform. Notably, many goods can be purchased or traded in either centralized or disconnected markets. Different spectrum frequencies are auctioned in a centralized (=combinatorial) auction, as in the FCC auction, yet also in disconnected (=non-combinatorial) auctions. Mineral rights, oil and gas royalties, dairy products and aquarian animals are each offered in global, online platforms,[1] but can also be purchased in separate markets that sell only milk, not butter, or only one species of fish, for instance. Another example are financial securities, which are in the focus of this article. Traditionally, different securities are traded in separate markets, one for each security. Most financial markets, such as over-the-counter-markets, or the New York Stock Exchange are in this sense *disconnected*. Some more recent exchanges, on the other hand, let customers place "contingent orders", "whose execution depend upon the execution and/or price of another security".[2] In such a *centralized* market participants are allowed make the demand for one good contingent on another. In other words, agents can ask and offer packages of the goods. Interlinking preferences across goods is by design not possible when each good is traded in a disconnected market. A dealer who bids for the 3-month German bond, for example, cannot make his

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[1]Examples of such platforms are: TheMinearlAuction, Global Dairy Trade, Aquabid.

[2]Definition from `http://www.investopedia.com/terms/c/contingentorder.asp`. For more detailed explanation of contingent orders and related "advanced trading types" see `https://www.fidelity.com/learning-center/trading-investing/trading/conditional-order-types`.

choice contingent on the price of the 1-year French bond. More generally, agents are constrained in the way they can display their joint preferences for the goods when markets are disconnected. They cannot freely maximize their gains from trade. Intuitively, the degree of efficiency is bound to hinge on the *market's structure* (centralized or disconnected). I show that this need not be the case. When the residual supplies of the goods are perfectly correlated the equilibrium allocation of disconnected markets is *identical* to the allocation of a centralized market. My counter-intuitive finding suggests that the market structure may be *irrelevant* when the underlying uncertainty that governs the strategic pricing process for each good is perfectly correlated. Extending this result to large markets in which agents are price-takers shows that disconnected markets can be *fully efficient*.

The irrelevance result provides guidance for the design of markets. While new technology has made it feasible to centralize separated markets, integrating them remains challenging for policy makers. They face national and institutional constraints. What is more, centralization often requires cross-border collaboration, further complicating the integration process. In general, combining disconnected markets involves some cost: some are transitory (like learning costs to adapt to a new system), some others are permanent. Existing ownership structures have to be broken. Market makers who centralize the system (intermediaries) take away parts of the total surplus, and might even distort the outcome by their strategic manipulations. My irrelevance result suggests when it is not worthwhile to pay these costs, because such policy intervention would have no or negligible effects on both the volume of trade and efficiency; or when we can expect advantages of separation, such as cross-market competition that can reduce trading fees and stimulate innovation, to dominate its disadvantages. It also helps one to understand why some markets remain disconnected even though centralization has long become technologically feasible. The markets of equity and fixed income securities are good examples. These are identical products which are traded in dozens of trading venues, none with dominating market shares. Why does the market structure not converge towards centralization? My result suggests a simple answer. The gains from market integration are not high enough to force changes in the existing market structure, because the fundamentals that drive the price for these identical products are highly correlated.

In the model, $n > 2$ agents, each with an independent private type, have joint-preferences over two perfectly divisible goods of potentially random exogenous total supply. These goods are traded in either a *centralized* or *disconnected* market. The later consists of two standard uniform-price double auctions which are run simultaneously for each good. In each auction agents submit demand schedules specifying a price for each quantity they demand or supply. The market clears at the price where aggregate demand meets aggregate supply, and each agent buys or sells what he demanded or offered at this price. When the market is centralized an agent is allowed to bid for bundles. More precisely, the rules of the standard uniform-price double auction are extended to allow the demand for one good to depend on the price of the other good. Holding all other rules of the game fixed allows me to focus on the effect of centralizing disconnected markets. If I were to compare the separate uniform-price auctions to some other combinatorial auction, I would no longer be able to separate the effect of centralization from those coming from changes of other rules of the transaction. The Irrelevance Theorem for markets with strategic agents ($n < \infty$), and its extension to large markets ($n \to \infty$), then builds on a comparison of the traded equilibrium quantities across market structures. These are the allocations of the unique symmetric, linear ex-post equilibria, whenever they exist, and of the corresponding Bayesian Nash Equilibrium otherwise.

My research topic fits into the literature that compares the performance of decentralized, or fragmented, markets with centralized markets. Decentralized markets are typically studied in (*i*) search (*ii*) bargaining or (*iii*) network models. Most contributions highlight different types of inefficiencies in decentralized markets. Using search theory, Miao (2006), for instance, shows that social welfare improves with monopolistic market making (*i*); Elliott and Nava (2015) argue in favor of centralized clearinghouses to resolve pervasive inefficiencies of disconnected matching markets (*ii*); while Elliott (2015) extends Kranton and Minehart (2001)'s pioneering network model of trade to quantify the efficiency losses (*iii*). In setting up an auction model, I take a different perspective on decentralized markets than previous studies (*i*-*iii*). It highlights a different aspect of decentralization, namely that demand for a good offered in one market cannot be made contingent on the price of another good. To avoid confusion, I call such a market disconnected, rather than decentralized. It consists of simultaneous multi-unit auctions. Therewith my work relates to a growing literature put forward by computer scientists. Motivated by Bikhchandani (1999), who warned that "simultaneous sealed bid auctions are likely to be inefficient under incomplete information" (p. 212), they quantify the efficiency of simultaneous auctions of heterogeneous goods by computing the "price of anarchy" (=the maximum ratio between the social welfare under an optimal allocation and the welfare at an equilibrium). While, most work concentrates on single-item auctions (e.g. Feldman et al. (2015a)), Syrgkanis and Tardos (2013) show that m simultaneously run uniform-price auctions achieve "at least" $\frac{e-1}{4e} \approx 0.158$ of the expected optimal effective welfare.[3] Malamud and Rostek (2014)'s findings are orthogonal to this literature. In independent work, they develop a framework that is similar to mine to study the potential of decentralizing the exchange of financial securities to improve efficiency.[5] They show that it can be strictly welfare improving to break up a centralized structure, modeled as a multi-asset uniform-price double auction. In their model, any change in market structure affects efficiency.

Coming from many different directions and using a wide variety of techniques, all of these articles agree that social welfare in centralized and decentralized markets differs. My Irrelevance Theorem goes against this broad consensus. Even though it is specific to particular applications, it is in the spirit of famous general theorems that tell us when "market structure" in different formats is irrelevant: Sah and Stiglitz (1987) and Dasgupta (1988) establish conditions under which the number of firms (=market structure) does not matter for technological innovation; Modigliani and Miller (1958) prove that the financial structure of the firm (=market structure) does not necessarily matter for the creation of value; Weber (1983) shows that the realized price of any auction game that sells identical objects (=market structure) is the realized price of the previous auction;

[3] Feldman et al. (2015b) suggest that such inefficiency washes out in the limit as markets grow large. This is a different environment than studied in this paper, where strategic pricing decisions of individual agents have an impact on the outcome.

and Vickery (1961) proves that some rules of the auction (=market structure) are irrelevant for the seller's expected revenue. Building on the Revenue Equivalence Theorem, Biais (1993) then demonstrates that centralized and fragmented markets with risk-averse agents who compete for a single market order (=market structure) may give rise to the same expected ask (bid) price.

My main methodological contribution belongs to the literature on multi-unit auctions of perfectly divisible goods. I rely on existing research on multi-unit auctions with perfectly divisible goods, so called "share auctions".[4] Share auctions were introduced by Wilson (1979) for single-sided transactions, and closely relate to Klemperer and Meyer (1989), Kyle (1989), Vives (2011), Rostek and Weretka (2012)'s work on uniform-price double auctions. More specifically, I draw on insights by Du and Zhu (2012), whose framework has been used in other articles in the finance literature, so for instance by Duffie and Zhu (2016). They make assumptions on the traders' utility functions that allow them to solve for ex-post equilibria of an isolated uniform-price double auction, as well as a multi-assets double auction. My ex-post equilibria are derived based on the same assumptions. This literature typically considers an auction in isolation neglecting possible interconnections across auction markets. While we have some understanding of how agents behave in multi-unit auctions that trade or sell either one good, or multiple goods within the same transaction, the existing published literature is - to the best of my knowledge - silent about strategic incentives of agents that participate in separate multi-unit auctions that offer related goods.[5] My necessary optimality condition for the Bayesian Nash Equilibrium of this complex game holds for a broad class of utilities and any differentiable distribution functions and enables me to explain the strategic incentives that lie behind the equilibrium. Moreover, it has a straight-forward extension to the other most frequently used (sealed-bid) multi-unit auction format, the pay-as-bid auction.

The remainder of the article is structured as follows. Having set-up the model in section 1, section 2 explains the bidding incentives of strategic agents based on first-order conditions (Lemma 1, 2), and provides existence as well as uniqueness results for symmetric, linear equilibria (Proposition 1, 2). A comparison of the equilibrium allocations across market structures leads to the Irrelevance Theorem stated in section 3. Before concluding in section 5, I extend the result to large markets with price-taking agents in section 4. All proofs are given in the appendix. Random variables will be denoted in **bold** throughout the article.

[4]Similar to the frequent assumption in the literature on single-unit auctions that the set of available prices is dense, the assumption of perfect divisibility is a continuous approximation of a discrete set of quantities - which across economic disciplines has long been recognized as a valuable alternative when discrete problems are intractable (Woodward (2015)). With imperfect divisibility of goods or buyers who can submit only a maximal amount of bids the analysis becomes more complex due to discontinuities and rationing. This has been demonstrated recently by Hortaçsu and McAdams (2010) and Kastl (2011, 2012).

[5]Independent to my own work Malamud, Rostek and Yoon are currently working on a related paper.

1 Framework

$n > 2$ agents trade two perfectly divisible goods, indexed $m = 1, 2$, in a centralized market or disconnected market.[6] The centralized market is modeled as a multi-good uniform-price double auction, the disconnected market consist of two separate single-good uniform-price double auctions which are run simultaneously. The total, exogenous supply in each market, $\{\boldsymbol{Q_1}, \boldsymbol{Q_2}\}$ may random and potentially correlated:

$$\begin{pmatrix} \boldsymbol{Q_2} \\ \boldsymbol{Q_2} \end{pmatrix} \sim \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right).$$

Assuming $Var(\boldsymbol{Q_1}) = Var(\boldsymbol{Q_2}) \equiv \sigma^2$ creates symmetric market conditions across goods. Relaxing this assumption complicates the algebraic derivations without bringing further insights. Setting $\sigma = 0$ and $\mu_m = Q_m$ with $|Q_m| < \infty$ leaves us with total supply quantities that are commonly known to all traders. When $Q_m = 0$ there is no exogenous supply.

Each agent has private information. He draws a private type s_i, which captures individual preferences or personal evaluations of risk. If the agent is part of a large financial institution, it may also reflect orders from individual customers. For simplicity it is only one-dimensional, and iid:

$$\boldsymbol{s_i} \sim (\mu_s, \sigma_s^2) \ iid \text{ across } i \text{ and } \boldsymbol{Q_1}, \boldsymbol{Q_2}.$$

By using a framework of independent private values, I break with the traditional view according to which the demand of financial securities is driven by common values. While the price of a security stabilizes in the long run, so that its value is common to all agents, it fluctuates a lot in the short run. In a fast moving financial market, individual factors might, therefore, explain demand more adequately. Empirical evidence for this view is provided by Hortaçsu and Kastl (2012) with data from single-sided treasury auctions.

Notice that I have not specified any particular distribution. In fact, all of my main results will hold for arbitrary distributions that are commonly known among agents, have differentiable distribution functions and fulfill the few specifications that I have mentioned so far. This stands in contrast to most of the related literature, which imposes the normal distribution so as to derive linear equilibria (e.g. Kyle (1989), Vives (2011), Rostek and Weretka (2012)). In this particular regard, my works is more general.

All agents submit a pair of differentiable demand functions, denoted $\{x_{i,1}(\cdot, s_i), x_{i,2}(\cdot, s_i)\}$ in the disconnected and $\{\bar{x}_{i,1}(\cdot, \cdot, s_i), \bar{x}_{i,2}(\cdot, \cdot, s_i)\}$ in the centralized market, which are decreasing in their first argument. Each demand schedule specifies how much the agent is willing to buy $0 < q_m$ or sell $q_m < 0$ at what price(s). Only finite quantity offers $q_m \in [\underline{q}_m, \overline{q}_m]$, $-\infty < \underline{q}_m < 0 < \overline{q}_m < \infty$, are accepted. This is a purely technical assumption. It rules out that the market clears at infinitely high or low prices - an event that can theoretically occur when either the types or the total supply

[6]With $n = 2$ agents the non-existence of equilibria has long been recognized in the literature when marginal utility is decreasing (e.g. Kyle (1989) from Ausubel et al. (2014), Du and Zhu (2016)).

have unbounded support. With bounded supports, bounding the demand becomes unnecessary.

$$x_{i,m}(\cdot, s_i) : \mathbb{R} \rightarrow [\underline{q}_m, \overline{q}_m] \quad \text{for } m = 1, 2 \qquad \text{in disconnected auctions} \qquad (1)$$
$$\bar{x}_{i,m}(\cdot, \cdot, s_i) : \mathbb{R}^2 \rightarrow [\underline{q}_m, \overline{q}_m] \quad \text{for } m = 1, 2 \qquad \text{in centralized auction} \qquad (\bar{1})$$

Demand functions map from price(s) into the space of quantities. When analyzing equilibrium behavior, it will be more intuitive to work with bidding functions: $b_{i,m}(\cdot, s_i)$, $\bar{b}_{i,m}(\cdot, \cdot, s_i)$. These are inverse demands, specifying a price per quantity/ies.

Once all agents have submitted their demands, the market for each good, say 1, clears at price p_1^* where aggregate demand meets total supply.

$$x_{i,1}(p_1^*, s_i) = Q_1 - \textstyle\sum_{j \neq i} x_{j,1}(p_1^*, s_j) \qquad \text{in disconnected auctions} \qquad (2)$$
$$\bar{x}_{i,1}(\bar{p}_1^*, \bar{p}_2^*, s_i) = Q_1 - \textstyle\sum_{j \neq i} \bar{x}_{j,1}(\bar{p}_1^*, \bar{p}_2^*, s_j) \qquad \text{in centralized auction} \qquad (\bar{2})$$

Each agent then buys or sells what he asked for at this price, abbreviated by $q_{i,1}^* \equiv x_{i,1}(p_1^*, s_i)$ and $\bar{q}_{i,1}^* \equiv \bar{x}_{i,1}(\bar{p}_1^*, \bar{p}_2^*, s_i)$. He makes a total payment of

$$TP(p_1^*, p_2^*, q_{i,1}^*, q_{i,2}^*) \equiv p_1^* q_{i,1}^* + p_2^* q_{i,2}^* \qquad \text{in disconnected auctions} \qquad (3)$$
$$\overline{TP}(\bar{p}_1^*, \bar{p}_2^*, \bar{q}_{i,1}^*, \bar{q}_{i,2}^*) \equiv \bar{p}_1^* \bar{q}_{i,1}^* + \bar{p}_2^* \bar{q}_{i,2}^* \qquad \text{in centralized auction} \qquad (\bar{3})$$

In order to determine the optimal strategy, each agent maximizes his net payoff. It is defined as the total utility the agent receives from the goods minus his total payment. Owning quantities q_1, q_2, type s_i receives a utility of

$$U(q_1, q_2, s_i) = \sum_{m=1,2} \left\{ s_i q_m - \frac{1}{2} \lambda q_m^2 \right\} - \delta q_1 q_2 \quad \text{with } \lambda > 0, |\delta| \leq \lambda, \ \lambda + \delta > 0. \qquad (4)$$

This utility function is simple and intuitive:[7] From winning amount q_m the agent obtains a marginal value s_i. Holding an "inventory" q_m of the illiquid asset is costly for the trader. He pays a cost of $\frac{1}{2}\lambda q_m^2$. It may be related to regulatory capital or collateral requirements, or represent an expected cost of being forced to raise liquidity by quickly disposing of remaining inventory into an illiquid market (Duffie and Zhu (2016)). When $\delta \neq 0$, the utility function displays an additional factor: $\delta q_1 q_2$. Its meaning is best understood by analyzing the agent's partial utility of q_m

$$\frac{\partial U(q_1, q_2, s_i)}{\partial q_m} = s_i - \lambda q_m - \delta q_{-m} \quad \text{for } m = 1, 2; -m \neq m. \qquad (5)$$

This partial utility is the agent's "true marginal willingness to pay" for a quantity q_m given that he obtains quantity q_{-m}. It decreases in the amount of good m ($\lambda > 0$), and decreases or increases

[7] In the relatively sparse literature that considers multiple assets, one often finds this utility function expressed in its matrix notation

$$U(q_1, q_2, s_i) = \begin{pmatrix} s_i & s_i \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} q_1 & q_2 \end{pmatrix} \Delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad \text{where } \Delta \equiv \begin{pmatrix} \lambda & \delta \\ \delta & \lambda \end{pmatrix}$$

Furthermore, related literature that only consider a single asset, frequently assumes a quadratic cost $\frac{1}{2}\lambda q_m^2$ (see Vives (2011), Rostek and Weretka (2012), Duffie and Zhu (2016) and others).

in the quantity of the other good $-m$ depending on the sign of δ. This parameter measures the relation across goods. Whenever $\delta > 0$ the agent is willing to pay *less* for any given amount q_m, *the more* he purchases of good $-m$. The goods are substitutes. They are perfect substitutes when $\delta = \lambda$. Then the marginal utility decreases by the same amount regardless of which good the agent purchases. On the other hand, when $\delta < 0$, the agent values the same quantity q_m *more*, *the more* he owns of the other good $-m$. In this case, goods are complements. Setting $\delta = 0$ I could shut down any interconnection between goods to be back to the case of an isolated auction. However, this case is uninteresting. With no relation between the goods there are no strategic effects across goods. The allocation of the centralized and disconnected market trivially coincides. I therefore focus on $\delta \neq 0$ throughout the article.

It is the simple functional form of the utility function that makes the model tractable. In particular, a linear marginal willingness to pay with deterministic slope coefficients gives rise to a linear equilibrium. My optimality conditions for the Bayesian Nash Equilibrium in the simultaneous double auctions (Lemma 4), however, holds for any utility function that is twice differentiable and has continuous cross-partial derivatives.

Definition 1. *In the disconnected market, a pure-strategy BNE is a pair* $\{b^*_{i,1}(\cdot, s_i), b^*_{i,2}(\cdot, s_i)\}$ *that maximizes expected total surplus for all* $\forall i \in I$*.*

$$\max_{b_{i,1}(\cdot,s_i),b_{i,2}(\cdot,s_i)} \mathbb{E}[U(\boldsymbol{q^*_{i,1}}, \boldsymbol{q^*_{i,2}}, s_i)] - \mathbb{E}[TP(\boldsymbol{p^*_1}, \boldsymbol{p^*_2}, \boldsymbol{q^*_{i,1}}, \boldsymbol{q^*_{i,2}})] \text{ with } \boldsymbol{p^*_m} = b^*_{i,m}(\boldsymbol{q^*_{i,m}}, s_i) \text{ for } m = 1, 2.$$

The definition for the centralized market is analogous, with the difference that both functions now depend on both quantities, i.e. $\{\bar{b}^*_{i,1}(\cdot, \cdot, s_i), \bar{b}^*_{i,2}(\cdot, \cdot, s_i)\}$.

Given linear true marginal willingness to pay, it is natural to look for BNE that are linear. "Linear equilibria are tractable, particularly in the presence of private information, have desirable properties like simplicity, and have proved to be very useful as a basis for empirical analysis" (Vives (2011), p. 1920). Studying them is the standard in the related, theoretic literature (e.g. Kyle (1989), Vives (2011), Rostek and Weretka (2012), Du and Zhu (2012), Malamud and Rostek (2014)). Support comes from the empirical literature on single-sided multi-unit auction by Hortaçsu (2002). Using data from Turkish treasury auctions he shows that linear demands fit actual bidding behavior quite closely. Since all agents are (ex-ante) symmetric and derive utility from both goods, I will solve for symmetric, linear BNE in which all agents are active in both markets (Proposition 1 and 2).

I will be particularly interested in ex-post equilibria. Such equilibria are Bayesian Nash equilibria which are robust in the sense that no agent wishes he would have chosen differently once all uncertainty resolves. This is because every agent would choose the same strategy even if he could observe the private types of all of his competitors and the total amount for sale. Nobody regrets his choice ex-post. This implies that we do not need to worry about strategic effects of a secondary (or after) market. Such markets are prominent in particular in the finance sector. Anticipating of such effects could ruin the equilibrium when taken into account.

Definition 2. *An ex post equilibrium is a profile of strategies such that there exists no profile of types or total supply for which some agent would have an incentive to deviate.*

2 Equilibria

I now state, compare and explain the equilibrium strategies in the different environments. I start by describing how traders choose their equilibrium demand schedules. Understanding how choices are made lays the ground for the Irrelevance Theorem. Since equilibria will be linear, I derive the intuition for this case. More precisely, I give necessary conditions that characterize a linear Bayesian Nash Equilibrium. It is ex-post optimal in the centralized market, and the disconnected auctions when total supply is deterministic. More general optimality conditions for a (not necessarily linear) BNE in the disconnected auctions are given in Lemma 4 in Appendix I. These hold under very mild assumptions on the functional form of the utility.

I begin by explaining bidding incentives in the disconnected market, say auction 1. To explain the the agent's bidding incentives for good 1, let all other agents $j \neq i$ play the equilibrium strategies $\{b^*_{j,1}(\cdot, s_j), b^*_{j,1}(\cdot, s_j)\}$. Assume agent i, himself, behaves in auction 2 as he will in equilibrium $b^*_{i,2}(\cdot, s_i)$. He knows that this makes him win $\boldsymbol{q^*_{i,2}}$, implicitly characterized by market clearing

$$\boldsymbol{q^*_{i,2}} = \boldsymbol{Q_2} - \sum_{j \neq i} x_{j,2}(\boldsymbol{p}^*_2, \boldsymbol{s_j}) \quad \text{with } \boldsymbol{p}^*_2 = b^*_{i,2}(\boldsymbol{q^*_{i,2}}, s_i) \tag{2}$$

However, since both auctions take place simultaneously and the bidder neither knows the types of his competitors $\boldsymbol{s_{-i}}$ nor the total supply $\boldsymbol{Q_2}$ ex-ante, he does not know how much he will win in auction 2, when choosing his strategy in auction 1. In that auction, he takes the submitted demand schedules of all others as given. What count for his choice is not the total, but the residual supply

$$\boldsymbol{RS_{i,1}}(p_1) = \boldsymbol{Q_1} - \sum_{j \neq i} x^*_{j,1}(p_1, \boldsymbol{s_j}) \qquad \text{in price-quantity space} \tag{6}$$

$$q_1 = \boldsymbol{Q_1} - \sum_{j \neq i} x^*_{j,1}(\boldsymbol{p^{RS}_{i,1}}(q_1)) \qquad \text{in quantity-price space} \tag{7}$$

It is continuous and upward-sloping by the assumption that all bidding functions are continuous and decreasing. Moreover, since $\boldsymbol{s_{-i}}$ and $\boldsymbol{Q_1}$ are random, the residual supply is random. This makes it difficult for the agent. If he knew the realization of the supply, he would simply pick the point on the residual supply curve that maximizes his net payoff. To determine his optimal price offers, he goes through all possible realizations of the residual supply curve for good 1, $p^{RS}_{i,1}(q_1)$. The optimal bid-offer $b^*_{i,1}(q_1, s_i)$ equates its expected marginal utility with its expected marginal payment and clears the market: $p^{RS}_{i,1}(q_1) = b^*_{i,1}(q_1, s_i)$. Hereby, the agent takes the best guess about how much he will obtain in the other auction, by taking the conditional expectation. Lemma 1 summarizes.

Lemma 1. *A linear BNE with bidding functions* $b^*_{i,1}(\cdot, s_i), b^*_{i,2}(\cdot, s_i)$ *that are strictly decreasing in quantity must satisfy* $b^*_{i,m}(q_m, s_i) = p^{RS}_{i,m}(q_m)$*, and*

$$\mathbb{E}\left[\frac{\partial U(q_m, \boldsymbol{q^*_{i,-m}}, s_i)}{\partial q_m}\middle| q_m\right] = \mathbb{E}\left[\frac{\partial TP(p^{RS}_{i,m}(q_m), \boldsymbol{p^*_{-m}}, q_m, \boldsymbol{q^*_{i,-m}})}{\partial q_m}\middle| q_m\right] \tag{8}$$

for all q_m *and* $m = 1, 2, -m \neq m$.

The bidding behavior in the centralized market is different. In search for the optimal strategy, the agent now goes through all possible *pairs* of realizations of residual supply curves $\{\bar{p}^{RS}_{i,1}(q_1, q_2), \bar{p}^{RS}_{i,2}(q_2, q_1)\}$. Say a particular pair realizes and that offering prices $\{\bar{b}^*_{i,1}(q_1, q_2, s_i), \bar{b}^*_{i,2}(q_2, q_1, s_i)\}$ makes agent i win $\{q_1, q_2\}$. For each bid-offer to be optimal it must be that marginal utility from winning the bid, that is winning q_m, must equate the marginal payment for both goods $m = 1, 2$, and clear both markets simultaneously: $\bar{b}^*_{i,1}(q_1, q_2, s_i) = \bar{p}^{RS}_{i,1}(q_1, q_2)$, $\bar{b}^*_{i,2}(q_2, q_1, s_i) = \bar{p}^{RS}_{i,2}(q_2, q_1)$. Lemma 2 summarizes.

Lemma 2. *A linear BNE with two bidding functions* $\bar{b}^*_{i,1}(\cdot, \cdot, s_i), \bar{b}^*_{i,2}(\cdot, \cdot, s_i)$*, that are strictly decreasing in the first argument, must satisfy* $\bar{b}^*_{i,1}(q_1, q_2, s_i) = \bar{p}^{RS}_{i,1}(q_1, q_2)$*,* $\bar{b}^*_{i,2}(q_2, q_1, s_i) = \bar{p}^{RS}_{i,2}(q_2, q_1)$ *and*

$$\left[\frac{\partial U(q_1, q_2, s_i)}{\partial q_m}\right] = \left[\frac{\partial \overline{TP}(\bar{p}^{RS}_{i,1}(q_1, q_2), \bar{p}^{RS}_{i,2}(q_2, q_1), q_1, q_2)}{\partial q_m}\right] \tag{9}$$

for all q_1, q_2 *and* $m = 1, 2$.

Lemma 1 and 2 give necessary conditions for linear BNE. They help us understand the strategic considerations that drive the equilibrium bidding choice, if such equilibria exist. The next two propositions establish their functional form, and provide existence and uniqueness results. Proposition 1 continues with the centralized market. It is a variant of Du and Zhu (2012)'s Proposition 3.[8]

Proposition 1. *There exists a linear BNE in which traders submit*

$$\bar{b}^*_m(q_m, q_{-m}, s_i) = s_i - \left(\frac{n-1}{n-2}\right)\{\lambda q_m + \delta q_{-m}\} \text{ for } m = 1, 2, -m \neq m. \tag{10}$$

It is the unique symmetric ex-post equilibrium, in which all submit linear functions in both markets.

[8] When the agent has a type with a common value component: $\gamma s_i + \kappa \sum_{j \neq i} s_j$ with $\gamma + (n-1)\kappa = 1$, and total supply quantities are fixed, i.e. $\mathbb{E}[\boldsymbol{Q_m}] = Q_m$ for $m = 1, 2$, $\sigma = 0$, Du and Zhu (2012) show that there is an ex-post equilibrium in which traders submit

$$\bar{b}^*_m(q_m, q_{-m}, s_i) = s_i - \left(\frac{n-1}{n\gamma - 2}\right)\{\lambda q_m + \delta q_m - \kappa Q_m\} \text{ for } m = 1, 2; -m \neq m \tag{38}$$

It can be shown that Lemma 2 carries over to that case, replacing $\mathbb{E}[\boldsymbol{Q_m}] = Q_m$ for $m = 1, 2$ in formula (14).

In equilibrium each agent shades his true marginal willingness to pay $\frac{\partial U(q_1,q_2,s_i)}{\partial q_m} = s_i - \lambda q_m - \delta q_{-m}$. He can influence the market-clearing price with positive probability. In the optimum all traders use their individual market power by shading bids for higher quantities more strongly. Similar to an oligopolist they reduce their demand. This strategic behavior is well-understood in the literature for multi-unit auctions that sell one divisible good to agents with multi-unit demand (see Ausubel et al. (2014)). It carries over to double auctions with multiple goods where the true demand is multi-dimensional.

The equilibrium strategy in a centralized market is very similar to the one in a disconnected market in which the residual supply curves are perfectly correlated across goods. Before analyzing this case, let me formally define what this condition means. To do so, it helps to have a better understanding of each residual supply curve. Given all other agents $j \neq i$ play linear strategies of the following form

$$x^*_{j,m}(p_m, s_j) = o_m + a_m s_j - c_m p_m \text{ with } o_m, a_m \in \mathbb{R}, c_m > 0 \tag{11}$$

it is linear:

$$RS_{i,m}(p_m, \boldsymbol{Z_{i,m}}) = \boldsymbol{Z_{i,m}} - (n-1)o_m + (n-1)c_m p_m \tag{12}$$

$$\text{with } \boldsymbol{Z_{i,m}} \equiv \boldsymbol{Q_m} - a_m \sum_{j \neq i} \boldsymbol{s_j}. \tag{13}$$

Definition 3. *(i) Fix p_1, p_2. The residual supply quantities at those prices are perfectly correlated iff $\boldsymbol{Z_{i,1}}$ and $\boldsymbol{Z_{i,2}}$ are. (ii) The residual supply curves are perfectly correlated iff the residual supply quantities are for all prices.*

Lemma 3 tells us when the residual supply curves are perfect correlated in the current framework, in which all agents have one private type and participate in both markets.

Lemma 3. *The residual supply curves are perfectly correlated iff $\sigma = 0$ or $\rho = 1$ and $a_1 = a_2$.*

When the total amount for sale is known to all agents ($\sigma = 0$), the only random source that shifts the residual supply is $\sum_{j \neq i} \boldsymbol{s_j}$. Since it is common to both curves, they are perfectly correlated. On the other hand, with perfectly correlated exogenous supply ($\rho = 1$), the residual supply curves are perfectly correlated when the type's coefficient a_m is the same across markets. The following proposition shows that this is the case in equilibrium.[8]

Proposition 2. *Let the residual supply curves be perfectly correlated across markets.*
(i) In a linear BNE, in which all are active in both markets, traders submit for $m = 1, 2, -m \neq m$

$$b^*_m(q_m, s_i) = \bar{b}^*_m(q_m, q_m, s_i) + \left(\frac{\delta}{n}\right)(\mu_m - \mu_{-m}). \tag{14}$$

(ii) Its existence is guaranteed when total supply is deterministic ($\sigma = 0$, $\mu_m = Q_m$ for $m = 1, 2$). Then it is the unique symmetric ex-post equilibrium, in which all submit linear functions in both markets.

In a disconnected auction, the price offer for good m can - by design - not depend on the amount the agents has of good $-m$. The agent is forced to substitute q_{-m} in $\bar{b}^*_m(q_m, q_{-m}, s_i)$ by q_m. This means that he can no longer interlink his submitted demands explicitly. What he can do however, is to make his submitted demand in market m dependent on the expected total supply of the other market, μ_1, μ_2. This allows the agent to *implicitly* interlink his submitted demand across markets, even though the market rules prevent him from *explicitly* connecting his preferences.

Notably, the equilibrium is linear, even though its underlying optimality condition given in Lemma 1 (with quadratic utility) hinges on a conditional expectation, $\mathbb{E}\left[\boldsymbol{q^*_{i,-m}} \middle| q_m\right]$, which are typically not linear. The solution is independent of any particular distribution, because i's winning quantity in market $-m$ is a linear function of i's winning quantity in market m when the residual supply curves are perfect correlated. As a result, both conditional expectations are linear. Now, one might wonder how agents behave in a disconnected market in which the residual supply curves are not perfectly correlated. Without linear mapping between the equilibrium winning quantities of both auctions, an appropriate distributional assumptions is needed to ensure that the conditional expectation of the winning quantity of the other auction is a linear function. Only then there can be a linear equilibrium. Here I choose the standard distribution used in the literature, the normal distribution. Otherwise, I keep the same distributional assumptions, i.e. $\boldsymbol{s_i} \sim N(\mu_s, \sigma_s^2), iid$ and $\begin{pmatrix} \boldsymbol{Q_2} \\ \boldsymbol{Q_2} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$.

Proposition 3. *Define* $\rho^i(\alpha) \equiv \frac{\rho\sigma^2 + \alpha^2(n-1)\sigma_s^2}{\sigma^2 + \alpha^2(n-1)\sigma_s^2}$. *Let* $|\rho^i(\alpha)\delta| \leq \lambda$. *In a symmetric BNE in which all are active in both markets, traders submit for* $m = 1, 2, -m \neq m$

$$\beta^*_m(q_m, s_i) = \epsilon(\alpha) + \alpha s_i - \gamma(\alpha) q_m \text{ with } \alpha = 1 - \delta\alpha\left(\frac{1}{n}\right)(n-1)[1 - \rho^i(\alpha)] \tag{15}$$

$$\gamma(\alpha) = \left(\frac{n-1}{n-2}\right)\left(\lambda + \delta\rho^i(\alpha)\right)$$

$$\epsilon(\alpha) = \delta\left(\frac{1}{n}\right)\left[(\rho^i(\alpha)\mu_m - \mu_{-m}) + \alpha(n-1)\mu_s[1 - \rho^i(\alpha)]\right]$$

When the residual supply curves are not perfectly correlated, the agent no longer knows exactly how much he will win in the other auction $-m$ conditional on winning a particular amount in m. The best he can do is to exploit the correlation across residual supplies, that is - by market clearing - his winning quantities (Lemma 1). This is the reason for which the correlation $\rho^i(\alpha)$ of winning quantities, or equivalently of residual supplies, now plays a key role in his choice. Otherwise, the strategy is very similar to the one under perfect correlation. This becomes transparent when comparing the coefficients of the linear functions of Proposition 2 and 3 (see Figure 1).

Not surprisingly, the function form of the BNE under the normal distribution coincides with Proposition 2 when winning quantities are perfectly correlated, $\rho^i(\alpha) = 1$. The later was derived for arbitrary distributions for the case in which residual supply curves - and therewith winning quantities - are perfectly correlated. It does not otherwise.

Corollary 1. $\rho^i(\alpha) = 1$ *iff* $\beta^*_m(q_m, s_i) = b^*_m(q_m, s_i)$.

Figure 1: $b^*_m(q_m, s_i) = \epsilon + \alpha s_i - \gamma q_m$ with

Prop. 2: perf. corr.	Prop. 3: w/o perf. corr.
$\epsilon = \delta\left(\frac{1}{n}\right)[\mu_m - \mu_{-m}]$	$\epsilon = \delta\left(\frac{1}{n}\right)\left[\rho^i(\alpha)\mu_m - \mu_{-m} + (1-\rho^i(\alpha))\alpha(n-1)\mu_s\right]$
$\alpha = 1$	$\alpha = 1 - \delta\alpha\left(\frac{1}{n}\right)(n-1)(1-\rho^i(\alpha))$
$\gamma = \left(\frac{n-1}{n-2}\right)(\lambda+\delta)$	$\gamma = \left(\frac{n-1}{n-2}\right)\left(\lambda + \delta\rho^i(\alpha)\right)$

3 Irrelevance Theorem

A comparison of the equilibrium allocations across market structures leads to the Irrelevance Theorem. It is counter-intuitive. In the centralized market, traders with joint preferences over the goods for sale are allowed to bid for bundles and can therewith jointly maximize their total surplus. Instead, in a disconnected market, their demand schedule can only depend on the price of the security traded in that market. By design of the transaction, agents are *always* constrained in the way they can display their preferences. One would therefore expect that the equilibrium allocation of the centralized market must differ from the one of the disconnected market.

Irrelevance Theorem. *The equilibrium allocation $\{q^*_{i,1}, q^*_{i,2}\}_{i=1}^n$ of symmetric, linear equilibria in centralized and disconnected markets coincide if the residual supply curves are perfectly correlated across goods.*

To understand why the market structure can be irrelevant recall the intuition that was laid out to explain equilibrium behavior. While preferences are two-dimensional a the submitted demand is one-dimensional in a disconnected auction. The agent picks an optimal point on each possible supply curve, taking the expectation of what will happen in the other market (condition (8)). On the contrary, in the centralized auction the agent is free to pick a *pair* of points on each *pair* of realizations of residual supply curves (condition (9)). In choosing how much he trades of one good the agents knows exactly how much he will trade of the other good. There is no need to take an expectation. This means that the trader can make a relatively "better informed" decision in the centralized market, unless the residual supply curves are perfectly correlated. In my set-up where all traders participate in both markets, this case occurs either with fixed ($\sigma = 0$) or random but perfectly correlated total supply ($\rho = 1$). Then a realization of the residual supply curve of good 1, which corresponds to some optimal choice for good 1, maps one-to-one to some realization of the curve of good 2, which in turn corresponds to an optimal choice in auction 2. Conditional on observing the realization in auction 1, the agent knows exactly how much he will win in the other auction 2. The inherent constraint that he faces in an disconnected auction becomes irrelevant. He deals with the same amount of uncertainty in either market structure. As a consequence, he trades as much as he does in the centralized market.

This intuition should generalize to many other environments that are not considered on a formal level. Say there are some underlying uncertainties about good m. So far they come from iid private information of the competitors $\boldsymbol{s_{-i}}$, and random exogenous total supply $\boldsymbol{Q_m}$, but they could also come from affiliated or common values of strategic agents, or from noise traders, etc. The key is that, for given strategies of the other agents, these underlying uncertainties aggregate to some random variable $\boldsymbol{Z_{i,m}}$ that governs the residual supply for good m: $RS_{i,m}(p_m, \boldsymbol{Z_{i,m}})$.[9] In such a more general setting, I expect the equilibrium allocation of the disconnected and the centralized market to coincide not only when $\boldsymbol{Z_{i,1}}$ and $\boldsymbol{Z_{i,2}}$ are perfectly correlated, but more generally when they move one-to-one. This is a weaker condition than perfect correlation, as it allows the realizations of both variables to be interlinked by some deterministic function $f(\cdot)$ that is not necessarily linear: $\boldsymbol{Z_{i,2}} = f(\boldsymbol{Z_{i,1}})$. Perfect correlation, instead, defines a linear relation: $\boldsymbol{Z_{i,2}} = r + g\boldsymbol{Z_{i,1}}$ with $g \pm \sqrt{\frac{Var(\boldsymbol{Z_{i,2}})}{Var(\boldsymbol{Z_{i,1}})}}$ and $r = \mathbb{E}[\boldsymbol{Z_{i,2}}] - g\mathbb{E}[\boldsymbol{Z_{i,1}}]$. Without it, equilibria will no longer be linear. While it is mathematically much more challenging to solve for non-linear equilibria, my intuition does not rely on linearity. I therefore conjecture that the market structure is irrelevant whenever residual supply curves move one-to-one.

3.1 Welfare Implications

The equivalence between equilibrium quantities has important implications for the total amount traded and efficiency. Since all agents trade the same quantities across market structures, the aggregated amount of trade is equivalent. Furthermore, either market structure achieves the same level of welfare:

$$W^* \equiv \sum_i \lambda_i U(q^*_{i,1}, q^*_{i,2}, s_i) \quad \text{,with type-specific welfare weight } \lambda_i,$$

is independent of whether the market is centralized or disconnected. Due to strategic demand reduction the outcome under either market structure is inefficient. This is a well-known weakness of uniform-price auctions with strategic bidders (see Ausubel et al. (2014)): Agents with high (low) valuations obtain less (more) than what would be efficient because they shade more (less) at the market-clearing price. The more market participants, the lower the impact of each individual agent, the lower the incentives to reduce demand strategically. In the limit, as $n \to \infty$, the individual impact on the market's outcome vanishes completely. The inefficiency washes out. The following section shows that disconnected markets give rise to the fully efficient allocation when residual supply curves are perfectly correlated across goods (Irrelevance Theorem for large markets).

4 Large Market

So far, I have considered markets in which agents behave strategically. They take their individual effect on the clearing price into account when setting their demand schedules. When markets are large, each agent has a negligible effect on the market's outcome. He is a price-taker. By sending

[9] Recall: In the above set-up with with linear equilibria, $x^*_{i,m}(p_m, s_i) = o_m + a_m s_i - c_m p_m$ this aggregate random variable was $\boldsymbol{Z_{i,m}} \equiv \boldsymbol{Q_m} - a_m \sum_{j \neq i} \boldsymbol{s_j}$.

the number of market participants $n \to \infty$, I can reproduce a world of perfect competition, shutting down strategic behavior. In the limit, all agents become price-takers. Alternatively, I could determine "price-taking equilibria", as in Vives (2011). Those are equilibria, in which agents are price-takers by assumption. With quadratic utility the following equilibria are the unique price-taking equilibria.

Corollary 2. *Let $n \to \infty$, $m = 1, 2, -m \neq m$. The price-taking agents choose*
(i) In the centralized market

$$b_m^*(q_m, q_{-m}, s_i) = \left[\frac{\partial U(q_1, q_2, s_i)}{\partial q_m}\right] = s_i - \delta q_m - \lambda q_{-m} \tag{16}$$

(ii) In the disconnected market

$$b_m^*(q_m, s_i) = \lim_{n\to\infty} \mathbb{E}\left[\frac{\partial U(q_m, \boldsymbol{q^*_{i,-m}}, s_i)}{\partial q_m}\middle| q_m\right] = s_i - (\delta + \lambda) q_m \quad \textit{under perf. corr.} \tag{17}$$

In the centralized market, all agents submit their true marginal willingness to pay: $s_i - \lambda q_m - \delta q_{-m}$. The outcome is fully efficient as a result. Strategic demand reduction no longer distorts the allocation of quantities. The Irrelevance Theorem for large markets reveals that disconnected markets can be fully efficient.

Irrelevance Theorem (Large Markets)**.** *When agents are price-takers, the allocation of disconnected markets coincides with the fully efficient allocation of the centralized market as long as residual supply curves are perfectly correlated across goods.*

This finding contradicts a common understanding among economists according to which a social planner would never choose to separate markets when agents have joint preferences over goods. In fact, if there is only a negligible cost to centralize disconnected markets, a social planner would refrain from breaking up the existing market structure, in case the uncertainty that governs the market price of each good is perfectly correlated across markets. Intuitively, there is enough information in each disconnected market to achieve first-best.

5 Conclusion

I provide a novel Irrelevance Theorem. It shows under which conditions strategic traders with joint preferences over bundles of goods trade the exact same amount in disconnected and centralized markets. Only in the later they can freely represent their true two-dimensional preferences. I argue that the inherent constraint that agents face in a disconnected market is non-binding whenever the underlying uncertainty that drives pricing decisions is perfectly correlated across markets. In that case there is no informational differences that could lead to differences in allocations of the two market structures. Large disconnected markets, in which agents are price-takers, turn out to be fully efficient.

Besides broad significance for the design of trading markets, my findings could have concrete policy implications. Recently Budish et al. (2015) proposed to reform high-frequency trading markets. They advocate to replace the continuous limit order book which causes an inefficient race in high-frequency trading with frequently held batch uniform-price double auctions. Their model only has one good and thus abstracts from strategic substitution, or arbitrage effects across markets. Such effects could in principle have adverse consequences on the equilibrium dynamics. My irrelevance result tells us when we do not have to care about cross-market effects. It should, however, be used with caution in the evaluation of real-life markets. Rather than representing any particular market as realistically as possible, my model should be taken as insightful theoretic benchmark. It points to an important extreme case, highlighting one particular factor. Other factors should not be forgotten, when evaluating the performance of real-life markets which are much more complex than any theoretic model. In this regard, my irrelevance theorem is similar to other irrelevance statements, including the most influential ones. It tells us what we have to care about, or more broadly, what goes wrong when the condition under which the theorem holds is violated. Typically, those conditions are extreme. In my case it is perfect correlation of residual supply. Other irrelevance theorems build on other knife-edge assumptions, so are zero transaction costs, for instance, necessary for allocation of resources to be invariant to the assignment of private property rights (Coase Theorem).

In future work, I aim to generalize the Irrelevance Theorem to apply in more environments, including those that give rise to non-linear, potentially asymmetric equilibria. As as starting point, I would like to enrich the framework in letting agents have two-dimensional types with a common value component, and study effects of asymmetric market participation. In addition, my general first-order conditions of the disconnected auctions serve as theoretic foundation for a related empirical project of mine. In collaboration with Jason Allen and Jakub Kastl I structurally estimate the interdependencies in primary dealer's demand for government securities with different maturities in pay-as-bid auctions.

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Appendix

The appendix is split into two parts. **Appendix I** (pp. 1-10) states and proves general necessary optimality conditions for disconnected markets that hold for non-linear equilibria and many utility functions (Lemma 4). **Appendix II** (pp. 11-27) has all proofs corresponding to formal statements given in the main text.

Appendix I: General opt. condition for discon. market

The goal of this section is to state and prove the necessary condition for a BNE in the disconnected market which consists of a set of two bidding functions $\{b^*_{i,1}(\cdot, s_i), b^*_{i,2}(\cdot, s_i)\}_{i=1}^n$, or equivalently demand functions $\{x^*_{i,1}(\cdot, s_i), x^*_{i,2}(\cdot, s_i)\}_{i=1}^n$, that are differentiable and strictly decreasing in quantities, respectively prices (Lemma 4).

This condition holds for a broad class of utility functions. More precisely, I require $U(\cdot, \cdot, s_1)$ to be twice differentiable with continuous cross-partial derivatives. For convenience I denote[10]

$$\mu_1(q_1, q_2, s_i) \equiv \frac{\partial U(q_1, q_2, s_i)}{\partial q_1},\ \mu_2(q_2, q_1, s_i) \equiv \frac{\partial U(q_1, q_2, s_i)}{\partial q_2}, \mu(q_1, q_2, s_i) \equiv \frac{\partial^2 U(q_1, q_2, s_i)}{\partial q_1 \partial q_2}$$

Remark: Twice differentiability is enough to give a necessary condition, but it is not enough to guarantee that there is a function that satisfies this condition point-wise, and is a maximum not minimum. While I do not make this claim here, I nevertheless would like to comment on additional assumptions that would be needed. For one, the partial derivative must be decreasing so that the solution can maximize the objective function. Second, recall that solution is assumed to be differentiable and strictly decreasing in quantity. The first property can only be fulfilled if all distribution functions are differentiable. The latter is more complicated. It will depend on the shape of the partial derivative and the conditional distribution of the clearing-price quantity, denoted $\boldsymbol{q^*_{i-m}}$. The later in turn hinges on the particular distribution of the types and the total supply quantities. Formally, $\mathbb{E}\left[\mu_m(q_m, \boldsymbol{q^*_{i-m}}, s_i)|q_m\right] < 0$ for $m = 1, 2, -m \neq m$.

The necessary condition (Lemma 4) will be stated using the joint and marginal distribution over i's clearing price quantities. Before stating the lemma, let me define this distribution. To do so, it helps to recall the definition of i's clearing-price quantity in auction m, $\boldsymbol{q^*_{i,m}}$. It is for $m = 1, 2$ implicitly defined by market clearing

$$\boldsymbol{q^*_{i,m}} = \boldsymbol{Q_m} - \sum_{j \neq i} x^*_{j,m}(\boldsymbol{p^*_m}, \boldsymbol{s_j}) \text{ with } \boldsymbol{p^*_m} = b_{i,m}(\boldsymbol{q^*_{i,m}}, s_i) \tag{2}$$

It is the (random) point on the (random) strictly increasing residual supply curve at price $\boldsymbol{p^*_m} = b_{i,m}(\boldsymbol{q^*_{i,m}}, s_i)$. Its support, $[\underline{q}^*_{i,m}, \overline{q}^*_{i,m}]$. It depends on the bid-offer for this amount. The maximal

[10] By Schwarz's Theorem, the cross-partial derivative is symmetric, i.e. $\frac{\partial^2 U(q_1,q_2,s_i)}{\partial q_2 \partial q_1} = \frac{\partial^2 U(q_1,q_2,s_i)}{\partial q_1 \partial q_2} \frac{\partial^2 U(q_1,q_2,s_i)}{\partial q_2 \partial q_1} \equiv \mu(q_1, q_2, s_i)$.

possible support is given by the maximal amounts one agent can sell $|\underline{q}_m|$ and buy $\overline{q}_m$ in the market by the rules of the transaction:

$$[\underline{q}^*_{i,m}, \overline{q}^*_{i,m}] \subseteq [\underline{q}_m, \overline{q}_m] \text{ for any } b_{i,m}(\cdot, s_i).$$

Definition 4. *Define the joint distribution over i's clearing price quantities as the probability that agent i receives at most quantity q_1 and at most quantity q_2 when bidding $b_{i,1}(q_1, s_i) = p_1, b_{i,2}(q_2, s_i) = p_2$ as*

$$G^i(q_1, q_2, p_1, p_2) \equiv \Pr\left(\boldsymbol{q^*_{i,1}} \leq q_1 \text{ and } \boldsymbol{q^*_{i,2}} \leq q_2\right) \tag{18}$$

Analogously, define the marginal distribution of i's clearing price quantity in market $m = 1, 2$ by

$$G^i_m(q_m, p_m) = \Pr\left(\boldsymbol{q^*_{i,m}} \leq q_m)\right) \tag{19}$$

And the conditional distribution

$$G^i_{2|1}(q_2, p_2|q_1, p_1) = \Pr\left(\boldsymbol{q^*_{i,2}} \leq q_2 \middle| \boldsymbol{q^*_{i,1}} \leq q_1\right). \tag{20}$$

I denote the corresponding joint and marginal density functions by g^i, g^i_m and $g^i_{2|1}$, and oftentimes abbreviate $b_{i,m}(q_m, s_i) = b_{i,m}$.

Lemma 4. *A BNE with pairs of strictly decreasing, differentiable bidding functions must for all $q_m, m = 1, 2$ satisfy*

$$\mathbb{E}\left[\frac{\partial U(q_m, \boldsymbol{q^*_{i,-m}}, s_i)}{\partial q_m} \middle| q_m\right] - b^*_{i,m}(q_m, s_i) = -q_m \left[\frac{\frac{\partial G^i_m(q_m, b^*_{i,m}(q_m, s_i))}{\partial q_m}}{\frac{\partial G^i_m(q_m, b^*_{i,m}(q_m, s_i))}{\partial p_m}}\right] \tag{21}$$

(ii) When $b^{*-1}_{i,m}(\cdot, s_i) = x^*_{i,m}(\cdot, s_i)$ is additive separable in s_i the necessary condition can be stated as

$$\mathbb{E}\left[\frac{\partial U(q_m, \boldsymbol{q^*_{i,-m}}, s_i)}{\partial q_m} \middle| q_m\right] - b^*_{i,m}(q_m, s_i) = +q_m \left[\frac{\partial RS_{i,m}(b^*_{i,m}(q_m, s_i))}{\partial p_m}\right]^{-1} \tag{22}$$

Proof of Lemma 4

The proof involves lengthly algebraic derivations. To not get lost in the equations, I lay out the core of the argument in the next section.

5.1 The Core of the Argument

Taking behavior of all others as given, agent i chooses two bidding functions $b^*_{i,1}(\cdot, s_i), b^*_{i,2}(\cdot, s_i)$ that maximize his objective function:

$$\mathcal{V}(b_{i,1}(\cdot, s_i), b_{i,2}(\cdot, s_i)) \equiv \mathbb{E}\left[U(\boldsymbol{q^*_{i,1}}, \boldsymbol{q^*_{i,2}}, s_i) - \sum_{m=1,2} \boldsymbol{q^*_{i,m} p^*_m}\right] \tag{$\mathcal{V}$}$$

By definition of an equilibrium there cannot be another pair of functions different from $\{b^*_{i,1}(\cdot,s_i), b^*_{i,2}(\cdot,s_i)\}$ which generates a higher payoff for agent i:

$$\{b^*_{i,1}(\cdot,s_i), b^*_{i,2}(\cdot,s_i)\} \in \arg\max_{b_{i,1}(\cdot,s_i),b_{i,2}(\cdot,s_i)} \mathcal{V}(b_{i,1}(\cdot,s_i), b_{i,2}(\cdot,s_i)) \tag{JM}$$

Since $\{b^*_{i,1}(\cdot,s_i), b^*_{i,2}(\cdot,s_i)\}$ must be the solution to i's maximization problem each function must solve the agent's maximization problem holding fixed the other:

$$\Rightarrow b^*_{i,m}(\cdot,s_i) \in \arg\max_{b_{i,m}(\cdot,s_i)} \mathcal{O}(b_{i,m}(\cdot,s_i)) \tag{M}$$

$$\text{with } \mathcal{O}(b_{i,m}(\cdot,s_i)) \equiv \mathcal{V}(b_{i,m}(\cdot,s_i), b^*_{i,-m}(\cdot,s_i)) \tag{O}$$

for $m = 1$ or 2 and $m \neq -m$. Otherwise there would be another pair of functions that would generate a higher payoff for the agent, so that $\{b^*_{i,1}(\cdot,s_i), b^*_{i,2}(\cdot,s_i)\}$ could not be the solution of the joint maximization problem (JM).

The rest of the prove derives the first-order condition of maximization problem (M). It is complicated for two main reasons. First we are maximizing over a function, not just a point. Second the objective function is the expected value of the agent's total surplus, which depends non-trivially on the bidding function we are trying to determine. Techniques of calculus of variation can be used to solve the optimization. The first step is to ex-press the objective function $\mathcal{O}(b_{i,m}(\cdot,s_i))$ in a format that explicitly states its dependence of the slope of the bidding function:

Auxiliary Lemma 1. *Denote* $\dot{b}_{i,m}(q_m,s_i) = \frac{\partial b_{i,m}(q_m,s_i)}{\partial q_m}$.

$$\mathcal{O}(b_{i,m}(\cdot,s_i)) = \int_{\underline{q}_m}^{\overline{q}_m} \mathcal{F}(q_m, b_{i,m}(q_m,s_i), \dot{b}_{i,m}(q_m,s_i))dq_m \tag{O}$$

$\mathcal{F}(\cdot,\cdot,\cdot)$ that is is continuous in its three arguments and has continuous partial derivatives with respect to the second and third, and takes the following form

$$\begin{aligned}\mathcal{F}(q_1, b_{i,1}(q_1,s_i), \dot{b}_{i,1}(q_1,s_i)) \equiv &\left[\mu_1(q_1,\overline{q}_2,s_i) - b_{i,1}(q_1,s_i) - q_1\dot{b}_{i,1}(q_1,s_i)\right][1 - G^i_1(q_1,b_{i,1})] \\ &- \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1,q_2,s_i)[1 - G^i(q_1,q_2,b_{i,1},b_{i,2})]dq_2 + const\end{aligned} \tag{$\mathcal{F}$}$$

Section 5.2 proves this auxiliary lemma. Section 5.3 then solves maximization problem (M). In a nutshell, its solution $b^*_{i,m}(\cdot,s_i)$ is then characterized by the Euler Equation. It is known in the literature of variational calculus (e.g. Kamien and Schwartz (1993), pp. 14-16):

$$\mathcal{F}_{b_{i,m}}(q_m, b^*_{i,m}(q_m,s_i), \dot{b}^*_{i,m}(q_m,s_i)) = \frac{d}{dq_m}\mathcal{F}_{\dot{b}_{i,m}}(q_m, b^*_{i,m}(q_m,s_i), \dot{b}^*_{i,m}(q_m,s_i))$$

where $\mathcal{F}_{b_{i,m}}$, and $\mathcal{F}_{\dot{b}_{i,m}}$ denotes the partial derivative of $\mathcal{F}(\cdot,\cdot,\cdot)$ w.r.t. the second and third argument. Rearranging the Euler Equation will give rise to the optimality condition of the lemma 4.

5.2 Proof of Auxiliary Lemma 1

Section 5.2.1 re-expresses the bidder's objective function $\mathcal{V}$. In a second step I fix function $b_{i,2}(\cdot, s_i) = b^*_{i,2}(\cdot, s_i)$ to obtain $\mathcal{O}(b_{i,m}(\cdot, s_i)) = \int_{\underline{q}_m}^{\overline{q}_m} \mathcal{F}(q_m, b_{i,m}(q_m, s_i), \dot{b}_{i,m}(q_m, s_i), s_i)dq_m$ and show that $\mathcal{F}(\cdot, \cdot, \cdot)$ has the claimed properties (section 5.2.2).

Several times throughout the proof, I will rely on the Fundamental Theorem of Calculus and Fubini's Theorem. The first applies because all functions are integrable w.r.t. $q_1 \times q_2$ and all integrals take finite values. Here I am relying on the assumption that no bidder can supply or demand infinite amounts which bounds the quantity space $[\underline{q}_m, \overline{q}_m]$ for $m = 1, 2$, and thus avoids complications due to unboundedness. Similarly, Fubini's Theorem is valid whenever applied because the function inside integrals will be defined on the closed interval $[\underline{q}_m, \overline{q}_m]$ for $m = 1$ or 2. Moreover it is continuous since I am assuming that all involved functions are differentiable.

5.2.1 Step 1: Simplifying $\mathcal{V}$

The simplification of i's objective involves several rounds of integration by parts. The goal is to express everything in everything in terms of distribution functions, rather than densities. I start with the expected utility.

(a) Re-expressing the expected utility

Using the introduced distribution of i's clearing-price quantities, the expected utility is

$$\mathbb{E}\left[U(\boldsymbol{q}^*_1, \boldsymbol{q}^*_2, s_i)\right] = \int_{\underline{q}^*_2}^{\overline{q}^*_2} \int_{\underline{q}^*_1}^{\overline{q}^*_1} U(q_1, q_2, s_i) g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1 dq_2$$

Now, since i's clearing-price quantity depends on what price he offers for this amount, its support depends on the bid choice. This is inconvenient because will want to determine the optimal bid choice. Luckily there is a clever way around this complication. Since for a given bid choice, $g^i_m(q_m, b_{i,m}(q_m, s_i)) = g^i(q_1, q_2, b_{i,1}(q_2, s_i), b_{i,2}(q_2, s_i)) = 0$ for $q_m \notin \left[\underline{q}^*_m, \overline{q}^*_m\right]$, $m = 1$ or 2, I can extend the bounds of the integrals. They are then independent of i's bid choice.

$$\mathbb{E}\left[U(\boldsymbol{q}^*_1, \boldsymbol{q}^*_2, s_i)\right] = \int_{\underline{q}_2}^{\overline{q}_2} \int_{\underline{q}_1}^{\overline{q}_1} U(q_1, q_2, s_i) g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1 dq_2$$

In the first round of the simplification, integrate the inner integral by parts, taking the derivative of $U(q_1, q_2, s_i)$ and integrating $g^i(q_1, q_2, b_{i,1}, b_{i,2})$ w.r.t. q_1.

$$\begin{aligned}\int_{\underline{q}_2}^{\overline{q}_2} \int_{\underline{q}_1}^{\overline{q}_1} U(q_1, q_2, s_i) g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1 dq_2 &= \int_{\underline{q}_2}^{\overline{q}_2} \left[U(q_1, q_1, s_i) \int_{\underline{q}_1}^{q_1} g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1 \Bigg|_{q_1=\underline{q}_1}^{q_1=\overline{q}_1} \right] dq_l \\ &\quad - \int_{\underline{q}_2}^{\overline{q}_2} \left[\int_{\underline{q}_1}^{\overline{q}_1} \left[\mu_1(q_1, q_2, s_i) \int_{\underline{q}_1}^{q_1} g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1 \right] dq_1 \right] dq_2\end{aligned}$$

Evaluate the first term at its bounds of integration. Since $\int_{\underline{q}_1}^{q_1} g^i(q_1, q_2, b_{i,1}, b_{i,2}) = 0$ and $\int_{\underline{q}_1}^{\overline{q}_1} g^i(q_1, q_2, b_{i,1}, b_{i,2}) = \int_{\underline{q}_{i,1}^*}^{\overline{q}_{i,1}^*} g^i(q_1, q_2, b_{i,1}, b_{i,2}) = g^i(q_2, b_{i,2})$ by definition of a marginal distribution we obtain

$$
\begin{aligned}
&= \int_{\underline{q}_2}^{\overline{q}_2} U(\overline{q}_1, q_2, s_i) g_2^i(q_2, b_{i,2}) dq_2 \\
&- \int_{\underline{q}_2}^{\overline{q}_2} \left[\int_{\underline{q}_1}^{\overline{q}_1} \left[\mu_1(q_1, q_2, s_i) \int_{\underline{q}_1}^{q_1} g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1 \right] dq_1 \right] dq_2
\end{aligned}
$$

I label the first term by A and the second by B.

A) Consider term A and integrate by parts w.r.t. q_2

$$
A = \int_{\underline{q}_2}^{\overline{q}_2} U(\overline{q}_1, q_2, s_i) g_2^i(q_2, b_{i,2}) dq_2 = U(\overline{q}_1, q_2, s_i) G_2^i(q_2, b_{i,2}) \Big|_{\underline{q}_2}^{\overline{q}_2} - \int_{\underline{q}_2}^{\overline{q}_2} \mu_2(\overline{q}_2, q_2, s_i) G_2^i(q_2, b_{i,2}) dq_2
$$

Since $G_2^i(\overline{q}_2, b_{i,2}) = 1$ and $G_2^i(\underline{q}_2, b_{i,2}) = 0$ for all $b_{i,2}$, this is

$$
A = U(\overline{q}_1, \overline{q}_2, s_i) - \int_{\underline{q}_2}^{\overline{q}_2} \mu_2(\overline{q}_2, q_2, s_i) G_2^i(q_2, b_{i,2}) dq_2 \tag{A}
$$

B) Now consider term B. Applying Fubini's Theorem, I can revert the order of integration of the two outer integrals:

$$
B = \int_{\underline{q}_1}^{\overline{q}_1} \left[\int_{\underline{q}_2}^{\overline{q}_2} \left[\mu_1(q_1, q_2, s_i) \int_{\underline{q}_1}^{q_1} g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1 \right] dq_2 \right] dq_1
$$

In the following I simplify the inner integral (corresponding to dq_2) by parts, repeating the same exercise as of the very first step. I integrate $\int_{\underline{q}_1}^{q_1} g^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_1$ and take the derivative of $\mu_1(q_1, q_2, s_i)$ w.r.t. q_2.

$$
B = \int_{\underline{q}_1}^{\overline{q}_1} \left[\mu_1(q_1, q_2, s_i) G^i(q_1, q_2, b_{i,1}, b_{i,2}) \Big|_{q_2 = \underline{q}_2}^{q_2 = \overline{q}_2} - \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2, s_i) G^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_2 \right] dq_1
$$

Again the first term simplifies, now using that for any $b_{i,2}, b_{i,1}$, $G^i(q_2, \overline{q}_2, b_{i,1}, b_{i,2}) = G_1^i(q_1, b_{i,1})$ and $G^i(q_1, \underline{q}_2, b_{i,1}, b_{i,2}) = 0$. I obtain

$$
B = \int_{\underline{q}_1}^{\overline{q}_1} \mu_1(q_1, \overline{q}_2, s_i) G_1^i(q_1, b_{i,1}) dq_1 - \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2, s_i) G^i(q_1, q_2, b_{i,1}, b_{i,2}) dq_2 \tag{B}
$$

Combining A - B, the expected utility reads with $m = 1, 2, -m \neq m$

$$
\begin{aligned}
\mathbb{E}\left[U(\boldsymbol{q}_1^*, \boldsymbol{q}_2^*, s_i)\right] = &- \sum_m \int_{\underline{q}_m}^{\overline{q}_m} \mu_m(q_m, \overline{q}_{-m}, s_i) G_m^i(q_m, b_{i,m}) dq_m \\
&+ \int_{\underline{q}_1}^{\overline{q}_2} \int_{\underline{q}_2}^{\overline{q}_2} \left[\mu(q_1, q_2, s_i) G^i(q_1, q_2, b_{i,1}, b_{i,2})\right] dq_2 dq_1 + U(\overline{q}_1, \overline{q}_2, s_i)
\end{aligned}
$$

Applying the Fundamental Theorem of Calculus one can re-express the expression as

$$\mathbb{E}\left[U(\boldsymbol{q}_1^*, \boldsymbol{q}_2^*, s_i)\right] = +\sum_m \int_{\underline{q}_m}^{\overline{q}_m} \mu_m(q_m, \overline{q}_{-m}, s_i)[1 - G_m^i(q_m, b_{i,m})]dq_m$$
$$- \int_{\underline{q}_1}^{\overline{q}_2} \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2, s_i)[1 - G^i(q_1, q_2, b_{i,1}, b_{i,2})]dq_2 dq_1 - U(\underline{q}_1, \underline{q}_2, s_i) \tag{EU}$$

(b) Re-expressing the expected payments

Let me label the expected payment in auction m by

$$\mathbb{E}\left[B_{i,m}(\boldsymbol{q^*_{i,m}})\right] = \mathbb{E}[\boldsymbol{q^*_{i,m} p^*_m}] = \mathbb{E}\left[\boldsymbol{q^*_{i,m}} b_{i,m}(\boldsymbol{q^*_{i,m}}, s_i)\right]$$

Using the distribution of i's clearing price quantities, extending the integral in the same way as above we have

$$\mathbb{E}\left[B_{i,m}(\boldsymbol{q^*_{i,m}})\right] = \int_{\underline{q}_m}^{\overline{q}_m} q_m b_{i,m}(q_m, s_i) g_m^i(q_m, b_{i,m}) dq_m$$

Integrating by parts we obtain

$$\mathbb{E}\left[B_{i,m}(\boldsymbol{q^*_{i,m}})\right] = q_m b_{i,m}(q_m, s_i) G_m^i(q_m, b_{i,m})\Big|_{\underline{q}_m}^{\overline{q}_m} - \int_{\underline{q}_m}^{\overline{q}_m} \left[q_m b_{i,m}(q_m, s_i)\right]' G_m^i(q_m, b_{i,m}) dq_m$$

Since $G_m^i(\overline{q}_m, b_{i,m}) = 1, G_m^i(\underline{q}_m, b_{i,m}) = 0$ for all $b_{i,m}$ this simplifies to

$$\mathbb{E}\left[B_{i,m}(\boldsymbol{q^*_{i,m}})\right] = \int_{\underline{q}_m}^{\overline{q}_m} \left[q_m b_{i,m}(q_m, s_i)\right]' [1 - G_m^i(q_m, b_{i,m})] dq_m$$
$$\mathbb{E}\left[B_{i,m}(\boldsymbol{q^*_{i,m}})\right] = \int_{\underline{q}_m}^{\overline{q}_m} \left[b_{i,m}(q_m, s_i) + q_m \left(\frac{\partial b_{i,m}(q_m, s_i)}{\partial q_m}\right)\right] [1 - G_m^i(q_m, b_{i,m})] dq_m \tag{EB_m}$$

→ The objective function

Combining (EU) -$\sum_m$ (EB_m) the objective function

$$\mathcal{V}(b_{i,1}(\cdot, s_i), b_{i,2}(\cdot, s_i)) \equiv \mathbb{E}\left[U(\boldsymbol{q^*_{i,1}}, \boldsymbol{q^*_{i,2}}, s_i) - \sum_{m=1,2} \boldsymbol{q^*_{i,m} p^*_m}\right] \tag{$\mathcal{V}$}$$

can be written as

$$\mathcal{V}(b_{i,1}(\cdot, s_i), b_{i,2}(\cdot, s_i)) =$$
$$+\sum_m \int_{\underline{q}_m}^{\overline{q}_m} \mu_m(q_m, \overline{q}_{-m}, s_i) - \left[b_{i,m}(q_m, s_i) + q_m \left(\frac{\partial b_{i,m}(q_m, s_i)}{\partial q_m}\right)\right] [1 - G_m^i(q_m, b_{i,m})] dq_m$$
$$- \int_{\underline{q}_1}^{\overline{q}_1} \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2, s_i)[1 - G^i(q_1, q_2, b_{i,1}, b_{i,2})]dq_2 dq_1 - U(\underline{q}_1, \underline{q}_2, s_i) \tag{$\mathcal{V}$}$$

5.2.2 Step 2: Deriving $\mathcal{F}$

For notational convenience set $m = 1, -m = 2$. The other auction is analogous.

Fix $b_{i,2}(\cdot, s_i) = b^*_{i,2}(\cdot, s_i)$ and recall that

$$\mathcal{O}(b_{i,1}(\cdot, s_i)) \equiv \mathcal{V}(b_{i,1}(\cdot, s_i), b^*_{i,2}(\cdot, s_i)) \tag{O}$$

and denote $\left(\frac{\partial b_{i,m}(q_m, s_i)}{\partial q_m}\right) = \dot{b}_{i,m}(q_m, s_i)$ for $m = 1, 2$.

A straightforward mathematical manipulation rearranges $(\mathcal{V})$ with $b_{i,2}(\cdot, s_i) = b^*_{i,2}(\cdot, s_i)$ to

$$\mathcal{O}(b_{i,1}(\cdot, s_i)) = \int_{\underline{q}_1}^{\overline{q}_1} \mathcal{F}(q_1, b_{i,1}(q_1, s_i), \dot{b}_{i,1}(q_1, s_i))dq_1 \tag{O}$$

with

$$\begin{aligned}\mathcal{F}(q_1, b_{i,1}(q_1, s_i), \dot{b}_{i,1}(q_1, s_i)) \equiv &\left[\mu_1(q_1, \overline{q}_2, s_i) - b_{i,1}(q_1, s_i) - q_1\dot{b}_{i,1}(q_1, s_i)\right][1 - G^i_1(q_1, b_{i,1})] \\ &- \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2, s_i)[1 - G^i(q_1, q_2, b_{i,1}, b_{i,2})]dq_2 + const \end{aligned} \tag{$\mathcal{F}$}$$

where[11]

$$const \equiv \left[\frac{1}{\overline{q}_1 - \underline{q}_1}\right] \left(\int_{\underline{q}_2}^{\overline{q}_2} \left[\mu_2(q_2, \overline{q}_1, s_i) - b^*_{i,2}(q_2, s_i) - q_2\dot{b}^*_{i,2}(q_2, s_i)\right] [1 - G^i_2(q_2, b^*_{i,2})]dq_2 - U(\underline{q}_1, \underline{q}_2, s_i)\right)$$

This leaves us with the claimed functional form of $\mathcal{F}$. This function is continuous in its three arguments, and has continuous partial derivatives with respect to the second and third because $b_{i,1}(\cdot, s_i), b_{i,2}(\cdot, s_i)$ as well as all distribution functions are differentiable, and the the utility function has continuous partial and cross-partial derivatives by assumption.
This completes the proof of the auxiliary lemma.

5.3 Solving Maximization Problem (M)

The proof derives the necessary condition of the following maximization problem, where

$$\max_{b_{i,1}(\cdot, s_i)} \int_{\underline{q}_1}^{\overline{q}_2} \mathcal{F}(q_1, b_{i,1}(q_1, s_i), \dot{b}_{i,1}(q_1, s_i))dq_1$$

Since $\mathcal{F}$ is continuous in its three arguments and has continuous partial derivatives with respect to the second and third it is a standard problem of variational calculus. Its solution $b^*_{i,1}(\cdot, s_i)$: $[\underline{q}_1, \overline{q}_1] \to \mathbb{R}$ must satisfy the Euler Equation for all quantity points $q_1 \in [\underline{q}_1, \overline{q}_1]$:

$$\mathcal{F}_{b_{i,1}}(q_1, b^*_{i,1}(q_1, s_i), \dot{b}^*_{i,1}(q_1, s_i)) = \frac{d}{dq_1}\mathcal{F}_{\dot{b}_{i,1}}(q_1, b^*_{i,1}(q_1, s_i), \dot{b}^*_{i,1}(q_1, s_i)) \tag{23}$$

[11] To obtain the constant, I have extended the integral $\int_{\underline{q}_1}^{\overline{q}_1} dq_1$, to also go over the part of the objective function that are independent of q_1. It is the part in round brackets in *const*. To undo the integration I divide by $[\overline{q}_1 - \underline{q}_1] = \int_{\underline{q}_1}^{\overline{q}_1} dq_1$.

The remainder of the proof derives this condition and rearranges it to become the statement of the lemma.

Let me abbreviate $\mathcal{F}_{b_{i,1}} \equiv \mathcal{F}_{b_{i,1}}(q_1, b^*_{i,1}(q_1,s_i), \dot{b}^*_{i,1}(q_1,s_i))$ and $\mathcal{F}_{\dot{b}_{i,1}} = \mathcal{F}_{\dot{b}_{i,1}}(q_1, b^*_{i,1}(q_1,s_i), \dot{b}^*_{i,1}(q_1,s_i))$. Recalling

$$\mathcal{F}(q_1, b_{i,1}(q_1,s_i), \dot{b}_{i,1}(q_1,s_i)) \equiv \left[\mu_1(q_1,\overline{q}_2,s_i) - b_{i,1}(q_1,s_i) - q_1\left(\frac{\partial b_{i,1}(q_1,s_i)}{\partial q_1}\right)\right][1 - G^i_1(q_1,b_{i,1})]$$
$$- \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1,q_2,s_i)[1 - G^i(q_1,q_2,b_{i,1},b^*_{i,2})]dq_2 + const \qquad (\mathcal{F})$$

The two partial derivatives evaluated at the solution are

$$\mathcal{F}_{b_{i,1}} = \left[\mu_1(q_2,\overline{q}_2,s_i) - b^*_{i,1}(q_1,s_i) - q_1\left(\frac{\partial b^*_{i,1}(q_1,s_i)}{\partial q_1}\right)\right](-1)\left(\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial b_{i,1}}\right) - [1 - G^i(q_1,q_2,b^*_{i,1},b^*_{i,2})]$$
$$- \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1,q_2,s_i)(-1)\left(\frac{\partial G^i(q_1,q_2,b^*_{i,1},b^*_{i,2})}{\partial b_{i,1}}\right)dq_2 \qquad (24)$$

And

$$\mathcal{F}_{\dot{b}_{i,1}} = -q_1[1 - G^i_1(q_1,b^*_{i,1})] \qquad (25)$$

The total derivative evaluated at the solution is therefore

$$\frac{d}{dq_1}\mathcal{F}_{\dot{b}_{i,1}} = -[1 - G^i_1(q_1,b^*_{i,1})] - q_1\left[(-1)\left(\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial q_1}\right) + (-1)\left(\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial b_{i,1}}\right)\left(\frac{\partial b^*_{i,1}(q_1,s_i)}{\partial q_1}\right)\right] \qquad (26)$$

The Euler Equation is given by (24) = (26). Simplifying it gives

$$[\mu_1(q_1,\overline{q}_2,s_i) - b^*_{i,1}(q_1,s_i)](-1)\left(\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial b_{i,1}}\right) + \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1,q_2,s_i)\left(\frac{\partial G^i(q_1,q_2,b^*_{i,1},b^*_{i,2})}{\partial b_{i,1}}\right)dq_2 = q_1\left(\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial q_1}\right)$$

Apply the Fundamental Theorem of Calculus to replace

$$\mu_1(q_1,\overline{q}_2,s_i) = \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1,q_2,s_i)dq_2 + \mu_1(q_2,\underline{q}_2,s_i)$$

I rearrange this necessary condition to to obtain

$$\mu_1(q_1,\underline{q}_2,s_i) - \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1,q_2,s_i)\left[1 - \frac{\left(\frac{\partial G^i(q_1,q_2,b^*_{i,1},b^*_{i,2})}{\partial b_{i,1}}\right)}{\left(\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial b_{i,1}}\right)}\right] - b^*_{i,1}(q_1,s_i) = -q_1\left[\frac{\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial q_1}}{\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial b_{i,1}}}\right]$$

Using

$$\left[1 - \frac{\left(\frac{\partial G^i(q_1,q_2,b^*_{i,1},b^*_{i,2})}{\partial b_{i,1}}\right)}{\left(\frac{\partial G^i_1(q_1,b^*_{i,1})}{\partial b_{i,1}}\right)}\right] = 1 - G^i_{2|1}(q_2, b^*_{i,2}|q_1, b^*_{i,1})$$

The condition becomes (now no longer abbreviating $b^*_{i,m} \equiv b^*_{i,m}(q_m, s_i)$)

$$\mu_1(q_1, \underline{q}_2, s_i) - \int_{\underline{q}_2}^{\overline{q}_2} \mu(q_1, q_2, s_i) \left[1 - G^i_{2|1}(q_2, b^*_{i,2}(q_2, s_i)|q_1, b^*_{i,1}(q_1, s_i))\right] - b^*_{i,1}(q_1, s_i) = -q_1 \left[\frac{\frac{\partial G^i_1(q_1, b^*_{i,1}(q_1,s_i))}{\partial q_1}}{\frac{\partial G^i_1(q_1, b^*_{i,1}(q_1,s_i))}{\partial b_{i,1}(q_1,s_i)}}\right]$$

Integrating by parts, using $G^i_{2|1}(\overline{q}_2, b_2(\overline{q}_2, s_i)|q_1, b_1(q_1, s_i)) = 1$ and $G^i_{2|1}(\underline{q}_2, b_2(\underline{q}_2, s_i)|q_1, b_1(q_1, s_i)) = 0$ for any bids at any points, shows that the first two terms are the conditional expectation of the partial utility. Denoting $b_{i,1}(q_1, s_i) = p_1$ following the notation of definition 4 of i's distribution function we obtain

$$\mathbb{E}\left[\frac{\partial U(q_1, \boldsymbol{q_2^*}, s_i)}{\partial q_1}\middle| q_1\right] - b^*_{i,1}(q_1, s_i) = -q_1 \left[\frac{\frac{\partial G^i_1(q_1, b^*_{i,1}(q_1,s_i))}{\partial q_1}}{\frac{\partial G^i_1(q_1, b^*_{i,1}(q_1,s_i))}{\partial p_1}}\right] \tag{21}$$

This completes the proof of (i) of the lem.

To prove the second part of the lemma, assume that the demand function $x^*_{i,m}(\cdot, s_i)$ is additively separable in the type s_i, that is, take the following form

$$x^*_{i,m}(p_m, s_i) = \eta^*_{i,m}(s_i) + y^*_{i,m}(p_m) \text{ for } m = 1, 2. \tag{27}$$

with $y^*_{i,m}(\cdot)$ being differentiable and strictly decreasing. The following shows how optimality condition (21) can be expressed as

$$\mathbb{E}\left[\frac{\partial U(q_1, \boldsymbol{q^*_{i,2}}, s_i)}{\partial q_1}\middle| q_1\right] - b^*_{i,1}(q_1, s_i) = +q_1 \left[\frac{\partial RS_{i,1}(b^*_{i,1}(q_1, s_i))}{\partial p_1}\right]^{-1} \tag{22}$$

in that case. This is because the slope of the residual supply is deterministic with additively separability in the type. Given all other players choose such an equilibrium strategy $x^*_{j,m}(\cdot, s_j)$ of form (27), this residual supply takes the following form

$$\boldsymbol{RS}_{i,1}(p_1) = \boldsymbol{Q_1} - \sum_{j \neq i} \eta^*_{j,1}(\boldsymbol{s_j}) - \sum_{j \neq i} y^*_{j,1}(p_1) \tag{28}$$

To simplify the RHS of condition (21) recall

$$G^i(q_1, p_1) \equiv \Pr(\boldsymbol{q^*_{i,1}} \leq q_1) \tag{29}$$

$$G^i(q_1, p_1) \overset{(2)}{=} \Pr\left(\boldsymbol{Z_{i,1}} \leq q_1 + \sum_{j \neq i} y^*_{j,1}(p_1)\right) \tag{30}$$

with random variable $\boldsymbol{Z_{i,1}} \equiv \boldsymbol{Q_1} - \sum_{j \neq i} \eta^*_{j,1}(\boldsymbol{s_j})$. Denote its distribution by $F_{Z_{i,1}}$ and its density $f_{Z_{i,1}}$. Then

$$G^i(q_1, p_1) = F_{Z_{i,1}}\left(q_1 + \sum_{j \neq i} y^*_{j,1}(p_1)\right) \tag{31}$$

Using this new random variable, the partial derivative of G^i are

$$\frac{\partial G_1^i(q_1, p_1)}{\partial q_1} = f_{Z_{i,1}}\left(q_1 + \sum_{j \neq i} y_{j,1}^*(p_1)\right) \tag{32}$$

$$\frac{\partial G_1^i(q_1, p_1)}{\partial p_1} = f_{Z_{i,1}}\left(q_1 + \sum_{j \neq i} y_{j,1}^*(p_1)\right)\left(\frac{\partial \sum_{j \neq i} y_{j,1}^*(p_1)}{\partial p_1}\right)$$

Since $\left(\frac{\partial RS_{i,1}(b_1(q_1,s_i))}{\partial p_1}\right) \overset{(28)}{=} -\left(\frac{\partial \sum_{j \neq i} y_{j,1}^*(p_1)}{\partial p_1}\right)$ this is

$$\frac{\partial G_1^i(q_1, p_1)}{\partial p_1} = -f_{Z_i}\left(q_1 + \sum_{j \neq i} y_{j,1}^*(p_1)\right)\left(\frac{\partial RS_{i,1}(p_1)}{\partial p_1}\right) \tag{33}$$

Dividing (32) by (33) and evaluating both expressions at $p_1 = b_{i,1}^*(q_1, s_i)$ we obtain the following simplified necessary condition

$$\mathbb{E}\left[\frac{\partial U(q_1, \boldsymbol{q^*_{-m}}, s_i)}{\partial q_1}\middle| q_1\right] - b_{i,1}^*(q_1, s_i) = +q_1\left[\frac{\partial RS_{i,1}(b_{i,1}^*(q_1, s_i))}{\partial p_1}\right]^{-1} \tag{22}$$

$\square$

Appendix II: Proofs for the Main Text

6 Proof of Lemma 1

Lemma 1 follows from lemma 4, part (*ii*) because a linear function is additively separable in s_i. The condition in that case is

$$\mathbb{E}\left[\left.\frac{\partial U(q_1, \boldsymbol{q^*_{-m}}, s_i)}{\partial q_1}\right| q_1\right] - p_1 = q_1\left[\frac{\partial RS_{i,1}(p_1)}{\partial p_1}\right]^{-1} \text{ and } p_1 = b^*_{i,1}(q_1, s_i) \tag{22}$$

The following shows that the RHS of the necessary condition can be expressed as conditional expectation of the partial derivative of the total payment evaluated at market clearing, $\mathbb{E}\left[\left.\frac{\partial TP(p^{RS}_{i,1}(q_1), \boldsymbol{p^*_2}, q_1, \boldsymbol{q^*_{i,2}})}{\partial q_1}\right| q_1\right]$.

To see this, we just have to invert the residual supply curve. It is linear and strictly increasing by assumption, so that its inverse is defined: $\left[\frac{\partial RS_{i,1}(p_1)}{\partial p_1}\right]^{-1} = \left[\frac{\partial p^{RS}_{i,1}(q_1)}{\partial q_1}\right]$. Now, by market clearing the bid-offer must lie on the residual supply: $p_1 = p^{RS}_{i,1}(q_1)$. Taking $p_1 = p^{RS}_{i,1}(q_1)$ on the RHS necessary condition (22) becomes

$$\mathbb{E}\left[\left.\frac{\partial U(q_1, \boldsymbol{q^*_2}, s_i)}{\partial q_1}\right| q_1\right] = p^{RS}_{i,1}(q_1) + q_1\left[\frac{\partial p^{RS}_{i,1}(q_1)}{\partial q_1}\right] \text{ and } p^{RS}_{i,1}(q_1) = b^*_{i,1}(q_1, s_i) \tag{36'}$$

Notice that this condition is analogous to (36) in the centralized market. Now, by definition of the total payment, $TP(p^{RS}_{i,1}(q_1), \boldsymbol{p^*_2}, q_1, \boldsymbol{q^*_{i,2}}) \equiv q_1 p^{RS}_{i,1}(q_1) + \boldsymbol{q^*_{i,2}}\boldsymbol{p^*_2}$, so that the RHS is

$$\mathbb{E}\left[\left.\frac{\partial U(q_1, \boldsymbol{q^*_2}, s_i)}{\partial q_1}\right| q_1\right] = \left[\frac{\partial TP(p^{RS}_{i,1}(q_1), \boldsymbol{p^*_2}, q_1, \boldsymbol{q^*_{i,2}})}{\partial q_1}\right] \text{ and } p^{RS}_{i,1}(q_1) = b^*_{i,1}(q_1, s_i) \tag{34}$$

As we have seen the RHS is independent of $\boldsymbol{p^*_2}, \boldsymbol{q^*_{i,2}}$. We can, but don't have to take the conditional expectation also on the RHS. Both writings are equivalent.

$$\mathbb{E}\left[\left.\frac{\partial U(q_1, \boldsymbol{q^*_2}, s_i)}{\partial q_1}\right| q_1\right] = \mathbb{E}\left[\left.\frac{\partial TP(p^{RS}_{i,1}(q_1), \boldsymbol{p^*_2}, q_1, \boldsymbol{q^*_{i,2}})}{\partial q_1}\right| q_1\right] \text{ and } p^{RS}_{i,1}(q_1) = b^*_{i,1}(q_1, s_i) \tag{8}$$

□

7 Proof Lemma 2

Assume the equilibrium bidding functions (inverse demand functions), $\{\bar{b}^*_{i,1}(\cdot,\cdot,s_i), \bar{b}^*_{i,2}(\cdot,\cdot,s_i)$ are for all i linear that are strictly decreasing in the first argument. Their inverses, the demand functions are $\{\bar{x}^*_{i,1}(\cdot,\cdot,s_j), \bar{x}^*_{i,2}(\cdot,\cdot,s_i)\}\}$.

The prove derives the necessary condition of a linear ex-post equilibrium. Since all ex-post equilibria are (by definition) BNE, it is a necessary condition of a linear BNE.

Take the perspective of agent i and fix a profile of private types $(s_1, ..., s_n)$ and total supply quantities (Q_1, Q_2). Assume that all other agents play their equilibrium strategy $\{\bar{x}^*_{i,j}(\cdot,\cdot,s_j), \bar{x}^*_{i,2}(\cdot,\cdot,s_j)\}$. Agent i trades against two fixed residual supply curves

$$\overline{RS}_{i,m}(p_m, p_{-m}) = Q_m - \sum_{j\neq i} \bar{x}^*_{j,m}(p_m, p_{-m}, s_j)$$

His task is to pick an optimal point on each curve. In other words, he chooses a price that lies on this residual supply curve in each market. Denoting, $\vec{s}_i \equiv \begin{pmatrix} s_i & s_i \end{pmatrix}'$, $\vec{q} \equiv \begin{pmatrix} q_1 & q_2 \end{pmatrix}'$, $\vec{p} \equiv \begin{pmatrix} p_1 & p_2 \end{pmatrix}'$, and $\vec{RS}_i(\vec{p}) \equiv \begin{pmatrix} \overline{RS}_{i,1}(p_1,p_2) & \overline{RS}_{i,2}(p_2,p_1) \end{pmatrix}'$, the agent solves the following maximization problem

$$\max_{\vec{p}} \pi(\vec{p}, \vec{s}_i) = \max_{\vec{p}} \{U(\vec{q}, \vec{s}_i) - \vec{p}' \cdot \vec{q}\} \quad \text{with } \vec{q} = \vec{RS}_i(\vec{p}) \tag{35}$$

Inserting the assumed form of the utility function (4) in matrix notation, the agent's maximization problem reads

$$\max_{\vec{p}} \pi(\vec{p}, \vec{s}_i) = \max_{\vec{p}} \left\{ (\vec{s}_i - \vec{p})' \cdot \vec{RS}_i(\vec{p}) - \frac{1}{2} \vec{RS}_i(\vec{p})' \Delta \vec{RS}_i(\vec{p}) \right\} \qquad \text{with } \Delta \equiv \begin{pmatrix} \lambda & \delta \\ \delta & \lambda \end{pmatrix}$$

The optimal prices $\vec{p}^*$ must fulfill the first-order condition

$$0 = -\vec{RS}_i(\vec{p}^*) + \left(\frac{\partial \vec{RS}_i(\vec{p}^*)}{\partial \vec{p}} \right)' \left(\vec{s}_i - \vec{p}^* - \Delta \vec{RS}_i(\vec{p}^*) \right) \tag{FOC}$$

and clear the market: $\vec{RS}_i(\vec{p}^*) = \vec{x}^*_i(\vec{p}^*, s_i)$.

The condition given in the main text is stated in terms bidding functions $\vec{b}^*_i(\vec{q}, s_i)$ (with quantities as independent variables), instead of demand functions $\vec{x}^*_i(\vec{p}^*, s_i)$ (with prices as independent variables). To derive the optimality condition of the lemma, let me re-state the first-order condition. To do so, recall that $\vec{p}^{RS}_i(\vec{q})$ denotes the residual supply curve in the price-quantity space. By the chain-rule and basic rules of matrix transposition and inversion:

$$\left[\left(\frac{\partial RS_i(\vec{p}^*)}{\partial \vec{p}} \right)' \right]^{-1} = \left(\frac{\partial p^{RS}_i(\vec{q})}{\partial \vec{q}} \right)' \quad \text{at } \vec{p}^* = p^{RS}_i(\vec{q})$$

Then, reverting prices and quantities in the agent's strategy, $\vec{p}^* = \vec{b}^*_i(\vec{q}, s_i)$ with $\vec{q} \equiv \vec{x}^*_i(\vec{p}^*, s_i)$, in addition to $\left(\frac{\partial U(\vec{q}, s_i)}{\partial \vec{q}}\right)' = \vec{s}_i - \Delta \vec{q}$, the (FOC) rearranges to

$$\left(\frac{\partial U(\vec{q}, s_i)}{\partial \vec{q}} \right)' = \vec{p}^{RS}_i(\vec{q}) + \left(\frac{\partial \vec{p}^{RS}_i(\vec{q})}{\partial \vec{q}} \right)' \vec{q} \quad \text{at } \vec{p}^{RS}_i(\vec{q}) = \vec{b}^*_i(\vec{q}, s_i) \tag{36}$$

Equivalently,

$$\begin{pmatrix} \frac{\partial U(q_1,q_2,s_i)}{\partial q_1} \\ \frac{\partial U(q_1,q_2,s_i)}{\partial q_2} \end{pmatrix} = \begin{pmatrix} \bar{p}^{RS}_{i,1}(q_1,q_2) \\ \bar{p}^{RS}_{i,2}(q_2,q_1) \end{pmatrix} + \begin{pmatrix} \frac{\partial \bar{p}^{RS}_{i,1}(q_1,q_2)}{\partial q_1} & \frac{\partial \bar{p}^{RS}_{i,2}(q_2,q_1)}{\partial q_1} \\ \frac{\partial \bar{p}^{RS}_{i,1}(q_1,q_2)}{\partial q_2} & \frac{\partial \bar{p}^{RS}_{i,2}(q_2,q_1)}{\partial q_2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad \text{at } \vec{p}^{RS}_i(\vec{q}) = \vec{b}^*_i(\vec{q}, s_i)$$

For market $m = 1$ we have

$$\left(\frac{\partial U(q_1, q_2, s_i)}{\partial q_1}\right) = \bar{p}_{i,1}^{RS}(q_1, q_2) + q_1 \left(\frac{\partial \bar{p}_{i,1}^{RS}(q_1, q_2)}{\partial q_1}\right) + q_2 \left(\frac{\partial \bar{p}_{i,2}^{RS}(q_2, q_1)}{\partial q_1}\right) \quad \text{at } \bar{p}_{i,1}^{RS}(q_1, q_2) = \bar{b}_{i,1}^*(q_1, q_2, s_i)$$

By definition of total payment, $\overline{TP}(p_1, p_2, q_1, q_2) \equiv q_1 p_1 + q_2 p_2$, this is

$$\left(\frac{\partial U(q_1, q_2, s_i)}{\partial q_1}\right) = \left[\frac{\partial \overline{TP}(\bar{p}_{i,1}^{RS}(q_1, q_2), \bar{p}_{i,2}^{RS}(q_2, q_1), q_1, q_2)}{\partial q_1}\right] \quad \text{at } \bar{p}_{i,1}^{RS}(q_1, q_2) = \bar{b}_{i,1}^*(q_1, q_2, s_i) \tag{9}$$

The analogous equation must hold for $m = 2$.

□

8 Proof of Proposition 1

The proof is analogous to Du and Zhu (2012)'s proof of proposition 3. There are two key differences. First I consider an environment of independent private values. To replicate it in Du and Zhu (2012)'s proof, set $\alpha = 1, \beta = 0$. Secondly, I allow total supply to be random, while Du and Zhu (2012) assume that the total amount for sale is fixed. To account for this difference it suffices to *let agent i go through all possible realizations of the residual supply* instead of *all realization of type profiles of his competitors s_{-i}* in Du and Zhu (2012)'s proof. The algebra remains unchanged, which is why I do not reproduce the proof here. The reader is referred to Du and Zhu (2012) pp. 26-27. They derive the following unique equilibrium that fulfills the first-order condition (9) of Lemma 7 in a guess and verify approach.

$$\bar{x}_m^*(p_m, p_{-m}, s_i) = \left(\frac{1}{\lambda^2 - \delta^2}\right)\left(\frac{n-2}{n-1}\right)[\lambda(s_i - p_m) - \delta(s_i - p_{-m})] \quad \text{as demand} \tag{37}$$

$$\bar{b}_m^*(q_m, q_{-m}, s_i) = s_i - \left(\frac{n-1}{n-2}\right)\{\lambda q_m + \delta q_{-m}\} \quad \text{as bidding function} \tag{38}$$

It is the unique solution that solves the FOC (9).

To prove existence it is straightforward to show that the second-order condition is satisfied as long as Δ is positive semi definite. It is by assumption $\lambda > 0, |\delta| < \lambda$ in my set-up.

□

9 Proof of Lemma 3

The goal is to show that $Corr(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}}) = \pm 1$ iff $\sigma = 0$ or $\rho = 1$ and $a_1 = a_2$. Determining the correlation, this is easy to see. By definition

$$Corr(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}}) \equiv \frac{Cov(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}})}{\sqrt{Var(\boldsymbol{Z_{i,1}})Var(\boldsymbol{Z_{i,2}})}} \text{ with}$$

$$Cov(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}}) = Cov(\boldsymbol{Q_1}, \boldsymbol{Q_1}) - a_1 a_2 (n-1)^2 Var(\boldsymbol{s_i}) \ ; \ Var(\boldsymbol{Z_{i,m}}) = Var(\boldsymbol{Q_m}) + a_m^2 (n-1)^2 Var(\boldsymbol{s_i})$$

since $Cov\left(\boldsymbol{Q_m}, \sum_{j\neq i} \boldsymbol{s_j}\right) = 0$ by the assumption that total supply is independent of types. Replacing $Var(\boldsymbol{Q_1}) = \sigma_1^2, Var(\boldsymbol{Q_2}) = \sigma_2^2, Var(\boldsymbol{s_i}) = \sigma_s^2$, $Corr(\boldsymbol{Q_1}, \boldsymbol{Q_2}) = \rho$, the correlation becomes

$$Corr(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}}) = \frac{\rho\sigma^2 + a_1 a_2 (n-1)^2 \sigma_s^2}{\sqrt{[\sigma^2 + a_1^2(n-1)^2\sigma_s^2]\,[\sigma^2 + a_2^2(n-1)^2\sigma_s^2]}} \tag{39}$$

From here we easy that the correlation can only be perfect if $\sigma = 0$ or $\rho = 1$ and $a_1 = a_2$.

□

10 Proof of Proposition 2

The proposition consists of two statements. (i) gives the equilibrium function that agents choose if there is a symmetric, linear BNE (proof in section 10.2). (ii) says that such an equilibrium exists and is the unique symmetric, linear ex-post equilibrium if total supply is deterministic (proof in section 10.3).

10.1 Core of the argument

To derive the function of an equilibrium, I follow a guess and verify approach. I guess that there is an linear equilibrium, in which all bidders submit a strategy of the following form for $m = 1, 2$ with $c_m > 0$:

$$x_m^*(p_m, s_i) = o_m + a_m s_i - c_m p_m \qquad \text{in terms of prices} \tag{40}$$

$$b_m^*(q_m, s_i) = \left(\frac{o_m}{c_m}\right) + \left(\frac{a_m}{c_m}\right) s_i - \left(\frac{1}{c_m}\right) q_m \qquad \text{in terms of prices} \tag{41}$$

if residual supply curves are perfectly correlated, i.e.

$$\boldsymbol{Z_{i,m}} \equiv \boldsymbol{Q_m} - a_m \sum_{j\neq i} \boldsymbol{s_j} \tag{13}$$

are perfectly correlated (definition 3). Hereby I rely on the following alternative definitions of perfect correlation.[12]

Definition 5. *$\boldsymbol{Z_{i,1}}$ is perfectly correlated with $\boldsymbol{Z_{i,2}}$ iff $\boldsymbol{Z_{i,2}} = r + g\boldsymbol{Z_{i,1}}$ where*

$$g \equiv \pm\sqrt{\frac{Var(\boldsymbol{Z_{i,2}})}{Var(\boldsymbol{Z_{i,1}})}} = \pm\sqrt{\frac{\sigma^2 + a_2^2(n-1)^2\sigma_s}{\sigma^2 + a_1^2(n-1)^2\sigma_s}} \tag{42}$$

and

$$r \equiv \mathbb{E}[\boldsymbol{Z_{i,2}}] - g\mathbb{E}[\boldsymbol{Z_{i,1}}] = (\mu_2 - g\mu_1) - (n-1)\mu_s(a_2 - ga_1) \tag{43}$$

[12] The variance and expectation of $\boldsymbol{Z_{i,m}}$ can easily be computed from the primitives

$$\begin{pmatrix}\boldsymbol{Q_2}\\ \boldsymbol{Q_2}\end{pmatrix} \sim \left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \sigma^2\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right) \text{ and } \boldsymbol{s_i} \sim (\mu_s, \sigma_s^2), iid \text{ across } i \text{ and w.r.t. } \boldsymbol{Q_1}, \boldsymbol{Q_2}.$$

in combination with $\boldsymbol{Z_{i,m}}$'s definition (13).

The proof of statement (i) relies on on the first-order conditions as stated in Lemma 4 with $b^*_{i,m}(\cdot, s_i) = b^*_m(\cdot, s_i)\forall i$. Each condition characterizes i's best responds in market m to all others choosing the guessed linear function (40), given that he himself plays as in equilibrium in the other market $-m$. Now, for the guessed strategy to constitute a symmetric equilibrium, it must be optimal for i to choose also it in responds to his competitors playing it. Using the assumption that residual supply curves are perfectly correlated, which pins down a (deterministic) linear mapping between i's winning quantity in market m and $-m$, I can determine the equilibrium coefficients $\{o_m, a_m, c_m\}$ by matching the coefficients of i's best responds to the linear guess. This gives the unique function that agents choose in a symmetric, linear BNE, in which all are active in both markets.

To prove statement (ii), I then verifies that this candidate is an ex-post equilibrium if total supply is deterministic, i.e. with $\sigma = 0, \mu_m = Q_m$ for $m = 1, 2$.

10.2 Proof of (i)

Agent i's optimal choice in market m as responds to all others choosing the guessed equilibrium strategy, and given he plays as in equilibrium in the other market $-m$, is characterized by his first-order condition. It is specified in Lemma 4 part (ii), with $b^*_{i,m}(\cdot, s_i) = b^*_m(\cdot, s_i)\forall i$ for $m = 1, 2, -m \neq m$:

$$\mathbb{E}\left[\frac{\partial U(q_m, \boldsymbol{q^*_{i,-m}}, s_i)}{\partial q_m}\bigg| q_m\right] - b^*_m(q_m, s_i) = q_m \left[\frac{\partial RS_m(b^*_m(q_m, s_i))}{\partial p_m}\right]^{-1} \tag{22}$$

With linear partial utility $\frac{\partial U(q_m, q_{-m}, s_i)}{\partial q_m} = s_i - \lambda q_m - \delta q_m$, and given all others playing the linear strategy (40), it simplifies to

$$s_i - \lambda q_1 - \delta \mathbb{E}\left[\boldsymbol{q^*_{i,2}}\big| q_1\right] - b^*_1(q_1, s_i) = q_1 \left[\frac{1}{(n-1)c_1}\right] \tag{22.1'}$$

$$s_i - \lambda q_2 - \delta \mathbb{E}\left[\boldsymbol{q^*_{i,1}}\big| q_2\right] - b^*_2(q_2, s_i) = q_2 \left[\frac{1}{(n-1)c_2}\right] \tag{22.2'}$$

The RHS depends on the conditional expectation of i's winning quantity in the other auction. It looks as if the solution will depend on particular distribution functions; but it does not! The reason is that i's winning quantity in market 2 is a (linear) function of i's winning quantity in market 1 (and vice versa) when $\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}}$ are perfectly correlated. In other words, simultaneous market clearing and perfect correlation imply that we can ex-press $\boldsymbol{q^*_{i,2}}$ as a function of $\boldsymbol{q^*_{i,1}}$ (and vice versa). The conditional expectations $\mathbb{E}\left[\boldsymbol{q^*_{i,-m}}\big| q_m\right]$ are, thus, independent of the particular distribution, both linear functions of q_m. The following derives this linear function. With it I then can solve for the coefficients of the guessed linear equilibrium.

To express i's winning quantity in one market as function of the other, I first the closed form solution of i's equilibrium winning quantity in the guessed linear equilibrium ((40), resp. (41)).

By definition he wins in $m = 1, 2$

$$\boldsymbol{q^*_{i,m}} = \boldsymbol{Q_m} - \sum_{j \neq i} x^*_m(\boldsymbol{p^*_m}, \boldsymbol{s_j}) \text{ with } \boldsymbol{p^*_m} = b^*_m(\boldsymbol{q^*_{i,m}}, s_i) \tag{2}$$

Inserting the assumed linear functional forms (40) for $j \neq i$

$$x^*_m(p_m, s_i) = o_m + a_m s_i - c_m p_m \tag{40}$$

and (41) for i

$$b^*_m(q_m, s_i) = \left(\frac{o_m}{c_m}\right) + \left(\frac{a_m}{c_m}\right) s_i - \left(\frac{1}{c_m}\right) q_m \tag{41}$$

and solving for $\boldsymbol{q^*_{i,m}}$ gives

$$\boldsymbol{q^*_{i,2}}(\boldsymbol{Q_m}, \boldsymbol{s_{-i}}) = a_m \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n}\left[\boldsymbol{Q_m} - a_m \sum_{j \neq i} \boldsymbol{s_j}\right] \tag{44}$$

By definition of $\boldsymbol{Z_{i,m}}$

$$\boldsymbol{q^*_{i,m}}(\boldsymbol{Z_{i,m}}) = a_m \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n}\boldsymbol{Z_{i,m}} \text{ for } m = 1, 2 \tag{45}$$

Consider $m = 2$. Since $\boldsymbol{Z_{i,1}}$ and $\boldsymbol{Z_{i,2}}$ are perfectly correlated we can replace $\boldsymbol{Z_{i,2}} = r + g\boldsymbol{Z_{i,1}}$

$$\boldsymbol{q^*_{i,2}}(\boldsymbol{Z_{i,1}}) = a_2 \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n}\{r + g\boldsymbol{Z_1}\} \tag{46}$$

$\boldsymbol{Z_{i,1}}$ in turn is a function of $\boldsymbol{q^*_{i,1}}$. This is because in equilibrium both markets must clear simultaneously. Formally, agent i must win

$$\boldsymbol{q^*_{i,1}}(\boldsymbol{Z_{i,1}}) = a_1 \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n}\boldsymbol{Z_{i,1}} \tag{45}$$

in market 1.

$$\Leftrightarrow \frac{1}{n}\boldsymbol{Z_{i,1}}(\boldsymbol{q^*_{i,1}}) = \boldsymbol{q^*_{i,1}} - a_1 \left(1 - \frac{1}{n}\right) s_i \tag{47}$$

Inserting (47) into (46) leaves us with

$$\boldsymbol{q^*_{i,2}}(\boldsymbol{q^*_{i,1}}) = a_2 \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n}r + g\left[\boldsymbol{q^*_{i,1}} - a_1 \left(1 - \frac{1}{n}\right) s_i\right]$$

$$\boldsymbol{q^*_{i,2}}(\boldsymbol{q^*_{i,1}}) = [a_2 - g a_1] \left(1 - \frac{1}{n}\right) s_i + g\boldsymbol{q^*_{i,1}} + \frac{1}{n}r \tag{48}$$

An analogous argument, now using $\boldsymbol{Z_{i,1}} = \frac{\boldsymbol{Z_{i,2}} - r}{g}$, instead of $\boldsymbol{Z_{i,2}} = r + g\boldsymbol{Z_{i,1}}$, gives

$$\boldsymbol{q^*_{i,1}}(\boldsymbol{q^*_{i,2}}) = \left[a_1 - \frac{1}{g}a_2\right] \left(1 - \frac{1}{n}\right) s_i + \frac{1}{g}\boldsymbol{q^*_{i,2}} - \frac{1}{n}\frac{r}{g} \tag{49}$$

Having expressed each winning quantity as a function the other, we can solve for the equilibrium candidate based on the first-order conditions.

With (49) and (48) the first-order conditions (22.1') and (22.2') rearrange to

$$b_1^*(q_1, s_i) = -\delta \frac{1}{n} r + \left[1 - \delta [a_2 - g a_1] \left(1 - \frac{1}{n}\right)\right] s_i - \left[\lambda + \delta g + \frac{1}{(n-1)c_1}\right] q_1 \tag{22.1'}$$

$$b_2^*(q_2, s_i) = +\delta \frac{1}{n} \frac{r}{g} + \left[1 - \delta \left[a_1 - \frac{1}{g} a_2\right] \left(1 - \frac{1}{n}\right)\right] s_i - \left[\lambda + \delta \frac{1}{g} + \frac{1}{(n-1)c_2}\right] q_2 \tag{22.2'}$$

(22.1') characterizes the agent's optimal bid price for quantity q_1 in market 1, given all others choose according to the equilibrium strategy and he behaves as in equilibrium in the other market. (22.2') is the analogous for market 2.

Now, for the guessed strategy to be indeed an equilibrium, it must be optimal for agent i to choose in both markets choose according to the guess, i.e.

$$b_1^*(q_1, s_i) = \left(\frac{o_1}{c_1}\right) + \left(\frac{a_1}{c_1}\right) s_i - \left(\frac{1}{c_1}\right) q_1 \tag{41.1}$$

$$b_2^*(q_2, s_i) = \left(\frac{o_2}{c_2}\right) + \left(\frac{a_2}{c_2}\right) s_i - \left(\frac{1}{c_2}\right) q_2 \tag{41.2}$$

The equilibrium candidate can therefore be determined by matching coefficients of (22.1') and (22.2') with (41.1) and (41.2). Matching the coefficients of q_1, then q_2, and thereafter of s_i for market 1 followed by market 2 gives

$$\left[\lambda + \delta g + \frac{1}{(n-1)c_1}\right] = \frac{1}{c_1} \qquad \Rightarrow c_1 = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda + \delta g}\right) \tag{50}$$

$$\left[\lambda + \delta \frac{1}{g} + \frac{1}{(n-1)c_2}\right] = \frac{1}{c_2} \qquad \Rightarrow c_2 = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda + \delta \frac{1}{g}}\right) \tag{51}$$

$$\left[1 - \delta [a_2 - g a_1] \left(1 - \frac{1}{n}\right)\right] = \frac{a_1}{c_1} \qquad \Rightarrow a_1(a_2, c_1) = c_1 \left(\frac{n - \delta a_2 (n-1)}{n - \delta g c_1 (n-1)}\right) \tag{52}$$

$$\left[1 - \delta \left[a_1 - \frac{1}{g} a_2\right] \left(1 - \frac{1}{n}\right)\right] = \frac{a_2}{c_2} \qquad \Rightarrow a_2(a_1, c_2) = c_2 \left(\frac{n - \delta a_1 (n-1)}{n - \delta \frac{1}{g} c_2 (n-1)}\right) \tag{53}$$

Solving (50), (51), (52), (53) with g as defined

$$g = \pm \sqrt{\frac{\sigma^2 + a_2^2 (n-1)^2 \sigma_s}{\sigma^2 + a_1^2 (n-1)^2 \sigma_s}} \tag{42}$$

pins down the solution. Since g nontrivially depends on a_1, a_2, it is an ugly system of equations. Instead of solving it by brute force, I guess that the solution is symmetric with $a_1 = a_2 \neq 0$, and verify that it is indeed.

According to auxiliary lemma 2 stated below $a_1 = a_2 \neq 0 \Leftrightarrow g = 1$. Now, when $g = 1$,

$$c_1 = c_2 = c = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda+\delta}\right)$$

With $c_1 = c_2$ and $g = 1$, the conditions for $a_2(a_1, c)$ and $a_1(a_2, c)$ become perfectly symmetric.

$$a_2(a_1, c) = c\left(\frac{n - \delta a_1 (n-1)}{n - \delta c (n-1)}\right)$$
$$a_1(a_2, c) = c\left(\frac{n - \delta a_2 (n-1)}{n - \delta c (n-1)}\right)$$

This implies that the solution is indeed symmetric, $a_1 = a_2$, as I have guessed.

Now it is easy to solve for a. Just insert $a_1 = a_2 = a, g = 1, c_1 = c_2 = c$ into the condition:

$$a = c\left(\frac{n - \delta a (n-1)}{n - \delta c (n-1)}\right)$$
$$a\left[n - \delta c (n-1)\right] = c\left[n - \delta a (n-1)\right]$$
$$a\left[n - \delta c (n-1) + c\delta (n-1)\right] = cn$$
$$\Rightarrow a = \frac{cn}{n} = c = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda+\delta}\right) \neq 0 \text{since } n > 2, \lambda + \delta > 0. \tag{54}$$

Finally, with $g = 1$ and $a_1 = a_2$, we know from Definition 5

$$r \stackrel{(43)}{=} (\mu_2 - g\mu_1) - (n-1)\mu_s(a_2 - ga_1) = \mu_2 - \mu_1 \tag{55}$$

Inserting (54), (55) and $g = 1$ into (22.1), (22.2) leaves us with the following equilibrium candidate for $m = 1, 2, m \neq -m$

$$b^*_m(q_m, s_i) = s_i - \left(\frac{n-1}{n-2}\right)(\lambda+\delta)\, q_m + \delta\left(\frac{1}{n}\right)(\mu_m - \mu_{-m}) \tag{14}$$

Since $n > 2, \delta + \lambda > 0$ it is strictly decreasing in quantity and can be inverted. The submitted demand in equilibrium is

$$x^*_m(p_m, s_i) = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda+\delta}\right)\left\{s_i - p_m + \delta\left(\frac{1}{n}\right)(\mu_m - \mu_{-m})\right\} \tag{14b}$$

This function is the unique function that satisfies the necessary condition of a symmetric, linear BNE. It holds under perfectly correlated residual supply curves.

Auxiliary Lemma 2. $a_1 = a_2 \neq 0 \Leftrightarrow g = 1$.

Proof. Let $g = 1$. By definition (42)

$$1 = \sqrt{\frac{\sigma^2 + a_2^2(n-1)^2\sigma_s}{\sigma^2 + a_1^2(n-1)^2\sigma_s}} \Leftrightarrow 1 = \frac{\sigma^2 + a_2^2(n-1)^2\sigma_s}{\sigma^2 + a_1^2(n-1)^2\sigma_s} \Leftrightarrow a_2^2 = a_1^2 \Leftrightarrow a_2 = a_1$$

Now let $a_1 = a_2 = a \neq 0$. The following shows that it cannot be that $Corr(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}})$ takes value -1. Then it follows immediately from the definition (42) that

$$g = \sqrt{\frac{\sigma^2 + a^2(n-1)^2\sigma_s}{\sigma^2 + a^2(n-1)^2\sigma_s}} = 1$$

So let me how that the correlation between $Corr(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}}) \neq -1$ when $a_1 = a_2 \neq 0$. To do so, I first find an expression for this correlation. Recall that

$$Corr(\boldsymbol{Z_{i,1}}, \boldsymbol{Z_{i,2}}) = \frac{\rho\sigma^2 + a_1 a_2 (n-1)^2 \sigma_s^2}{\sqrt{[\sigma^2 + a_1^2(n-1)^2\sigma_s^2]\,[\sigma^2 + a_2^2(n-1)^2\sigma_s^2]}} \tag{39}$$

The following shows by contradiction that the correlation cannot be -1.

$$\begin{aligned} -1 &= \frac{\rho\sigma^2 + a^2(n-1)^2\sigma_s^2}{[\sigma^2 + a^2(n-1)^2\sigma_s^2]} \\ -1\left[\sigma^2 + a^2(n-1)^2\sigma_s^2\right] &= \rho\sigma^2 + a^2(n-1)^2\sigma_s^2 \\ \sigma^2(1+\rho) &= -2a^2(n-1)^2\sigma_s^2 \end{aligned}$$

The LHS is strictly negative since $a \neq 0$, while the lowest value of the LHS is 0 since $\rho \geq -1$. The equation cannot hold.

$\square$

10.3 Proof of *(ii)*

Let total supply be deterministic, i.e. $\sigma = 0, \mu_m = Q_m$ for $m = 1, 2$. Since all agents participate in both markets and each has only one private type, this gives us perfect correlation between the residual supply curves.

To show that the equilibrium candidate is an ex-post equilibrium, I will show that the agent has no incentive to deviate, after observing the types of the others and the realized total supply (ex-post).

The idea of the proof is simply. For some fix profile of private types $(s_1, ..., s_n)$, and total supply quantities Q_1, Q_2, I show that agent i has no profitable deviation from the equilibrium candidate if all others play this strategy $\{x_1^*(\cdot, s_j), x_2^*(\cdot, s_j)\}$. I do so by solving his maximization problem for this fixed realization of types and total quantities. More precisely I show that the equilibrium guess satisfies the first and second order condition of this ex-post maximization. It is analogous to the centralized market of section 7, with the important difference that demands (bid-offers) now depend on just the price (quantity) of the market at hand. Since strategies are one-dimensional, I refrain from using the matrix notation as I did for the centralized market.

Take the perspective of agent i. Knowing $(s_1, ..., s_n)$, and total supply quantities Q_1, Q_2 agent i trades against two fixed residual supply curves

$$RS_{i,m}(p_m) = Q_m - \sum_{j \neq i} x_m^*(p_m, s_j) \quad \text{for } m = 1, 2. \tag{56}$$

His task is to pick an optimal point on each curve. In other words, he chooses a price that lies on this residual supply curve in each market. He does so maximizing his payoff of winning $\{q_1, q_2\} = \{RS_{i,1}(p_1), RS_{i,2}(p_2)\}$ at prices $\{p_1, p_2\}$.

$$\max_{p_1,p_2} \pi(p_1, p_2, s_i) = \max_{p_1,p_2} \left\{ U(q_1, q_2, s_i) - \sum_{m=1,2} p_m q_m \right\} \quad \text{with } q_m = RS_{i,m}(p_m) \text{ for } m = 1,2 \tag{57}$$

Inserting the assumed form of the utility function (4) the agent's maximization problem reads

$$\max_{p_1,p_2} \pi(p_1, p_2, s_i) = \max_{p_1,p_2} \sum_{m=1,2} \left\{ (s_i - p_m)\, RS_{i,m}(p_m) - \frac{1}{2}\lambda \left(RS_{i,m}(p_m)\right)^2 \right\} - \delta RS_{i,1}(p_1) RS_{i,2}(p_2) \tag{MP}$$

10.3.1 Verifying the FOC

For market 1 the first-order condition is

$$0 = -\, x_{i,1}(p_1^*, s_i) + \left(\frac{\partial RS_{i,1}(p_1^*)}{\partial p_1} \right) (s_i - p_1^* - \lambda x_{i,1}(p_1^*, s_i) - \delta x_{i,2}(p_2^*, s_i)) \tag{FOC}$$

where I have already used that markets must clear at the optimum, i.e. $RS_{i,m}(p_m^*) = x_{i,m}(p_m^*, s_i)$ for $m = 1, 2$. Since both markets must clear simultaneously. Only if he chooses

$$x_{i,2}^*(p_2^*, s_i) = x_{i,1}^*(p_1^*, s_i) + \frac{1}{n}(Q_2 - Q_1) \tag{48'}$$

market 2 can clear. This constraint is identical to condition (48) with $a_1 = a_2, g = 1, r = Q_2 - Q_1$.[13]

Inserting the equilibrium candidate,

$$x_{i,1}^*(p_1^*, s_i) = c \left[s_i - p_1^* + \delta \frac{1}{n}(Q_1 - Q_2) \right] \text{ with } c = \left(\frac{n-1}{n-2} \right) (\lambda + \delta) \tag{14b}$$

[13] With deterministic total supply, the condition can also be derived in a different way: Insert the the sum over the types of all others

$$\sum_{j \neq i} s_j = \left[\frac{a_1 \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n} Q_1 - x_1(p_1^*, s_i)}{a_1} \right] \Leftrightarrow x_1(p_1^*, s_i) \equiv q_{i,1}^* \overset{(44)}{=} a_1 \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n} \left[Q_1 - a_1 \sum_{j \neq i} s_j \right] \tag{58}$$

into

$$x_2(p_2^*, s_i) \equiv q_{i,2}^* \overset{(44)}{=} a_2 \left(1 - \frac{1}{n}\right) s_i + \frac{1}{n} \left[Q_2 - a_2 \sum_{j \neq i} s_j \right] \tag{59}$$

Impose $a_1 = a_2$ (as in equilibrium) and we obtain

$$x_2(p_2^*, s_i) = x_1(p_1^*, s_i) + \frac{1}{n}(Q_2 - Q_1) \tag{49'}$$

and the constraint that both markets must clear simultaneously (48') into (FOC) it is easy to verify that the candidate fulfills this first-order condition:

$$\begin{aligned}0 = & -c\left[s_i - p_1^* + \delta\frac{1}{n}(Q_1 - Q_2)\right] \\ & + (n-1)c\left(s_i - p_1^* - \lambda c\left[s_i - p_1^* + \delta\frac{1}{n}(Q_1 - Q_2)\right] - \delta c\left[s_i - p_1^* + \delta\frac{1}{n}(Q_1 - Q_2)\right] - \delta\frac{1}{n}(Q_2 - Q_1)\right)\end{aligned}$$

Simplifying

$$\left[s_i - p_1^* + \delta\frac{1}{n}(Q_2 - Q_1)\right][c + \lambda c(n-1)c + \delta c(n-1)c] = (n-1)c\left[s_i - p_1^* - \delta\frac{1}{n}(Q_1 - Q_2)\right]$$

$$[1 + (n-1)c(\lambda + \delta)] = (n-1)$$

At the solution with $c = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda+\delta}\right)$

$$0 = 0$$

10.3.2 Verifying the SOC

To verify that the found strategy is indeed a maximum I verify the second order condition. The agent has no profitable deviation if

1. $\frac{\partial^2 \pi(p_1^*, p_2^*, s_i)}{\partial^2 p_m} < 0$ for $m = 1, 2$
2. $|H(p_1^*, p_2^*, s_i)| \equiv \begin{vmatrix} \frac{\partial^2 \pi(p_1^*, p_2^*, s_i)}{\partial^2 p_1} & \frac{\partial^2 \pi(p_1^*, p_2^*, s_i)}{\partial p_1 p_2} \\ \frac{\partial^2 \pi(p_1^*, p_2^*, s_i)}{\partial p_2 p_1} & \frac{\partial^2 \pi(p_1^*, p_2^*, s_i)}{\partial^2 p_2} \end{vmatrix} > 0$

The following shows that the second order condition is fulfilled for any large number of agents iff $n > 2$ and $|\delta| \leq \lambda, \lambda + \delta > 0$. Both holds by assumption.

The second derivative of maximization problem (MP) for $m = 1$ is

$$\frac{\partial^2 \pi(p_1, p_2, s_i)}{\partial^2 p_1} = -\left(\frac{\partial RS_{i,1}(p_1)}{\partial p_1}\right) + \left(-1 - \lambda\frac{\partial RS_{i,1}(p_1)}{\partial p_1}\right)\left(\frac{\partial RS_{i,1}(p_1)}{\partial p_1}\right)$$

At the solution

$$\begin{aligned}\frac{\partial^2 \pi(p_1^*, p_2^*, s_i)}{\partial^2 p_1} &= -(n-1)c + (-1 - \lambda(n-1)c)\,(n-1)c \\ &= -(n-1)c\left\{1 + [1 + \lambda(n-1)c]\right\} \\ &= -(n-1)c\,[2 + \lambda(n-1)c] \\ &= -\left(\frac{n-2}{\lambda+\delta}\right)\left[2 + \lambda\left(\frac{n-2}{\lambda+\delta}\right)\right] \\ &= -\left(\frac{n-2}{\lambda+\delta}\right)\left(\frac{2\delta + \lambda n}{\lambda+\delta}\right) \\ &< 0 \;\; \text{holds since } n > 2 \text{ and } |\delta| \leq \lambda,\ \lambda + \delta > 0.\end{aligned}$$

The cross-partial derivative is

$$\frac{\partial^2 \pi(p_1, p_2, s_i)}{\partial p_1 p_2} = -\delta \left(\frac{\partial RS_{i,1}(p_1)}{\partial p_1}\right) \left(\frac{\partial RS_{i,2}(p_2)}{\partial p_2}\right)$$

At the solution

$$\frac{\partial^2 \pi(p_1^*, p_2^*, s_i)}{\partial p_1 p_2} = -\delta(n-1)^2 c^2$$

By symmetry of the problem, the hessian therefore is

$$H(p_1^*, p_2^*, s_i) = \begin{vmatrix} -(n-1)c\,[2+\lambda(n-1)c] & -\delta(n-1)^2c^2 \\ -\delta(n-1)^2c^2 & -(n-1)c\,[2+\lambda(n-1)c] \end{vmatrix}$$

The determinant of the hessian matrix is

$$Det(H(p_1^*, p_2^*, s_i)) > 0 \Leftrightarrow \{(n-1)c\,[2+\lambda(n-1)c]\}^2 - \delta^2(n-1)^4c^4 > 0$$

Since $(n-1)c > 0$

$$\Leftrightarrow [2+\lambda(n-1)c]^2 > \delta^2(n-1)^2c^2$$

Taking the square root

$$\Leftrightarrow [2+\lambda(n-1)c] > \delta(n-1)c$$
$$\Leftrightarrow 2 > (\delta-\lambda)(n-1)c$$

At the solution

$$\Leftrightarrow 2 > \left(\frac{\delta-\lambda}{\lambda+\delta}\right)(n-2)$$

We know $n > 2$ (since $n > 2$). Then this condition holds independent of how many agents there are as long as $|\delta| \leq \lambda$, and $\lambda + \delta > 0$.

$\square$

11 Proof of Proposition 3

The proof is analogous to the proof of Lemma 2. I follow a guess and verify strategy, guessing that the equilibrium will take the following form

$$\beta_m^*(q_m, s_i) = \epsilon_m + \alpha_m s_i - \gamma_m q_m$$

Under the linear guess and with the quadratic utility function, FOC (22) reads

$$s_i - \lambda q_m - \delta \mathbb{E}[\boldsymbol{q^*_{i,-m}}|q_m] - \beta_m(q_m, s_i) = \left[\frac{q_m}{(n-1)\frac{1}{\gamma_m}}\right] \tag{60}$$

Under the distributional assumption $\binom{\boldsymbol{Q_2}}{\boldsymbol{Q_2}} \sim N\left(\binom{\mu_1}{\mu_2}, \sigma^2\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right)$ and $\boldsymbol{s_i} \sim N(\mu_s, \sigma_s^2), iid$ across i and $\boldsymbol{Q_1}, \boldsymbol{Q_2}$, one can determine an expression for $\mathbb{E}[\boldsymbol{q^*_{i,-m}}|q_m]$. The first step is to determine the joint distribution of $\boldsymbol{q^*_{i,1}}, \boldsymbol{q^*_{i,2}}$. As each is a linear transformation of normally distributed independent variables, $\boldsymbol{q^*_{i,m}} = \frac{1}{n}\left[\boldsymbol{Q_m} - \alpha_m \sum_{j\neq i} \boldsymbol{s_j} + (n-1)\alpha_m s_i\right]$, they are jointly normally distributed:

$$\begin{pmatrix} \boldsymbol{q^*_{i,1}} \\ \boldsymbol{q^*_{i,2}} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{q_1^i} \\ \mu_{q_2^i} \end{pmatrix}, \begin{pmatrix} \sigma^2_{q_1^i} & \rho^i \sigma_{q_1^i}\sigma_{q_2^i} \\ \rho^i \sigma_{q_1^i}\sigma_{q_2^i} & \sigma^2_{q_2^i} \end{pmatrix}\right)$$

with for $m = 1, 2, -m \neq m$

$$\mu_{q_m^i} \equiv \left(\frac{1}{n}\right)\{\mu_m + \alpha_m(n-1)[s_i - \mu_s]\}$$

and

$$\sigma_{q_m^i} \equiv \left(\frac{1}{n}\right)\sqrt{\alpha_m^2(n-1)\sigma_s^2 + \sigma^2}$$

and

$$\rho^i \equiv \frac{\alpha_m \alpha_{-m}(n-1)\sigma_s^2 + \rho\sigma^2}{\sqrt{[\alpha_m^2(n-1)\sigma_s^2 + \sigma^2][\alpha_{-m}^2(n-1)\sigma_s^2 + \sigma^2]}}$$

Jointly normal random variables have a linear conditional expectation of the following form

$$\mathbb{E}[\boldsymbol{q^*_{i,-m}}|q_m] = \mu_{q_{-m}^i} + \rho^i\left(\frac{\sigma_{q_{-m}^i}}{\sigma_{q_m^i}}\right)\left(q_m - \mu_{q_m^i}\right). \tag{61}$$

Inserting the expression $\mathbb{E}[\boldsymbol{q^*_{i,-m}}|q_m]$ into the FOC, (60), and rearranging gives

$$\beta_m^*(q_m, s_i) = E_m(\alpha_m, \alpha_{-m}) + A_m(\alpha_m, \alpha_{-m})s_i - C_m(\alpha_m, \alpha_{-m}, \gamma_m)q_m \quad \text{with}$$

$$E_m(\alpha_m, \alpha_{-m}) = -\delta\left(\frac{1}{n}\right)\left[[\mu_{-m} - \alpha_{-m}(n-1)\mu_s] - \left(\frac{\alpha_m\alpha_{-m}(n-1)\sigma_s^2 + \rho\sigma^2}{\alpha_m^2(n-1)\sigma_s^2 + \sigma^2}\right)[\mu_m - \alpha_m(n-1)\mu_s]\right]$$

$$A_m(\alpha_m, \alpha_{-m}) = 1 - \delta\left(\frac{1}{n}\right)(n-1)\left[\alpha_{-m} - \left(\frac{\alpha_m\alpha_{-m}(n-1)\sigma_s^2 + \rho\sigma^2}{\alpha_m^2(n-1)\sigma_s^2 + \sigma^2}\right)\alpha_m\right]$$

$$C_m(\alpha_m, \alpha_{-m}, \gamma_m) = \left[\frac{(n-1)\frac{1}{\gamma_m}\lambda + 1}{(n-1)\frac{1}{\gamma_m}}\right] + \delta\left(\frac{\alpha_m\alpha_{-m}(n-1)\sigma_s^2 + \rho\sigma^2}{\alpha_m^2(n-1)\sigma_s^2 + \sigma^2}\right)$$

In the symmetric equilibrium we must have

$$E_m(\alpha_m, \alpha_{-m}) = \epsilon_m \tag{62}$$

$$A_m(\alpha_m, \alpha_{-m}) = \alpha_m \tag{63}$$

$$C_m(\alpha_m, \alpha_{-m}, \gamma_m) = \gamma_m \tag{64}$$

for both markets. In this symmetric environment, it can be shown that the solution must be symmetric: $\epsilon_1 = \epsilon_2 = \epsilon, \alpha_1 = \alpha_2 = \alpha, \gamma_1 = \gamma_1 = \gamma$. Here I use a short-cut and simply impose symmetry. Denote the correlation of the winning quantities as

$$\rho^i(\alpha) \equiv \frac{\rho\sigma^2 + \alpha^2(n-1)\sigma_s^2}{\sigma^2 + \alpha^2(n-1)\sigma_s^2} \tag{65}$$

The equilibrium parameter then must solve

$$E(\alpha) = \epsilon \Leftrightarrow \epsilon(\alpha) = -\delta\left(\frac{1}{n}\right)\left[[\mu_{-m} - \alpha(n-1)\mu_s] - \rho^i(\alpha)[\mu_m - \alpha(n-1)\mu_s]\right] \tag{62}$$

$$A(\alpha) = \alpha \Leftrightarrow \alpha = 1 - \delta\left(\frac{1}{n}\right)(n-1)\alpha\left[1 - \rho^i(\alpha)\right] \tag{63}$$

$$C(\alpha,\gamma) = \gamma \Leftrightarrow \gamma = \left[\frac{(n-1)\frac{1}{\gamma}\lambda + 1}{(n-1)\frac{1}{\gamma}}\right] + \delta\rho^i(\alpha) \Leftrightarrow \gamma(\alpha) = \left(\frac{n-1}{n-2}\right)\left(\lambda + \delta\rho^i(\alpha)\right) \tag{64}$$

Slightly rearranging the expressions for the equilibrium coefficients, this gives the functional form of the proposition:

$$\beta^*_m(q_m, s_i) = \epsilon(\alpha) + \alpha s_i - \gamma(\alpha)p_m \text{ with } \alpha = 1 - \delta\alpha\left(\frac{1}{n}\right)(n-1)[1-\rho^i(\alpha)] \tag{15}$$

$$\gamma(\alpha) = \left(\frac{n-1}{n-2}\right)\left(\lambda + \delta\rho^i(\alpha)\right)$$

$$\epsilon(\alpha) = \delta\left(\frac{1}{n}\right)\left[[\rho^i(\alpha)\mu_m - \mu_{-m}] + \alpha(n-1)\mu_s[1-\rho^i(\alpha)]\right]$$

□

12 Proof of Corollary 2

(i) **Centralized market**

Recall the optimality condition of Lemma 2

$$\left[\frac{\partial U(q_1,q_2,s_i)}{\partial q_m}\right] = \left[\frac{\partial \overline{TP}(\bar{p}^{RS}_{i,1}(q_1,q_2), \bar{p}^{RS}_{i,2}(q_2,q_1), q_1, q_2)}{\partial q_m}\right] \text{ at } \bar{p}^{RS}_{i,1}(q_1,q_2) = \bar{b}^*_{i,1}(q_1,q_2,s_i) \tag{9}$$

In its proof I have shown that for $m = 1$ the condition is equivalent to

$$\left[\frac{\partial U(q_1,q_2,s_i)}{\partial q_1}\right] = \bar{p}^{RS}_{i,1}(q_1,q_2) + q_1\left(\frac{\partial \bar{p}^{RS}_{i,1}(q_1,q_2)}{\partial q_1}\right) + q_2\left(\frac{\partial \bar{p}^{RS}_{i,2}(q_2,q_1)}{\partial q_1}\right) \text{ at } \bar{p}^{RS}_{i,1}(q_1,q_2) = \bar{b}^*_{i,1}(q_1,q_2,s_i)$$

Given all others choose a linear strategy

$$x^*_m(p_m, p_{-m}, s_i) = \bar{o}_m + \bar{a}_m - \bar{c}_m p_m - \bar{e}_m p_{-m} \text{ with } \bar{e}_m \neq \bar{c}_m > 0$$

We can equivalently write

$$\left[\frac{\partial U(q_1,q_2,s_i)}{\partial q_1}\right] = \bar{b}^*_{i,1}(q_1,q_2,s_i) + q_1\left(\frac{1}{n-1}\right)\left(\frac{\bar{c}_2}{\bar{c}_1\bar{c}_2 - \bar{e}_1\bar{e}_2}\right) - q_2\left(\frac{1}{n-1}\right)\left(\frac{\bar{e}_2}{\bar{c}_1\bar{c}_2 - \bar{e}_1\bar{e}_2}\right)$$

Sending $n \to \infty$ the last two terms disappear, and we obtain the statement of the corollary. The existence result of proposition 1 carries over.

(ii) **Disconnected markets**

The proof is analogous to showing (i). Given linear strategies

$$x_m^*(p_m, s_i) = o_m + a_m s_i - c_m p_m \text{ with } c_m > 0$$

For $m = 1$, the necessary condition of Lemma 1 can be written as

$$\mathbb{E}\left[\frac{\partial U(q_1, \boldsymbol{q^*_{i,2}}, s_i)}{\partial q_1}\middle| q_1\right] = b^*_{i,1}(q_1, s_i) + q_1\left(\frac{1}{n-1}\right)\left(\frac{1}{c_1}\right)$$

Sending $n \to \infty$, the last term vanishes. The condition becomes

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{\partial U(q_1, \boldsymbol{q^*_{i,2}}, s_i)}{\partial q_1}\middle| q_1\right] = b^*_{i,1}(q_1, s_i)$$

With perfect correlation we know from proposition 2, that this is $b^*_{i,1}(q_1, s_i) \to \bar{b}^*_{i,1}(q_1, q_1, s_i) = s_i - (\delta + \lambda)q_m$ as $n \to \infty$. The existence result of this proposition carries over.

$\square$

13 Proof of the Irrelevance Theorem

Proving the Irrelevance theorem based on the functional form of the linear equilibria of the propositions is straightforward. It suffices to compare the winning quantities in these ex-post equilibria. Those are the quantities, the bidder demands at the clearing prices.

In the disconnected market, the market clears

$$\begin{aligned} Q_m &= \sum_i x_m^*(p_m^*, s_i) \\ Q_m &= n o_m + a \sum_i s_i - n c p_m^* \text{ with } a = c = \left(\frac{n-1}{n-2}\right)\left(\frac{1}{\lambda+\delta}\right), o_m = \frac{\delta}{n}(\mu_m - \mu_{-m}) \\ \Rightarrow p_m^* &= \frac{o_m}{c} + \left(\frac{1}{n}\right)\sum_i s_i - \frac{1}{c}\frac{Q_m}{n} \end{aligned} \tag{66}$$

This makes agent i win $x_m^*(s_i, p_m^*) \equiv q^*_{i,m}$

$$\begin{aligned} x_m^*(s_i, p_m^*) &= o_m + a s_i - c p_m^* \\ &= o_m + a s_i - c\left[\frac{o_m}{c} + \left(\frac{1}{n}\right)\sum_i s_i - \frac{1}{c}\frac{Q_m}{n}\right] \\ \Rightarrow q^*_{i,m} &= \left(\frac{n-1}{n-2}\right)\left(\frac{1}{\lambda+\delta}\right)\left[s_i - \frac{1}{n}\sum_i s_i\right] + \frac{Q_m}{n} \end{aligned} \tag{67}$$

In the centralized market, the markets clear

$$Q_1 = \sum_i \bar{x}_1^*(\bar{p}_1^*, \bar{p}_2^*, s_i)$$
$$Q_2 = \sum_i \bar{x}_2^*(\bar{p}_2^*, \bar{p}_1^*, s_i)$$

where both functions are defined in (37) with $\gamma = 1, \kappa = 0$. Solving this system of equations for the clearing prices gives for $m = 1, 2, -m \neq m$

$$\bar{p}_m^* = \sum_i s_i - \left(\frac{n-1}{n-2}\right)\left(\frac{1}{n}\right)\{\lambda Q_m + \delta Q_{-m}\} \tag{68}$$

Evaluating both submitted demand functions at the clearing prices determines how much agent i wins in equilibrium:

$$\bar{q}_{i,m}^* \equiv \bar{x}_m^*(p_m^*, p_{-m}^*, s_i) = \left(\frac{n-2}{n-1}\right)\left(\frac{1}{\lambda+\delta}\right)\left[s_i - \frac{1}{n}\sum_i s_i\right] + \frac{Q_m}{n} \tag{69}$$

Both winning quantities (67) and (69) we see that they coincide.

□

14 Proof of the Irrelevance Theorem for Large Markets

To show that the allocation of the linear equilibria is under both market structures fully efficient given perfectly correlated residual supply curves as $n \to \infty$ I compare the agent's winning quantity in equilibrium to the efficient allocation $\{q_{i,1}^e, q_{i,2}^e\}_{i=1}^n$ as $n \to \infty$.

We already know how much the agent wins in either linear equilibrium when $n < \infty$ (see section 13, proof of the Irrelevance Theorem). Sending $n \to \infty$ gives

$$\lim_{n\to\infty} q_{i,m}^* = \lim_{n\to\infty}\left\{\left(\frac{n-1}{n-2}\right)\left(\frac{1}{\lambda+\delta}\right)\left[s_i - \frac{1}{n}\sum_i s_i\right] + \frac{Q_m}{n}\right\} \tag{67}$$

Since types are iid by assumption, I can apply the law of large numbers to obtain

$$\lim_{n\to\infty} q_{i,m}^* = \left(\frac{1}{\lambda+\delta}\right)(s_i - \mu_s) \tag{70}$$

It is straightforward to show that this is efficient solution. For any n, the efficient solution $\{q_{i,1}^e, q_{i,2}^e\}_{i=1}^n$ solves

$$\max_{\{q_{i,1}, q_{i,2}\}_{i=1}^n} \sum_i U(q_{i,1}, q_{i,2}, s_i) \text{ s.t. } \sum_i q_{i,1} = Q_1 \text{ and } \sum_i q_{i,2} = Q_2$$

with $U(q_{i,1}, q_{i,2}, s_i) \overset{(4)}{=} \sum_m \left\{ s_i q_{i,m} - \frac{\lambda}{2} q_{i,m}^2 \right\} - \delta q_{i,1} q_{i,2}$. Denoting the Lagrange multipliers by γ_1, γ_2 the efficient allocation is characterized by

$$s_i - \lambda q_{i,1}^e - \delta q_{i,2}^e + \gamma_1 = 0 \; \forall i$$
$$s_i - \lambda q_{i,2}^e - \delta q_{i,1}^e + \gamma_2 = 0 \; \forall i$$

in addition to the binding feasibility constraints $Q_1 = \sum_i q_{i,1}^e$, $Q_2 = \sum_i q_{i,2}^e$. Solving for $q_{i,1}^e$ and $q_{i,2}^e$ gives

$$q_{i,m}^e = \left(\frac{1}{\lambda + \delta} \right) \left(s_i - \frac{1}{n} \sum_i s_i \right) + \frac{1}{n} Q_m \text{ for } m = 1, 2.$$

Applying the law of large numbers as above, we see that the solution coincides with the equilibrium allocation under either market structure:

$$\lim_{n \to \infty} q_{i,m}^e = \left(\frac{1}{\lambda + \delta} \right) (s_i - \mu_s) \overset{(70)}{=} \lim_{n \to \infty} q_{i,m}^* \text{ for } m = 1, 2.$$

□