On Top-connected Single-peaked and Partially Single-peaked Domains

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Abstract

We characterize all domains for which the set of unanimous and strategy-proof social choice functions coincides with the set of min-max rules. As an application of our result, we obtain a characterization of unanimous and strategy-proof social choice functions on maximal single-peaked domains (Moulin (1980), Weymark (2011)), minimally rich single-peaked domains (Peters et al. (2014)), maximal regular single-crossing domain (Saporiti (2009)), and distance based single-peaked domains. We further consider domains that exhibit single-peaked property only over a subset of alternatives. We call such domains top-connected partially single-peaked domains. We characterize the unanimous and strategy-proof social choice functions on such domains. As an application of this result, we obtain a characterization of the unanimous and strategy-proof social choice functions on multiple single-peaked domains (Reffgen (2015)), single-peaked domains on graphs, and several other domains of practical significance.

KEYWORDS: Strategy-proofness, min-max rules, min-max domains, single-peaked preferences, top-connectedness property, partially single-peaked preferences, partly dictatorial generalized median voter schemes.

JEL CLASSIFICATION CODES: D71, D82.

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1. Introduction

We consider a standard social choice problem where an alternative has to be chosen based on privately known preferences of the agents in a society. Such a procedure is known as *social choice function* (SCF). Agents are strategic in the sense that they misreport their preferences whenever it is strictly beneficial for them. An SCF is called *strategy-proof* if no agent can benefit by misreporting her preferences. Moreover, an SCF is called *unanimous* if whenever all the agents in the society agree on their best alternative, that alternative is chosen.

Most of the subject matter of social choice theory concerns the study of unanimous and strategy-proof SCFs for different admissible domains of preferences. In the seminal works by Gibbard (1973) and Satterthwaite (1975), it is shown that in a society with at least three alternatives, if the admissible domain of preferences for each individual is unrestricted, then every unanimous and strategy-proof SCF is *dictatorial*. An SCF is dictatorial if a particular individual in the society determines the outcome regardless of the preferences of others. The celebrated Gibbard-Satterthwaite theorem hinges crucially on the assumption that the admissible domain of each individual is unrestricted. However, it is well established that there are natural restrictions on such domains in many economic and political applications. For instance, the problem of locating a firm in a unidimensional spatial market (Hotelling (1929)), setting the rate of carbon dioxide emissions (Black (1948)), setting the level of public expenditure (Romer and Rosenthal (1979)), and so on admit naturally restricted preferences widely known as *single-peaked* preferences. The crucial property of a single-peaked preference is that there is a prior order over the alternatives such that the preference decreases as one moves away (with respect to the prior order) from her best alternative.

The study of single-peaked domains dates back to Black (1948). Moulin (1980) and Weymark (2011) have characterized the unanimous and strategy-proof SCFs defined over such a domain as *min-max rules*.[1,2] The characterization by Moulin (1980) and Weymark (2011) rests upon the crucial assumption that the domain is the *maximal* single-peaked domain, i.e., it contains all the single-peaked preferences with respect to a given prior order over the alternatives. However, demanding the existence of all single-peaked preferences is a strong prerequisite in many practical

[1] Barberà et al. (1993) and Ching (1997) provide equivalent presentations of this class of SCFs.

[2] A rich literature has developed around the single-peaked restriction by considering various generalizations and extensions (see Barberà et al. (1993), Demange (1982), Schummer and Vohra (2002), Nehring and Puppe (2007a), and Nehring and Puppe (2007b)).

situations and hence, it is justified to impose further restrictions on the maximal single-peaked domain.[3]

In continuity with the above discussion, we characterize the minimal subsets of a single-peaked domain so that every unanimous and strategy-proof SCF on it is a min-max rule. Such a domain contains $2m-2$ preferences whereas a maximal single-peaked domain contains 2^{m-1} preferences, where m is the number of alternatives. We also characterize the domains for which the set of unanimous and strategy-proof SCFs coincides with the set of min-max rules. We call such a domain a *min-max domain*. We show that a domain is a min-max domain if all the preferences in it are single-peaked and it satisfies the *top-connected* property. Top-connectedness property with respect to a prior order requires that for every two alternatives x and y that are adjacent (consecutive) in that prior order, there exists a preference that places x at the top and y at the second-ranked position.[4]

A regular single-crossing domain (Saporiti (2009)) is an example of a top-connected single-peaked domain.[5,6] Saporiti (2009) shows that an SCF is unanimous, anonymous, and strategy-proof on a *maximal single-crossing domain* if and only if it is a median rule. We extend Saporiti (2009)'s result by relaxing the anonymity assumption on the SCF and the maximality assumption on the domain. However, we assume the domains to be regular. In particular, we show that an SCF is unanimous and strategy-proof on a regular single-crossing domain satisfying top-connectedness property if and only if it is a min-max rule. Note that a maximal single-crossing domain requires $m(m-1)/2$ preferences, whereas a regular single-crossing domain satisfying top-connectedness property requires $2m-2$ preferences.

Although single-peaked domains are used to model many practical situations, several empirical studies (Niemi and Wright (1987), Feld and Grofman (1988), Pappi and Eckstein (1998)) fail to support the assumption that *all* the preferences of an agent are single-peaked. In view of this, we

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[3]Further examples of such restricted single-peaked domains include the preference restriction considered in models of voting (Tullock (1967), Arrow (1969)), taxation and redistribution (Epple and Romer (1991)), determining the levels of income redistribution (Hamada (1973), Slesnick (1988)), and measuring tax reforms in the presence of horizontal inequity (Hettich (1979)) and recently, Puppe (2015) shows that under mild conditions these domains form subsets of maximal single-peaked domains.

[4]The top-connectedness property is well studied in the literature (see Barberà and Peleg (1990), Aswal et al. (2003), Chatterji and Sen (2011), Chatterji et al. (2014), Chatterji and Zeng (2015), and Puppe (2015)).

[5]A domain is *regular* if every alternative appears as a top in some preference in the domain.

[6]Single-crossing domains appear in models of taxation and redistribution (Roberts (1977), Meltzer and Richard (1981)), local public goods and stratification (Westhoff (1977), Epple and Platt (1998), Epple et al. (2001)), coalition formation (Demange (1994), Kung (2006)), selecting constitutional and voting rules (Barberà and Jackson (2004)), and designing policies in the market for higher education (Epple et al. (2006)).

consider domains which satisfy single-peakedness only for a strict subset of alternatives. We call such domains *partially single-peaked domains*. We characterize the unanimous and strategy-proof SCFs on a class of partially single-peaked domains as *partly dictatorial generalized median voter schemes* (PDGMVS). Loosely put, a PDGMVS acts like a min-max rule over the subset of the domain where single-peakedness is satisfied and acts like a dictatorial rule everywhere else.

Our result generalizes those in Reffgen (2015). He introduces the notion of *multiple single-peaked domains* and characterizes the unanimous and strategy-proof SCFs on such domains. A multiple single-peaked domain is a union of several maximal single-peaked domains with respect to different prior orders over the alternatives. A plausible justification for such a domain restriction is provided by Niemi (1969) who argues that alternatives can be ordered differently using different criteria (which he calls an *impartial culture*) and it is not publicly known which individual uses what criterion. On one extreme, such a domain becomes an unrestricted domain if there is no consensus among the individuals on the prior order over the alternatives, and on the other extreme, it becomes a maximal single-peaked domain if all the individuals agree on a single prior order over the alternatives. We extend Reffgen (2015)'s result in two directions: (i) by requiring minimum knowledge about the prior orders over the alternatives as perceived by individuals and (ii) by requiring a minimal set of single-peaked preferences for each of these prior orders. We further show that this class of domains contains almost all domains for which the set of unanimous and strategy-proof SCFs coincides with the set of PDGMVS. Note that a multiple single-peaked domain with respect to k prior orders over the alternatives requires approximately (depending on the prior orders) $k \times 2^{m-1}$ preferences. On the contrary, the partially single-peaked domains that we consider require only $2m$ preferences.

Lastly, we consider group strategy-proofness. Barberà et al. (2010) provides a sufficient condition for the equivalence of strategy-proofness and group strategy-proofness on a domain. Top-connected single-peaked domains satisfy their condition, and hence we obtain a characterization of unanimous and group strategy-proof SCFs on these domains as a corollary of their result. However, our partially single-peaked domains do not satisfy their condition and consequently, we show that strategy-proofness and group strategy-proofness are equivalent on these domains. In particular, we provide a characterization of unanimous and group strategy-proof SCFs on such domains.

We conclude this section by relating our paper to the existing literature. For every *given* min-max rule, Serizawa (1995) and Barberà et al. (1999) characterize maximal domains under which it is strategy-proof. However, we characterize domains where *all* min-max rules are strategy-proof. Arribillaga and Massó (2016) provide necessary and sufficient conditions for the comparability of two min-max rules in terms of their vulnerability to manipulation. Our results identify the set of all min-max rules that remain strategy-proof on a domain that violate single-peakedness around the middle of the linear order. Chatterji et al. (2013) show that if a domain admits an anonymous (and hence non-dictatorial), tops-only, unanimous, and strategy-proof SCF, then it is a semi-single-peaked domain. Semi-single-peaked domains violate single-peakedness around the tails of the linear order. However, our results show that if the single-peakedness is violated around the middle of the linear order, then there is *no* unanimous, strategy-proof, and anonymous SCF. Thus, our results complement that in Chatterji et al. (2013).

The rest of the paper is organized as follows. We describe the usual social choice framework in Section 2. In Section 3, we study the unanimous and strategy-proof SCFs on the top-connected single-peaked domains. Section 4 studies the unanimous and strategy-proof SCFs on a class of partially single-peaked domains. Section 5 deals with group strategy-proofness and the last section concludes the paper. All the omitted proofs are collected in Appendix A and Appendix B.

2. Preliminaries

Let $N = \{1, ..., n\}$ be a set of at least two agents, who collectively choose an element from a finite set $X = \{a, a+1, \ldots, b-1, b\}$ of at least three alternatives, where a is an integer. For $x, y \in X$ such that $x \leq y$, we define the intervals $[x, y] = \{z \in X \mid x \leq z \leq y\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. A preference P over X is a complete, transitive, and asymmetric binary relation (also called a linear order) defined on X. We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} ranked alternative in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k-1$. By $\mathcal{D} \subseteq \mathbb{L}(X)$, we denote a domain of admissible preferences. An element $P_N = (P_1, \ldots, P_n) \in \mathcal{D}^n$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \{x \in X \mid r_1(P_i) = x \text{ for some } i \in N\}$. A domain $\mathcal{D}$ of preferences is *regular* if for all $x \in X$, there exists a preference $P \in \mathcal{D}$ such that $r_1(P) = x$. All the domains we consider in this paper are assumed to be regular. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

Definition 2.1. A *social choice function* (SCF) f on $\mathcal{D}^n$ is a mapping $f : \mathcal{D}^n \to X$.

Definition 2.2. An SCF $f : \mathcal{D}^n \to X$ is *unanimous* if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.

Definition 2.3. An SCF $f : \mathcal{D}^n \to X$ is *manipulable* if there exists an agent $i \in N$, a preference profile $P_N \in \mathcal{D}^n$, and a preference $P_i' \in \mathcal{D}$ such that $f(P_i', P_{N\setminus i}) P_i f(P_N)$. An SCF f is *strategy-proof* if it is not manipulable.

Definition 2.4. An SCF $f : \mathcal{D}^n \to X$ is called *dictatorial* if there exists an agent $i \in N$ such that for all preference profiles $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.

Definition 2.5. Two preference profiles P_N, P_N' are called *tops-equivalent* if $r_1(P_i) = r_1(P_i')$ for all $i \in N$.

Definition 2.6. An SCF $f : \mathcal{D}^n \to X$ is called *tops-only* if for any two tops-equivalent preference profiles $P_N, P_N' \in \mathcal{D}^n$, $f(P_N) = f(P_N')$.

Definition 2.7. A domain $\mathcal{D}$ is called *dictatorial* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial. A domain that is not dictatorial is called a *non-dictatorial* domain.

Definition 2.8. A domain $\mathcal{D}$ is called *tops-only* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is tops-only.

Definition 2.9. A preference $P \in \mathbb{L}(X)$ is called *single-peaked* if for all $x, y \in X$, $[x < y \leq r_1(P)$ or $r_1(P) \leq y < x]$ implies yPx. A domain $\mathcal{S}$ is called a *single-peaked domain* if each preference in it is single-peaked, and a domain $\bar{\mathcal{S}}$ is called *maximal single-peaked* if it contains all single-peaked preferences.

Definition 2.10. An SCF $f : \mathcal{D}^n \to X$ is called *uncompromising* if for all $P_N \in \mathcal{D}^n$, all $i \in N$, and all $P_i' \in \mathcal{D}$:

(i) if $r_1(P_i) < f(P_N)$ and $r_1(P_i') \leq f(P_N)$, then $f(P_N) = f(P_i', P_{-i})$, and

(ii) if $f(P_N) < r_1(P_i)$ and $f(P_N) \leq r_1(P_i')$, then $f(P_N) = f(P_i', P_{-i})$.

REMARK 2.1. If an SCF f satisfies uncompromisingness, then by definition, f is tops-only.

Definition 2.11. Let $\beta = (\beta_S)_{S \subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{D}^n \to X$ is called a *min-max rule with respect to* β if:

$$f^\beta(P_1, \ldots, P_n) = \min_{S \subseteq N}\{\max_{i \in S}\{r_1(P_i), \beta_S\}\}.$$

REMARK 2.2. Every min-max rule is uncompromising.[7]

Now, we introduce a few graph theoretic notions. A *directed graph* G is defined as a pair $\langle V, E\rangle$, where V is the set of nodes and $E \subseteq V \times V$ is the set of directed edges, and an *undirected graph* G is defined as a pair $\langle V, E\rangle$, where V is the set of nodes and $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$. For two graphs (directed or undirected) $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$, the graph $G_1 \cup G_2$ is defined as $G_1 \cup G_2 = \langle V_1 \cup V_2, E_1 \cup E_2\rangle$.

All the graphs we consider in this paper are of the kind $G = \langle X, E\rangle$, i.e., whose node set is the set of alternatives.

Definition 2.12. A directed (undirected) graph $G = \langle X, E\rangle$ is called the *directed (undirected) line graph* on X if $(x, y) \in E$ ($\{x, y\} \in E$) if and only if $|x - y| = 1$.

Definition 2.13. A *node-path* in a directed (undirected) graph $G = \langle X, E\rangle$ from a node x to a node y, denoted by $\pi_G(x, y)$, is defined as a sequence of nodes $x_1, \ldots, x_k$ such that $x_1 = x$, $x_k = y$, and $(x_i, x_{i+1}) \in E$ ($\{x_i, x_{i+1}\} \in E$) for all $i = 1, \ldots, k-1$.

Definition 2.14. A *cycle* in a directed (undirected) graph G is defined as a node-path from a node to itself such that all the edges involved in the path are distinct. A cycle in a directed (undirected) graph is called *essential* if it involves more than two nodes and it is not a union of two cycles.

Definition 2.15. Let $G = \langle X, E\rangle$ be a directed graph and let $x, y \in X$ be such that $x < y - 1$. Then, G is called a *directed (undirected) partial line graph with respect to* x, y if $G = G_1 \cup G_2$, where $G_1 = \langle X, E_1\rangle$ is the directed (undirected) line graph on X and $G_2 = \langle [x, y], E_2\rangle$ is a directed (undirected) graph such that for all $z \in \{x, y\}$ there is an essential cycle involving the node z.

Definition 2.16. The *top-graph* of a domain $\mathcal{D}$ is defined as the directed graph $\langle X, E\rangle$, where $(x, y) \in E$ if and only if there exists a preference $P \in \mathcal{D}$ such that $r_1(P) = x$ and $r_2(P) = y$.

Definition 2.17. An undirected graph $G = \langle X, E\rangle$ is called *connected* if for all $x, y \in X$, there is a node-path from x to y.

Definition 2.18. An undirected graph $G = \langle X, E\rangle$ is called a *tree* if for every two distinct nodes $x, y \in X$, there is a unique path from x to y in G. A *spanning tree* T of an undirected connected graph G is defined as a connected subgraph of G that is a tree. For an undirected connected graph G, we denote by $\mathcal{T}_G$ the set of all spanning trees of G.

[7] For details, see Weymark (2011).

3. Top-connected Single-peaked Domains

In this section, we introduce the notion of top-connected single-peaked domains and characterize the unanimous and strategy-proof SCFs on these domains. We begin with a few formal definitions.

Definition 3.1. A domain $\mathcal{D}$ satisfies *top-connectedness* property if for all $x, x+1 \in X$, there are preferences $P, P' \in \mathcal{D}$ such that $r_1(P) = r_2(P') = x$ and $r_2(P) = r_1(P') = x+1$.

Note that a domain satisfies top-connected property if and only if its top-graph is the directed line graph on X.

Definition 3.2. A domain $\hat{\mathcal{S}}$ is called a *top-connected single-peaked domain* if it is a single-peaked domain, and it satisfies top-connectedness property.

Note that a top-connected single-peaked domain with m alternatives can be constructed with $2m-2$ preferences. Also, since a maximal single-peaked domain is top-connected single-peaked, such a domain can contain at most 2^{m-1} preferences. Thus, the class of top-connected single-peaked domains is quite big ranging from domains with $2m-2$ preferences to domains with 2^{m-1} preferences. In what follows, we provide an example of a top-connected single-peaked domain with five alternatives.

Example 3.1. Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the domain in Table 1 is a top-connected single-peaked domain.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_4	x_1	x_4	x_4	x_2	x_5	x_2	x_5	x_3	x_3
x_4	x_4	x_1	x_4	x_5	x_5	x_5	x_2	x_1	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_5	x_1	x_1	x_1

Table 1: A top-connected single-peaked domain

In Figure 1, we present the top-graph of the domain in Example 3.1.

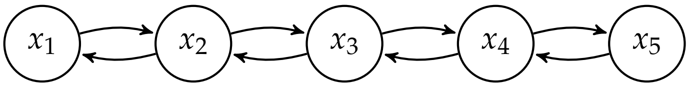


Figure 1: Top-graph of the domain in Example 3.1

3.1 Unanimous and Strategy-proof SCFs

In this subsection, we provide a characterization of the unanimous and strategy-proof SCFs on top-connected single-peaked domains.

Theorem 3.1. *Let $\hat{\mathcal{S}}$ be a top-connected single-peaked domain. Then, an SCF $f : \hat{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

The proof of the Theorem 3.1 is relegated to Appendix A.

The following corollary is immediate from Theorem 3.1.

Corollary 3.1 (Moulin (1980); Weymark (2011))**.** *Let $\bar{\mathcal{S}}$ be a maximal single-peaked domain. Then, an SCF $f : \bar{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.2 Min-max Domains

In this section, we introduce the notion of min-max domains and provide a characterization of these domains. A domain is called a min-max domain if the set of unanimous and strategy-proof SCFs coincides with the set of min-max rules. Below, we provide a formal definition of min-max domains.

Definition 3.3. A domain $\mathcal{D}$ is called a *min-max domain* if:

(i) every min-max rule on $\mathcal{D}^n$ is strategy-proof, and

(ii) every unanimous and strategy-proof SCF on $\mathcal{D}^n$ is a min-max rule.

Our next theorem provides a characterization of the min-max domains.

Theorem 3.2. *A domain $\mathcal{D}$ is a min-max domain if and only if $\mathcal{D}$ is a top-connected single-peaked domain.*

Proof. The proof of the if-part follows from Theorem 3.1. We proceed to prove the only-if part. Let $\mathcal{D}$ be a min-max domain. We show that $\mathcal{D}$ is a top-connected single-peaked domain. First, we show that $\mathcal{D}$ is a single-peaked domain. Assume for contradiction that there is $Q \in \mathcal{D}$ and

$x, y \in X$ such that $x < y < r_1(Q)$ and xQy. Consider the min-max rule f^β with respect to $(\beta_S)_{S \subseteq N}$ such that $\beta_S = x$ for all $\emptyset \subsetneq S \subsetneq N$. Consider $P_N \in \mathcal{D}^n$ such that $P_1 = Q$ and $r_1(P_i) = y$ for all $i \in N \setminus 1$. By the definition of f^β, $f^\beta(P_N) = y$. Consider $P'_1 \in \mathcal{D}$ with $r_1(P'_1) = x$. Again, by the definition of f^β, $f^\beta(P'_1, P_{N \setminus 1}) = x$. This means agent 1 manipulates at P_N via P'_1, which is a contradiction to the assumption that $\mathcal{D}$ is a min-max domain. Hence, $\mathcal{D}$ must be a single-peaked domain.

Now, we show that $\mathcal{D}$ satisfies top-connectedness property. Note that since $\mathcal{D}$ is single-peaked, $r_1(P) = a$ (or b) implies $r_2(P) = a + 1$ (or $b - 1$). Consider some $x \in X \setminus \{a, b\}$. Since $\mathcal{D}$ is single-peaked, for all $P \in \mathcal{D}$, $r_1(P) = x$ implies $r_2(P) \in \{x - 1, x + 1\}$. Without loss of generality, assume for contradiction to the top-connectedness property that for all $P \in \mathcal{D}$, $r_1(P) = x$ implies $r_2(P) = x - 1$. Consider the following SCF[8]:

$$f(P_N) = \begin{cases} x \text{ if } r_1(P_1) = x \text{ and } xP_j(x-1) \text{ for all } j \in N \setminus 1, \\ x - 1 \text{ if } r_1(P_1) = x \text{ and } (x-1)P_j x \text{ for some } j \in N \setminus 1, \\ r_1(P_1) \text{ otherwise.} \end{cases}$$

It is left to the reader to verify that f is unanimous and strategy-proof. We show that f is not uncompromising, which in turn means that f is not a min-max rule. Let $P_N \in \mathcal{D}^n$ be such that $r_1(P_1) = x$ and $r_1(P_j) = x - 1$ for some $j \neq 1$ and let $P'_1 \in \mathcal{D}$ be such that $r_1(P'_1) = x + 1$. Then, by the definition of f, $f(P_N) = x - 1$ and $f(P'_1, P_{N \setminus 1}) = x + 1$. However, this is a violation of uncompromisingness. This completes the proof of the only-if part. ■

3.3 Applications

3.3.1 Regular Single-crossing Domains

In this subsection, we introduce the notion of regular single-crossing domains and provide a characterization of the unanimous and strategy-proof SCFs on these domains. First, we present the formal definition of single-crossing domains.

Definition 3.4. A domain $\mathcal{D}$ is called a *single-crossing domain* if there is a linear order $\lhd$ on $\mathcal{D}$ such

[8]Here $\mathcal{D}$ satisfies the *unique seconds* property defined in Aswal et al. (2003) and the construction of the SCF f is motivated from the arguments employed in the proof of Theorem 5.1 in Aswal et al. (2003).

that for all $x, y \in X$ and all $P, \hat{P} \in \mathcal{D}$,

$$[x < y, P \lhd \hat{P}, \text{ and } x\hat{P}y] \Rightarrow xPy.$$

Definition 3.5. A single-crossing domain $\bar{\mathcal{S}}_c$ is called *maximal* if there is no single-crossing domain $\mathcal{D}$ such that $\bar{\mathcal{S}}_c \subsetneq \mathcal{D}$.

In what follows, we provide an example of a maximal regular single-crossing domain with five alternatives.

Example 3.2. Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the domain $\mathcal{D}$ in Table 2 is a maximal regular single-crossing domain with respect to the linear order $\lhd \in \mathbb{L}(\mathcal{D})$ given by $P_1 \lhd P_2 \lhd P_3 \lhd P_4 \lhd P_5 \lhd P_6 \lhd P_7 \lhd P_8 \lhd P_9 \lhd P_{10} \lhd P_{11}$. To see this, consider two alternatives, say x_2 and x_4. Then, $x_2 P x_4$ for all $P \in \{P_1, P_2, P_3, P_4, P_5, P_6\}$ and $x_4 P x_2$ for all $P \in \{P_7, P_8, P_9, P_{10}, P_{11}\}$. Therefore, $x_2 \hat{P} x_4$ for some $\hat{P} \in \mathcal{D}$ implies $x_2 P x_4$ for all $P \lhd \hat{P}$.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_1	x_4	x_4	x_4	x_2	x_5	x_5	x_3	x_3
x_4	x_4	x_4	x_1	x_5	x_5	x_5	x_2	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_1	x_1	x_1

Table 2: A maximal regular single-crossing domain

In the following lemma, we show that every regular single-crossing domain is single-peaked.

Lemma 3.1. *Every regular single-crossing domain $\mathcal{S}_c$ is a single-peaked domain.*

Proof. Let $\mathcal{S}_c$ be a regular single-crossing domain. Let $\lhd \in \mathbb{L}(\mathcal{S}_c)$ be such that for all $x, y \in X$ and all $P, \hat{P} \in \mathcal{S}_c$,

$$[x < y, P \lhd \hat{P}, \text{ and } x\hat{P}y] \Rightarrow xPy.$$

We show that each $P \in \mathcal{S}_c$ is single-peaked. Without loss of generality, assume for contradiction that there are $x, y \in X$ and $Q \in \mathcal{S}_c$ such that $x < y < r_1(Q)$ and xQy. Since $x < y$ and xQy, by the definition of single-crossing domain, xPy for all $P \in \mathcal{S}_c$ with $P \lhd Q$. This, in particular, means $r_1(P) \neq y$ for all $P \in \mathcal{S}_c$ with $P \lhd Q$. Moreover, since $y < r_1(Q)$, by the definition of single-crossing domain, $r_1(Q)Py$ for all $P \in \mathcal{S}_c$ with $Q \lhd P$. This, in particular, means $r_1(P) \neq y$ for all $P \in \mathcal{S}_c$ with $Q \lhd P$. This, together with the fact that $r_1(Q) \neq y$, means $r_1(P) \neq y$ for all $P \in \mathcal{S}_c$, which is a contradiction to the regularity of $\mathcal{S}_c$. Therefore, $\mathcal{S}_c$ is single-peaked. ■

Our next lemma shows that every maximal regular single-crossing domain satisfies top-connected property.

Lemma 3.2. *Every maximal regular single-crossing domain $\bar{\mathcal{S}}_c$ satisfies top-connectedness property.*

Proof. Let $\bar{\mathcal{S}}_c$ be a maximal regular single-crossing domain. Then, by Lemma 3.1, $\bar{\mathcal{S}}_c$ is regular single-peaked. Take $x \in X \setminus \{a, b\}$. We show that there exist $P, P' \in \bar{\mathcal{S}}_c$ such that $r_1(P) = r_2(P') = x$ and $r_2(P) = r_1(P') = x + 1$. Without loss of generality, assume for contradiction that for all $P \in \bar{\mathcal{S}}_c$ with $r_1(P) = x$, $r_2(P) \neq x + 1$. Because $\bar{\mathcal{S}}_c$ is single-peaked, it must be that $x \neq a$ and $r_2(P) = x - 1$ for all $P \in \bar{\mathcal{S}}_c$ with $r_1(P) = x$. Let $\lhd \in \mathbb{L}(\bar{\mathcal{S}}_c)$ be such that for all $x, y \in X$ and all $P, \hat{P} \in \bar{\mathcal{S}}_c$,

$$[x < y, P \lhd \hat{P}, \text{ and } x\hat{P}y] \Rightarrow xPy.$$

Let $\hat{P} \in \bar{\mathcal{S}}_c$ be such that $r_1(\hat{P}) = x$ and for all $P \in \bar{\mathcal{S}}_c$ with $\hat{P} \lhd P$, $r_1(P) \neq x$. Consider the preference $\tilde{P}$ with $r_1(\tilde{P}) = x$ and $r_2(\tilde{P}) = x + 1$ such that for all $x, y \in X \setminus \{x, x + 1\}$, $x\tilde{P}y$ if and only if $x\hat{P}y$. Clearly, $\tilde{P} \notin \bar{\mathcal{S}}_c$. Since $\bar{\mathcal{S}}_c$ is single-peaked, it follows that $\bar{\mathcal{S}}_c \cup \tilde{P}$ is single-crossing with respect to the ordering $\lhd' \in \mathbb{L}(\bar{\mathcal{S}}_c \cup \tilde{P})$, where $\lhd'$ is obtained by placing $\tilde{P}$ just after $\hat{P}$ in the ordering $\lhd$, i.e., for all $P, P' \in \bar{\mathcal{S}}_c$, $P \lhd' P'$ if and only if $P \lhd P'$, and for all $P \in \bar{\mathcal{S}}_c$ with $\hat{P} \lhd P$, $\hat{P} \lhd' \tilde{P} \lhd' P$. However, this contradicts the maximality of $\bar{\mathcal{S}}_c$, which completes the proof. ■

The following corollaries are obtained from Theorem 3.1, Lemma 3.1, and Lemma 3.2. They characterize the unanimous and strategy-proof SCFs on the top-connected regular single-crossing domains and the maximal regular single-crossing domains. Note that a top-connected regular single-crossing domain with m alternatives can be constructed with $2m - 2$ preferences, whereas a maximal regular single-crossing domain requires $m(m-1)/2$ preferences.

Corollary 3.2. *Let $\mathcal{S}_c$ be a top-connected regular single-crossing domain. Then, an SCF $f : \mathcal{S}_c^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

Corollary 3.3. *Let $\bar{\mathcal{S}}_c$ be a maximal regular single-crossing domain. Then, an SCF $f : \bar{\mathcal{S}}_c^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.3.2 Minimally Rich Single-peaked Domains

In this subsection, we present a characterization of unanimous and strategy-proof SCFs on minimally rich single-peaked domains. The notion of minimally rich single-peaked domains is introduced in Peters et al. (2014). For the sake of completeness, we present below the formal definition of such domains.

Definition 3.6. A single-peaked preference P is called *left single-peaked* (*right single-peaked*) if for all $u < r_1(P) < v$, we have uPv (vPu). Moreover, a single-peaked domain $\mathcal{S}_m$ is called *minimally rich* if it contains all left and all right single-peaked preferences.

Clearly, a minimally rich single-peaked domain is a top-connected single-peaked domain. So, we have the following corollary from Theorem 3.1.

Corollary 3.4. *Let $\mathcal{S}_m$ be a minimally rich single-peaked domain. Then, an SCF $f : \mathcal{S}_m^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.3.3 Distance based Single-peaked Domains

In this subsection, we introduce the notion of single-peaked domains that are based on distances. Consider the situation where a public facility has to be developed at one of the locations $x_1, \ldots, x_m$. Suppose that there is a street connecting the locations, and for every two locations x_i and x_{i+1}, there are two types of distances, a forward distance from x_i to x_{i+1} and a backward distance from x_{i+1} to x_i. An individual bases her preferences on such distances, i.e., whenever a location is strictly closer than another to her most preferred location, she prefers the former to the latter. Moreover, ties are broken on both sides. We show that such a domain is a top-connected single-peaked domain under some condition on the distances. Below, we present a formal definition of such domains.

Consider the directed line graph $G = \langle X, E \rangle$ on X. A function $d : E \to (0, \infty)$ is called a *distance function* on G. Given a distance function d, define the distance between two nodes $x, y \in X$ as $d(x,y) = d(x, x+1) + \ldots + d(y-1, y)$ if $y > x$ and $d(x,y) = d(x, x-1) + \ldots + d(y+1, y)$ if $y < x$. A distance function satisfies *adjacent symmetry* if $d(x, x+1) = d(x, x-1)$ for all $x \in X \setminus \{a, b\}$. A preference $P \in \mathbb{L}(X)$ *respects a distance function* d if for all $x, y \in X$, $d(r_1(P), x) < d(r_1(P), y)$ implies xPy. A domain $\mathcal{D}$ *respects a distance function* d if $\mathcal{D} = \{P \in \mathbb{L}(X) \mid P \text{ respects } d\}$.

Below, we provide an example of a distance based single-peaked domain.

Example 3.3. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. The directed line graph $G = \langle X, E \rangle$ on X and the adjacent symmetric distance function d on E are as given below:

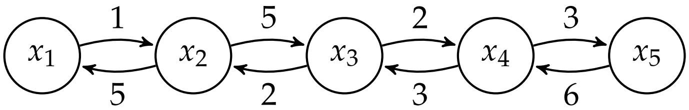


Figure 2: The directed line graph G on X and an adjacent symmetric distance function on G

Then, the single-peaked domain based on the distance function d is:

$$\begin{aligned}\mathcal{D} = &\{x_1x_2x_3x_4x_5, x_2x_3x_1x_4x_5, x_2x_1x_3x_4x_5, x_3x_4x_2x_5x_1, x_3x_2x_4x_5x_1, x_4x_5x_3x_2x_1,\\ &x_4x_3x_5x_2x_1, x_5x_4x_3x_2x_1\}.\end{aligned}$$

In the following lemma, we show that a single-peaked domain based on an adjacent symmetric distance function is a top-connected single-peaked domain.

REMARK 3.1. Let $G = \langle X, E \rangle$ be the directed line graph on X and let $d : E \to (0, \infty)$ be an adjacent symmetric distance function. Suppose a domain $\mathcal{D}$ respects d. Then, $\mathcal{D}$ is a top-connected single-peaked domain.

The following corollary is obtained from Theorem 3.1 and Remark 3.1.

Corollary 3.5. *Let $G = \langle X, E \rangle$ be the directed line graph on X and let $d : E \to (0, \infty)$ be an adjacent symmetric distance function. Suppose a domain $\mathcal{D}$ respects d. Then, $f : \mathcal{D}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

4. PARTIALLY SINGLE-PEAKED DOMAINS

In this section, we consider a particular class of non-single-peaked domains. These domains exhibit single-peaked property only over a strict subset of alternatives. We call such domains partially single-peaked domains which are formally defined below.

Definition 4.1. Let $x, y \in X$ such that $x < y - 1$. Then, a domain $\tilde{\mathcal{S}}$ is called *partially single-peaked* with respect to x, y if:

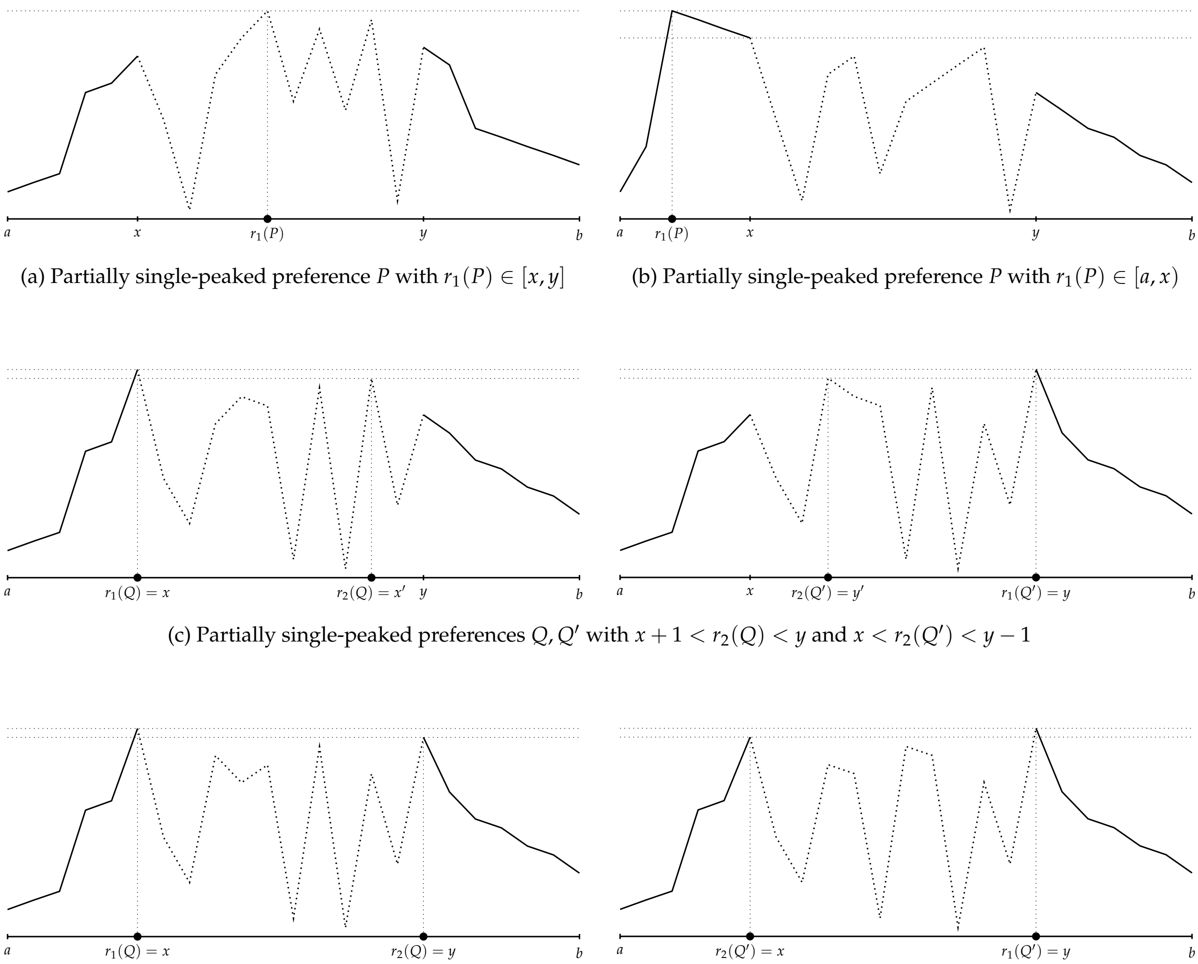
(i) for all $P \in \tilde{\mathcal{S}}$ with $r_1(P) \in [x, y]$ and all $u, v \notin (x, y)$, $\left[v < u \leq r_1(P) \text{ or } r_1(P) \leq u < v\right]$ implies uPv,

(ii) for all $P \in \tilde{\mathcal{S}}$ with $r_1(P) \notin [x,y]$ and all $u,v \in X$ such that $u \notin (x,y)$, $[v < u \leq r_1(P)$ or $r_1(P) \leq u < v]$ implies uPv, and

(iii) there exist $Q, Q' \in \tilde{\mathcal{S}}$ with $r_1(Q) = x$ and $r_1(Q') = y$ such that either $[r_2(Q) \in (x+1, y)$ and $r_2(Q') \in (x, y-1)]$ or $[r_2(Q) = y$ and $r_2(Q') = x]$.

Condition (i) in Definition 4.1 implies that if the top alternative of a preference in a partially single-peaked domain lies in the interval $[x,y]$, then it maintains single-peaked property over the alternatives in the interval $[a,x]$ and in the interval $[y,b]$. Note that Condition (i) does not impose any restriction on the relative ordering of an alternative in $[x,y]$ and an alternative outside $[x,y]$. The interpretation of Condition (ii) is as follows. Consider a preference P in a partially single-peaked domain such that $r_1(P) \notin [x,y]$. Suppose, for instance, $r_1(P) \in [a,x)$. Then, P maintains single-peaked property over the alternatives in the interval $[a, r_1(P)]$. Moreover, if an alternative u lies in the interval $(r_1(P), x]$ or in the interval $[y,b]$, then it is preferred to any alternative v in the interval $(u,b]$. Note that if u lies in the interval $(r_1(P), x]$, then it is preferred to an alternative in $[x,y]$. Thus, Condition (ii) imposes some restriction on the relative ordering of an alternative in $[x,y]$ and an alternative outside $[x,y]$. Further, note that Conditions (i) and (ii) do not impose any restriction on the relative ordering of two alternatives in the interval $[x,y]$. Finally, Condition (iii) ensures that the intervals $[a,x]$ and $[y,b]$ are the maximal intervals over which every preference in a partially single-peaked domain maintains the single-peaked property. To see this, first note that both Q and Q' are non-single-peaked preferences. Q violates single-peakedness for the alternatives $x+1$ and $r_2(Q)$, and Q' violates single-peakedness for the alternatives $y-1$ and $r_2(Q')$. Therefore, the intervals $[a,x]$ and $[y,b]$ are the maximal intervals over which every preference in a partially single-peaked domain maintains the single-peaked property. In Section 4.2, we show that the particular restrictions on the second ranked alternatives of Q and Q' given by Condition (iii) are necessary for our results.

We illustrate the notion of partially single-peaked domains in Figure 3. Figure 3(a) and Figure 3(b) present a partially single-peaked preference P with $r_1(P) \in [x,y]$ and $r_1(P) \in [a,x)$, respectively. Figure 3(c) presents the partially single-peaked preferences Q and Q' with $r_1(Q) = x$, $r_2(Q) \in (x+1, y)$, $r_1(Q') = y$, and $r_2(Q') \in (x, y-1)$ while Figure 3(d) presents partially single-peaked preferences Q and Q' with $r_1(Q) = x$, $r_2(Q) = y$, $r_1(Q') = y$, and $r_2(Q') = x$. Note that all these preferences are single-peaked over the intervals $[a,x]$ and $[y,b]$. Furthermore, for the preference depicted in Figure 3(a), there is no restriction on the alternatives in the interval (x,y), and for that shown in Figure 3(b), there is no restriction on the alternatives in the interval (x, y) except that x is preferred to all the alternatives in $(x, b]$. Also, for the preferences in Figures 3(c) and 3(d), there is no restriction on the alternatives in (x, y) other than the one on the second ranked alternatives.



(a) Partially single-peaked preference P with $r_1(P) \in [x, y]$

(b) Partially single-peaked preference P with $r_1(P) \in [a, x)$

(c) Partially single-peaked preferences Q, Q' with $x + 1 < r_2(Q) < y$ and $x < r_2(Q') < y - 1$

(d) Partially single-peaked preferences Q, Q' with $r_2(Q) = y$ and $r_2(Q') = x$

Figure 3: Partially single-peaked preferences

In the following, we define a top-connected partially single-peaked domain.

Definition 4.2. A domain $\tilde{\mathcal{S}}$ is called a *top-connected partially single-peaked domain* with respect to alternatives x, y with $x < y - 1$ if:

(i) $\tilde{\mathcal{S}}$ is a partially single-peaked domain with respect to x, y, and

(ii) $\tilde{\mathcal{S}}$ contains a top-connected single-peaked domain.

We interpret Definition 4.2 in terms of its top-graph. Let G be the top-graph of a top-connected partially single-peaked domain with respect to alternatives x and y. Then, G is a directed partial line graph with respect to x and y since $G = G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the directed line graph on X and $G_2 = \langle [x, y], E_2 \rangle$ is a directed graph such that $(x, r_2(Q), r_2(Q) - 1, \ldots, x)$ is an essential cycle involving x and $(y, r_2(Q'), r_2(Q') + 1, \ldots, y)$ is an essential cycle involving y. In Example 4.1, we present a top-connected partially single-peaked domain with seven alternatives, and in Figure 4, we present the top-graph of the domain.

Example 4.1. Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7$. Then, the domain in Table 3 is a top-connected partially single-peaked domain with respect to x_3 and x_6. To see this, first consider a preference with its top alternative in the set $\{x_3, x_4, x_5, x_6\}$, say P_7. Note that $x_3 P_7 x_2 P_7 x_1$ and $x_6 P_7 x_7$, which means P_7 is single-peaked over the subsets of alternatives $\{x_1, x_2, x_3\}$ and $\{x_6, x_7\}$. Moreover, the position of x_5 is completely unrestricted (here at the bottom) in P_7. Next, consider a preference with the top alternative in the set $\{x_1, x_2, x_3\}$, say P_2. Note that P_2 is single-peaked over the sets $\{x_1, x_2, x_3\}$ and $\{x_6, x_7\}$. Further, note that x_3 is preferred to the alternatives x_4, x_5, x_6, x_7, and that there is no restriction on the relative ordering of the alternatives x_4 and x_5 (here $x_5 P_2 x_4$). Finally, consider the preferences Q and Q'. Since $r_1(Q) = x_3$, $r_2(Q) = x_5$, $r_1(Q') = x_6$, and $r_2(Q') = x_4$, they satisfy Condition (iii) in Definition 4.1.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	Q	Q'
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_3}$	$\mathbf{x_6}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_7}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_3	x_1	x_4	x_2	x_3	x_5	x_3	x_3	x_4	x_4	x_5	x_5	x_2	x_3
x_4	x_6	x_4	x_4	x_5	x_5	x_2	x_2	x_2	x_6	x_3	x_3	x_4	x_4	x_6	x_7
x_5	x_5	x_5	x_5	x_6	x_6	x_1	x_6	x_1	x_7	x_2	x_2	x_3	x_3	x_1	x_2
x_6	x_7	x_6	x_6	x_7	x_1	x_7	x_1	x_6	x_2	x_7	x_7	x_2	x_2	x_7	x_1
x_7	x_4	x_7	x_7	x_1	x_7	x_5	x_7	x_7	x_1	x_1	x_1	x_1	x_1	x_4	x_5

Table 3: A top-connected partially single-peaked domain

The top-graph G of the domain in Example 4.1 is given in Figure 4. Note that $G = G_1 \cup G_2$, where G_1 is the directed line graph on $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and G_2 is a directed graph on $\{x_3, x_4, x_5, x_6\}$ containing two essential cycles (x_3, x_5, x_4, x_3) and (x_6, x_4, x_5, x_6) involving the nodes x_3 and x_6, respectively.

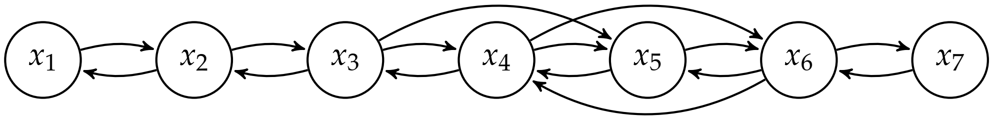


Figure 4: Top-graph of the Domain in Example 4.1

4.1 UNANIMOUS AND STRATEGY-PROOF SCFS

In this subsection, we characterize the unanimous and strategy-proof SCFs on top-connected partially single-peaked domains as partly dictatorial generalized median voter schemes. A formal definition of such SCFs is presented below:

Definition 4.3. Let $x, y \in X$ be such that $x < y - 1$. Then, a min-max rule $f^\beta : \mathcal{D}^n \to X$ with parameters $\beta = (\beta_S)_{S \subseteq N}$ is a *partly dictatorial generalized median voter scheme* (PDGMVS) with respect to x, y if there exists an agent $d \in N$, called the *partial dictator* of f^β, such that $\beta_d \in [a, x]$ and $\beta_{N \setminus d} \in [y, b]$.

REMARK 4.1. Reffgen (2015) defines PDGMVS in a different fashion but it can be shown that their definition is equivalent to Definition 4.3.[9]

The following lemma justifies why the agent d in Definition 4.3 is called the partial dictator.

Lemma 4.1. *Let $x, y \in X$ be such that $x < y - 1$ and let $f^\beta : \mathcal{D}^n \to X$ be a PDGMVS with respect to x, y. Let agent d be the partial dictator of f^β. Then,*

(i) $f^\beta(P_N) \in [a, x]$ *if* $r_1(P_d) \in [a, x)$,

(ii) $f^\beta(P_N) \in [y, b]$ *if* $r_1(P_d) \in (y, b]$, *and*

(iii) $f^\beta(P_N) = r_1(P_d)$ *if* $r_1(P_d) \in [x, y]$.

Proof. We prove (i) and (iii), the proof of (ii) can be established using symmetric arguments. Assume for contradiction that $r_1(P_d) \in [a, x)$ and $f^\beta(P_N) > x$. Since f^β is a min-max rule, f^β is uncompromising. Therefore, $f^\beta(P'_d, P_{N \setminus d}) = f^\beta(P_N)$, where $r_1(P'_d) = a$. By the definition of f^β, we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_i) = b$ for all $i \neq d$. Because $f^\beta(P_N) > x$, we have $f^\beta(P'_N) > x$. However, by the definition of f^β, $f^\beta(P'_N) = \beta_d$. Since $\beta_d \in [a, x]$, this is a contradiction. This completes the proof of (i).

[9]For details see the proof of Theorem 3.1 in Reffgen (2015).

Now, we prove (iii). Without loss of generality, assume for contradiction that $r_1(P_d) \in [x, y]$ and $f^\beta(P_N) > r_1(P_d)$. Using similar logic as in the proof of (i), we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_d) = a$ and $r_1(P'_i) = b$ for all $i \neq d$. Since $f^\beta(P'_N) = \beta_d \in [a, x]$, this is a contradiction. This completes the proof of (iii). ■

The following theorem characterizes the unanimous and strategy-proof SCFs on top-connected partially single-peaked domains.

Theorem 4.1. *Let $x, y \in X$ be such that $x < y - 1$ and let $\tilde{\mathcal{S}}$ be a top-connected partially single-peaked domain with respect to x, y. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a PDGMVS with respect to x, y.*

The proof of the Theorem 4.1 is relegated to Appendix B.

Our next corollary is a consequence of Lemma 4.1 and Theorem 4.1. It characterizes a class of dictatorial domains, and thereby it generalizes the celebrated Gibbard-Satterthwaite (Gibbard (1973), Satterthwaite (1975)) results. Note that our dictatorial result is independent of those in Aswal et al. (2003), Sato (2010), Pramanik (2015) and so on.

Corollary 4.1. *Let $\mathcal{D}$ be a top-connected partially single-peaked domain with respect to a and b. Then, $\mathcal{D}$ is a dictatorial domain.*

4.2 A Result on Partial Necessity

In this subsection, we consider domains for which the set of unanimous and strategy-proof SCFs coincides with the set of PDGMVS. We call such domains PDGMVS domains. A formal definition is given below.

Definition 4.4. A domain $\mathcal{D}$ is called a *PDGMVS domain* if there are $x, y \in X$ with $x < y - 1$ such that:

(i) every PDGMVS with respect to x, y on $\mathcal{D}^n$ is strategy-proof, and

(ii) every unanimous and strategy-proof SCF on $\mathcal{D}^n$ is a PDGMVS with respect to x, y.

Conditions (i), (ii), and (iii) in Definition 4.1 are obviously strong conditions. Are they necessary for PDGMVS domains? The question appears to be extremely difficult to resolve completely. However, the following lemma shows that Conditions (i) and (ii) in Definition 4.1 are necessary, and the subsequent discussion shows that Condition (iii) is also close to being necessary in an appropriate sense.

Lemma 4.2. *Let $\mathcal{D}$ be a PDGMVS domain. Then, $\mathcal{D}$ satisfies Conditions (i) and (ii) in Definition 4.1.*

Proof. First, we show that $\mathcal{D}$ satisfies Condition (i) in Definition 4.1. Without loss of generality, assume that there exists $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [x, y]$ such that $u\tilde{P}v$ for some $u < v \leq x$. Consider the PDGMVS f^β such that:

$$\beta_S = \begin{cases} v \text{ if } S = \{1\}, \\ x - 1 \text{ if } \{1\} \subsetneq S, \\ r_1(P_1) \text{ if } 1 \notin S. \end{cases}$$

Note that agent 1 is the partial dictator of f^β. Consider the preference profile $P_N \in \mathcal{D}^n$ such that $r_1(P_1) = a$, $P_2 = \tilde{P}$, and $r_1(P_j) = b$ for all $j \neq 1, 2$. Then, by the definition of f^β, $f^\beta(P_N) = v$. Let $P_2' \in \mathcal{D}$ be such that $r_1(P_2') = u$. Then, $f^\beta(P_2', P_{N\setminus 2}) = u$. Since $u\tilde{P}v$, agent 2 manipulates at P_N via P_2'.

Now, we show that $\mathcal{D}$ satisfies Condition (ii) in Definition 4.1. Without loss of generality, assume for contradiction that there exist $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [a, x)$ and $u, v \in X$ with $u \notin (x, y)$ such that $[v \prec u \preceq r_1(P)$ or $r_1(P) \preceq u \prec v]$ and $v\tilde{P}u$. If $v \in [a, x]$, then using similar argument as in the proof of the necessity of Condition (i), it follows that there is a PDGMVS on $\mathcal{D}^n$ that is manipulable. So, assume $r_1(\tilde{P}) \leq u \leq x < v$. Consider the PDGMVS $f^\beta : \mathcal{D}^n \to X$ such that:

$$\beta_S = \begin{cases} u \text{ if } 1 \in S \text{ and } S \neq N, \\ b \text{ if } 1 \notin S \text{ and } S \neq N. \end{cases}$$

Let $P_N \in \mathcal{D}^n$ be such that $P_1 = \tilde{P}$ and $r_1(P_j) = b$ for all $j \neq 1$. Then, $f^\beta(P_N) = u$. Let $P_1' \in \mathcal{D}$ be such that $r_1(P_1') = b$. Then, $f^\beta(P_1', P_{N\setminus 1}) = v$. Since $v\tilde{P}u$, agent 1 manipulates at P_N via P_1'. ∎

Coming to Condition (iii) in Definition 4.1, it is to be noted that it can be violated in many ways, we consider those domains obtained through mild violations of the same and show that there do exist unanimous and strategy-proof SCFs on such domains that are not PDGMVS.

Recall that Condition (iii) requires two non-single-peaked preferences Q and Q' in $\mathcal{D}$ such that $r_1(Q) = x$, $r_2(Q) = x'$, $r_1(Q') = y$, and $r_2(Q') = y'$, where either $[x' \in (x, y-1)$ and $y' \in (x+1, y)]$ or $[x' = x$ and $y' = x]$. Suppose a domain $\mathcal{D}$ satisfies Conditions (i) and (ii) in Definition 4.1. Suppose further that $\mathcal{D}$ contains a non-single-peaked preference of the form Q, but no preference of the form Q'. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain $\mathcal{D}$ that is not a PDGMVS.

Example 4.2. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$. By $P = x_1x_2x_3x_4x_5$, we mean a preference P such that $x_1Px_2Px_3Px_4Px_5$. Consider the domain:

$$\mathcal{D} = \{x_1x_2x_3x_4x_5, x_1x_3x_4x_5x_2, x_2x_1x_3x_4x_5, x_2x_3x_4x_5x_1, x_3x_2x_1x_4x_5, x_3x_4x_5x_2x_1, x_4x_3x_2x_1x_5, x_4x_5x_3x_2x_1, x_5x_4x_3x_2x_1\}.$$

Note that $\mathcal{D} \setminus \{x_1x_3x_4x_5x_2\}$ is a top-connected single-peaked domain and the preference $x_1x_3x_4x_5x_2$ is of the form Q with $x = x_1$ and $x' = x_3$. However, there is no preference in $\mathcal{D}$ of the form Q'. In Table 4, we present a two-agent SCF that is unanimous and strategy-proof but not a PDGMVS.

$P_1 \backslash P_2$	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5

Table 4: A unanimous and strategy-proof SCF which is not a PDGMVS

It is left to the reader to verify that f is unanimous and strategy-proof. Note that f violates tops-onlyness at the preference profiles $(x_3x_4x_5x_2x_1, x_1x_2x_3x_4x_5)$ and $(x_3x_4x_5x_2x_1, x_1x_3x_4x_5x_2)$, and hence f is not a PDGMVS.

Now, suppose that $\mathcal{D}$ contains two non-single-peaked preferences Q and Q' which do *not* satisfy Condition (iii) in Definition 4.1 for their second ranked alternatives. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain $\mathcal{D}$ that is not a PDGMVS.

Example 4.3. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$. Consider the domain:

$$\begin{aligned}\mathcal{D} =\{&x_1x_2x_3x_4x_5, x_1x_3x_4x_5x_2, x_2x_1x_3x_4x_5, x_2x_3x_4x_5x_1, x_3x_2x_1x_4x_5, x_3x_4x_5x_2x_1, x_4x_3x_2x_1x_5,\\ &x_4x_5x_3x_2x_1, x_5x_4x_3x_2x_1, x_5x_1x_4x_3x_2\}.\end{aligned}$$

Let $Q = x_1x_3x_4x_5x_2$ and $Q' = x_5x_1x_4x_3x_2$. Note that $\mathcal{D} \setminus \{Q, Q'\}$ is a top-connected single-peaked domain. Further, since $r_2(Q) = x_3$ and $r_2(Q') = x_1$, Q and Q' do not satisfy Condition (iii) in Definition 4.1. In Table 5, we present a two-agent SCF that is unanimous and strategy-proof but not a PDGMVS.

P_1 \ P_2	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$	$x_5x_1x_4x_3x_2$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_1
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_1
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5
$x_5x_1x_4x_3x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5

Table 5: A unanimous and strategy-proof SCF which is not a PDGMVS

It is left to the reader to verify that f is unanimous and strategy-proof. Note that f violates tops-onlyness at the preference profiles $(x_3x_4x_5x_2x_1, x_1x_2x_3x_4x_5)$ and $(x_3x_4x_5x_2x_1, x_1x_3x_4x_5x_2)$, and hence f is not a PDGMVS.

4.3 APPLICATIONS

4.3.1 MULTIPLE SINGLE-PEAKED DOMAIN

In this subsection, we consider a well-known class of domains called multiple single-peaked domains and present a characterization of the unanimous and strategy-proof SCFs on such domains. However, before formally defining such domains below we first provide a formal definition of such domains. First, we define the notion of single-peaked domain with respect to an arbitrary order over X.

Definition 4.5. Let $\prec \in \mathbb{L}(X)$ be a prior order over X. Then, a preference $P \in \mathbb{L}(X)$ is *single-peaked with respect to* $\prec$ if for all $x, y \in X$, $[x \prec y \preceq r_1(P)$ or $r_1(P) \preceq y \prec x]$ implies yPx. A domain $\mathcal{S}_\prec$ is called a *single-peaked domain with respect to* $\prec$ if each preference in it is single-peaked with respect to $\prec$, and a domain $\bar{\mathcal{S}}_\prec$ is called *maximal single-peaked with respect to* $\prec$ if it contains all single-peaked preferences with respect to $\prec$.

Similarly, one can define the notion of top-connected partially single-peaked domains and PDGMVS with respect to some prior order $\prec$ over X. For the sake of completeness, we provide the formal definitions of these concepts in what follows.

For $x, y \in X$ such that $x \preceq y$, we define the intervals $[x, y]_\prec = \{z \in X \mid x \preceq z \preceq y\}$, $[x, y)_\prec = [x, y]_\prec \setminus \{y\}$, $(x, y]_\prec = [x, y]_\prec \setminus \{x\}$, and $(x, y)_\prec = [x, y]_\prec \setminus \{x, y\}$. Also, for $\prec \in \mathbb{L}(X)$ and $x \in X$, by $x^+_\prec$ and $x^-_\prec$ we denote the alternatives that appear just after and just before x in the ordering $\prec$, respectively. More formally, $x^+_\prec = y$ if $x \prec y$ and there does not exist $z \in X$ with $x \prec z \prec y$, and $x^-_\prec = y$ if $y \prec x$ and there does not exist $z \in X$ with $y \prec z \prec x$.

Definition 4.6. Let $\prec \in \mathbb{L}(X)$ be a prior order over X and let $x, y \in X$ be such that $x \prec y^-_\prec$. Then, a domain $\tilde{\mathcal{S}}$ is called *partially single-peaked with respect to* $\prec$ *and* x, y if:

(i) for all $P \in \tilde{\mathcal{S}}$ with $r_1(P) \in [x, y]_\prec$ and all $u, v \notin (x, y)_\prec$, $\big[v \prec u \preceq r_1(P)$ or $r_1(P) \preceq u \prec v\big]$ implies uPv,

(ii) for all $P \in \tilde{\mathcal{S}}$ with $r_1(P) \notin [x, y]_\prec$ and all $u, v \in X$ such that $u \notin (x, y)_\prec$, $\big[v \prec u \preceq r_1(P)$ or $r_1(P) \preceq u \prec v\big]$ implies uPv, and

(iii) there exist $Q, Q' \in \tilde{\mathcal{S}}$ with $r_1(Q) = x$ and $r_1(Q') = y$ such that either $\big[r_2(Q) \in (x^+_\prec, y)_\prec$ and $r_2(Q') \in (x, y^-_\prec)_\prec\big]$ or $\big[r_2(Q) = y$ and $r_2(Q') = x\big]$.

Definition 4.7. Let $\prec \in \mathbb{L}(X)$ be a prior order over X and let $x, y \in X$ be such that $x \prec y^-_\prec$. A domain $\tilde{\mathcal{S}}$ is called a *top-connected partially single-peaked domain with respect to* $\prec$ *and* x, y if:

(i) $\tilde{\mathcal{S}}$ is a partially single-peaked domain with respect to $\prec$ and x, y, and

(ii) $\tilde{\mathcal{S}}$ contains a top-connected single-peaked domain with respect to $\prec$.

Definition 4.8. Let $\prec \in \mathbb{L}(X)$ be a prior order over X. Further, let $\beta = (\beta_S)_{S \subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = \max_\prec(X)$, $\beta_N = \min_\prec(X)$, and (iii) for all

$S \subseteq T$, $\beta_T \preceq \beta_S$. Then, an SCF $f^\beta : \mathcal{D}^n \to X$ is called a *min-max rule with respect to* $\prec$ *and* β if:

$$f^\beta(P_1, \ldots, P_n) = \min_{S \subseteq N}\{\max_{i \in S}\{r_1(P_i), \beta_S\}\},$$

where all the minimums and maximums are taken with respect to the prior order $\prec$.

Definition 4.9. Let $x_1 \prec \ldots \prec x_m$ be a prior order over X. Let $x, y \in X$ be such that $x \prec y^-_\prec$. Then, a min-max rule $f^\beta : \mathcal{D}^n \to X$ with parameters $(\beta_S)_{S \subseteq N}$ is a *partly dictatorial generalized median voter scheme with respect to* $\prec$ *and* x, y if there exists an agent $d \in N$ called the *partial dictator* of f^β such that $\beta_d \in [x_1, x]_\prec$ and $\beta_{N \setminus d} \in [y, x_m]_\prec$.

Definition 4.10. Let $\mathcal{L} = \{\prec_1, \ldots, \prec_q\}$, where $\prec_k \in \mathbb{L}(X)$ for all $k \leq q$ be a set of q prior orders over X. A domain is called a *multiple single-peaked domain with respect to* $\mathcal{L}$, denoted by $\mathcal{S}_\mathcal{L}$, if $\mathcal{S}_\mathcal{L} = \bigcup\limits_{k \in \{1, \ldots, q\}} \bar{\mathcal{S}}_{\prec_k}$, where $\bar{\mathcal{S}}_{\prec_k}$ is the maximal single-peaked domain with respect to the prior order $\prec_k$. A multiple single-peaked domain with respect to $\mathcal{L}$ is called *trivial* if $\bar{\mathcal{S}}_\prec = \bar{\mathcal{S}}_{\prec'}$ for all $\prec, \prec' \in \mathcal{L}$.

Definition 4.11. Let $\mathcal{S}_\mathcal{L}$ be a non-trivial multiple single-peaked domain with respect to a set of prior orders $\mathcal{L}$. Then, alternatives $u, v \in X$ are called *break-points* of $\mathcal{S}_\mathcal{L}$ if:

(i) for all preferences $P \in \mathcal{S}_\mathcal{L}$, all $\prec \in \mathcal{L}$, and all $c, d \in X$ such that $\{c, d\} \cap [u, v]_\prec = \emptyset$, $\big[d \prec c \preceq r_1(P)$ or $r_1(P) \preceq c \prec d\big]$ implies cPd, and

(ii) there exist $\prec \in \mathcal{L}$ and $P, P' \in \mathcal{S}_\mathcal{L}$ such that $u \prec v^-_\prec$, $r_1(P) = u$, $r_2(P) \in (u^+_\prec, v]_\prec$, $r_1(P') = v$, and $r_2(P') \in [u, v^-_\prec)_\prec$.

REMARK 4.2. Let $\mathcal{S}_\mathcal{L}$ be a non-trivial multiple single-peaked domain with respect to $\mathcal{L}$. Then, the break-points of $\mathcal{S}_\mathcal{L}$ are unique.[10]

REMARK 4.3. Let u, v be the break-points of a non-trivial multiple single-peaked domain $\mathcal{S}_\mathcal{L}$ and let $\prec \in \mathcal{L}$. Suppose $\min\limits_\prec(X) = x_1$ and $\max\limits_\prec(X) = x_m$. Then, u, v induce the partition $\{X_L, X_M, X_R\}$ of X, where $X_L = [x_1, u)_\prec$, $X_M = [u, v]_\prec$, and $X_R = (v, x_m]_\prec$. Such a partition is called the *maximal common decomposition* of X. It can be verified that the maximal decomposition does not depend on the choice of the prior order $\prec \in \mathcal{L}$. Reffgen (2015) calls the sets X_L, X_M, and X_R as the *left component*, the *middle component*, and the *right component* of alternatives respectively.

[10]The proof of this fact is available upon request.

In the following, we illustrate the notion of break-points of a non-trivial multiple single-peaked domain by means of an example.

Example 4.4. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be the set of alternatives. Consider the set of prior orders $\mathcal{L} = \{\prec_1, \prec_2, \prec_3, \prec_4\}$, where $\prec_1 = x_1x_2x_3x_5x_4x_6x_7$, $\prec_2 = x_1x_2x_5x_4x_3x_6x_7$, $\prec_3 = x_1x_2x_3x_4x_5x_6x_7$, and $\prec_4 = x_1x_2x_4x_3x_5x_6x_7$. Let $\mathcal{S}_{\mathcal{L}}$ be the multiple single-peaked domain with respect to $\mathcal{L}$. Clearly, $\mathcal{S}_{\mathcal{L}}$ is non-trivial since $\bar{\mathcal{S}}_{\prec_1} \neq \bar{\mathcal{S}}_{\prec_2}$. We claim $u = x_2$ and $v = x_6$ are the break points of $\mathcal{S}_{\mathcal{L}}$. To see this, note that $\prec_1$, $\prec_2$, $\prec_3$, and $\prec_4$ have same relative orderings over the sets $\{x_1, x_2\}$ and $\{x_6, x_7\}$. Therefore, Condition (i) in Definition 4.1 is satisfied for all $P \in \bar{\mathcal{S}}_{\prec_i}$ and all $i = 1, 2, 3, 4$. For Condition (ii), consider the prior order $\prec_2$. Then, $x_2 \prec_2 x_3 \prec_2 x_6$ and there are preferences $P, P' \in \bar{\mathcal{S}}_{\prec_2} \subseteq \mathcal{S}_{\mathcal{L}}$ such that $r_1(P) = x_2$, $r_2(P) = x_5$, $r_1(P') = x_6$, and $r_2(P') = x_3$. To obtain the maximal common decomposition, consider a prior order in $\mathcal{L}$, say $\prec_1$. Then, x_2 and x_6 induce the maximal common decomposition of X given by $X_L = \{x_1\}$, $X_M = \{x_2, x_3, x_4, x_5, x_6\}$, and $X_R = \{x_7\}$. It can be verified that the maximal common decomposition of X does not change if we consider some prior order in $\mathcal{L}$ other than the order $\prec_1$.

REMARK 4.4. Let $\mathcal{S}_{\mathcal{L}}$ be a non-trivial multiple single-peaked domain with break-points u, v and let $\prec \in \mathcal{L}$ be arbitrary. Then, $\mathcal{S}_{\mathcal{L}}$ is a top-connected partially single-peaked domain with respect to $\prec$ and u, v.

Corollary 4.2 (Reffgen (2015))**.** *Let $\mathcal{S}_{\mathcal{L}}$ be a non-trivial multiple single-peaked domain with break-points u, v and let $\prec \in \mathcal{L}$ be arbitrary. Then, an SCF $f : \mathcal{S}^n_{\mathcal{L}} \to X$ is unanimous and strategy-proof if and only if it is a PDGMVS with respect to $\prec$ and u, v.*

4.3.2 Single-peaked Domains on Graphs

In this subsection, we introduce the notion of single-peaked domains on graphs. All the graphs considered in this subsection are undirected.

Definition 4.12. Let $T = \langle X, E \rangle$ be a tree. A domain is called *single-peaked with respect to T*, denoted by $\mathcal{D}_T$, if for all $P \in \mathcal{D}_T$ and all distinct $x, y \in X$,

$$[x \in \pi_T(r_1(P), y)] \implies [xPy].$$

Definition 4.13. Let $G = \langle X, E \rangle$ be a undirected connected graph. A domain is called *single-peaked with respect to G*, denoted by $\mathcal{D}_G$, if $\mathcal{D}_G = \cup_{T \in \mathcal{T}_G} \mathcal{D}_T$.

Note that if T is the undirected line graph on X, then $\mathcal{D}_T$ is a maximal single-peaked domain. In our next lemma, we show that if a domain is single-peaked with respect to an undirected partial line graph, then it is a top-connected partially single-peaked domain.

Lemma 4.3. *Let $x < y - 1$ and let G be an undirected partial line graph with respect to x and y. Then, $\mathcal{D}_G$ is a top-connected partially single-peaked domain with respect to x, y.*

Proof. Let G be an undirected partial line graph with respect to x, y, where $x < y - 1$. We show that $\mathcal{D}_G$ is a top-connected partially single-peaked domain. Let $G = G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the undirected line graph on X and $G_2 = \langle [x, y], E_2 \rangle$ is an undirected graph such that for all $z \in \{x, y\}$ there is a cycle involving z.

First, we show that $\mathcal{D}_G$ is partially single-peaked. Take $P \in \mathcal{D}_G$ with $r_1(P) \in [x, y]$ and take $u, v \in X \setminus (x, y)$. Suppose $[v < u \leq r_1(P)$ or $r_1(P) \leq u < v]$. Consider an arbitrary spanning tree T of G. Then, by the definition of G, $u \in \pi_T(r_1(P), v)$. Therefore, uPv. Using similar logic, it follows that $\mathcal{D}_G$ satisfies Condition (ii) in Definition 4.1. Finally, we show that there are $Q, Q' \in \mathcal{D}_G$ satisfying Condition (iii) in Definition 4.1. Let C be a cycle in G_2 involving the node x. Then, there must be an edge $\{x, x'\}$ in C such that $x' \in (x + 1, y]$. Consider the tree $T = \langle X, E \rangle$ such that $E = (E_1 \setminus \{x, x + 1\}) \cup \{x, x'\}$. Since $G_1 = \langle X, E_1 \rangle$ is the undirected line graph on X, T is a spanning tree of G. Because $\{x, x'\} \in E$, there is a preference $P \in \mathcal{D}_T$ with $r_1(P) = x$ and $r_2(P) = x'$. Therefore, P satisfies Condition (iii) in Definition 4.1 for Q. Similarly, it can be shown that there is $P' \in \mathcal{D}_G$ that satisfies Condition (iii) in Definition 4.1 for Q'. Now, suppose $x' = y$. That means there is an edge in T connecting x and y. Therefore, there is $P' \in \mathcal{D}_T$ such that $r_1(P') = y$ and $r_2(P') = x$. Therefore, P and P' satisfy the restrictions on the second ranked alternatives of Q and Q' given by Condition (iii) in Definition 4.1, respectively.

Now, we show that $\mathcal{D}_G$ contains a top-connected single-peaked domain. Since G_1 is the undirected line graph on X, $\mathcal{D}_{G_1}$ is a top-connected single-peaked domain. Moreover, since G_1 is a spanning tree of G, $\mathcal{D}_{G_1} \subseteq \mathcal{D}_G$. This completes the proof of the lemma. ∎

Now, we have the following corollary.

Corollary 4.3. *Let $x, y \in X$ be such that $x < y - 1$ and let $G = \langle X, E \rangle$ be a undirected partial line graph with respect to x, y. Suppose $\mathcal{D}_G$ is the single-peaked domain with respect to G. Then, an SCF $f : \mathcal{D}_G^n \to X$ is unanimous and strategy-proof if and only if it is a PDGMVS.*

5. Group Strategy-proofness

In this section, we consider group strategy-proofness and obtain a characterization of the unanimous and group strategy-proof SCFs on top-connected single-peaked domains and top-connected partially single-peaked domains. We begin with the definition of group strategy-proofness.

Definition 5.1. An SCF $f : \mathcal{D}^n \to X$ is called *group manipulable* if there is a preference profile P_N, a non-empty coalition $C \subseteq N$, and a preference profile $P'_C \in \mathcal{D}^{|C|}$ of the agents in C such that $f(P'_C, P_{N\setminus C}) P_i f(P_N)$ for all $i \in C$. An SCF $f : \mathcal{D}^n \to X$ is called *group strategy-proof* if it is not group manipulable.

Barberà et al. (2010) established a sufficient condition that ensures the equivalence of strategy-proofness and group strategy-proofness on a domain. It can be easily verified that top-connected single-peaked domains satisfy their sufficient condition. Thus, we have the following corollary.

Corollary 5.1. *Let $\hat{\mathcal{S}}$ be a top-connected single-peaked domain. Then, an SCF $f : \hat{\mathcal{S}}^n \to X$ is unanimous and group strategy-proof if and only if it is a min-max rule.*

In the following theorem, we present a characterization of the unanimous and group strategy-proof SCFs on top-connected partially single-peaked domains. It is worth mentioning that these domains do not satisfy the sufficient condition for the equivalence of strategy-proofness and group strategy-proofness provided in Barberà et al. (2010).

Theorem 5.1. *Let $x, y \in X$ be such that $x < y - 1$ and let $\tilde{\mathcal{S}}$ be a top-connected partially single-peaked domain with respect to x, y. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and group strategy-proof if and only if it is a PDGMVS with respect to x, y.*

Proof. Let $x, y \in X$ be such that $x < y - 1$ and let $\tilde{\mathcal{S}}$ be a top-connected partially single-peaked domain with respect to x, y. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a PDGMVS with respect to x, y with partial dictator d. It is enough to show that f is group strategy-proof. Clearly, no group can manipulate f at a preference profile $P_N \in \tilde{\mathcal{S}}^n$, where $r_1(P_d) \in [x, y]$. Consider a preference profile $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, x)$. We show that f is group strategy-proof at P_N. By the definition of PDGMVS, $f(P_N) \in [a, x]$. Let $C' = \{i \in N \mid r_1(P_i) \leq f(P_N)\}$ and $C'' = \{i \in N \mid r_1(P_i) > f(P_N)\}$. Suppose a coalition C manipulates f at P_N. Then, there is $P'_C \in \tilde{\mathcal{S}}^{|C|}$ such that $f(P'_C, P_{N\setminus C}) P_i f(P_N)$ for all $i \in C$. If $f(P'_C, P_{N\setminus C}) < f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C'' = \emptyset$. However, by the definition of PDGMVS, $f(P'_C, P_{N\setminus C}) \geq f(P_N)$ for all $C \subseteq C'$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. Again, if $f(P'_C, P_{N\setminus C}) > f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C' = \emptyset$. However, by the definition of PDGMVS, $f(P'_C, P_{N\setminus C}) \leq f(P_N)$ for all $C \subseteq C''$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. The proof for the case where $r_1(P_d) \in (y, b]$ follows from symmetric argument. ■

6. Conclusion

In this paper, we have introduced a class of restricted domains which we call top-connected single-peaked domains and have characterized the unanimous and strategy-proof SCFs on such domains as min-max rules. Outstanding examples of top-connected single-peaked domains are maximal single-peaked domains (Moulin (1980), Weymark (2011)), minimally rich single-peaked domains (Peters et al. (2014)), distance based single-peaked domains, and regular single-crossing domains (Saporiti (2009)). Further, we have introduced the notion of min-max domains, for which the set of unanimous and strategy-proof SCFs coincides with the set of min-max rules. We show that a domain is a min-max domain if and only if it is a top-connected single-peaked domain.

Next, we have considered domains that violate single-peaked property over a subset of alternatives. We call such domains top-connected partially single-peaked domains. For such a domain, there are two alternatives x, y with $x < y - 1$ such that the domain satisfies single-peaked property over $[a, x]$ and $[y, b]$, and violate the property over (x, y). We have shown that an SCF is unanimous and strategy-proof on such a domain if and only if it is a PDGMVS with respect to x, y. We have also shown that the top-connected partially single-peaked domains includes almost all domains with the property that the set of unanimous and strategy-proof SCFs coincides with the set of PDGMVS. Outstanding examples of top-connected partially single-peaked domains are multiple single-peaked domains (Reffgen (2015)) and single-peaked domains on graphs. Our result on partial necessity shows that top-connected partially single-peaked domains are the minimal subsets of a multiple single-peaked domain such that every unanimous and strategy-proof SCF is a PDGMVS.

Finally, we have considered group strategy-proofness. It follows from Barberà et al. (2010) that strategy-proofness and group strategy-proofness are equivalent on top-connected single-peaked domains. However, their result does not apply on top-connected partially single-peaked domains. In this paper, we have shown that strategy-proofness and group strategy-proofness are equivalent on these domains, and in particular, we have provided a characterization of unanimous and group strategy-proof SCFs on such domains.

APPENDIX A. PROOF OF THEOREM 3.1

Proof. (If-part) Suppose $f : \hat{\mathcal{S}}^n \to X$ is a min-max rule. Then, f is unanimous and strategy-proof on the maximal single-peaked domain (Weymark (2011)). Since every top-connected single-peaked domain is a subset of the maximal single-peaked domain, f is unanimous and strategy-proof on $\hat{\mathcal{S}}^n$.

(Only-if part) Let $f : \hat{\mathcal{S}}^n \to X$ be a unanimous and strategy-proof SCF. We show that f is a min-max rule. In what follows, we establish a few properties of f in the following sequence of lemmas.

In the following lemma, we show that the outcome of f at every preference profile $P_N \in \hat{\mathcal{S}}^n$ must lie in-between $\min(\tau(P_N))$ and $\max(\tau(P_N))$.

Lemma A.1. *It must be that* $f(P_N) \in [\min(\tau(P_N)), \max(\tau(P_N))]$ *for all* $P_N \in \hat{\mathcal{S}}^n$.

Proof. Assume to the contrary that $f(P_N) \notin [\min(\tau(P_N)), \max(\tau(P_N))]$ for some $P_N \in \hat{\mathcal{S}}^n$. Without loss of generality, assume that $f(P_N) = x < \min(\tau(P_N))$. Then, $f(P_N) = x < x+1 \leq \min(\tau(P_N)) \leq r_1(P_i)$ for all $i \in N$. Since P_i is single-peaked, this means $(x+1)P_i x$ for all $i \in N$. For each $i \in N$, consider $P'_i \in \hat{\mathcal{S}}$ such that $r_1(P'_i) = x+1$ and $r_2(P'_i) = x$. Then, by strategy-proofness, $f(P'_i, P_{N\setminus i}) = x$. By moving the agents $i \in N$ from the preference P_i to the preference P'_i one-by-one and applying strategy-proofness at every step, we have $f(P_N) = f(P'_1, P_{N\setminus 1}) = f(P'_1, P'_2, P_{N\setminus\{1,2\}}) = \ldots = f(P'_1, \ldots, P'_{n-1}, P_n) = x$. However, by unanimity, $f(P'_1, \ldots, P'_n) = x+1$. This means agent n manipulates at $(P'_1, \ldots, P'_{n-1}, P_n)$ via P'_n, a contradiction. This completes the proof. ■

Our next lemma and its corollary establish a restricted version of uncompromisingness. The implication of the lemma is as follows. Consider a preference profile P_N. Fix an alternative $y \in X$. Construct another preference profile P'_N where each agent with top-ranked alternatives at P_N on the left (right) of y move to a preference with top-ranked alternative y, while all other agents keep their preferences unchanged. Then, (i) if $f(P_N)$ was on the right (left) of y, then $f(P'_N) = f(P_N)$, and (ii) if $f(P_N)$ was on the left (right) of y, then $f(P'_N) = y$.

Lemma A.2. *Let* $P_N, P'_N \in \hat{\mathcal{S}}^n$ *and* $y \in X$ *be such that for all* $i \in N$, *if* $r_1(P_i) < y$ *then* $r_1(P'_i) = y$, *otherwise* $P_i = P'_i$. *Then,* $f(P'_N) = \max\{f(P_N), y\}$.

Proof. Suppose $f(P_N) = x$. Suppose further that $y \leq x$. Note that if $y \leq \min(P_N)$, then $P'_N = P_N$, and hence, by Lemma A.1, $y \leq f(P_N)$. Therefore, there is nothing to prove. Suppose $\min(P_N) < y$. Let $i \in N$ be such that $r_1(P_i) = \min(P_N)$. Take $P'_i \in \hat{\mathcal{S}}$ such that $r_1(P'_i) = y$. We show that $f(P'_i, P_{N\setminus i}) = x$. Suppose $f(P'_i, P_{N\setminus i}) > x$. Since P'_i is single-peaked and $r_1(P'_i) \leq x < f(P'_i, P_{N\setminus i})$, it must be that $xP'_i f(P'_i, P_{N\setminus i})$. This means agent i manipulates at $(P'_i, P_{N\setminus i})$ via P_i, a contradiction. Now suppose $f(P'_i, P_{N\setminus i}) < x$. Since $r_1(P_i) < r_1(P'_i)$, we have $\min(P_N) \leq \min(P'_i, P_{N\setminus i})$. Because $r_1(P_i) = \min(P_N)$ and $\min(P_N) \leq \min(P'_i, P_{N\setminus i})$, by Lemma A.1, it must be that $r_1(P_i) \leq f(P'_i, P_{N\setminus i})$. Since P_i is single-peaked and $r_1(P_i) \leq f(P'_i, P_{N\setminus i}) < x$, it follows that $f(P'_i, P_{N\setminus i})P_i x$. This means agent i manipulates at P_N via P'_i, a contradiction. Therefore, $f(P'_i, P_{N\setminus i}) = x$. Now, if $y \leq \min(P'_i, P_{N\setminus i})$, then by the condition of the lemma, $P'_N = (P'_i, P_{N\setminus i})$, and the proof is complete. Suppose $\min(P'_i, P_{N\setminus i}) < y$. Consider $j \in N$ such that $r_1(P_j) = \min(P'_i, P_{N\setminus i})$. Let P'_j be such that $r_1(P'_j) = y$. Using similar argument as before, it follows that $f(P'_i, P'_j, P_{N\setminus\{i,j\}}) = f(P'_i, P_{N\setminus i}) = f(P_N)$. Continuing in this manner, it follows that $f(P'_N) = f(P_N)$. This completes the proof of the lemma for $y \leq x$.

Now, suppose $y > x$. Let $y = x + k$ for some positive integer k. Let $\hat{P}_N \in \hat{\mathcal{S}}^n$ be such that $r_1(\hat{P}_i) = x$ for all $i \in N$ with $r_1(P_i) \leq x$ and $\hat{P}_i = P_i$ for all other agents. By strategy-proofness, $f(\hat{P}_N) = x$. Let $N_x = \{i \in N \mid r_1(\hat{P}_i) = x\}$. Suppose $\bar{P} \in \hat{\mathcal{S}}$ is such that $r_1(\bar{P}) = x + 1$ and $r_2(\bar{P}) = x$. Take $i \in N_x$ and let $\bar{P}_i = \bar{P}$. Then, by strategy-proofness, $f(\bar{P}_i, \hat{P}_{N\setminus i}) \in \{x, x+1\}$ as otherwise agent i manipulates at $(\bar{P}_i, \hat{P}_{N\setminus i})$ via $\hat{P}_i$. Using similar argument, $f(\bar{P}_i, \bar{P}_j, \hat{P}_{N\setminus\{i,j\}}) \in \{x, x+1\}$, where $i, j \in N_x$ and $\bar{P}_j = \bar{P}$. Continuing in this manner, we have $f(\bar{P}_N) \in \{x, x+1\}$, where $\bar{P}_i = \bar{P}$ for all $i \in N_x$ and $\bar{P}_i = \hat{P}_i$ for all $i \in N \setminus N_x$. However, $\min(\bar{P}_N) = x + 1$. Hence, by Lemma A.1, $f(\bar{P}_N) = x + 1$. Suppose $N_{x+1} = \{i \in N \mid r_1(\bar{P}_i) = x + 1\}$. Let $\tilde{P} \in \hat{\mathcal{S}}$ be such that $r_1(\tilde{P}) = x + 2$ and $r_2(\tilde{P}) = x + 1$. Further, let $\tilde{P}_N \in \hat{\mathcal{S}}^n$ be such that $\tilde{P}_i = \tilde{P}$ if $i \in N_{x+1}$ and $\tilde{P}_i = \bar{P}_i$ otherwise. Then, by using arguments similar to the above, we have $f(\tilde{P}_N) = x + 2$. Continuing in this manner, we have $f(P''_N) = x + k$, where $P''_N \in \hat{\mathcal{S}}^n$ is such that for all $i \in N$, if $r_1(P_i) < y$ then $r_1(P''_i) = y$ and $r_2(P''_i) = y - 1$, otherwise $P_i = P''_i$. By strategy-proofness, this means $f(P'_N) = x + k$, which completes the proof of the lemma for $y > x$. ■

Corollary A.1. *Let $P_N, P'_N \in \hat{\mathcal{S}}^n$ and $y \in X$ be such that for all $i \in N$, if $r_1(P_i) > y$ then $r_1(P'_i) = y$, otherwise $P_i = P'_i$. Then, $f(P'_N) = \min\{f(P_N), y\}$.*

Our next lemma shows that f is uncompromising.[11]

Lemma A.3. *The SCF f is uncompromising.*

Proof. Let $P_N \in \hat{\mathcal{S}}^n$, $i \in N$, and $P'_i \in \hat{\mathcal{S}}$ be such that $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \le f(P_N)$. It is sufficient to show $f(P'_i, P_{N\setminus i}) = f(P_N)$. Suppose $f(P_N) = x$, $r_1(P_i) = y$, and $r_1(P'_i) = y'$. Assume for contradiction that $f(P'_i, P_{N\setminus i}) = x' \neq x$. By strategy-proofness, it must be that $x' < y$ as otherwise agent i manipulates either at P_N via P'_i or at $(P'_i, P_{N\setminus i})$ via P_i. Consider $\bar{P}_N \in \hat{\mathcal{S}}^n$ such that $r_1(\bar{P}_j) = y$ for all $j \in N$ with $r_1(P_j) \le y$, and $\bar{P}_j = P_j$ for all other agents. Since $f(P_N) = x$, by Lemma A.2, we have $f(\bar{P}_N) = \max\{x, y\} = x$. On the other hand, since $f(P'_i, P_{N\setminus i}) = x'$, by Lemma A.2, we have $f(\bar{P}_N) = \max\{x', y\} = y$, a contradiction. This completes the proof of the lemma. ■

The following lemma establishes that f is a min-max rule.

Lemma A.4. *The SCF f is a min-max rule.*

Proof. For all $S \subseteq N$, let $(P^a_S, P^b_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(P^a_i) = a$ for all $i \in S$ and $r_1(P^b_i) = b$ for all $i \in N \setminus S$. Define $\beta_S = f(P^a_S, P^b_{N\setminus S})$ for all $S \subseteq N$. Clearly, $\beta_S \in X$ for all $S \subseteq N$. By unanimity, $\beta_\varnothing = b$ and $\beta_N = a$. Also, by uncompromisingness, $\beta_S \le \beta_T$ for all $T \subseteq S$.

Take $P_N \in \hat{\mathcal{S}}^n$. We show $f(P_N) = \min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\}$. Suppose $S_1 = \{i \in N \mid r_1(P_i) < f(P_N)\}$, $S_2 = \{i \in N \mid f(P_N) < r_1(P_i)\}$, and $S_3 = \{i \in N \mid r_1(P_i) = f(P_N)\}$. By uncompromisingness, $\beta_{S_1\cup S_3} \le f(P_N) \le \beta_{S_1}$. Now, consider the expression $\min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\}$. Take $S \subseteq S_1$. Then, by Condition (iii) in Definition 2.11, $\beta_{S_1} \le \beta_S$. Since $r_1(P_i) < f(P_N)$ for all $i \in S$ and $f(P_N) \le \beta_{S_1} \le \beta_S$, we have $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} = \beta_S$. Similarly, for all $S \subseteq N$ such that $S \cap S_2 \neq \varnothing$, we have $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} > f(P_N)$. Consider $S \subseteq N$ such that $S \cap S_2 = \varnothing$ and $S \cap S_3 \neq \varnothing$. Then, $S \subseteq S_1 \cup S_3$, and hence, $\beta_{S_1\cup S_3} \le \beta_S$. Therefore, $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} = \max\{f(P_N), \beta_S\} \ge \max\{f(P_N), \beta_{S_1\cup S_3}\}$. Since $\beta_{S_1\cup S_3} \le f(P_N)$, we have $\max\{f(P_N), \beta_{S_1\cup S_3}\} = f(P_N)$. Combining all these, we have $\min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\} = \min\{\beta_{S_1}, f(P_N)\}$. Because $f(P_N) \le \beta_{S_1}$, we have $\min\{\beta_{S_1}, f(P_N)\} = f(P_N)$. This completes the proof. ■

The proof of the only-if part of Theorem 3.1 follows from Lemmas A.1 - A.4. ■

[11] Since every SCF satisfying uncompromisingness is tops-only, Lemma A.3 shows that a top-connected single-peaked domain is a tops-only domain. It can be easily verified that top-connected single-peaked domains fail to satisfy the sufficient conditions for a domain to be tops-only identified in Chatterji and Sen (2011) and Chatterji and Zeng (2015).

APPENDIX B. PROOF OF THEOREM 4.1

Proof. (If-part) Let $x, y \in X$ be such that $x < y - 1$ and let f^β be a PDGMVS on $\tilde{\mathcal{S}}^n$ with respect to x, y. Then, f^β is unanimous by definition. We show that f^β is strategy-proof. Let d be the partial dictator of f^β. Since $f^\beta(P_N) = r_1(P_d)$ whenever $r_1(P_d) \in [x, y]$, f^β cannot be manipulated at a preference profile $P_N \in \tilde{\mathcal{S}}^n$, where $r_1(P_d) \in [x, y]$. Take $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, x)$. Then, by Lemma 4.1, $f^\beta(P_N) \in [a, x]$. Take $i \in N$ such that $r_1(P_i) \leq f^\beta(P_N)$. By the definition of f^β, $f^\beta(P'_i, P_{N \setminus i}) \geq f^\beta(P_N)$ for all $P'_i \in \tilde{\mathcal{S}}$. Since $f^\beta(P_N) \leq x$, by the definition of partially single-peaked domain, $r_1(P_i) \leq f^\beta(P_N)$ means $f^\beta(P_N) P_i z$ for all $z > f^\beta(P_N)$. Therefore, agent i cannot manipulate f^β at P_N. By symmetric argument, it follows that agent i cannot manipulate f^β at a preference profile where $r_1(P_i) \geq f^\beta(P_N)$. Using similar logic, it follows that f^β is strategy-proof when $r_1(P_d) \in (y, b]$. This completes the proof of the if-part.

(Only-if part) Let $x, y \in X$ be such that $x < y - 1$ and let $\tilde{\mathcal{S}}$ be a top-connected partially single-peaked domain with respect to x, y. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a unanimous and strategy-proof SCF. We show that f is a PDGMVS with respect to x, y. Let $\hat{\mathcal{S}}$ be a top-connected single-peaked domain contained in $\tilde{\mathcal{S}}$. Such a domain must exist by Definition 4.2. By Theorem 3.1, f restricted to $\hat{\mathcal{S}}^n$ must be a min-max rule. We establish a few properties of f in the following sequence of lemmas.

Our next lemma shows that f satisfies tops-onlyness for a particular type of preference profiles. It says the following. Let c be an arbitrary alternative. Consider a preference profile P_N where each P_i is single-peaked with the top alternative either x or c. Construct another preference profile P'_N where some agents with top alternatives x change their preferences (possibly to a non-single-peaked preference) keeping x at the top, while all other agents keep their preferences unchanged. Suppose the outcome of f at P_N is c. Then, the outcome of f at P'_N must be c.

Lemma B.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(P_S, P_{N \setminus S}) \in \hat{\mathcal{S}}^n$ and $(P'_S, P_{N \setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $r_1(P_i) = x$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Then, $f(P_S, P_{N \setminus S}) = c$ implies $f(P'_S, P_{N \setminus S}) = c$.*

Proof. Take S such that $\emptyset \subsetneq S \subsetneq N$. We prove the lemma using induction on $|c - x|$. By unanimity, the lemma holds for $c = x$. Suppose the lemma holds for all c such that $|c - x| \leq k$. We prove the lemma for all c such that $|c - x| = k + 1$. Take c such that $|c - x| = k + 1$. Let $(P_S, P_{N \setminus S}) \in \hat{\mathcal{S}}^n$ and $(P'_S, P_{N \setminus S}) \in \tilde{\mathcal{S}}^n$ be two tops-equivalent preference profiles such that $r_1(P_i) = x$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Suppose $f(P_S, P_{N \setminus S}) = c$. We show $f(P'_S, P_{N \setminus S}) = c$. We show this for $x < c$, the proof for the case $x > c$ is similar. Since $x < c$ and $|c - x| = k + 1$, we have $c = x + k + 1$. Let $(P_S, \hat{P}_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(\hat{P}_j) = x + k$ and $r_2(\hat{P}_j) = x + k + 1$ for all $j \in N \setminus S$. Because f is a min-max rule on $\hat{S}^n$ and $f(P_S, P_{N\setminus S}) = x + k + 1$, we have $f(P_S, \hat{P}_{N\setminus S}) = x + k$. Since $(P_S, \hat{P}_{N\setminus S})$ and $(P'_S, \hat{P}_{N\setminus S})$ are tops-equivalent and $r_1(\hat{P}_j) = x + k$ for all $j \in N \setminus S$, we have by the induction hypothesis $f(P'_S, \hat{P}_{N\setminus S}) = x + k$. For all $j \in N \setminus S$, let $\bar{P}_j \in \hat{\mathcal{S}}$ be such that $r_1(\bar{P}_j) = x + k + 1$ and $r_2(\bar{P}_j) = x + k$. Since $f(P'_S, \hat{P}_{N\setminus S}) = x + k$, by moving the agents $j \in N \setminus S$ from $\hat{P}_j$ to $\bar{P}_j$ one-by-one and applying strategy-proofness at every step, we have $f(P'_S, \bar{P}_{N\setminus S}) \in \{x + k, x + k + 1\}$. We claim $f(P'_S, \bar{P}_{N\setminus S}) = x + k + 1$. Assume for contradiction that $f(P'_S, \bar{P}_{N\setminus S}) = x + k$. Recall that $P_i \in \hat{\mathcal{S}}$ for all $i \in S$. Since $(x + k)P_i(x + k + 1)$ for all $i \in S$, by moving the agents $i \in S$ from P'_i to P_i one-by-one and applying strategy-proofness at every step, we have $f(P_S, \bar{P}_{N\setminus S}) \leq x + k$. Since $r_1(P_j) = r_1(\bar{P}_j) = x + k + 1$ for all $j \in N \setminus S$, by strategy-proofness, we have $f(P_S, P_{N\setminus S}) \neq x + k + 1$. This contradicts our assumption that $f(P_S, P_{N\setminus S}) = x + k + 1$. Therefore, $f(P'_S, \bar{P}_{N\setminus S}) = x + k + 1$. Since $r_1(P_j) = r_1(\bar{P}_j) = x + k + 1$ for all $j \in N \setminus S$, we have by strategy-proofness, $f(P'_S, P_{N\setminus S}) = x + k + 1$. This completes the proof. ■

Corollary B.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(P_S, P_{N\setminus S}) \in \hat{\mathcal{S}}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $r_1(P_i) = y$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Then, $f(P_S, P_{N\setminus S}) = c$ implies $f(P'_S, P_{N\setminus S}) = c$.*

Our next lemma shows that the outcome of f at a boundary preference profile cannot be strictly in-between x and y.[12]

Lemma B.2. *Let $P_N \in \tilde{\mathcal{S}}^n$ be such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Then, $f(P_N) \notin (x, y)$.*

Proof. Assume for contradiction that $f(P_N) = z \in (x, y)$ for some $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Let $S = \{i \in N \mid r_1(P_i) = a\}$. Then, it must be that $\emptyset \subsetneq S \subsetneq N$ as otherwise we are done by unanimity. Let $r_2(Q) = x'$ and $r_2(Q') = y'$, where $Q, Q' \in \tilde{\mathcal{S}}$ are as given in Condition (iii) of Definition 4.1. We distinguish a few cases based on the relative positions of x', y', and z.

CASE 1. Suppose $x' \in (x + 1, y - 1)$, $y' \in (x + 1, y - 1)$, and $z \in (x, y'] \cup [x', y)$. We consider the case where $z \in (x, y']$, the proof for the case where $z \in [x', y)$ follows from symmetric argument. Let $P'_N \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = y'$ for all $i \in S$ and $r_1(P'_j) = y - 1$ for all

[12] A *boundary preference profile* is one where the top alternative of each agent is either a or b.

$j \in N \setminus S$ and let $\hat{P}_N \in \hat{\mathcal{S}}^n$ be such that $r_1(\hat{P}_i) = x$ for all $i \in S$ and $r_1(\hat{P}_j) = x+1$ for all $j \in N \setminus S$. Suppose that for all $j \in N \setminus S$, $r_2(P'_j) = y$. Because f is a min-max rule on $\hat{\mathcal{S}}^n$ and $f(P_S, P_{N\setminus S}) = z$, we have $f(P'_S, P'_{N\setminus S}) = y'$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = x+1$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = x+1$, by Lemma B.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = x+1$, where $Q_i = Q$ for all $i \in S$. Consider the preference profile $(Q'_S, P'_{N\setminus S})$, where $Q'_i = Q'$ for all $i \in S$. Note that $f(P'_S, P'_{N\setminus S}) = y'$, $r_1(Q') = y$, and $r_2(Q') = y'$. Therefore, by moving the agents $i \in S$ from P'_i to Q' one-by-one and using strategy-proofness at every step, we have $f(Q'_S, P'_{N\setminus S}) \in \{y, y'\}$. We claim $f(Q'_S, P'_{N\setminus S}) = y$. Assume for contradiction that $f(Q'_S, P'_{N\setminus S}) = y'$. Since yP'_jy' for all $j \in N \setminus S$, by moving the agents $j \in N \setminus S$ from P'_j to Q' one-by-one and applying strategy-proofness at every step, we have $f(Q'_S, Q'_{N\setminus S}) \neq y$. However, this contradicts unanimity. So, $f(Q'_S, P'_{N\setminus S}) = y$. For all $i \in S$, let $\tilde{P}_i \in \hat{\mathcal{S}}$ be such that $r_1(\tilde{P}_i) = y$. By strategy-proofness, $f(\tilde{P}_S, P'_{N\setminus S}) = y$. Since f is a min-max rule on $\hat{\mathcal{S}}^n$, this means $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = y$. For all $i \in S$, let $\tilde{P}'_i \in \hat{\mathcal{S}}$ be such that $r_1(\tilde{P}'_i) = x'$. Because $(\tilde{P}_S, \hat{P}_{N\setminus S}), (\tilde{P}'_S, \hat{P}_{N\setminus S}) \in \hat{\mathcal{S}}^n$ and f is a min-max rule on $\hat{\mathcal{S}}^n$, $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = y$ implies $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = x'$. Note that $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = x'$, $r_1(Q) = x$, and $r_2(Q) = x$. Therefore, by moving agents $i \in S$ from $\tilde{P}'_i$ to Q one-by-one and applying strategy-proofness at every step, we have $f(Q_S, \hat{P}_{N\setminus S}) \in \{x, x'\}$. However, $\{x+1\} \cap \{x, x'\} = \emptyset$ by our assumption. This is a contradiction to our earlier finding that $f(Q_S, \hat{P}_{N\setminus S}) = x+1$. This completes the proof for Case 1.

CASE 2. Suppose $x' \in (x+1, y-1)$, $y' \in (x+1, y-1)$, $y' < x'-1$, and $z \in (y', x')$. Let $P'_N \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = x'$ for all $i \in S$ and $r_1(P'_j) = y$ for all $j \in N \setminus S$ and let $\hat{P}_N \in \hat{\mathcal{S}}^n$ be such that $r_1(\hat{P}_i) = x$ for all $i \in S$ and $r_1(\hat{P}_j) = y'$ for all $j \in N \setminus S$. Because f is a min-max rule on $\hat{\mathcal{S}}^n$ and $f(P_S, P_{N\setminus S}) = z$, we have $f(P'_S, P'_{N\setminus S}) = x'$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = y'$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = y'$, by Lemma B.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = y'$, where $Q_i = Q$ for all $i \in S$. Again, as $f(P'_S, P'_{N\setminus S}) = x'$, by Lemma B.1, we have $f(P'_S, Q'_{N\setminus S}) = x'$, where $Q'_i = Q'$ for all $j \in N \setminus S$. Note that $f(Q_S, \hat{P}_{N\setminus S}) = y'$, $r_1(Q') = y$, and $r_2(Q') = y'$. Therefore, by moving agents $j \in N \setminus S$ from $\hat{P}_j$ to Q' one-by-one and using strategy-proofness at every step, we have $f(Q_S, Q'_{N\setminus S}) \in \{y, y'\}$. Further, note that $f(P'_S, Q'_{N\setminus S}) = x'$, $r_1(Q) = x$, and $r_2(Q) = x'$. So, by moving agents $i \in S$ from P'_i to Q one-by-one and using strategy-proofness at every step, we have $f(Q_S, Q'_{N\setminus S}) \in \{x, x'\}$. However, by our assumption $\{x, x'\} \cap \{y, y'\} = \emptyset$, which is a contradiction. This completes the proof for Case 2.

CASE 3. Suppose $x' = y$, $y' = x$, and $z \in (y', x')$. Let $P'_N \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = x$ for all $i \in S$ and $r_1(P'_j) = y$ for all $j \in N \setminus S$. Because f is a min-max rule on $\hat{\mathcal{S}}^n$ and $f(P_S, P_{N\setminus S}) = z$, we have $f(P'_S, P'_{N\setminus S}) = z$. Take $i \in N$ and consider the preference profile $(Q_i, P'_{S\setminus i}, P'_{N\setminus S})$, where $Q_i = Q$.

Since $r_1(P'_i) = r_1(Q_i) = x$ and $f(P'_S, P'_{N\setminus S}) \neq x$, by strategy-proofness, $f(Q_i, P'_{S\setminus i}, P'_{N\setminus S}) \neq x$. Continuing in this manner, it follows that $f(Q_S, P'_{N\setminus S}) \neq x$, where $Q_i = Q$ for all $i \in S$. Moreover, since $r_2(Q_i) = y$ for all $i \in S$ and $r_1(P'_j) = y$ for all $j \in N \setminus S$, by unanimity and strategy-proofness, $f(Q_S, P'_{N\setminus S}) \in \{x, y\}$. Since $f(Q_S, P'_{N\setminus S}) \neq x$, this means $f(Q_S, P'_{N\setminus S}) = y$. Let $Q'_j = Q'$ for all $j \in N \setminus S$. As $f(Q_S, P'_{N\setminus S}) = y$ and $r_1(Q') = y$, by strategy-proofness, $f(Q_S, Q'_{N\setminus S}) = y$. Now, if we first move the agents $j \in N \setminus S$ from P'_j to Q' and then move the agents $i \in S$ from P'_i to Q, then it follows from similar logic that $f(Q_S, Q'_{N\setminus S}) = x$. Since $x \neq y$, this is a contradiction to our earlier finding that $f(Q_S, Q'_{N\setminus S}) = y$. This completes the proof for Case 3, and hence, the proof of the lemma. ■

Let $(\beta_S)_{S\subseteq N}$ be the parameters of f restricted to $\hat{\mathcal{S}}^n$. In Lemma B.3 and Lemma B.4, we establish a few properties of these parameters.

Lemma B.3. *For all $S \subseteq N$, $\beta_S \in [a, x]$ if and only if $\beta_{N\setminus S} \in [y, b]$.*

Proof. Take $S \subseteq N$. It is enough to show that $\beta_S \in [a, x]$ implies $\beta_{N\setminus S} \in [y, b]$. Assume for contradiction that $\beta_S, \beta_{N\setminus S} \in [a, x]$. Let $Q' \in \tilde{\mathcal{S}}$ with $r_1(Q') = y$ be as given in Condition (iii) of Definition 4.1. Suppose $r_2(Q') = y'$. Take $z \in (y', y)$. Let $(P_S, P_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(P_i) = a$ for all $i \in S$ and $r_1(P_j) = b$ for all $j \in N \setminus S$. Since f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule, $f(P_S, P_{N\setminus S}) = \beta_S \in [a, x]$. Let $(P'_S, P'_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = y'$ for all $i \in S$ and $r_1(P'_j) = z$ for all $j \in N \setminus S$. Since $f(P_S, P_{N\setminus S}) \in [a, x]$, by uncompromisingness of f restricted to $\hat{\mathcal{S}}^n$, we have $f(P'_S, P'_{N\setminus S}) = y'$. Because $r_1(Q') = y$ and $r_2(Q') = y'$, by moving the agents $i \in S$ one-by-one from P'_i to Q' and applying strategy-proofness at every step, we have $f(Q'_S, P'_{N\setminus S}) \in \{y, y'\}$, where $Q'_i = Q'$ for all $i \in S$.

Now, let $(\bar{P}_S, \bar{P}_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(\bar{P}_i) = b$ for all $i \in S$ and $r_1(\bar{P}_j) = a$ for all $j \in N \setminus S$. Again, since f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule, $f(\bar{P}_S, \bar{P}_{N\setminus S}) = \beta_{N\setminus S} \in [a, x]$. Recall that for $j \in N \setminus S$, $P'_j \in \hat{\mathcal{S}}$ with $r_1(P'_j) = z$. Consider $(P''_S, P'_{N\setminus S}) \in \hat{\mathcal{S}}^n$ such that $r_1(P''_i) = y$ for all $i \in S$. Since $f(\bar{P}_S, \bar{P}_{N\setminus S}) \in [a, x]$, by uncompromisingness of f restricted to $\hat{\mathcal{S}}^n$, we have $f(P''_S, P'_{N\setminus S}) = z$. Because $r_1(P''_i) = y = r_1(Q')$ for all $i \in S$, by Lemma B.1, it follows that $f(Q'_S, P'_{N\setminus S}) = z$. However, as $z \notin \{y, y'\}$, this is a contradiction to our earlier finding that $f(Q'_S, P'_{N\setminus S}) \in \{y, y'\}$. ■

The following lemma says that there is exactly one agent i such that $\beta_i \in [a, x]$.

Lemma B.4. *It must be that $|\{i \in N \mid \beta_i \in [a, x]\}| = 1$.*

Proof. Suppose there are $i \neq j \in N$ such that $\beta_i, \beta_j \in [a,x]$. By Lemma B.3, $\beta_i \in [a,x]$ implies $\beta_{N\setminus i} \in [y,b]$. Since $j \in N \setminus i$ and $\beta_T \leq \beta_S$ for all $S \subseteq T$, $\beta_{N\setminus i} \in [y,b]$ implies $\beta_j \in [y,b]$, a contradiction. Hence, there can be at most one agent $i \in N$ such that $\beta_i \in [a,x]$.

Suppose $\beta_i \in [y,b]$ for all $i \in N$. By Lemma B.3, this means $\beta_{N\setminus i} \in [a,x]$ for all $i \in N$. Therefore, there must be $S \subseteq N$ such that $\beta_S \in [a,x]$ and for all $S' \subsetneq S$, $\beta_{S'} \in [y,b]$. By unanimity, $S \neq \emptyset$. If S is singleton, say i for some $i \in N$, then $\beta_i \in [a,x]$ and we are done. So assume that there are $j \neq k \in S$.

Consider the preference profile $P_N \in \hat{\mathcal{S}}^n$ such that $r_1(P_i) = x$ for all $i \in S \setminus j$, $r_1(P_j) = x+1$, $r_2(P_j) = x$, and $r_1(P_i) = x'$ for all $i \notin S$. Since $\beta_S \in [a,x]$ and $\beta_{S'} \in [y,b]$ for all $S' \subsetneq S$, it follows from the definition of min-max rule that $f(P_N) = x+1$. Let $P'_k \in \hat{\mathcal{S}}$ be such that $r_1(P'_k) = x'$. Since $\beta_{S\setminus k} \in [y,b]$ and f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule, it follows that $f(P'_k, P_{N\setminus k}) = x'$. Consider the preference profile $(Q_k, P_{N\setminus k})$, where $Q_k = Q$. Note that $f(P'_k, P_{N\setminus k}) = x'$, $r_1(Q_k) = x$, and $r_2(Q_k) = x'$. Therefore, by strategy-proofness, we have $f(Q_k, P_{N\setminus k}) \in \{x, x'\}$. Suppose $f(Q_k, P_{N\setminus k}) = x$. Because $f(P_N) = x+1$ and $r_1(P_k) = x$, this means agent k manipulates at P_N via Q_k. So, $f(Q_k, P_{N\setminus k}) = x'$. Let $P'_j \in \hat{\mathcal{S}}$ be such that $r_1(P'_j) = x$. Since $\beta_S \in [a,x]$ and x is the top-ranked alternative of the agents in S at preference profile $(P'_j, P_{N\setminus j})$, we have $f(P'_j, P_{N\setminus j}) = x$. As $r_1(P_k) = r_1(Q_k) = x$, this means $f(P'_j, Q_k, P_{N\setminus\{j,k\}}) = x$. Note that $f(Q_k, P_{N\setminus k}) = x'$, $r_1(P_j) = x+1$, and $r_2(P_j) = x$. Hence, agent j manipulates at $(Q_k, P_{N\setminus\{k\}})$ via P'_j. This completes the proof of the lemma. ■

REMARK B.1. By Lemma B.3 and Lemma B.4, it follows that f restricted to $\hat{\mathcal{S}}^n$ is a PDGMVS.

Our next lemma establishes that f is uncompromising.[13] First, we introduce few notations that we use in the proof of the lemma. For $P_N \in \tilde{\mathcal{S}}^n$, let $\tilde{N}(P_N) = \{i \in N \mid P_i \notin \hat{\mathcal{S}}\}$ be the set of agents who do not have single-peaked preferences at P_N. Moreover, for $0 \leq l \leq n$, let $\tilde{\mathcal{S}}^n_l = \{P_N \in \tilde{\mathcal{S}}^n \mid |\tilde{N}(P_N)| \leq l\}$ be the set of preference profiles where at most l agents have non-single-peaked preferences. Note that $\tilde{\mathcal{S}}^n_0 = \hat{\mathcal{S}}^n$ and $\tilde{\mathcal{S}}^n_n = \tilde{\mathcal{S}}^n$.

Lemma B.5. *The SCF f is uncompromising.*

Proof. Since $\tilde{\mathcal{S}}^n_0 = \hat{\mathcal{S}}^n$, f restricted to $\tilde{\mathcal{S}}^n_0$ is uncompromising. Suppose f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising for some $k < n$. We show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is uncompromising. It is

[13]Since every SCF satisfying uncompromisingness is tops-only, Lemma B.5 shows that a top-connected partially single-peaked domain is a tops-only domain. As in the case of Lemma A.3, it can be easily verified that top-connected partially single-peaked domains fail to satisfy the sufficient conditions for a domain to be tops-only identified in Chatterji and Sen (2011) and Chatterji and Zeng (2015).

enough to show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only. To see this, note that if f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only, then f is uniquely determined on $\tilde{\mathcal{S}}^n_{k+1}$ by its outcomes on $\hat{\mathcal{S}}^n$. Therefore, since f restricted to $\hat{\mathcal{S}}^n$ is uncompromising, f is uncompromising on $\tilde{\mathcal{S}}^n_{k+1}$.

Take $P_N \in \tilde{\mathcal{S}}^n_{k+1}$ and $j \in \tilde{N}(P_N)$. Let $\hat{P}_j \in \hat{\mathcal{S}}$ be such that $r_1(\hat{P}_j) = r_1(P_j)$. Then, P_N and $(\hat{P}_j, P_{N\setminus j})$ are tops-equivalent and $(\hat{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$. It is sufficient to show that $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$. Assume for contradiction that $f(P_N) \neq f(\hat{P}_j, P_{N\setminus j})$. Assume without loss of generality that the partial dictator of f restricted to $\hat{\mathcal{S}}^n$ is agent 1. By the induction hypothesis, agent 1 is the partial dictator of f restricted to $\tilde{\mathcal{S}}^n_k$, i.e., for all $P_N \in \tilde{\mathcal{S}}^n_k$, if $r_1(P_1) \in [a, x)$ then $f(P_N) \in [a, x]$, if $r_1(P_1) \in (y, b]$ then $f(P_N) \in [y, b]$, and if $r_1(P_1) \in [x, y]$ then $f(P_N) = r_1(P_1)$. We distinguish a few cases based on the position of the top-ranked alternative of agent 1.

CASE 1. Suppose $r_1(P_1) \in [a, x) \cup (y, b]$. We consider the case where $r_1(P_1) \in [a, x)$ as the proof for the case where $r_1(P_1) \in (y, b]$ follows from symmetric arguments. Since $r_1(P_1) \in [a, x)$, we have $f(\hat{P}_j, P_{N\setminus j}) \in [a, x]$. Because $\hat{P}_j$ is single-peaked, if $f(\hat{P}_j, P_{N\setminus j}) < f(P_N) \leq r_1(\hat{P}_j)$ or $r_1(\hat{P}_j) \leq f(P_N) < f(\hat{P}_j, P_{N\setminus j})$, then agent j manipulates at $(\hat{P}_j, P_{N\setminus j})$ via P_j. Moreover, since $f(\hat{P}_j, P_{N\setminus j}) \in [a, x]$, if $f(P_N) < f(\hat{P}_j, P_{N\setminus j}) \leq r_1(\hat{P}_j)$ or $r_1(P_j) \leq f(\hat{P}_j, P_{N\setminus j}) < f(P_N)$, then by the definition of top-connected partially single-peaked domain, agent j manipulates at $(P_j, P_{N\setminus j})$ via $\hat{P}_j$. Now, suppose $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < f(P_N)$. Let $\bar{P}_j \in \hat{\mathcal{S}}$ be such that $r_1(\bar{P}_j) = f(P_N)$. Since f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising and $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < r_1(\bar{P}_j)$, we have $f(\bar{P}_j, P_{N\setminus j}) = f(\hat{P}_j, P_{N\setminus j})$. Because $r_1(\bar{P}_j) = f(P_N)$, it follows that agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Using similar logic, it can be shown that $f(P_N) < r_1(\hat{P}_j) < f(\hat{P}_j, P_{N\setminus j})$ leads to a manipulation by agent j. Therefore, we have $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [a, x)$.

CASE 2. Suppose $r_1(P_1) \in [x, y]$. This means $f(\hat{P}_j, P_{N\setminus j}) = r_1(P_1)$. Consider $\bar{P}_j \in \hat{\mathcal{S}}$ such that $r_1(\bar{P}_j) = f(P_N)$. Since $(\bar{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$, by the induction hypothesis, we have $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1)$. Because $r_1(\bar{P}_j) = f(P_N)$ and $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1) \neq f(P_N)$, agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Therefore, $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [x, y]$.

This completes the proof of the lemma by induction. ■

Now, we complete the proof of the only-if part of Theorem 4.1. Since f is uncompromising on $\tilde{\mathcal{S}}^n$ and f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule with parameters $(\beta_S)_{S\subseteq N}$ satisfying the properties as stated in Lemma B.3 and Lemma B.4, it follows that f is a PDGMVS. ■

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