A SAS® Macro for the Generalized RAS Algorithm

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A SAS Macro for Implementing the Generalized RAS Algorithm for Constraining Matrices of Mixed Sign

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Note:

SAS® macro %GRAS for the algorithm can be found at https://sourceforge.net/p/constrainingarrays/code/ci/master/tree/gras.sas.

ABSTRACT

The problem of constraining matrices of mixed sign to controls of possibly mixed sign arises in input-output matrices in economics and net migration in demography. The recently developed Generalized RAS (GRAS) algorithm is presented to solve these problems. The GRAS algorithm produces a unique solution that minimizes an entropy-like function. The algorithm is applied to a well-known example and compared to the solution originally obtained using a generalization of the Akers-Siegel procedure.

KEYWORDS

matrix scaling; matrix raking; matrix balancing; mixed signs; GRAS; SAS; plus-minus problem

1. Introduction

Controlling matrices of mixed sign is a problem occurring in economics and demography often by constraining input-output matrices and net migration matrices, respectively. This paper describes the Generalized RAS Algorithm (GRAS) as a method to find unique solutions. GRAS is applied to net migration data within Japan from Shryock et al. (1973, 711). GRAS is compared to Shryock et al.'s (1973, 709-712) use of the Akers-Siegel (1965) "plus-minus" procedure. They display two iterations of the Akers-Siegel procedure applied to the columns and rows of a matrix. The use of the Akers-Siegel procedure appears to have been motivated by the then lack of alternatives. Junius and Oosterhaven (2003) introduced the GRAS algorithm to solve this problem in a theoretically sound manner. Lenzen, Gallego and Wood (2007), Temurshoev, Miller and Bouwmeester (2013), and Lenzen, Moran, Geschke and Kanemoto (2014) improved GRAS so that it always produces an exact solution no matter the inputs.

GRAS is a derivative of the RAS algorithm, also known as matrix scaling, matrix raking, iterative proportionate fitting, among others. The RAS algorithm can be applied to nonnegative matrices with nonnegative controls, provided that a feasible solution exists. When that feasible solution exists, Bregman (1967) proved its uniqueness. A feasible solution always exists when the inputs are all positive. The existence of one or more zeroes in the input matrix can create infeasibility. Fagan and Greenberg (1984) describe using a succession of linear programs to determine infeasibility caused by the presence of these zeroes. Instead, GRAS can be used on the original problem with a negative output element indicating infeasibility. Bregman (1967) proved that the RAS algorithm solves a minimum entropy measure of change. That is, RAS minimizes the loss of information from controlling the input matrix. See Schneider and Zenios (1990) for a history, properties and interpretations of the RAS algorithm.

While RAS is restricted in its application and is not always feasible, GRAS can take any real value in its inputs and produce a unique solution. This solution may require changing the signs of some elements of the input matrix. GRAS solves a variant of the minimum entropy problem. When GRAS's inputs are all positive (or negative), it reduces to the RAS algorithm. These problems are always RAS-feasible.

2. The RAS Algorithm

Given nonnegative $m \times n$ input and output matrices $A = [a_{ij}]$ and $X = [x_{ij}]$, an m-dimensional vector of row marginals u and an n-dimensional vector of row marginals v, all of which are nonnegative, the RAS algorithm solves the equations

$$\sum_{j=1}^{n} x_{ij} = u_i, i = 1, \ldots, m \tag{1a}$$

and

$$\sum_{i=1}^{m} x_{ij} = v_i, j = 1, \dots, n$$

(1b)

so as to minimize the entropy, E, of the changes, $E = \sum_{i,j}\left[x_{ij}\left(\frac{x_{ij}}{a_{ij}} - 1\right)\right]$, where the constant 1 is arbitrary.

Let tol be the convergence criterion. Its value is the required precision. The RAS algorithm is implemented by:

Step 0. (Initialization) Set $k = 0$ and $A^0 = A$.
Step 1. (Row Raking) For $i = 1,2, \dots, m$, define $r_i^k = u_i / \sum_{j=1}^{n} a_{ij}^k$, where u_i is the marginal for row i. Define the $m \times n$ matrix B by the elements $b_{ij} = r_i^k a_{ij}^k$ for $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$.
Step 2. (Column Raking) For $j = 1,2, \dots, n$ define $s_j^k = v_j / \sum_{i=1}^{m} a_{ij}^k$, where v_j is the marginal for column j. Define matrix A^{k+1} by the elements $a_{ij}^{k+1} = s_j^k b_{ij}$ for $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$.
Step 3. (Convergence Test) Compute $maxdiff = \max_{i,j}\left|a_{ij}^{k+1} - a_{ij}^k\right|$. If $maxdiff < tol$, then output A^{k+1}. Otherwise, repeat from Step 1.

The result has the form

$$A^{k+1} = R^k A S^k,$$

(2)

where $R^k = \mathrm{diag}\left(R_1^k, \dots, R_m^k\right)$, $S^k = \mathrm{diag}\left(S_1^k, \dots, S_n^k\right)$, $R_i^k = \prod_{l=1}^{k} r_i^k$ and $S_i^k = \prod_{l=1}^{k} s_i^k$. The reasoning for the algorithm's name is obvious.

While the RAS algorithm iterates the A^k to convergence, Eq. (2) shows that it can be viewed as the result of iterating the rakes. This intuition is the foundation of the GRAS algorithm.

3. The GRAS Algorithm

Given the same inputs as the RAS algorithm, but without any restriction on sign, the GRAS algorithm first separates A into its positive elements, $P = \left[p_{ij}\right]$, and absolute values of its negative elements, $N = \left[n_{ij}\right]$, with zeroes filling out the other elements:

$$A = P - N.$$

(3)

Given A and vectors of row and column marginals u and v, all of any sign, GRAS finds the matrix X that minimizes the generalized entropy function $E = \sum_{i,j} \log\left\{\left|x_{ij}\right|\left(\left|\frac{x_{ij}}{a_{ij}}\right| - 1\right)\right\}$, where

$$X = RPS - R^{-1}NS^{-1},$$

(4)

where $R = \text{diag}(R_1, \dots, R_m)$, $S = \text{diag}(S_1, \dots, S_m)$ with corresponding inverses R^{-1} and S^{-1}. Zero elements on the diagonals are replaced by 1.

Suppressing superscripts, let

$$p_i(S) = \sum_j p_{ij} S_j \,, p_j(R) = \sum_i p_{ij} R_i \,, n_i(S) = \sum_j \frac{p_{ij}}{S_j}, n_j(s) = \sum_i \frac{p_{ij}}{R_i},$$

(5)

where all values are taken from the just completed iteration.

Let tol be the convergence criterion. Because it is the criterion for the rakes, its preferred value is not obvious. Temushoev et al. (2013) use 10^{-6}. The GRAS algorithm uses the following steps:

Step 0 (Initialization): Set P equal to the positive elements of A, zero elsewhere. Set $-N$ equal to the negative elements of A, zero elsewhere. Set the row rakes, $r_i, i = 1, \dots, m$, and column rakes, $s_j, j = 1, \dots, n$ equal to 1.
Step 1 (Column Rakes): For $j = 1, \dots, n$, compute

$$S_j = \begin{cases} \dfrac{v_j + \left[v_j^2 + 4p_j(R)n_j(R)\right]^{\frac{1}{2}}}{2p_j(R)} & \text{for } p_j(R) > 0 \\ -\dfrac{n_j(R)}{v_j} & \text{for } p_j(R) = 0 \end{cases}.$$

Step 2 (Column Sign Changes): Compute the current value of X using Eq. (4). For $j = 1, \dots, n$, if sign $\sum_i x_{ij} \neq \text{sign}(v_j)$, then set $S_j = -S_j$.

Step 3 (Row Rakes): For $i = 1, \dots, m$, compute

$$R_i = \begin{cases} \dfrac{u_i + [u_i^2 + 4p_i(S)n_i(S)]^{\frac{1}{2}}}{2p_i(S)} & \text{for } p_i(S) > 0 \\ -\dfrac{n_i(S)}{u_i} & \text{for } p_i(S) = 0 \end{cases}.$$

Step 4 (Column Sign Changes): Compute the current value of X using Eq. (4). For $i = 1, \dots, m$, if $\text{sign}\left(\sum_j x_{ij}\right) \neq \text{sign}(u_i)$, then set $R_i = -R_i$.

Step 5 (Convergence Test): Compute $maxdiff = \max_j \left|S_j^{k+1} - S_j^k\right|$, where k refers to the previous iteration. If $maxdiff < tol$, then repeat Steps 1 and 3 and output X. Otherwise, repeat from Step 1.

4. An RAS-infeasible Problem Solved by GRAS

Let $A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$, $u = (3,1)$ and $v = (2,2)$. It is easy to show that this problem is RAS-infeasible. Satisfying the first row marginal requires $R_1 = 1$. Thus, $X_{12} = 3$. However there is no nonnegative value X_{22} such that $X_{22} + 3 = 1$. Thus, the problem is RAS-infeasible.

Removing the nonnegativity constraint results in the rakes $R = \text{diag}(1, -1)$ and $S = \text{diag}(-1,1)$ with solution $X = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$.

Sinkhorn (1964) proves that the RAS algorithm is feasible for all positive A. The presence of zeroes in the example, together with its constraints, made it infeasible. Fagan and Greenberg (1984) propose determining feasibility using a sequence of linear programs. This example suggests first solving a possibly infeasible problem using GRAS, then checking the solution for negative entries. If any exist, excluding machine zeroes, then the problem is RAS-infeasible.

5. A Net Migration Problem

Shryock et al. (1973) present a problem involving constraining a net migration problem. They do two iterations of the Akers-Siegel procedure, each iteration consisting of constraining rows then columns. The problem is shown in Table 1, with the marginals at right and bottom.

Table 1: Reported Net Migration Within Japan, 1955-60

Region	1955-56	1956-57	1957-58	1958-59	1959-60	1955-60, adjusted
Hokkaido	-561	-3,715	25,566	-583	-11,509	-52,976
Tohoku	-80,810	-102,454	-92,620	-96,156	-119,310	-583,301
Kanto	208,016	241,799	237,025	253,926	283,776	1,218,828
Chubu	-57,369	-56,726	-72,701	-56,320	-33,060	-251,318
Kinki	77,287	125,944	90,937	100,310	136,377	551,007
Chugoka	-39,182	-46,038	-46,995	-53,327	-61,643	-329,777
Shikoku	-35,808	-53,560	-46,803	-45,301	-60,257	-296,668
Kyushu	-79,313	-115,441	-101,406	-113,161	-184,552	-788,929
All Japan, adjusted	-104,715	-91,963	-97,550	-105,037	-133,869	

Source: Shryock et al. (1973, 711).

The results after their constraining process are shown in Table 2.

Table 2: Controlled Net Migration in Japan, 1955-60, as Published by Shryock et al. (1973)

Region	1955-56	1956-57	1957-58	1958-59	1959-60	Sum
Hokkaido	-9,754	-15,233	10,411	-9,436	-28,956	-52,968
Tohoku	-100,815	-120,738	-110,992	-115,532	-135,234	-583,311
Kanto	196,706	242,683	233,931	249,940	295,558	1,218,818
Chubu	-54,400	-50,811	-66,220	-51,434	-28,482	-251,347
Kinki	75,982	131,414	93,307	102,649	147,669	551,021
Chugoka	-54,944	-60,983	-63,300	-72,018	-78,536	-329,781
Shikoku	-46,240	-65,335	-58,057	-56,341	-70,698	-296,671
Kyushu	-111,250	-152,959	-136,629	-152,867	-235,192	-788,897
Sum	-104,715	-91,962	-97,549	-105,039	-133,871	

Source: Shryock et al. (1973, 711).

The results of running GRAS with a tolerance of 10^{-7} are in Table 3. Six iterations were required.

Table 3: Net Migration in Japan, 1955-60, Controlled by GRAS

Region	1955-56	1956-57	1957-58	1958-59	1959-60
Hokkaido	-2264	-13999	6714	-2257	-41170
Tohoku	-102041	-120822	-110380	-116488	-133570
Kanto	194606	242217	234952	247612	299442
Chubu	-54966	-50759	-65741	-51770	-28083
Kinki	75124	131081	93656	101630	149517
Chugoka	-55613	-61026	-62953	-72616	-77570
Shikoku	-46821	-65405	-57758	-56829	-69854
Kyushu	-112739	-153250	-136041	-154320	-232580

The unrounded GRAS solution is exact within a small rounding error. GRAS's great advantages compared to the Akers-Siegel procedure lie in its sound theoretic basis and automatic running. To the contrary, Shryock et al. (1973, 709) state that they did Hokkaido separately, including an unspecified additive adjustment, because of a weakness in the Akers-Siegel procedure that also causes nonunique solutions (Akers and Siegel, 1965). GRAS lacks this sort of instability because it produces unique solutions. Moreover, GRAS's multipliers for positive and negative data are deterministically related, being reciprocals of one another, while the Akers-Siegel procedure's multipliers have no predefined relationship.

6. A Perturbed Version of the Net Migration Problem

Perturbing the net migration problem above leads to additional insights. Starting with Table 1, the row control for Kanto is decreased by 251,328 and added to Chubu, changing the latter's sign and resulting in a value of 10. The new problem is shown in Table 4.

Table 4: Perturbed Net Migration Within Japan, 1955-60

Region	1955-56	1956-57	1957-58	1958-59	1959-60	1955-60, adjusted
Hokkaido	-561	-3,715	25,566	-583	-11,509	-52,976
Tohoku	-80,810	-102,454	-92,620	-96,156	-119,310	-583,301
Kanto	208,016	241,799	237,025	253,926	283,776	967500
Chubu	-57,369	-56,726	-72,701	-56,320	-33,060	10
Kinki	77,287	125,944	90,937	100,310	136,377	551,007
Chugoka	-39,182	-46,038	-46,995	-53,327	-61,643	-329,777
Shikoku	-35,808	-53,560	-46,803	-45,301	-60,257	-296,668
Kyushu	-79,313	-115,441	-101,406	-113,161	-184,552	-788,929
All Japan, adjusted	-104,715	-91,963	-97,550	-105,037	-133,869	

The GRAS solution to the new problem is shown in Table 5.

Table 5: Solution to Perturbed Net Migration Problem

	1955-56	1956-57	1957-58	1958-59	1959-60
Hokkaido	-2370	-14277	6417	-2302	-40443
Tohoku	-104636	-120678	-113105	-116382	-128500
Kanto	150419	192210	181735	196435	246702
Chubu	2	2	3	2	1
Kinki	72992	130756	91064	101349	154846
Chugoka	-57047	-60974	-64529	-72575	-74651
Shikoku	-48039	-65364	-59217	-56809	-67240
Kyushu	-116036	-153637	-139918	-154754	-224584

Perhaps the most notable aspect of Table 5 is that Chubu's data have changed sign and are in rough proportion to the absolute values of the input data. An examination of the rakes is revealing. The column rakes are shown Tables 6; the row rakes in Table 7.

Table 6: Column Rakes

Period	1955-56	1956-57	1957-58	1958-59	1959-60
Original	0.83	0.89	0.88	0.87	0.94
Perturbed	0.66	0.72	0.70	0.70	0.79

Table 7: Row Rakes

Region	Original	Perturbed
Hokkaido	0.30	0.36
Tohoku	0.95	1.18
Kanto	1.12	1.10
Chubu	1.25	-39278.24
Kinki	1.17	1.44
Chugoka	0.85	1.05
Shikoku	0.92	1.14
Kyushu	0.84	1.04

The column rakes decrease across the board per Table 6. This appears to be driven by the need for Kanto's greater reduction to offset the loss of negative values in Chubu. Table 7 is particularly revealing. All the original rakes are all positive and within 70% of 1. The perturbed rakes are approximately the same except for Chubu. Chubu's rake is negative and its absolute value is four orders of magnitude larger. Since Chubu's data are negative, they are divided by this rake to obtain, to a first approximation, Table 5's results. This demonstrates GRAS's ability to handle sign changes and differences in magnitudes.

6. Conclusion

The recently developed GRAS algorithm solves the matrix balancing problem for any combination of real inputs. It is applied to a well-known demographic problem with satisfactory results. Moreover, it solves problems involving sign changes and changes in the order of magnitude of results. GRAS has the advantages of being theoretically sound and completely automated.

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