Violations of Uniform Partner Ranking Condition in Two-way Flow Strict Nash Networks

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27 January 2017

VIOLATIONS OF UNIFORM PARTNER RANKING CONDITION IN TWO-WAY FLOW STRICT NASH NETWORKS

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March 27, 2017

Abstract– **The paper of Charoensook ((2015), [3]) extends the results of the original model of two-way flow information sharing network of Bala and Goyal ((2000),[1]), given that a condition called Uniform Partner Ranking is satisfied. In this technical note, we study what happen to these results when this condition is violated. By providing some examples, we conclude that a certain degree of agent homogeneity needs to exist in order that the results of [3] remains satisfied.**

Index Terms– **Network Formation, Strict Nash Network, Two-way Flow Network, Branching Network, Agent Heterogeneity, Information Network**

I Introduction

A game-theoretic model of network formation assume that networks are form based upon self-interest agents who choose to establish costly connections or links with each other in order to exchange some benefits (eg., his private information) . The original two-way flow model of Bala and Goyal (2000, [1]), BG henceforth, further has in mind a situation in which each agent pays for all information that he wishes to acquire by (i) solely bears the cost of link establishment used for communication, and (ii) promises to share his own private piece of information with others. Since this model assumes agent homogeneity, it has inspired many extensions that allow for the existence of agent heterogeneity. An interesting paper in this literature is that of Charoensook (2015, [3]) that generalizes the original results of BG and that of [4] and [2]. Importantly, the generalization of [3] is achieved through imposing a condition called Uniform Partner Ranking on the characteristics of the structure of link establishment cost in order that the shapes of SNNs can be predicted.

Naturally, this raises the question of what happen when this condition - Uniform Parter Ranking - is violated. In this technical note, we contribute to this literature by proposing some answers to this question. Specifically, we provide some examples that show that (i) the results of [3] can still hold even if the Uniform Partner Ranking condition, UPR henceforth, is violated, (ii) only partial results still hold, and (iii) even partial results do not hold. Through these examples we conclude that a certain degree of agent homogeneity needs to exist in order that the results of [3] remain to hold.

We provide a brief introduction to related literature here. The literature in game-theoretic model of network formation is invented by two papers - [7] and [1]. These two papers are quite different in terms of basic assumptions on the nature of benefits that each agent possesses. On one hand, [7] assumes the benefits that each agent possess may not necessarily be nonrival. Therefore, a link is formed and the benefits are shared only if both agents agree. For an elaborate review of the literature of network formation, [6] and [5] provide a through introduction.

On the other hand, the original two-flow flow of network formation of BG assumes that each agent owns a unique piece of private information that is non-rival in the sense that each does not mind sharing his information with other agents. He can independently choose to establish a link with any other agent in the network by bearing a link establishment cost on his own. In this paper, Nash and Strict Nash equilibrium in pure strategies are adopted to predict the appearance of equilibrium networks. which are called Nash networks and Strict Nash networks, SNNs henceforth, respectively. An important assumption is that Link establishment cost is assumed to be identical across all agents. Thus, agent homogeneity is assumed in BG.

Several works in the literature extend this BG model to cases at which link formation cost is heterogeneous across agents. How this heterogeneity is imposed, though, varies among existing literature. A paper that is of our interest is that of Charoensook (2015, [2]) since it establish a result that generalize the

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models of [1], [4], and [2]. This generalized result assumes that link formation cost satisfies a condition called Uniform Partner Ranking. Simply put, this condition states that agents may pay different levels of link formation cost. However, each of them has the *same ranking* in terms of partner preference. That is, if an agent i finds that linking to j is cheaper than linking to k, all other agents find likewise. This condition results in the fact that every non-empty component of an SNN has at most one agent who receives more than one link. Our paper, therefore, contributes to the literature by investigating what happen to SNNs when this Uniform Partner Ranking is violated.

The paper proceeds as follows. In the next two sections, model specifications and the definition of SNN as an equilibrium prediction criterion is introduced. We then proceed to the main results section by giving examples of Strict Nash networks that violate the Uniform Partner Ranking condition. Finally, in the conclusion section we discuss on the insights from these examples.

II The Model

II.I Basic Setting

Let $N = \{1, ..., n\}$ be the set of all agents in the network. Consider an agent $i \in N$, i establishes a link with an agent j by paying the link formation cost $c_{i,j}$. The incentive of i is to acquire the information of j. Notice that $c_{i,j}$ depends on both the identity of i and j. This is where agent heterogeneity is introduced in our model. Whenever a link to j is established by i, we say that i is a link sender and j is a link receiver. Furthermore, we say that i accesses j.

Individual's strategy and network representation. Let $g_{i,j} = 1$ represents the fact that i accesses j and $g_{i,j} = 0$ represents the fact that i does not access j. A strategy of i, represented by g_i, is $g_i = \{g_{i,1}, ..., g_{i,i-1}...g_{i,n}\}$. A strategy profile is, therefore, $g = (g_1, ..., g_n)$. Since all links form the network, we set g also represents the network. Graphically, we let an agent i be presented by a node i. A point from node i to node j then represents the fact that i accesses j

Structure of information flow. Information flow is two-way in the sense that if i has an entry to the information j then j also has an entry to the information of i. i has an entry to the information of j whenever a path between i and j exists. Formally, let $\bar{g}_{ij} = max\left\{g_{ij}, g_{ji}\right\}$. A path between i and j or $ij-$path in a network g is then defined as a sequence $P_{i,j}(g) = \bar{g}_{i,j_1}, \bar{g}_{j_1 j_2}, ..., \bar{g}_{j_m j}$ such that each element in this sequence is 1. If an $ij-$ path exists, i is said to observe j.

Individual's payoff. Let $N^d(i;\bar{g})$ and $N(i;\bar{g})$ be the set of all agents that i accesses and observes respectively. Let $V_{i,j}$ be the value of information of j that i receives. Then, the payoff of i in g is defined as:

$$\Pi_i(g) = \sum_{j \in N(i;g)} V_{i,j} - \sum_{j \in N^d(i;g)} c_{i,j} \tag{1a}$$

Graph-theoretic notations. Consider a network g. A network is connected if i observes j for for all $i, j \in N$ and $i \neq j$. A subnetwork g' is a subset of a network g, ie., $g' \subset g$. A component of a network is a subnetwork that is maximally connected. That is, i observe j if and only if i and j belong to the same component. A network is said to be minimal if every path between i and j is unique. That is, there exists one and only one path through which i observes j. An agent who observes no other agent is said to be isolated. If all agents in the network are isolated, the network is said to be an empty network.

B_i **and branching networks.** The definitions of these terms are borrowed from [2]. A branching network is a minimally connected such that there is a unique agent i who receives no link and every other agent receives exactly one link. That is, a branching network rooted at i is a minimally connected network such that $|I_i(g)| = 0$ and $|I_j(g)| = 1$ for all $j \neq i$ and $j \in N$ (I and O should be defined somewhere !!!).

To define B_i network, we first introduce the following notations. Let $Q_{N'} = N' \cup j \in N|apathfromitojexist$ (a path needs to be defined somewhere!!!). A point contrabasis of a network g, $B(g)$, is a minimal set of players such that $Q_B(g) = N$. Intuitively, $Q_B(g)$ carries the intuition that there is a set of agents that can be used to observe all other agents through the existence of the path between an agent in this set and an agent outside of this set. An i-point contrabasis, $B_i(g)$, is a point contrabasis of g such that all players $j \in B_i(g)$ accesses i. Finally, A network g is a B_i-network if $|I_i(g)|2$, $|I_j(g)| < 2$ for all $j \neq i$, and $I_i(g) = B_i(g)$.

II.II The Definitions of Nash Network

Consider a network g. Let g_{-i} be the set of all links in g that i does not establish. That is, $g_{-i} = g \backslash g_i$. Put differently, a union of g_{-i} and g_i is exactly the network g. These notations are used to define the following terms.

Definition 1 (Best response). *A strategy g_i is a <u>best response</u> of i to g_{-i} if*

$$\Pi_i(i; g_i \oplus g_{-i}) \geq \Pi_i\left(i; g_i' \oplus g_{-i}\right), \quad \textit{for all } g_i' \in G_i$$

Definition 2 (Nash network). *A network g is a <u>Nash network</u> if g_i is a best response to g_{-i} for every agent $i \in N$.*

Moreover, if the inequality is strict for all $i \in N$, Nash network is a *Strict Nash Network*. We abbreviate the term Strict Nash Network by SNN .

II.III Cost Structure and the Uniform Parter Ranking Condition

A cost structure $\mathcal{C}$ is defined as a collection of all link formation costs $\mathcal{C} = \{c_{i,j} : i, j \in N, i \neq j\}$. We use this definition to define the following two terms, which are borrowed from [3].

Definition 3 (Better Partner)**.** *Consider a set $X \subset N$ and agents $j, k \in X$, j is **at least as good a partner** as k with respect to the set X if $c_{i,j} \leq c_{i,k}$ for any $i \in X, i \neq j \neq k$. Moreover, if the inequality is strict then j is said to be a **better partner** than k with respect to the set X.*

That is, if j is at least as good a partner as k with respect to set of agents X, then every agent in the set X finds that accessing j is at least as costly as accessing k. Put differently, all agents in X 'rank' j as a preferred partner than k in terms of costliness of link establishment. The Uniform Partner Ranking condition below simply adds that the set X is exactly N and that all agents can be ranked.

Definition 4 (Uniform Partner Ranking)**.** *A cost structure $\mathcal{C}$ is said to satisfy **Uniform Partner Ranking** condition if for any distinct pair $j, k \in N$ it holds true that j is at least as good a partner as k or k is at least as a good a partner as j with respect to the set N.*

Intuitively, since all agents can be ranked if $\mathcal{C}$ satisfies the UPR condition, there exists an agent who is ranked 'first' in the sense that he is at least as good a partner as every other agent. This leads to the following definition.

Definition 5 (Common Best Partner)**.** *An agent i^* is said to be Common Best Partner if i^* is at least as good a partner as i' with respect to the set N, where $i' \neq i^*$.*

III Main Results

III.I Case 1: UPR is violated but the results of Charoensook 2015 still hold

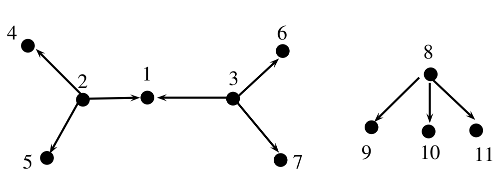

Figure 1: Example 1

agents	1	2	3	4	5	6	7	8	9	10
1	-	0.1	0.2	0.3	0.4	0.5	0.6	*20*	*20*	*20*
2	0.1	-	0.2	0.3	0.4	0.5	0.6	*20*	*20*	*20*
3	0.1	0.2	-	0.3	0.4	0.5	0.6	*20*	*20*	*20*
4	0.1	0.2	0.3	-	0.4	0.5	0.6	*20*	*20*	*20*
5	0.1	0.2	0.3	0.4	-	0.5	0.6	*20*	*20*	*20*
6	0.1	0.2	0.3	0.4	0.5	-	0.6	*20*	*20*	*20*
7	0.1	0.2	0.3	0.4	0.5	0.6	-	*20*	*20*	*20*
8	*20*	*20*	*20*	*20*	*20*	*20*	*20*	-	0.3	0.4
9	*20*	*20*	*20*	*20*	*20*	*20*	*20*	0.4	-	0.5
10	*20*	*20*	*20*	*20*	*20*	*20*	*20*	0.5	0.6	-

Table 1: Cost Structure for Example 1

Example 1. *Let $V_{i,j} = 1$ for all $i, j \in N$ and $i \neq j$. Let the cost structure be represented by the above table, where each row represents an agent i, each column represents an agent j, and each number in the table represents the cost $c_{i,j}$. This cost structure divides agents into two groups, where agents 1 to 7 belong to group I and agents 8 to 10 belongs to group II. Accordingly, the table is divided into four quadrants at agent 7. Observe further that link formation costs between agents from the same group are at most 0.6, while the link formation costs between agents across groups are set at 20. Hence, accessing an agent from the other group is never a best response. This cost structure, therefore, is reminiscent of the insider-outsider model of [4]. A major difference, though, is that in this example link formation cost $c_{i,j}$ is not identical among agents in the same group.*

It is easy to show that this cost structure violates UPR, yet every non-empty SNNs consists of non-empty components that are either branching or B_i. To show the violation, consider agent 1 and agent 8. We can see that $c_{1,2} < c_{1,7}$ but $c_{8,2} > c_{8,7}$. Therefore, UPR is violated. Indeed, this is due to the fact that agents 1 and 8 belong to different groups. Observe further that $V_{i,j} = 1$ and $c_{i,j} = 20$ for any i, j that do not belong to the same group. Therefore, agents that do not belong to the same group will not establish links with each other. On the contrary, it is easy to see that links between agents from the same group are established since $V_{i,j} = 1$ but $c_{i,j} =< 1$ for any i, j that belong to the same group. Consequently, it is guaranteed that every SNN has exactly two non-empty components, each is composed of agents from the same group.

Finally, it remains to be shown that each non-empty component of SNN is either branching or B_i. First, observe that UPR is not violated if we consider only agents from the same group. Indeed, all agents in Group I (II) have agent 1 (agent 8) as their common best partner, and each agent i finds that $c_{i,j} < c_{i,j+1}$ for any $i, j, j+1$ that belong to the same group. Therefore, inside each component, UPR is not violated. As a result, it can be predicted that each component of SNN is either branching or B_i. Figure 1 above illustrates an SNN based upon this cost structure.

agent	1	2	3	4	5
1	-	0.4	0.3	0.1	0.2
2	0.1	-	0.2	0.3	0.4
3	0.1	0.4	-	0.3	0.2
4	0.1	0.2	0.3	-	0.4
5	0.1	0.4	0.2	0.3	-

Table 2: Cost Structure for Example 2

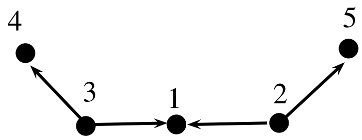

Figure 2: Example 2

Example 2. *Let the cost structure be represented by the above table and let $V_{i,j} = 1$ for all $i, j \in N$ and $i \neq j$. In this example, UPR is violated because $c_{4,2} = 0.2 < c_{4,3} = 0.3$ but $c_{5,2} = 0.4 > c_{5,3} = 0.2$. However, we have an SNN that is B_1. It is easy to see why UPR is violated but SNN remains a B_i network. First, observe that every agent (except agent 1) agrees that agent 1 is the common best partner. Therefore, agent 2 and agent 3 access agent 1 in this SNN.*

III.II Case 2: UPR is violated and the results of Charoensook 2015 do not hold

agents	1	2	3	4
1	-	7	8	9
2	0.1	-	5	5
3	0.1	5	-	5
4	5	5	0.1	-

Table 3: Cost Structure for Example 3

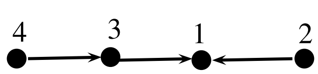

Figure 3: Example 3

Example 3. *Let the cost structure be represented by the above table and let $V_{i,j} = 1$ for all $i, j \in N$ and $i \neq j$. In this example, UPR is violated because $c_{1,2} = 7 < c_{1,3} = 8$ but $c_{4,2} = 5 < c_{4,3} = 0.1$. Indeed, agent 2 and 3 agree that agent 1 is the best partner. However, agent 4 has agent 3 as his best partner. This results in the fact that agent 4 accesses agent 3 in this SNN, while both agent 2 and agent 3 access agent 1. It is easy to see that this SNN is neither branching nor B_i. First, it is not branching because there is no agent who receives no link. Second, it is not B_i because a point contrabasis of this network is the set $\{2, 3, 4\}$ so that agent 2 cannot be a $2-$point contrabasis of this network.*

III.III Case 3: UPR is violated but the results of Charoensook 2015 partially hold

	1	2	3	4	5	6	7	8	9	10	11
1	-	0.1	0.2	0.3	0.4	0.5	0.6	20	20	20	20
2	0.1	-	0.2	0.3	0.4	0.5	0.6	20	20	20	20
3	0.1	0.2	-	0.3	0.4	0.5	0.6	20	20	20	20
4	0.1	0.2	0.3	-	0.4	0.5	0.6	20	20	20	20
5	0.1	0.2	0.3	0.4	-	0.5	0.6	20	20	20	20
6	0.1	0.2	0.3	0.4	0.5	-	0.6	20	20	20	20
7	0.1	0.2	0.3	0.4	0.5	0.6	-	20	20	20	20
8	20	20	20	20	20	20	20	-	7	8	9
9	20	20	20	20	20	20	20	7	-	8	9
10	20	20	20	20	20	20	20	7	8	-	9
11	20	20	20	20	20	20	20	7	8	9	-

Table 4: Cost Structure for Example 4

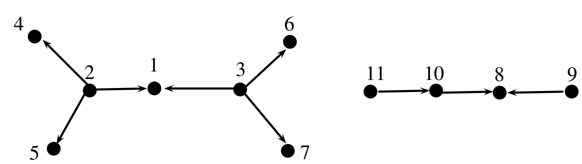

Figure 4: Example 4

Example 4. *The cost structure of this example is simply a combination of Example 1 and Example 3. Observe that the link formation costs of agent 1 to agent 7 is identical to that of example 1 and that the link formation costs of agent 8 to agent 11 is identical to that of example 3 (agent 1 to agent 4 in Example 3). We therefore divide agents into two groups, where agent 1 to agent 7 belong to the group I and agent 8 to agent 11 belong to group II. Observe further that link formation cost $c_{i,j}$ is set to be 20 if i and j belong to different groups. Similar to Example 1, we have an SNN that consists of two non-empty components, each is composed of agents from the same group. Moreover, the shape of each component is precisely that of Example 1 and Example 3. Consequently, we have an SNN such that one of its components is B_i and the other is neither branching or B_i. This entails that UPR is violated and the results of Charoensook 2015 hold only partially.*

IV Discussion and Conclusion

This paper shows various effects of the violation of UPR condition on Strict Nash networks. Let us summarize these effects as follows:

1. If it can be predicted that SNN consists of multiple components, and that we know which agent belongs to which component, the shape of each component depends merely on the cost structure pertaining to agents in this component. This insight can be seen from Example 1 and Example 4.

2. Following the first effect, whether the cost structure of all agents violates UPR or not does not matter. Indeed, if UPR is not violated when considering the cost structure pertaining to agents in the same component, the results of Charoensook 2015 still holds. Alternatively, it may partially hold in the sense that the shape of some components are predicted to be branching or B_i due to the fact that UPR is not violated inside that each of these components. This insight is illustrated in Example 1 and Example 4.

3. Even if the cost structure pertaining to agents in the same component does violate UPR, SNN can still be B_i. This insight is seen in Example 2.

4. In contrast to (3), there exists also cases such that a component of SNN is neither branching or B_i when the cost structure pertaining to agents in the same component violates UPR. This insight is seen in Example 3.

At this point, we further provide an important observation from point (3) and point (4) above. To do so, we first remark that in *both* Example 2 and Example 3 UPR is violated, yet only SNN in Example 3 remains B_i while SNN in Example is neither B_i nor branching. What explain this difference? In Example 2, we have that all agents (except agent 1) agree that agent 1 is their best common partner. However, this form of agreement between agents does not exist in Example 3, where agent 4 does not agree with agent 2 and agent 3 that agent 1 is the best partner. Therefore, we remark that some forms of agreement between all agents inside the component need to exist in order the results of the results of Charoensook 2015 - a component of SNN being branching or B_i - remains to hold.

Indeed, a similar argument is also applied to point 1 and 2 above, which illustrate that what matters is whether UPR is violated inside each component rather than the violation of UPR when considering all agents in the network. Since UPR in a component requires that all agents in the component agree on which agent is superior as a partner than which in terms link formation cost, one can interpret that some forms of agreement between all agents inside the component need to exist in order the results of the results of Charoensook 2015 - a component of SNN being branching or B_i - remains to hold.

Finally, we remark that these examples raise a question of what a necessary and sufficient condition for a component of SNN to be branching or B_i is. We leave this question as a research to be explored in the future.

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