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Abstract

A ranking rule (social welfare function) for a fixed population assigns a social preference to each profile of preferences. The rule satisfies "Positional Cancellation" if changes in the relative positions of two alternatives that cancel each other do not alter the social preference between the two. I show that the Borda rule is the only ranking rule that satisfies "Reversal" (a weakening of neutrality), "Positive Responsiveness," and "Pairwise Cancellation."

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* *URL:* `http://researchmap.jp/reiju/` (H. R. Mihara).

1 Introduction

A *ranking rule* ("social welfare function") aggregates individual preferences into a (collective) social preference. Arrow's Impossibility Theorem [1963] points out the difficulty of designing ranking rules: if a ranking rule for more than two alternatives satisfies the *Pareto* principle and *IIA* ("independence of irrelevant alternatives"), then it is dictatorial. There are several ways to escape from this negative conclusion.[1]

One way is to limit the number of alternatives to two. In that case, IIA becomes vacuous; the simple majority rule, for instance, satisfies Arrow's conditions.[2] In fact, it is the only rule (May, 1952) that satisfies *anonymity* (equal treatment of individuals), *neutrality* (equal treatment of alternatives), and *positive responsiveness* (if x is socially at least as good as y, then an increased support for x relative to y makes x socially preferred to y).

Another way is to discard IIA. This allows us to consider various *scoring rules*, which rank alternatives according to the total score received from all individuals: if an individual's preference is a linear order $a_1a_2\ldots a_m$ (where a_k is the kth ranked alternative), then a_1 receives s_{m-1} points, a_2 receives s_{m-2} points, ..., and a_m receives s_0 points from the individual ($s_{m-1} \geq s_{m-2} \geq \cdots \geq s_0$ and $s_{m-1} > s_0$). Scoring rules can be viewed as a generalization of simple majority rule to three or more alternatives.

A typical example of a scoring rule is the *Borda rule* [1781]. It refers to any scoring rule (viewed as a ranking rule) such that the difference between the consecutive scores s_k is constant ($s_{m-1} - s_{m-2} = s_{m-2} - s_{m-3} = \cdots = s_1 - s_0$). It is the unique scoring rule that minimizes "the kinds and number of paradoxes that can occur" (Saari, 1990). Also, it is the only scoring rule that (regardless of the population size) never top-ranks a Condorcet *loser* (an alternative beaten by all other alternatives in pairwise majority comparisons) (Fishburn and Gehrlein, 1976; Okamoto and Sakai, 2013). The objective of this paper is to present a set of conditions (axioms) that characterize the Borda rule.

May's three conditions (suitably modified for more than two alternatives) alone do not single out the Borda rule. (Every scoring rule satisfies anonymity and neutrality; if the scores are decreasing, it satisfies positive responsiveness.) Replacing anonymity, I show (Theorem 1) that within the class of all ranking rules, the conditions that I call "Reversal," "Positive

[1]For example, if there are *infinitely* many individuals, non-dictatorial rules exist that satisfy Arrow's conditions (e.g., Kirman and Sondermann (1972)). But as soon as we require the rules to be algorithmically computable (a requirement that any real-world voting rules satisfy), the negative conclusion prevails again (Mihara, 1997).

[2]The restriction to two alternatives can be relaxed, provided that transitivity of social preferences is replaced by a weaker requirement (such as "acyclicity") that ensures the existence of a maximal alternative. If the number of alternatives is less than an integer called the Nakamura number of the rue, then maximal alternatives always exist (Nakamura, 1979; Kumabe and Mihara, 2011).

Responsiveness," and "Pairwise Cancellation" characterize the Borda rule. One can substitute "Neutrality" for "Reversal" (Corollary 1). Also, none of the three conditions is redundant (Section 4.1).

Reversal is similar to May's version of neutrality: it requires that a reversal of the individuals' preferences between two alternatives reverse the social preference between them. Reversal is distinguished from the stronger (Lemma 2) *Neutrality* condition, usually defined for three or more alternatives. *Positive responsiveness* is similar to May's. I express an individual's increased support for x relative to y in terms of the differences between the lower-contour set ("worse set") of x and that of y.[3]

Positional Cancellation is the requirement that (nontrivial) *changes* in the relative positions of two alternatives that cancel each other do not alter the social preference between the two. For example, each profile in the following sequence should give the same social ranking of x and y:

$$\begin{aligned} R^1 &= (xaby, axby, ybax), \\ R^2 &= (axby, xaby, ybax), \\ R^3 &= (axby, xayb, ybxa), \\ R^4 &= (xbay, xayb, yabx). \end{aligned}$$

In going from R^1 to R^2, for example, the relative position of (x, y) decreases by 1 for individual 1 and increases by 1 for individual 2, while the preference of individual 3 is fixed. Positional Cancellation is a distinctive characteristic of the Borda rule, a defining characteristic within the class of scoring rules. However, we cannot restrict our attention to the scoring rules in proving Theorem 1, since the other two conditions (Reversal and Positive Responsiveness) do not imply that the rule is a scoring rule (Example 3).

Along the way to proving Theorem 1, I prove Proposition 1, which is of independent interest, asserting that these three conditions imply *Relative Positions*. The condition requires that changes in individual preferences that preserve the relative position of two alternatives do not alter the social preference between the two.[4] To see the difference between Positional Cancellation and Relative Positions, observe that the relative position of (x, y) is the same for R^1 as for R^4. Also, it is the same for $R^5 = (xaby, xbay, xaby)$ as for $R^6 = (xbay, xaby, xaby)$. Relative Positions immediately implies the conclusion that the social ranking of x and y is the same for R^1 as for R^4; also, it is the same for R^5 as for R^6. In contrast, Positional Cancellation implies the conclusion for R^1 and R^4, because there exists a sequence like (R^1, R^2, R^3, R^4) that begins with one and ends with the other. For two profiles to be comparable according to Positional Cancellation, there must

[3] This is analogous to the way (say, when defining *monotonic* social choice rules) we often express an increased support for x as an expansion of the lower-contour set of x.

[4] Relative Positions is a weakening of Arrow's IIA. It is equivalent to Saari's "Intensity IIA" (Saari, 1995, Definition 3.4.1).

exist a sequence like that. The idea is that a symmetrical treatment of two profiles is justified only if the change from one to the other produces identifiable *nontrivial marginal effects* that cancel each other. For this reason, Positional Cancellation is not sufficient for concluding that the social ranking of x and y is the same for R^5 as for R^6.[5] Combined with Reversal and Positive Responsiveness, however, the condition gives the conclusion.

I use Proposition 1 to prove (Lemma 4) that changes in individual preferences that preserve the differences (i.e., the sum of the individuals' relative positions) between the Borda scores of two alternatives do not alter the social preference between the two. Once this is proved, the proof of Theorem 1 is easy. What I actually do is to prove the theorem by way of a proposition (Proposition 2) that appears to be mathematically stronger.[6]

Pattanaik (2002) surveys the literature on the class of rules that include the scoring rules. Well-known characterizations of the Borda rule are obtained for variable sets of individuals: Young (1974) characterizes the Borda choice rule and Nitzan and Rubinstein (1981) characterize the Borda ranking rule. Their characterizations are sharply distinct from mine, because they include the "consistency" condition concerning the union of different sets of individuals. For a fixed set of individuals, Sato (2017) nicely complements the present paper. He characterizes the Borda *choice* rule by three conditions similar to mine, assuming *linearly ordered* preferences. In Section 4.2, I discuss the relation of these works with mine.

2 Framework

Let $N = \{1, \dots, n\}$ (where $n \geq 2$) denote a finite set of individuals. Let X be a finite set of (symbols labeling) alternatives, which has $\#X = m \geq 2$ alternatives. Let $\mathcal{R}$ be the set of *preferences*, i.e., complete and transitive binary relations on X. Let $\mathcal{P}$ be the set of linear orders on X (so $Q \in \mathcal{P}$ if and only if $Q \in \mathcal{R}$ and $[xQy$ and $yQx]$ implies $x = y$). A **profile** $R = (R_1, \dots, R_n)$ is an element of $\mathcal{R}^n$. When $R \in \mathcal{R}^n$, xR_iy means that "for individual i, x is at least as good as y." As usual, I write xP_iy ("i prefers x to y") if and only if xR_iy but not yR_ix. Also, xI_iy ("i is indifferent between x and y") if and only if xR_iy and yR_ix.

I represent each preference $Q \in \mathcal{R}$ by a string. For example, when

[5] Cancellations cannot occur with respect to (x, y): every individual can decrease the relative position of (x, y), but no individual can increase it. Considering transitivity of social preferences is not helpful, either. If the social preference for R^5 is $xyab$, Positional Cancellation implies that both x and y are preferred to a and b at R^6, but it does not determine the ranking of x and y.

[6] The proposition is closely related to the main theorem of a paper (Sato, 2017) that I noticed this month. The proof of the proposition is very similar to the one (which could have been shorter!) I gave to the Theorem 1, which was independently obtained.

$X = \{a, b, c, d, e, f, g\}$, (abusing the symbol) I write

$$Q = [abc][d][e][fg] = [abc]de[fg]$$

to express the preference that ranks a, b, and c (which are indifferent) first, d second, e third, f and g (which two are indifferent) last. More formally, an *indifference string* is a string $[x_1 \dots x_l]$ of (the symbols labeling) distinct alternatives in brackets, where $1 \leq l \leq m$. A **partial preference string** is a string $[x_1^1 \dots x_{l(1)}^1] \dots [x_1^k \dots x_{l(k)}^k]$ of indifference strings ($1 \leq l(1) + \cdots + l(k) \leq m$), in which each alternative $x \in X$ appears at most once. If each alternative $x \in X$ appears exactly once, then a partial preference string is called a (total) **preference string**. The partial preference string above represents any preference $Q \in \mathcal{R}$ satisfying $x_l^j Q x_{l'}^{j'}$ if and only if $j \geq j'$ for all l, l', j, j'. When an indifference string within a preference string consists of just one alternative, I write x instead of $[x]$.

A **ranking rule** is a mapping $G\colon \mathcal{R}^n \to \mathcal{R}$ that maps each profile $R = (R_1, \dots, R_n)$ to a *social preference* $G(R)$. Let $G^+(R)$ be the strict preference corresponding to $G(R)$: $xG^+(R)y$ if and only if $xG(R)y$ but not $yG(R)x$.

Given an individual $i \in N$, her preference $R_i \in \mathcal{R}$, and an alternative $x \in X$, let $b_i(x) = b(x, R_i) := \#\{a \in X : xP_i a\}$ be the *position* of x in R_i. Also, let

$$\delta(x, y) = \delta(x, y, R_i) := b(x, R_i) - b(y, R_i) = \#\{a : xP_i a\} - \#\{a : yP_i a\}$$

be the **position of** x **relative to** y. Note that for each x, y, R_i: (i) $\delta(y, x, R_i) = -\delta(x, y, R_i)$; (ii) if $xI_i y$, then $\delta(x, y, R_i) = 0$; (iii) if $xP_i y$, then $\delta(x, y, R_i)$ is the number of alternatives worse than x and at least as good as y. For example, if $X = \{a, b, c, d, e, f, g\}$ and $R_i = [abc]de[fg]$, then $\delta(a, b) = 0$, $\delta(a, d) = 1$, and $\delta(d, f) = 3$.

Lemma 1 *Let x, $y \in X$ be two distinct alternatives. Then, for any integer d, we have $d = \delta(x, y, Q)$ for some preference $Q \in \mathcal{R}$ if and only if d belongs to $[-m+1, m-1]$. That is,*

$$\{\delta(x, y, Q) : Q \in \mathcal{R}\} = \{d \in \mathbb{Z} : -m + 1 \leq d \leq m - 1\}.$$

Proof. For each k, where $0 \leq k \leq m - 2$, let u_k be a partial preference string consisting of k alternatives different from x and y, and v_k one consisting of the remaining $m - 2 - k$ alternatives. Given distinct x and y, the preference $Q = [xy]v_{m-2}$ gives $\delta(x, y, Q) = 0$; $Q = xu_k yv_k$ gives $\delta(x, y, Q) = k + 1 \in [1, m - 1]$; $Q = yu_k xv_k$ gives $\delta(x, y, Q) = -k - 1 \in [-m + 1, -1]$. ∎

When the positions $b(z, R_i)$ of alternatives $z \in X$ are viewed as "scores" that can be summed, we obtain the **Borda (ranking) rule**: it is a ranking rule $B\colon \mathcal{R}^n \to \mathcal{R}$ defined for each profile $R = (R_1, \dots, R_n)$ and for each pair $(x, y) \in X^2$ of alternatives by

$$xB(R)y \iff \sum_{i\in N} b(x, R_i) \geq \sum_{i\in N} b(y, R_i)$$

where $b(z, R_i) = \#\{a : zP_i a\}$ is the score received by $z \in X$ from i. The sum $\sum_{i\in N} b(z, R_i)$ is called the *Borda score* of z.

Observe that using the relative positions, one can rewrite the Borda rule as:

$$xB(R)y \iff \sum_{i\in N} \delta(x, y, R_i) \geq 0.$$

Observe also that the strict social preference $B^+(R)$ satisfies:

$$xB^+(R)y \iff \sum_{i\in N} \delta(x, y, R_i) > 0.$$

The Borda rule is a *scoring rule* (with $(s_0, s_1, \dots, s_{m-1}) = (0, 1, \dots, m-1)$) defined as follows:[7] A mapping $G\colon \mathcal{R}^n \to \mathcal{R}$ is a **scoring rule** if there is an m-tuple $(s_0, s_1, \dots, s_{m-1})$ of numbers (where $m = \#X$) such that $s_0 \leq s_1 \leq \dots \leq s_{m-1}$ with $s_0 < s_{m-1}$, and for all $R \in \mathcal{R}^n$ and x, $y \in X$,

$$xG(R)y \iff \sum_{i\in N} s_{b(x,R_i)} \geq \sum_{i\in N} s_{b(y,R_i)}.$$

To define the neutrality conditions, let $\pi\colon X \to X$ be a permutation on X. Given a profile $R \in \mathcal{R}^n$, denote by R^π the profile obtained from R by relabeling alternatives (or by reordering actual alternatives without relabeling them, under a different interpretation): for each i and each z, $w \in X$, $\pi(z)R_i^\pi\pi(w) \Leftrightarrow zR_iw$. If π is the *transposition* of z and w (so the other elements are fixed), I write $R^\pi = R^{zw}$. For example, if $R = abc[de]$, then $R^{ad} = dbc[ae]$. Note that for each x, y, R_i:

$$\delta(x, y, R_i^{xy}) = \delta(y, x, R_i) = -\delta(x, y, R_i).$$

I say that $G\colon \mathcal{R}^n \to \mathcal{R}$ is **Neutral** if for each $R \in \mathcal{R}^n$, for each permutation π, and for each (x, y), we have $xG(R)y \Leftrightarrow \pi(x)G(R^\pi)\pi(y)$. The Neutrality condition remains unaltered if "for each permutation π" in the definition is replaced by "for each transposition π of two alternatives z and w," since every permutation is a product of transpositions.[8] For the main result of the paper, the following weakening of Neutrality suffices:

[7] There are other definitions, but they usually agree on the subdomain $\mathcal{P}^n$ of linear orders.

[8] Unlike other variants, the Neutrality condition defined here is not a strengthening of Arrow's IIA (Independence of Irrelevant Alternatives) Arrow (1963), which I want to avoid in view of Arrow's theorem. It indeed is not, since the Borda rule satisfies this condition, but violates the IIA.

Definition 1 A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies **Reversal** if for each $R \in \mathcal{R}^n$ and for each $(x, y) \in X^2$, we have: $xG(R)y \Leftrightarrow yG(R^{xy})x$.

Lemma 2 *Every Neutral ranking rule satisfies Reversal. For each $m \geq 3$, there is a ranking rule that satisfies Reversal but violates Neutrality.*

The proof is in Appendix A.

To define positive responsiveness, Let R_j, $R'_j \in \mathcal{R}$ be preferences of $j \in N$ and x, $y \in X$ alternatives. I say that R'_j *has more support for* (x, y) *than* R_j if (i) $\{z : xP_jzR_jy\} \subseteq \{z : xP'_jzR'_jy\}$, (ii) $\{z : yP_jzR_jx\} \supseteq \{z : yP'_jzR'_jx\}$, and (iii) either $\subseteq$ in (i) is strict $\subsetneq$ or $\supseteq$ in (ii) is strict $\supsetneq$. I illustrate boundary cases for $X = \{x, y, a\}$: the preference xya has more support for (x, y) than yxa; xya than $[yx]a$; $[xy]a$ than yxa. Recall that $G^+(R')$ is the strict part of $G(R')$.

Definition 2 A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies **Positive Responsiveness** (or is *Positively Responsive*) if for each $(R, R') \in \mathcal{R}^n \times \mathcal{R}^n$ and for each $(x, y) \in X^2$, whenever for some j,

$$R'_j \text{ has more support for } (x, y) \text{ than } R_j$$
$$\text{and for all } i \neq j,\ R'_i = R_i,$$

we have

$$xG(R)y \Rightarrow xG^+(R')y.$$

Remark 1 If (a) R'_j has more support for (x, y) than R_j, then (b) $\delta(x, y, R'_j) > \delta(x, y, R_j)$. The converse is not true, however, as the following examples show: (i) $R_j = axbyc$ and $R'_j = bxacy$; (ii) $R_j = yabxc$ and $R'_j = ycxab$. One could replace (a) by (b) at the cost of making the Positive Responsiveness condition more demanding.

The following condition captures the distinctive characteristics of the Borda rule.

Definition 3 A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies **Positional Cancellation** if for each $(R, R') \in \mathcal{R}^n \times \mathcal{R}^n$ and for each $(x, y) \in X^2$, whenever for some j and l,

$$\delta(x, y, R'_j) = \delta(x, y, R_j) + 1,$$
$$\delta(x, y, R'_l) = \delta(x, y, R_l) - 1,$$
$$\text{for all } i \notin \{j, l\},\ R'_i = R_i,$$

we have

$$xG(R)y \Rightarrow xG(R')y.$$

By symmetry, "$\Rightarrow$" in the last line can be replaced by "$\Leftrightarrow$".

3 The Results

I first introduce two equivalent (Lemma 3) conditions, which are a consequence (Proposition 1) of Reversal, Positive Responsiveness, and Positional Cancellation. A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies **Relative Positions**, if for each $(R, R') \in \mathcal{R}^n \times \mathcal{R}^n$ and for each $(x, y) \in X^2$, whenever for all i, $\delta(x, y, R'_i) = \delta(x, y, R_i)$, we have $xG(R)y \Rightarrow xG(R')y$.[9] $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies **Relative Positions 2**, if for each $(R, R') \in \mathcal{R}^n \times \mathcal{R}^n$ and for each $(x, y) \in X^2$, whenever for some j and l (not necessarily distinct), $\delta(x, y, R'_j) = \delta(x, y, R_j)$ and $\delta(x, y, R'_l) = \delta(x, y, R_l)$, and for all $i \notin \{j, l\}$, $R'_i = R_i$, we have $xG(R)y \Rightarrow xG(R')y$.

Lemma 3 *A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies Relative Positions if and only if it satisfies Relative Positions 2.*

Proof. ($\Longrightarrow$). Obvious.

($\Longleftarrow$). Suppose that G satisfies Relative Positions 2. Suppose that $\delta(x, y, R'_i) = \delta(x, y, R_i)$ for all i and $xG(R)y$. We show that $xG(R')y$. We can assume $n \geq 3$ without loss of generality.

Let R^1, R^2, ..., R^{n-1} be the profiles defined by

$$
\begin{aligned}
R &= (R_1, R_2, R_3, R_4, R_5, \ldots, R_{n-1}, R_n)\\
R^1 &= (R'_1, R'_2, R_3, R_4, R_5, \ldots, R_{n-1}, R_n)\\
R^2 &= (R'_1, R'_2, R'_3, R_4, R_5, \ldots, R_{n-1}, R_n)\\
R^3 &= (R'_1, R'_2, R'_3, R'_4, R_5, \ldots, R_{n-1}, R_n)\\
&\cdots\\
R^{n-1} &= (R'_1, R'_2, R'_3, R'_4, R'_5, \ldots, R'_{n-1}, R'_n) = R'
\end{aligned}
$$

Since $xG(R)y$, Relative Positions 2 implies $xG(R^1)y$. Next, since $\delta(x, y, R^2_3) = \delta(x, y, R'_3) = \delta(x, y, R_3) = \delta(x, y, R^1_3)$, $R^2_j = R^1_j$ for all $j \neq 3$, and $xG(R^1)y$, Relative Positions 2 (for $j = l$) implies $xG(R^2)y$. Repeating the same argument, we obtain $xG(R^{n-1})y$; that is, $xG(R')y$. ∎

Proposition 1 *If a ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies Reversal, Positive Responsiveness, and Positional Cancellation, then it satisfies Relative Positions.*

Proof. Suppose that G satisfies the three conditions. By Lemma 3, it suffices to prove Relative Positions 2. Suppose $\delta(x, y, R'_j) = \delta(x, y, R_j)$, $\delta(x, y, R'_l) = \delta(x, y, R_l)$, for all $i \notin \{j, l\}$, $R'_i = R_i$, and $xG(R)y$. We show $xG(R')y$. There are three cases to consider.

[9] By symmetry, "$\Rightarrow$" can be replaced by "$\Leftrightarrow$".

Case 1:[10] $\delta(x, y, R_i) = m - 1$ *for all* i. We show $xG^+(R')y$. Suppose $yG(R')x$.

Claim. Let R^0 be any profile satisfying $\delta(x, y, R_i^0) = -m + 1$ for all i. Then $yG^+(R^0)x$.

To prove the Claim, suppose $xG(R^0)y$.

Given i, since $\delta(x, y, R'_i) = m - 1$, we can write $R'_i = xw[\dots y]$, where w is a partial preference string and $[\dots y]$ is an indifference string containing y (x is the only best element and y is one of the worst elements). Similarly, $R_i^0 = yw'[\dots x]$ for some w'. It is then easy to see that R'_i has more support for (x, y) than R_i^0.

Let R^1, R^2, ..., R^n be the profiles defined by

$$\begin{aligned} R^0 &= (R_1^0, R_2^0, R_3^0, \dots, R_{n-1}^0, R_n^0) \\ R^1 &:= (R'_1, R_2^0, R_3^0, \dots, R_{n-1}^0, R_n^0) \\ R^2 &:= (R'_1, R'_2, R_3^0, \dots, R_{n-1}^0, R_n^0) \\ &\dots \\ R^{n-1} &:= (R'_1, R'_2, R'_3, \dots, R'_{n-1}, R_n^0) \\ R^n &:= (R'_1, R'_2, R'_3, \dots, R'_{n-1}, R'_n) = R'. \end{aligned}$$

First, $R_1^1 = R'_1$ has more support for (x, y) than R_1^0; $R_i^1 = R_i^0$ for all $i \neq 1$. So $xG(R^0)y$ and Positive Responsiveness imply $xG^+(R^1)y$.

Second, $R_2^2 = R'_2$ has more support for (x, y) than $R_2^1 = R_2^0$; $R_i^2 = R_i^1$ for all $i \neq 2$. So $xG(R^1)y$ and Positive Responsiveness imply $xG^+(R^2)y$.

Repeating the same argument, we get $xG^+(R^n)y$; that is, $xG^+(R')y$. This contradicts the assumption (the first paragraph of Case 1) that $yG(R')x$. ‖

Let $R^0 := (R')^{xy}$ be obtained from R' by transposing x and y. Since $yG(R')x$, Reversal implies $xG(R^0)y$. On the other hand, since R^0 satisfies $\delta(x, y, R_i^0) = -m + 1$ for all i, the Claim implies $yG^+(R^0)x$. These conclusions contradict each other.

Case 2: $\delta(x, y, R_i) = -m + 1$ *for all* i. Case 1 implies that for (R', R) and for (y, x): if $\delta(y, x, R_i) = \delta(y, x, R'_i) = m - 1$ for all i, then $yG(R')x \Rightarrow yG^+(R)x$. Since the antecedent of this statement is true in this case, the consequent (whose contrapositive is "$xG(R)y \Rightarrow xG^+(R')y$") is true. The assumption $xG(R)y$ implies $xG^+(R')y$.

Case 3:[11] *Otherwise*. There are two cases:

Case 3.1: Either (a) $\delta(x, y, R_j) \neq m - 1$ & $\delta(x, y, R_l) \neq -m + 1$ or (b) $\delta(x, y, R_j) \neq -m + 1$ & $\delta(x, y, R_l) \neq m - 1$. Assume (a) without loss of

[10] Positional Cancellation is not used in this case.

[11] Neither Reversal nor Positive Responsiveness is used in this case.

generality. Using Lemma 1, choose any $R^1 \in \mathcal{R}^n$ satisfying that

$$\delta(x, y, R^1_j) = \delta(x, y, R_j) + 1$$
$$\delta(x, y, R^1_l) = \delta(x, y, R_l) - 1$$
$$\text{for all } i \notin \{j, l\},\ R^1_i = R_i.$$

Positional Cancellation, together with the assumption that $xG(R)y$ at the beginning of the proof, then implies $xG(R^1)y$. Other assumptions there and the choice of R^1 imply that

$$\delta(x, y, R'_j) = \delta(x, y, R_j) = \delta(x, y, R^1_j) - 1$$
$$\delta(x, y, R'_l) = \delta(x, y, R_l) = \delta(x, y, R^1_l) + 1$$
$$\text{for all } i \notin \{j, l\},\ R'_i = R_i = R^1_i.$$

Positional Cancellation, together with $xG(R^1)y$ obtained above, then implies $xG(R')y$. This is what we wanted to show.

Case 3.2: Otherwise, we have $n \geq 3$ and either $\delta(x, y, R_j) = \delta(x, y, R_l) = m - 1$ or $\delta(x, y, R_j) = \delta(x, y, R_l) = -m + 1$. Assume the first case (both are $m - 1$) without loss of generality. Since R does not belong to Case 1, there is a $k \notin \{j, l\}$ such that $\delta(x, y, R_k) \neq m - 1$. Choose such k. Using Lemma 1, we choose profiles R^1, R^2, $R^3 \in \mathcal{R}^n$ that are the same as R for the individuals other than the three (that is, $R^1_i = R^2_i = R^3_i = R_i$ for all $i \notin \{j, k, l\}$) as follows:

1. Choose $R^1 \in \mathcal{R}^n$ satisfying $\delta(x, y, R^1_j) = \delta(x, y, R_j) - 1$, $\delta(x, y, R^1_k) = \delta(x, y, R_k) + 1$, and $R^1_l = R_l$. Positional Cancellation, together with the assumption that $xG(R)y$, implies $xG(R^1)y$.

2. Choose $R^2 \in \mathcal{R}^n$ satisfying $R^2_j = R'_j$, $R^2_k = R^1_k$, and $\delta(x, y, R^2_l) = \delta(x, y, R^1_l) - 1$. Then $\delta(x, y, R^2_j) = \delta(x, y, R'_j) = \delta(x, y, R_j) = \delta(x, y, R^1_j) + 1$. Positional Cancellation, together with $xG(R^1)y$, implies $xG(R^2)y$.

3. Choose (uniquely defined) $R^3 \in \mathcal{R}^n$ satisfying $R^3_j = R^2_j$, $R^3_k = R_k$, and $R^3_l = R'_l$. Then $\delta(x, y, R^3_k) = \delta(x, y, R_k) = \delta(x, y, R^1_k) - 1 = \delta(x, y, R^2_k) - 1$ and $\delta(x, y, R^3_l) = \delta(x, y, R'_l) = \delta(x, y, R_l) = \delta(x, y, R^1_l) = \delta(x, y, R^2_l) + 1$. Positional Cancellation, together with $xG(R^2)y$, implies $xG(R^3)y$.

Since $R^3_j = R^2_j = R'_j$, $R^3_l = R'_l$, and $R^3_i = R_i = R'_i$ for all $i \notin \{j, l\}$ (the last equation coming from the assumption at the beginning of the proof), we have $R^3 = R'$. Therefore $xG(R^3)y$ implies $xG(R')y$. ∎

Lemma 4 *Let $G\colon \mathcal{R}^n \to \mathcal{R}$ be a ranking rule that satisfies Positional Cancellation and Relative Positions. Let $(R, R') \in \mathcal{R}^n \times \mathcal{R}^n$, $(x, y) \in X^2$, and $b_1, \ldots, b_n \in \mathbb{Z}$ satisfy the following equations:*

$$\delta(x, y, R'_1) = \delta(x, y, R_1) + b_1$$
$$\delta(x, y, R'_2) = \delta(x, y, R_2) + b_2$$
$$\cdots$$
$$\delta(x, y, R'_n) = \delta(x, y, R_n) + b_n$$
$$\sum_{i \in N} b_i = 0$$

(these equations are satisfied only if for each i, $-2m + 2 \leq b_i \leq 2m - 2$). Then

$$xG(R)y \Rightarrow xG(R')y.$$

Proof. The proof is by induction on the number $k := \sum_{i:b_i \geq 0} b_i \geq 0$.

If $k = 0$, then $b_i = 0$ for all i such that $b_i \geq 0$. Since $\sum b_i = k + \sum_{i:b_i<0} b_i = 0$ by assumption, $k = 0$ implies $b_i = 0$ for all i such that $b_i < 0$. It follows that for all i, $\delta(x, y, R'_i) = \delta(x, y, R_i)$. Relative Positions thus implies $xG(R)y \Rightarrow xG(R')y$.

Suppose the induction hypothesis: the assertion is true for $k = \bar{k}$. Suppose that $\sum_{i:b_i \geq 0} b_i = \bar{k} + 1$. Choose j such that $b_j > 0$. Since $\sum b_i = 0$, there is an l such that $b_l < 0$. Choose such l. Using Lemma 1, choose $R'' \in \mathcal{R}^n$ satisfying[12]

$$\delta(x, y, R''_j) = \delta(x, y, R_j) + b_j - 1$$
$$\delta(x, y, R''_l) = \delta(x, y, R_l) + b_l + 1$$
$$\text{for all } i \notin \{j, l\},\ R''_i = R'_i.$$

For $i \notin \{j, l\}$, observe that

$$\delta(x, y, R''_i) = \delta(x, y, R'_i) = \delta(x, y, R_i) + b_i$$

and let $c_i := b_i$. Let $c_j := b_j - 1 \geq 0$ and $c_l := b_l + 1 \leq 0$. Then $\sum c_i = 0$ and

$$\sum_{i:c_i \geq 0} c_i = (b_j - 1) + (\sum_{i:b_i \geq 0} b_i - b_j) = -1 + (\bar{k} + 1) = \bar{k}.$$

By the induction hypothesis, $xG(R)y \Rightarrow xG(R'')y$. Now, we have

$$\delta(x, y, R'_j) = \delta(x, y, R''_j) + 1$$
$$\delta(x, y, R'_l) = \delta(x, y, R''_l) - 1$$
$$\text{for all } i \notin \{j, l\},\ R'_i = R''_i.$$

[12] Such an R''_j exists since $-m + 1 \leq \delta(x, y, R_j) \leq \delta(x, y, R''_j) < \delta(x, y, R'_j) \leq m - 1$.

Positional Cancellation implies $xG(R'')y \Rightarrow xG(R')y$; hence $xG(R)y \Rightarrow xG(R')y$. ∎

Theorem 1 *A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies Reversal, Positive Responsiveness, and Pairwise Cancellation if and only if G is the Borda rule $B\colon \mathcal{R}^n \to \mathcal{R}$.*

Theorem 1, together with Lemma 2, implies the following:

Corollary 1 *A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies Neutrality, Positive Responsiveness, and Pairwise Cancellation if and only if G is the Borda rule B.*

The "$\Longleftarrow$" direction of Theorem 1 is immediate from the observation. The "$\Longrightarrow$" direction will be proved via Proposition 2 below. While the conditions in the proposition are eclectic, they do make the proposition easily comparable with a certain result in the literature (Section 4.2).

A ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies **PR1** if for each $(R, R') \in \mathcal{R}^n \times \mathcal{R}^n$ and for each $(x, y) \in X^2$, whenever for some j and for some $z \in X$,[13]

$$R'_j \text{ equals } R_j \text{ on } X \setminus \{z\},$$
$$zI_jx,$$
$$\text{for all } \xi \text{ such that } xP_j\xi\text{: } [z \neq y \;\&\; xP'_jzR'_j\xi] \text{ or } [z = y \;\&\; xP'_jzP'_j\xi],$$
$$\text{for all } i \neq j,\; R'_i = R_i,$$

we have $xG(R)y \Rightarrow xG^+(R')y$.

Let $R^{xy} \in \mathcal{R}^n$ be the profile obtained from R by transposing x and y. $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies **IARPPT** (Invariance under Average Relative Position Preserving Transpositions) if for each $R \in \mathcal{R}^n$ and for each $(x, y) \in X^2$, whenever $\sum_{i\in N} \delta(x, y, R_i^{xy}) = \sum_{i\in N} \delta(x, y, R_i)$, we have $xG(R)y \Rightarrow xG(R^{xy})y$.

Note that If G satisfies Positive Responsiveness, then it satisfies PR1. Also, if G satisfies the three conditions in Theorem 1, then it satisfies Relative Positions by Proposition 1; hence it satisfies IARPPT by Lemma 4 for $R' = R^{xy}$. It follows that Theorem 1 is immediate from the following:

Proposition 2 *If a ranking rule $G\colon \mathcal{R}^n \to \mathcal{R}$ satisfies Reversal (or Neutrality), PR1, and IARPPT, then G is the Borda rule B.*

[13] By "R'_j equals R_j on $X \setminus \{z\}$," I mean that for all x', $y' \in X \setminus \{z\}$, $x'R'_jy' \Leftrightarrow x'R_jy'$. In the first three lines below, z that is as good as x is pushed right below x (z is included in or placed above the indifference class just below that of x). Note that if these lines are satisfied, then R'_j has more support for (x, y) than R_j.

Proof. Suppose that G satisfies the three conditions. We show that[14]

(i) $\sum \delta(x, y, R_i) = 0 \Rightarrow xG(R)y$ & $yG(R)x$,

(ii) $\sum \delta(x, y, R_i) > 0 \Rightarrow xG^+(R)y$ and,

(iii) $\sum \delta(x, y, R_i) < 0 \Rightarrow yG^+(R)x$.

Case (i). Suppose that $\sum \delta(x, y, R_i) = 0$. To derive a contradiction, suppose $xG^+(R)y$ without loss of generality. By Reversal, $yG^+(R^{xy})x$.

Since $\delta(x, y, R_i^{xy}) = -\delta(x, y, R_i)$ for each i, we have $\sum \delta(x, y, R_i^{xy}) = -\sum \delta(x, y, R_i)$. Since the right hand side is zero by assumption, $\sum \delta(x, y, R_i^{xy}) = 0$. It follows that

$$\sum \delta(x, y, R_i^{xy}) = \sum \delta(x, y, R_i).$$

IARPPT, together with the assumption that $xG^+(R)y$, implies that $xG(R^{xy})y$. This contradicts the conclusion $yG^+(R^{xy})x$ in the first paragraph discussing this case.

Case (ii). Suppose that $\sum \delta(x, y, R_i) > 0$. We prove that $xG^+(R)y$ by induction on the number $k := \sum \delta(x, y, R_i)$ from 1 to $n(m-1)$.

First, suppose $k = 1$. Then there is a j such that $\delta(x, y, R_j) > 0$. Choose such j. Let $R'_j \in \mathcal{R}$ be obtained from R_j by moving a certain z satisfying xP_jzR_jy so that j is indifferent between x and z: if there is a $z \neq y$ satisfying the condition, choose any such z from the indifference class right below x (e.g., if $R_j = a[bx][cd][ye]f$, then $R'_j = a[bxc]d[ye]f$ and $R'_j = a[bxd]c[ye]f$ are obtained this way); otherwise let $z = y$ (e.g., if $R_j = a[bx]yd$, then $R'_j = a[bxy]d$). Let $R'_i = R_i$ for all $i \neq j$. Observe that $\delta(x, y, R'_j) = \delta(x, y, R_j) - 1$ and that (R', R) satisfies the assumptions in PR1. Then, since $\sum \delta(x, y, R'_i) = \sum \delta(x, y, R_i) - 1 = k - 1 = 0$, Case (i) implies $xG(R')y$ (as well as $yG(R')x$). PR1 then implies $xG^+(R)y$.

Suppose the induction hypothesis: the assertion is true for $k = \bar{k}$. Suppose that $\sum \delta(x, y, R_i) = \bar{k} + 1$. Choose j and $R' \in \mathcal{R}^n$ as in the case $k = 1$. Observe that $\delta(x, y, R'_j) = \delta(x, y, R_j) - 1$, that (R', R) satisfies the assumptions in PR1, and that $R_i = R'_i$ for all $i \neq j$. Then, since $\sum \delta(x, y, R'_i) = \sum \delta(x, y, R_i) - 1 = (\bar{k} + 1) - 1 = \bar{k}$, the induction hypothesis implies $xG^+(R')y$. PR1 then implies $xG^+(R)y$.

Case (iii). Suppose that $\sum \delta(x, y, R_i) < 0$. We prove that $yG^+(R)x$. Since $\delta(x, y, R_i^{xy}) = -\delta(x, y, R_i)$ for each i, we have $\sum \delta(x, y, R_i^{xy}) = -\sum \delta(x, y, R_i) > 0$. Case (ii) implies $xG^+(R^{xy})y$. By Reversal, $yG^+(R)x$. ∎

[14] It is important to note that the proof does not depend on any consequences (such as Proposition 1 or Lemma 4) of Positive Responsiveness or Positional Cancellation. If one can use the conclusion of Lemma 4, the proof of Case (ii) is easier: choose any R' satisfying $\sum \delta(x, y, R'_i) = \sum \delta(x, y, R_i) > 0$ and $\delta(x, y, R'_i) \geq 0$ for all i; use PR1 to compare R' with R^0 where every individual is indifferent between any two alternatives.

4 Discussion

4.1 Minimality of the set of axioms

The following examples show that none of the three axioms in Theorem 1 is redundant. I give examples for each set N of $n \geq 2$ individuals and for each set $\{a_1, \ldots, a_m\}$ of $m \geq 3$ of alternatives (with the exception of the second example in Example 3, where $m \geq 4$ is assumed).

Example 1 Ranking rules that *violate Reversal* but satisfies the other two. The first example is the constant rule that always gives the linear order $a_1 \ldots a_m$ as an output. The second example is the ranking rule G that equals the Borda rule except that it regards the alternative labeled a_m as a "forbidden choice." It is defined as follows: $xG^+(R)a_m$ for all x different from a_m; for any pair (x, y) of alternatives different from a_m, we have $xG(R)y \Leftrightarrow xB(R)y$.

Example 2 Ranking rules that *violate Positive Responsiveness* but satisfies the other two. The first example is the constant rule that always gives the total indifference $[a_1a_2 \ldots a_m]$. The second example is the inverse G of the Borda rule: $xG(R)y \Leftrightarrow yB(R)x$.

Example 3 Ranking rules that *violate Positional Cancellation* but satisfies the other two.

The first example is the non-anonymous rule that gives twice as much weight to individual 1 as to the other individuals. Recall that $b(x, R_i) = \#\{a : xP_ia\}$ and let $b_i(x, R_i) := 2b(x, R_i)$ if $i = 1$ and $b_i(x, R_i) := b(x, R_i)$ if $i \neq 1$. Define G by $xG(R)y \Leftrightarrow \sum_i b_i(x, R_i) \geq \sum_i b_i(y, R_i)$. Let $\delta_i(x, y, R_i) := b_i(x, R_i) - b_i(y, R_i)$. Then $xG(R)y \Leftrightarrow \sum_i \delta_i(x, y, R_i) \geq 0$. Reversal can be seen from the equation $\delta_i(x, y, R_i) = -\delta_i(x, y, R_i^{xy})$ for all i. To see Positive Responsiveness, consider the cases $j = 1$ and $j \neq 1$, noting that $\delta(x, y, R'_j) > \delta(x, y, R_j)$ implies $\delta_j(x, y, R'_j) > \delta_j(x, y, R_j)$. To see that Positional Cancellation is violated, let $R = (xayw, yaxw, yaxw, [xy]aw, \ldots, [xy]aw)$, where a is another alternative and w is a string consisting of the remaining alternatives. This gives $xG(R)y$ (and $yG(R)x$). Let $R' = (xyaw, yxaw, yaxw, [xy]aw, \ldots, [xy]aw)$. R and R' satisfy the hypothesis of Positional Cancellation. However, $yG^+(R')x$.

The second example is the scoring rule (which is an anonymous rule) with $(s_0, \ldots, s_{m-2}, s_{m-1}) = (0, \ldots, m-2, m-1/2)$. Suppose $m \geq 4$ and let $\tilde{b}(x, R_i) := m - 1/2$ if $b(x, R_i) = m - 1$ and $\tilde{b}(x, R_i) := b(x, R_i)$ otherwise. Define G by $xG(R)y \Leftrightarrow \sum_i \tilde{b}(x, R_i) \geq \sum_i \tilde{b}(y, R_i)$. Let $\tilde{\delta}(x, y, R_i) := \tilde{b}(x, R_i) - \tilde{b}(y, R_i)$, which is at most $\delta(x, y, R_i) + 1/2$ (if $\max R_i = \{x\}$ or $\{y\}$) and at least $\delta(x, y, R_i)$ (otherwise). Then $xG(R)y \Leftrightarrow \sum_i \tilde{\delta}(x, y, R_i) \geq 0$. Reversal can be seen from the equation $\tilde{\delta}(x, y, R_i) = -\tilde{\delta}(x, y, R_i^{xy})$ for all i. To see Positive Responsiveness, suppose the hypothesis. Then $\delta(x, y, R'_j) >$

$\delta(x, y, R_j)$ (implying $\delta(x, y, R'_j) \geq \delta(x, y, R_j) + 1$ since both sides are integers) and $\tilde{\delta}(x, y, R'_i) = \tilde{\delta}(x, y, R_i)$ for all $i \neq j$. Now $xG(R)y$ implies $\sum_i \tilde{\delta}(x, y, R_i) \geq 0$, which implies

$$-\sum_{i \neq j} \tilde{\delta}(x, y, R_i) \leq \tilde{\delta}(x, y, R_j). \tag{1}$$

The left-hand side of this inequality is equal to $-\sum_{i \neq j} \tilde{\delta}(x, y, R'_i)$. For the right-hand side, $\tilde{\delta}(x, y, R_j) \leq \delta(x, y, R_j) + 1/2 < \delta(x, y, R_j) + 1 \leq \delta(x, y, R'_j) \leq \tilde{\delta}(x, y, R'_j)$. Therefore, inequality (1) implies $\sum_i \tilde{\delta}(x, y, R'_i) > 0$; hence $xG^+(R')y$. To see that Positional Cancellation is violated, let $R = (ayxbw, abxyw, [xy]abw, \ldots, [xy]abw)$, where a and b are other alternatives and w is a string consisting of the remaining alternatives. This gives $xG(R)y$ (and $yG(R)x$). Let $R' = (yaxbw, axbyw, [xy]w', \ldots, [xy]w')$. R and R' satisfy the hypothesis of Positional Cancellation. However, $yG^+(R')x$.

4.2 Connection with results in the literature

[In a revised version, I am thinking of discussing the relationship between this work and related results, such as May (1952); Young (1974); Nitzan and Rubinstein (1981); Sato (2017).]

A The Proof of Lemma 2

Let $N = \{1, \ldots, n\}$ and $X = \{a_1, \ldots, a_m\}$, where $m \geq 3$. Let $\psi\colon \mathcal{R} \to \mathcal{R}$ be a function satisfying

$$\psi(Q) = \begin{cases} a_2 a_1 w & \text{if } Q \text{ is a linear order } a_1 a_2 w \text{ for some } w, \\ a_1 a_2 w & \text{if } Q \text{ is a linear order } a_2 a_1 w \text{ for some } w, \\ Q & \text{otherwise,} \end{cases}$$

where w denotes a (partial preference) *string* consisting of $a_3, \ldots, a_m$. Let $\mathbf{Fix} := \{Q : \psi(Q) = Q\}$ be the set of fixed points of ψ: it consists of the preferences belonging to the third case in the definition of ψ. If Q is a linear order, $Q \in \mathbf{Fix}$ iff either a_1 or a_2 ranks third or worse. *We show that the composition $\psi \circ B\colon \mathcal{R}^n \to \mathcal{R}$ of ψ and the Borda rule B is a non-Neutral ranking rule satisfying Reversal.* (In fact, instead of the Borda rule, we may use any Neutral ranking rule whose image contains a linear order.) Clearly, it is a ranking rule.

To show that $\psi \circ B$ satisfies Reversal, let $R \in \mathcal{R}^n$ and $x, y \in X$ be distinct alternatives. We want to show that

$$x(\psi \circ B)(R)y \iff y(\psi \circ B)(R^{xy})x. \tag{2}$$

Along the way, we will also show that $\psi \circ B$ is not Nuetral. Since B is Neutral, we have

$$xB(R)y \iff yB(R^{xy})x. \tag{3}$$

First, suppose that $B(R)$ is not a linear order. Then, since B is Neutral, $B(R^{xy})$ is not a linear order, either. (If z and w are indifferent at $B(R)$, then for any permutation π, $\pi(z)$ and $\pi(w)$ are indifferent at $B(R^{\pi})$.) The definition of ψ then implies $B(R)$, $B(R^{xy}) \in \mathbf{Fix}$; therefore,

$$B(R) = (\psi \circ B)(R) \text{ and } B(R^{xy}) = (\psi \circ B)(R^{xy}). \tag{4}$$

The conclusion (2) then follows from (3).

Next, suppose that the Borda ranking $B(R)$ is a linear order. Since B is Neutral, $B(R^{xy})$ is a linear order as well. Write R^{kl} for the profile $R^{a_k a_l}$ obtained from R by transposing a_k and a_l. There are four cases to consider.

Case 1: $(x, y) = (a_1, a_2)$ or (a_2, a_1). Suppose first that for some string w,

$$B(R) = a_1 a_2 w, \text{ implying } B(R^{12}) = a_2 a_1 w$$

since B is Neutral. The definition of ψ implies that

$$(\psi \circ B)(R) = a_2 a_1 w \text{ and } (\psi \circ B)(R^{12}) = a_1 a_2 w.$$

The equivalence (2) follows since both sides are false if $(x, y) = (a_1, a_2)$ and true if $(x, y) = (a_2, a_1)$ (note that $R^{xy} = R^{yx}$). The case where $B(R) = a_2 a_1 w$ for some string w is similar. In other cases, $B(R)$ belongs to **Fix** (i.e., a_1 or a_2 is third or worse). Since B is Neutral, $B(R^{12})$ also belongs to **Fix**. We therefore have (4). The conclusion (2) then follows from (3).

Case 2: $(x, y) = (a_1, a_k)$ or (a_k, a_1) for some $k \notin \{1, 2\}$. Suppose first that $B(R) = a_1 a_2 w$. As before, $B(R^{1k}) = a_k a_2 w'$, where w' is obtained from w by replacing a_k by a_1. Also, $(\psi \circ B)(R) = a_2 a_1 w$ and $(\psi \circ B)(R^{1k}) = a_k a_2 w'$; hence the equivalence (2) follows. Incidentally, the existence of profiles belonging to this case (the profile where every individual has the same preference $a_1 a_2 w$ is an example) exhibit that $\psi \circ B$ *is not Neutral*: if it were, then $(\psi \circ B)(R) = a_2 a_1 w$ would imply $(\psi \circ B)(R^{1k}) = a_2 a_k w'$. The cases $B(R) = a_2 a_1 w$, $a_2 a_k w$, $a_k a_2 w$ are similar. In other cases, both $B(R)$ and $B(R^{1k}) \in \mathbf{Fix}$ (if the latter were not true, then $B(R^{1k})$ could be written $a_1 a_2 w$ or $a_2 a_1 w$ for some w; this should lead to a contradiction). The rest of the proof runs as before.

Case 3: $(x, y) = (a_2, a_k)$ or (a_k, a_2) for some $k \notin \{1, 2\}$. Suppose first that $B(R) = a_1 a_2 w$. As before, $B(R^{2k}) = a_1 a_k w'$, where w' is obtained from w by replacing a_k by a_2. Also, $(\psi \circ B)(R) = a_2 a_1 w$ and $(\psi \circ B)(R^{2k}) = a_1 a_k w'$; hence the equivalence (2) follows. The cases $B(R) = a_2 a_1 w$, $a_1 a_k w$, $a_k a_1 w$ are similar. In other cases, both $B(R)$ and $B(R^{2k}) \in \mathbf{Fix}$. The rest of the proof runs as before.

Case 4: $(x, y) = (a_k, a_l)$ or (a_l, a_k) for some k, $l \notin \{1, 2\}$. Suppose first that $B(R) = a_1a_2w$. As before, $B(R^{kl}) = a_1a_2w'$, where w' is obtained from w by transposing a_k and a_l. Also, $(\psi \circ B)(R) = a_2a_1w$ and $(\psi \circ B)(R^{kl}) = a_2a_1w'$; hence the equivalence (2) follows. The case $B(R) = a_2a_1w$ is similar. In other cases, both $B(R)$ and $B(R^{kl}) \in \mathbf{Fix}$. The rest of the proof runs as before.

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