Bifurcation theory of a square lattice economy: Racetrack economy analogy in an economic geography model

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Abstract

Bifurcation theory for an economic agglomeration in a square lattice economy is presented in comparison with that in a racetrack economy. The existence of a series of equilibria with characteristic agglomeration patterns is elucidated. A spatial period doubling bifurcation cascade between these equilibria is advanced as a common mechanism to engender fewer and larger agglomerations in both economies. Analytical formulas for a break point, at which the uniformity is broken under reduced transport costs, are proposed for an economic geography model by synthetically encompassing both economies.

Keywords: Bifurcation, Economic geography model, Group theory, Replicator dynamics, Spatial period doubling

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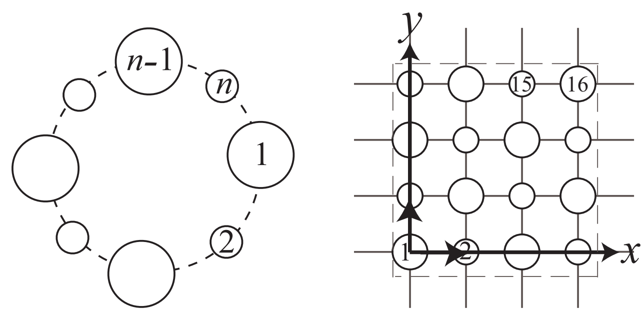
1. Introduction



(a) Racetrack economy (b) Square lattice economy

Figure 1: Two economic space models in the state of spatial period doubling.

A proper setting of a spatial platform is vital in the investigation of spatial economic agglomerations. A racetrack economy (Fig. 1(a)), which represents a series of places on a circle, is capable of representing some important agglomeration properties although this economy is essentially one-dimensional. This economy undergoes bifurcations to engender fewer and larger agglomerations (e.g., Krugman, 1993 [22]). The most characteristic behavior that has drawn attention is "spatial period doubling bifurcation" that leads to the alternating core and periphery patterns shown in Fig. 1(a) (see the related studies in Section 2).

A square lattice economy is often employed as a two-dimensional spatial platform.[4] The spatial period doubling pattern also exists in the lattice economy (Fig. 1(b)). Such coexistence of this pattern implies a role of the racetrack economy as an idealized one-dimensional counterpart of the agglomerations in two dimensions.

[4]Several studies of spatial agglomeration have been conducted on a square lattice; see, e.g., Clarke and Wilson (1983) [8], Weidlich and Haag (1987) [37], Munz and Weidlich (1990) [28], Brakman et al. (1999) [5]), and Stelder (2005) [35].

This paper aims to elucidate the mechanism of economic agglomeration in a square lattice, which turns out to be quite complicated (Section 8). In order to tackle such complexity, a racetrack economy analogy is proposed. The racetrack economy is endowed with a simpler spatial structure that is more easily treated analytically than the lattice economy. In particular, we would like to answer the following question: To what qualitative or quantitative extent can the racetrack economy serve as a platform for the agglomerations in two dimensions? While qualitative aspects of these agglomerations are described in a general setting by bifurcation theory, a qualitative measure of the agglomerations is presented for an economic geography model.

For a qualitative aspect, the progress of agglomeration by repeated bifurcations is studied comparatively in both economies.[5] As a novel contribution of this paper, a bifurcation theory in a square lattice is developed and cascades of spatial period doubling bifurcations leading to fewer and larger agglomerations are verified to exist.

For a quantitative aspect, a *break point*[6] is investigated comparatively for the two economies. When investment in transportation infrastructure is committed, the break point indexes the functioning of this investment. Formulas for this point in the square lattice are newly developed and are expressed so as to also encompass the racetrack economy by finding a linkage between these two economies.

[5]The mechanism of bifurcations in a racetrack economy was elucidated by the group-theoretic bifurcation analysis (Ikeda, Murota, and Akamatsu, 2012 [15]).

[6]The *break point* of the transport cost that produces a core–periphery pattern in a two-place economy was highlighted as a key concept (Fujita, Krugman, and Venables, 1999 [13]).

Whereas real economic activities accommodate models of various kinds, we refer to a specific economic geography model, i.e., that of Forslid and Ottaviano (2003) [11] in favor of its analytical tractability. There are unskilled workers who are immobile and equally distributed among places, and skilled ones who are footloose entrepreneurs seeking to maximize wages. By numerical comparative static analyses for both economies, the progress of agglomeration through successive emergence of spatial period doubling patterns is observed, thereby ensuring the validity of the racetrack economy analogy.

This paper is organized as follows. Related studies are presented in Section 2. Modeling of a spatial economy for an analytically solvable economic geography model is presented in Section 3. Symmetry of racetrack and lattice economies is described in Section 4. A theory of replicator dynamics is developed in Section 5. Bifurcating agglomeration patterns are predicted theoretically in Section 6. Formulas for break points are advanced in Section 7. Numerical examples are presented in Section 8.

2. Related studies

There are spatial platforms for economic activities of various kinds. The two-place economy has long been extensively employed.[7] There are several studies on three places.[8]

The racetrack economy was used to show the evolution of a regular lattice, for example, by Krugman (1993) [22]. Krugman (1996, p.91) [23] regarded the racetrack economy as one-dimensional and inferred its extendibility to a two-dimensional economy to engender hexagonal distributions. Tabuchi and Thisse (2011) [36] used a multi-industry model in a racetrack economy to show the emergence of central places, which denote a spatial alternation of a core place with a large population and a peripheral place with a small population. This economy undergoes a sequence of recurrent bifurcations, called the "spatial period doubling cascade," which has been observed ubiquitously for NEG models.[9]

A *break point* of the transport cost was introduced for the two-place economy (Fujita, Krugman, and Venables, 1999 [13]). The importance of the break point has come to acknowledged and its formulas have been derived for several spatial

[7]See, e.g., Krugman (1991) [21]; Fujita, Krugman, and Venables (1999) [13]; Baldwin et al. (2003) [4]; Mossay (2006) [26]; Oyama (2009) [30]; Fujishima (2013) [12].

[8]See, e.g., Krugman and Elizondo (1996) [24]; Mori and Nishikimi (2002) [27]; Ago, Isono, and Tabuchi, 2006 [1]; Castro, Correia-da-Silva, and Mossay, 2012 [6]; Commendatorea et al., 2014 [9].

[9]See, e.g., Picard and Tabuchi (2010) [33], Ikeda, Akamatsu, and Kono (2012) [15], Akamatsu, Takayama, and Ikeda (2012) [3], Akamatsu, Mori, and Takayama (2016) [2], and Osawa, Akamatsu, and Takayama (2017) [29].

economy models in several spatial platforms: a class of footloose-entrepreneur models (Pflüger and Südekum, 2008 [32]), the Pflüger model (2004) [31] in the racetrack economy for logit dynamics (Akamatsu, Takayama, and Ikeda, 2012 [3]), an analytically solvable model (Forslid and Ottaviano, 2003 [11]) in the racetrack economy for replicator dynamics (Ikeda et al., 2017a [18]), and the same model in the 6×6 hexagonal lattice for logit dynamics (Ikeda, Murota, and Takayama, 2017b [20]).

The bifurcation mechanism of the square lattice studied in this paper is based on that of a hexagonal lattice (Ikeda et al., 2012, 2014 [17, 19]; Ikeda and Murota, 2014 [16]). In comparison with previous studies on the racetrack economy, this paper treats this economy as a one-dimensional counterpart of two-dimensional agglomerations. Synthetic formulas that can encompass both the racetrack and the square lattice economies are proposed, whereas such formulas for these two economies have been derived up to now somewhat independently.

3. Modeling of the spatial economy

Modeling of the spatial economy is presented in this section. As a representative of spatial economy models, an analytically solvable core–periphery model by Forslid and Ottaviano (2003) [11] is used. The fundamental logic and governing equation of a multi-regional version of the model are presented based on work of Akamatsu, Mori, and Takayama (2016) [2], while details are given in Appendix A.

3.1. Basic assumptions

The economy of this model comprises K places (labeled $i = 1, \ldots, K$), two factors of production (skilled and unskilled labor), and two sectors (manufacturing, M, and agriculture, A). Both H skilled and L unskilled workers consume final goods of two types: manufacturing sector goods and agricultural sector goods. Workers supply one unit of each type of labor inelastically. Skilled workers are mobile among places, and the number of skilled workers in place i is denoted by λ_i ($\sum_{i=1}^{K} \lambda_i = H$). The total number H of skilled workers is normalized as $H = 1$. Unskilled workers are immobile and distributed equally across all places with unit density (i.e., $L = 1 \times K$).

Preferences U over the M- and A-sector goods are identical across individuals. The utility of an individual in place i is

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1 - \mu) \ln C_i^{\mathrm{A}} \qquad (0 < \mu < 1), \tag{1}$$

where μ is a constant parameter expressing the expenditure share of manufacturing sector goods, C_i^{A} stands for the consumption of the A-sector product in place i and

C_i^{M} represents the manufacturing aggregate in place i, which is defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_{j=1}^{K} \int_0^{n_j} q_{ji}(\ell)^{(\sigma-1)/\sigma} d\ell \right)^{\sigma/(\sigma-1)}, \tag{2}$$

where $q_{ji}(\ell)$ is the consumption in place i of a variety $\ell \in [0, n_j]$ produced in place j, n_j is the number of produced varieties at place j, and $\sigma > 1$ is the constant elasticity of substitution between any two varieties.

3.2. Iceberg form of transport cost

The transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from place i to place j ($\neq i$), only a fraction $1/T_{ij} < 1$ actually arrives ($T_{ii} = 1$). It is assumed that $T_{ij} = T_{ij}(\tau)$ is a function in a transport cost parameter $\tau > 0$ as

$$T_{ij} = \exp(\tau\, m(i, j)\, \tilde{L}), \tag{3}$$

where $m(i, j)$ is an integer expressing the shortest link between places i and j and $\tilde{L}$ is the distance unit. The spatial discounting factor

$$d_{ji} = T_{ji}^{1-\sigma} \tag{4}$$

represents friction between places j and i that decay in proportion to transportation distance. With the use of

$$r = \exp[-\tau(\sigma - 1)\tilde{L}] \tag{5}$$

($0 < r < 1$ for $\tau > 0$) expressing trade freeness, the spatial discounting factor $d_{ij} = T_{ij}^{1-\sigma}$ in (4) is expressed as $d_{ij} = r^{m(i,j)}$.

3.3. Market equilibrium

As worked out in Appendix A, the market equilibrium wage vector $\boldsymbol{w}$ is obtained as

$$\boldsymbol{w} = \frac{\mu}{\sigma}\left(I - \frac{\mu}{\sigma}\, D\Delta^{-1}\Lambda\right)^{-1} D\Delta^{-1}\mathbf{1} \tag{6}$$

with the notation

$$\begin{cases} \boldsymbol{w} = (w_i), \quad D = (d_{ij}), \quad \Delta = \mathrm{diag}(\Delta_1, \dots, \Delta_K), \\ \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_K), \quad \mathbf{1} = (1, \dots, 1)^\top. \end{cases} \tag{7}$$

The indirect utility v_i is expressed in terms of w_i and $\Delta_i = \sum_{k=1}^{K} d_{ki}\lambda_k$ as

$$v_i = \frac{\mu}{\sigma - 1}\ln\Delta_i + \ln w_i. \tag{8}$$

3.4. Spatial equilibrium

We introduce a spatial equilibrium, in which high skilled workers are allowed to migrate among places. A customary way to define such an equilibrium is to consider the following problem: Find $(\boldsymbol{\lambda}^*, \hat{v})$ satisfying

$$\begin{cases} (v_i - \hat{v})\lambda_i^* = 0, \quad \lambda_i^* \geq 0, \quad v_i - \hat{v} \leq 0, \quad (i = 1, \dots, K), \\ \sum_{i=1}^{K} \lambda_i^* = 1. \end{cases} \tag{9}$$

For the solution to this problem, $\hat{v}$ serves as the highest (indirect) utility. When the system is in a *spatial equilibrium*, no individual can improve his/her utility by changing his/her location unilaterally.

As guaranteed in Sandholm (2010) [34], it is possible to replace the problem to obtain a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics:

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \tau), \tag{10}$$

where $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = (F_i(\boldsymbol{\lambda}, \tau) \mid i = 1, \ldots, K)$, and

$$F_i(\boldsymbol{\lambda}, \tau) = (v_i(\boldsymbol{\lambda}, \tau) - \bar{v}(\boldsymbol{\lambda}, \tau))\lambda_i, \quad (i = 1, \ldots, K). \tag{11}$$

Here, $\bar{v} = \sum_{i=1}^{K} \lambda_i v_i$ is the average utility. Stationary points (rest points) $\boldsymbol{\lambda}^*(\tau)$ of the replicator dynamics (10) are defined as those points which satisfy the static governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}^*, \tau) = \mathbf{0}. \tag{12}$$

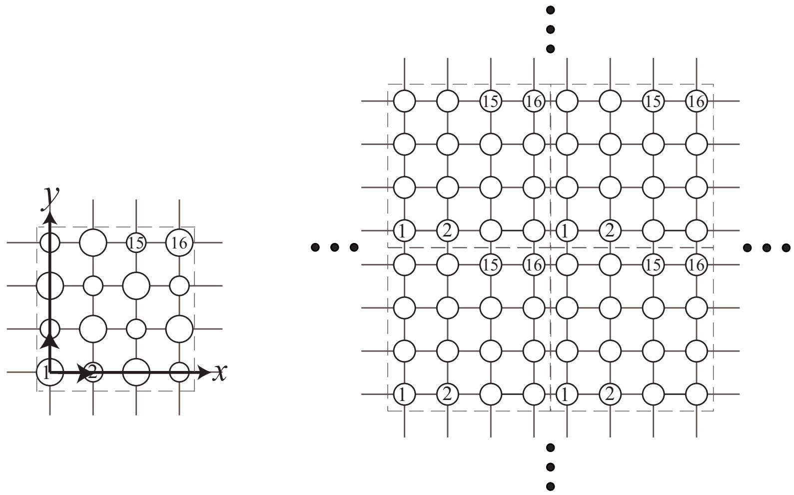
Using the eigenvalues of the Jacobian matrix

$$J(\boldsymbol{\lambda}^*, \tau) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*, \tau),$$

we classify stability as

$$\begin{cases} \text{linearly stable:} & \text{every eigenvalue has a negative real part,} \\ \text{linearly unstable:} & \text{at least one eigenvalue has a positive real part.} \end{cases}$$

A stationary point is asymptotically stable or unstable according to whether it is linearly stable or unstable.



(a) 4×4 square lattice (b) Periodically repeated 4×4 square lattice

Figure 2: A system of places on the 4×4 square lattice with periodic boundary conditions.

4. Symmetry of racetrack and lattice economies

In an investigation of bifurcating patterns of a symmetric system, we refer to group G that labels its symmetry. For the racetrack economy, a series of $K = n$ places (labeled $i = 1, \ldots, n$) is spread equally on the circumference of the circle and these places are connected by roads of the same length $\tilde{L}$. The symmetry of this economy located at the origin of the xy-plane is labeled by the dihedral group $\mathrm{D}_n = \langle s, r \rangle$, where s is the reflection $y \mapsto -y$, r is a $2\pi/n$ anticlockwise rotation around the origin, and $\langle \cdot \rangle$ is a group generated by the elements therein.

An $n \times n$ square lattice with periodic boundary conditions is introduced as a two-dimensional spatial platform. Nodes at a border of this lattice are connected periodically to those on the opposite border to cover an infinite space (Fig. 2(b)). Places of economic activities are located on the nodes, which are connected by roads of the same length $\tilde{L}$ along the lattice. The symmetry of the lattice is expressed by the group $\langle r, s, p_1, p_2 \rangle$, which is generated by the following four elements:[10] r: counterclockwise rotation about the origin at an angle of $\pi/2$, s: reflection $y \mapsto -y$, p_1: x-directional periodic translation at the length $\tilde{L}$, and p_2: y-directional one.

The flat earth equilibrium (uniform distribution) with $\boldsymbol{\lambda}^* = \frac{1}{K}(1, \ldots, 1)^\top$ exists in both the racetrack and the lattice economies. This equilibrium is invariant to $G = \mathrm{D}_n$ in the racetrack economy and to $G = \langle r, s, p_1, p_2 \rangle$ in the lattice economy.

[10]These four elements satisfy $r^4 = s^2 = (rs)^2 = p_1{}^n = p_2{}^n = e$, $p_2 p_1 = p_1 p_2$, $r p_1 = p_2 r$, $r p_2 = p_1^{-1} r$, $s p_1 = p_1 s$, $s p_2 = p_2^{-1} s$, where e is the identity element.

5. Existence, stability, and sustainability of trivial solutions

A bifurcation theory on the replicator dynamics is introduced. By virtue of its product form (11), this dynamics has a number of trivial solutions that retain spatial patterns when transport cost τ changes. After introducing classifications of stationary points, we formulate a symmetry condition for the existence of these trivial solutions and investigate the stability and sustainability of the trivial solutions as novel contributions of this paper.

5.1. Classifications of stationary points

Stationary points $(\boldsymbol{\lambda}, \tau)$ of the replicator dynamics are classified in preparation for the description of its bifurcation mechanism. First, these points are classified into an *interior solution*, for which all cities have positive population, and a *corner solution*, for which some cities have zero population.

A solution can be expressed, without loss of generality, by appropriately rearranging the order of independent variables $\boldsymbol{\lambda}$ as

$$\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \boldsymbol{\lambda}_+ \\ \boldsymbol{\lambda}_0 \end{bmatrix} \tag{13}$$

with $\boldsymbol{\lambda}_+ = \{\lambda_i > 0 \mid i = 1, \ldots, m\}$ and $\boldsymbol{\lambda}_0 = \mathbf{0}$. Note that $\boldsymbol{\lambda}_0$ is absent for an interior solution. The static governing equation (12) can be rearranged accordingly as

$$\hat{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{F}_+(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) \\ \boldsymbol{F}_0(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) \end{bmatrix} \tag{14}$$

with the rearranged Jacobian matrix

$$\hat{J} = \begin{bmatrix} J_+ & J_{+0} \\ O & J_0 \end{bmatrix}, \tag{15}$$

where

$$J_+ = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)\,\{\partial(v_i - \bar{v})/\partial\lambda_j \mid i, j = 1, \ldots, m\},$$

$$J_{+0} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)\,\{\partial(v_i - \bar{v})/\partial\lambda_j \mid i = 1, \ldots, m; j = m+1, \ldots, K\},$$

$$J_0 = \mathrm{diag}(v_{m+1} - \bar{v}, \ldots, v_K - \bar{v}).$$

A stable spatial equilibrium is given by a stable stationary solution, for which all eigenvalues of $\hat{J}$ are negative. Such stability condition is decomposed into two conditions:

$$\begin{cases} \text{Stability condition for } \boldsymbol{\lambda}_+: & \text{all eigenvalues of } J_+ \text{ are negative.} \\ \text{Sustainability condition for } \boldsymbol{\lambda}_0: & \text{all diagonal entries of } J_0 \text{ are negative.} \end{cases} \tag{16}$$

Next, critical points[11] are classified into a *break bifurcation point*[12] with singular J_+ and a *non-break point* with $v_i - \bar{v} = 0$ for some place i ($i = m+1, \ldots, K$); a sustain point is a special kind of non-break point. A bifurcating solution with reduced symmetry branches at a break point, whereas the populations of some places vanish at a non-break (sustain) point. A break point is a *simple bifurcation point*, a *double bifurcation point*, and so on, according to whether the number of zero eigenvalue(s) of the Jacobian matrix $\hat{J}$ is equal to one, two, and so on. A simple bifurcation is either *tomahawk* or *pitchfork*. Bifurcating solutions are unstable for the tomahawk and stable for the pitchfork.

Last, stationary points are classified into a *trivial solution*[13] $(\boldsymbol{\lambda}, \tau)$ with a con-

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[11] Critical points are those which have one or more zero eigenvalue(s) of the Jacobian matrix $\hat{J}$.

[12] There is another critical point, a limit point of τ, also with singular J_+ (Ikeda and Murota, 2014 [16]). Yet this kind of point does not play an important role in the discussion in this paper.

[13] Trivial solutions in a racetrack economy were studied in Castro, Correia-da-Silva, and Mossay (2012) [6] and Ikeda, Akamatsu, and Kono (2012) [15].

stant $\boldsymbol{\lambda}$ that exists for any $\tau \in (0, \infty)$ and a *non-trivial solution* $(\boldsymbol{\lambda}, \tau)$ for which $\boldsymbol{\lambda}$ changes with τ. The existence of trivial solutions of various kinds is a special feature of the replicator dynamics.

Proposition 1. *The flat earth (dispersion) equilibrium* $\boldsymbol{\lambda}^* = \frac{1}{K}(1, \ldots, 1)^\top$ *is a trivial equilibrium.*

Proof. Because we have $v_1 = \cdots = v_K = \bar{v}$ for this equilibrium, the conditions (9) for a spatial equilibrium are satisfied for any τ. □

5.2. Symmetry condition of a corner solution

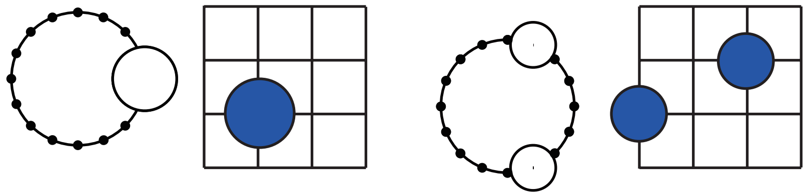
A corner solution with m identical agglomerated places, i.e.,

$$\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \boldsymbol{\lambda}_+ \\ \boldsymbol{\lambda}_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\mathbf{1} \\ \mathbf{0} \end{bmatrix} \tag{17}$$

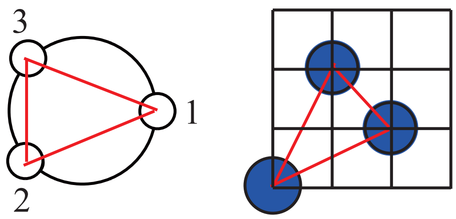
is given special attention in this paper. This is a core–periphery pattern with a two-level hierarchy: The population is agglomerated to m core places with identical populations, while other peripheral places have no populations. An atomic monocenter for $m = 1$ in Fig. 3(a) and twin places for $m = 2$ in (b) serve as simple examples of such a solution.

Assumption 1. *The corner solution with m identical agglomerated places in* (17) *is invariant to group G and there is a set of permutation matrices* $T_+(g)$ $(g \in G)$ *that permutes any two entries of* $\boldsymbol{\lambda}_+$.

Trivial solutions have several characteristics as expounded in the following Proposition and Corollary (see Appendix B for the proof).



(a) Atomic monocenter (trivial) (b) Twin places (trivial)



(c) Non-trivial corner solutions

Figure 3: Trivial and non-trivial corner solutions.

Proposition 2. *A corner solution* $(\lambda_+, \lambda_0, \tau) = (\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau)$ *that satisfies Assumption 1 is a trivial solution.*

Corollary 1. *An atomic monocenter*[14] $(m = 1)$ *and twin places* $(m = 2)$ *are trivial solutions.*

The corner solutions with m identical agglomerated places in (17) are not always trivial solutions. For example, the spatial patterns shown in Fig. 3(c) are not trivial solutions and there is no guarantee that they are solutions (Appendix B).

5.3. Stability and sustainability of trivial solutions

Prior to the description of stability and sustainability of trivial solutions, we first refer to the two-place economy (Fujita, Krugman, and Venables, 1999 [13]).

[14] An atomic monocenter (concentration) was shown to be a trivial solution in a racetrack economy in Castro, Correia-da-Silva, and Mossay (2012) [6] and Ikeda, Akamatsu, and Kono (2012) [15].

A trivial solution with $\boldsymbol{\lambda} = (1/2, 1/2)^\top$ is stable for $\tau > \tau_\mathrm{B}$, where τ_B is a break point. On the other hand, the core–periphery pattern $\boldsymbol{\lambda} = (1, 0)^\top$ is sustainable for $\tau < \tau_\mathrm{S}$, where τ_S is a sustain point.

In general, a trivial equilibrium possibly has a few non-break points (Section 8); accordingly, a sustain point is defined as the non-break point with the smallest τ value, which is set as τ_S. We introduce the following assumption, which is in line with the agglomeration behavior (Section 8) of the economic geography model (Section 3).

Assumption 2. *For a trivial equilibrium except for the flat earth equilibrium and the atomic monocenter,*[15] *there are τ_B and τ_S so that the stability condition of the core places in* (16) *is satisfied for $\tau > \tau_\mathrm{B}$ and the sustainability condition of the periphery places in* (16) *is satisfied for $\tau < \tau_\mathrm{S}$.*

Then we can consider the following classification:

$$\begin{cases} \text{Well-posed trivial solution:} \quad \tau_\mathrm{B} < \tau_\mathrm{S}, \\ \text{Ill-posed trivial solution:} \quad \tau_\mathrm{B} > \tau_\mathrm{S}. \end{cases} \tag{18}$$

Proposition 3. *Under Assumption 2, a well-posed trivial solution is a stable spatial equilibrium in the range $\tau_\mathrm{B} < \tau < \tau_\mathrm{S}$, while an ill-posed trivial solution is not a stable spatial equilibrium for any τ.*

[15] The flat earth equilibrium does not have a sustain point, while the atomic monocenter does not have a break bifurcation point.

6. Bifurcation mechanism of spatial period doubling cascades

Spatial period doubling cascades of the racetrack and the lattice economies are investigated in this section and are demonstrated in Section 8 to be predominant in the progress of agglomeration in the economic geography model (Section 3). It is ensured that spatial period doubling patterns of these economies are always trivial solutions. A bifurcation mechanism of the emergence of these patterns in the lattice economy is newly presented and is meshed consistently with the previous results in the racetrack economy (Ikeda, Akamatsu, and Kono, 2012 [15]). We focus on repeated occurrences of bifurcations engendering *spatial period doubling patterns* (Figs. 4(a) and (b)) and prove that these patterns are trivial solutions.[16]

6.1. Racetrack economy: spatial period doubling

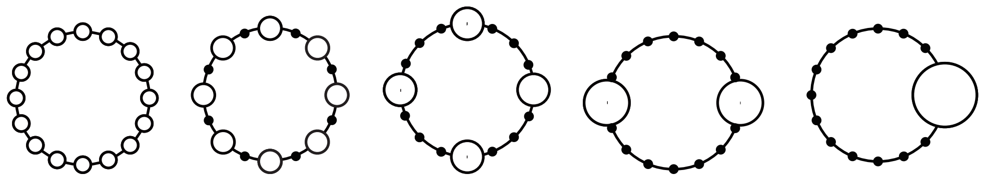
Bifurcation rules for a spatial period doubling cascade starting from the flat earth equilibrium $\boldsymbol{\lambda}^* = \frac{1}{n}(1, \ldots, 1)^\top$ en route to an atomic monocenter are presented. When n is even, at a simple break bifurcation point on the flat earth equilibrium, a solution curve bifurcates in the direction of an eigenvector

$$\boldsymbol{\eta}_{\mathrm{Ra}} = (1, -1, \ldots, 1, -1)^\top \tag{19}$$

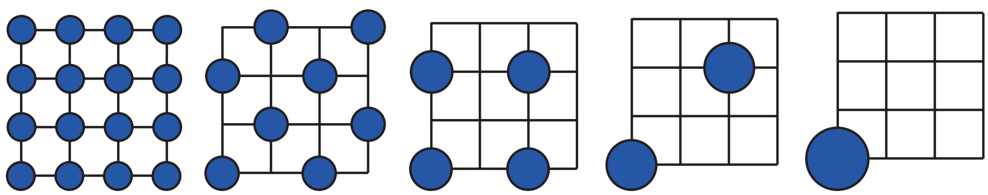
of the Jacobian matrix J. A bifurcating state has the following population distribution:

$$\boldsymbol{\lambda} = (1/n + a, 1/n - a, \ldots, 1/n + a, 1/n - a)^\top, \quad -1/n \leq a \leq 1/n. \tag{20}$$

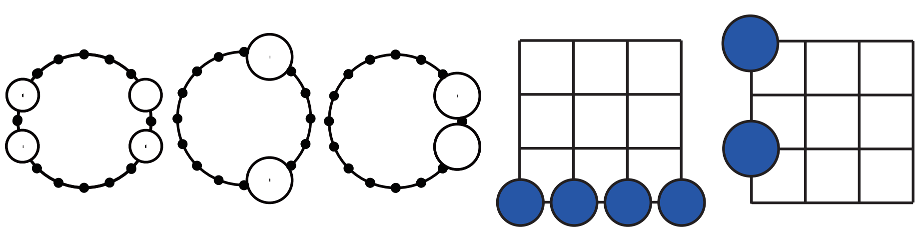
[16]There are trivial solutions other than spatial period doubling ones as depicted in Fig. 4(c).



(a) Spatial period doubling trivial solutions: racetrack economy ($n = 16$; $\tilde{T} = T/\tilde{L}$)



(b) Spatial period doubling trivial solutions: lattice economy
($n = 4$; $\tilde{T}_{xy} = T_{xy}/\tilde{L}$, $\tilde{T}_{\text{dia}} = T_{\text{dia}}/\tilde{L}$)



(c) Non-doubling trivial solutions

Figure 4: Spatial period doubling and non-doubling trivial solutions.

This represents a state in which concentrating places and extinguishing places alternate along the circle and, in turn, form a chain of spatially repeated core–periphery patterns *a la* Christaller (1933) [7] and Lösch (1940) [25].

We consider a case where the concentrating and the extinguishing proceed until reaching a non-break (sustain) point with a spatial period doubling pattern:

$$\boldsymbol{\lambda}_{\mathrm{Ra}} = (2/n, 0, \dots, 2/n, 0)^\top, \quad \text{i.e.,} \quad \hat{\boldsymbol{\lambda}} = (2/n, \dots, 2/n; 0, \dots, 0)^\top = \begin{bmatrix} \frac{2}{n}\mathbf{1} \\ \mathbf{0} \end{bmatrix}, \quad (21)$$

which is invariant to group $\mathrm{D}_{n/2}$.

For $n = 2^k$ ($k = 2, 3, \dots$) places, at a simple break bifurcation point, a secondary bifurcating solution branches in the direction of an eigenvector

$$\boldsymbol{\eta}_{\mathrm{Rb}} = (1, 0, -1, 0; \dots; 1, 0, -1, 0)^\top, \quad \text{i.e.,} \quad \hat{\boldsymbol{\eta}} = (1, -1, \dots, 1, -1; 0, 0, \dots, 0, 0)^\top. \quad (22)$$

Thereafter, a simple break point and a non-break (sustain) point can occur recurrently until reaching an atomic monocenter. Agglomeration patterns produced in this recurrence are expressed as

$$\lambda_i = \begin{cases} \dfrac{1}{2^m} & \text{for } i = 1, 1 + 2^{k-m}, \dots, 1 + (2^m - 1)2^{k-m} \ (m = 1, \dots, k-1), \\ 0 & \text{otherwise,} \end{cases}$$

and are called spatial period doubling patterns. The symmetries of these patterns are labeled by a set of groups

$$\mathrm{D}_n \ \to \ \mathrm{D}_{n/2} \ \to \ \cdots \ \to \ \mathrm{D}_1, \quad (23)$$

where (→) denotes spatial period doubling at a simple break bifurcation.

For example, Figure 4(a) depicts spatial period doubling patterns for $n = 16$ places. The core (agglomerated) places shown by (◯) are located equidistantly and the spatial period T between these places is doubled repeatedly as the number of these places decreases from 16, 8, 4, 2, to 1.

Proposition 4. *The spatial period doubling patterns of the racetrack economy are trivial solutions.*

Proof. For these patterns, group G in Proposition 2 is chosen as one of the groups in (23) to ensure the existence of a group permuting any two core places with none-zero and identical population. This ensures Assumption 1, and, in turn, Proposition 2, thereby proving that the patterns are trivial solutions. □

6.2. Lattice economy I: half spatial period doubling

A bifurcation rule of a spatial period doubling cascade of the lattice economy is presented below, while details of group-theoretic analysis are given in Appendix C. When n is even, at a simple break bifurcation point on the flat earth equilibrium, a bifurcating solution branches in the direction of an eigenvector

$$\boldsymbol{\eta}_{\mathrm{La}} = \{\cos(\pi(n_1 - n_2)) \mid n_1, n_2 = 1, \ldots, n\} = \boldsymbol{\eta}_{\mathrm{Ra}} \otimes \boldsymbol{\eta}_{\mathrm{Ra}} \qquad (24)$$

of the Jacobian matrix J (see Appendix C.2 for the proof), where $\boldsymbol{\eta}_{\mathrm{R}}$ is the spatial period doubling eigenvector of the racetrack economy in (19) and ($\otimes$) is the tensor product. This pattern $\boldsymbol{\eta}_{\mathrm{La}}$ represents period doubling in the horizontal and the vertical directions and has the symmetry of $\langle r, s, p_1 p_2, p_1^{-1} p_2 \rangle$. The lattice economy is linked to the racetrack economy via the tensor product structure in (24). Such a linkage is called herein a *squared tensor product linkage*.

We consider a case where the concentrating and the extinguishing proceed until reaching a non-break (sustain) point with a spatial period doubling pattern

$$\boldsymbol{\lambda}_{\mathrm{La}} = \boldsymbol{\lambda}_{\mathrm{Ra}} \otimes \boldsymbol{\lambda}_{\mathrm{Ra}} = (2/n, 0, \ldots, 2/n, 0) \otimes (2/n, 0, \ldots, 2/n, 0), \tag{25}$$

which is invariant to group $\langle r, s, p_1 p_2, p_1^{-1} p_2 \rangle$.

When $n = 2^m$ ($m = 2, 3, \ldots$), from the spatial period doubling pattern in (25), another doubling pattern branches in the following direction:

$$\boldsymbol{\eta}_{\mathrm{Lb}} = \boldsymbol{\eta}_{\mathrm{Rb}} \otimes \boldsymbol{\eta}_{\mathrm{Rb}} = (1, 0, -1, 0; \ldots; 1, 0, -1, 0) \otimes (1, 0, -1, 0; \ldots; 1, 0, -1, 0), \tag{26}$$

which is invariant to group $\langle r, s, p_1^2, p_2^2 \rangle$ (see Appendix C.2). In this manner, a series of spatial period doubling trivial solutions is engendered. As shown, for example, in Fig. 4(b) ($n = 4$), as the number of core (agglomerated) places decreases from 16, 8, 4, 2, to 1, there emerges a series of spatial period doubling patterns associated with a set of groups

$$\langle r, s, p_1, p_2 \rangle \;\rightarrow\; \langle r, s, p_1 p_2, p_1^{-1} p_2 \rangle \;\rightarrow\; \langle r, s, p_1^2, p_2^2 \rangle \;\rightarrow\; \cdots \;\rightarrow\; \mathrm{D}_2. \tag{27}$$

Proposition 5. *The spatial period doubling patterns of the lattice economy are trivial solutions.*

Proof. For these patterns, group G in Lemma 2 is chosen as one of the groups in (27) to ensure the existence of a group permuting any two core places with identical and none-zero populations. This proves that the patterns are trivial solutions. □

This lattice economy has a spatial period T_{xy} in the x- and y-directions and another spatial period T_{dia} in the two diagonal directions.[17] In the spatial pe-

[17] The diagonal distance is not measured by the road distance but by the Euclidean distance.

riod doubling cascade in (27), the spatial period doubling of T_{xy} and that of $T_{\rm dia}$ take place alternatively. This kind of spatial period doubling is called herein *half spatial period doubling* as half of the periods are doubled each time (see, e.g., Fig. 4(b)).

6.3. Lattice economy: full spatial period doubling

There are other kinds of bifurcation cascades. When $n = 2^m$ $(m = 2, 3, \ldots)$, from a double bifurcation point on the flat earth equilibrium, a bifurcating solution curve branches in the direction of the eigenvector in (26) (Appendix C.3):

$$\boldsymbol{\eta}_{\rm Lb} = \boldsymbol{\eta}_{\rm Rb} \otimes \boldsymbol{\eta}_{\rm Rb}. \tag{28}$$

There are two series of spatial period doubling bifurcation cascades associated with a series of groups

$$\langle r, s, p_1, p_2\rangle \Rightarrow \langle r, s, p_1^2, p_2^2\rangle \Rightarrow \langle r, s, p_1^4, p_2^4\rangle \Rightarrow \cdots \Rightarrow {\rm D}_2, \tag{29}$$

$$\langle r, s, p_1p_2, p_1^{-1}p_2\rangle \Rightarrow \langle r, s, (p_1p_2)^2, (p_1^{-1}p_2)^2\rangle \Rightarrow \cdots \Rightarrow \langle r, s, (p_1p_2)^{n/2}, (p_1^{-1}p_2)^{n/2}\rangle, \tag{30}$$

where $(\Rightarrow)$ indicates spatial period doubling at a double bifurcation point. This is called *full spatial period doubling* as spatial periods in all four directions are doubled.

Figure 5 depicts the mixed occurrence of half and full doubling for $n = 4$. Twice repeated occurrences of half doubling correspond to a single occurrence of full doubling. Such a mixture of half doubling and full doubling makes the progress of agglomeration of the lattice economy more complex than that of the racetrack economy.

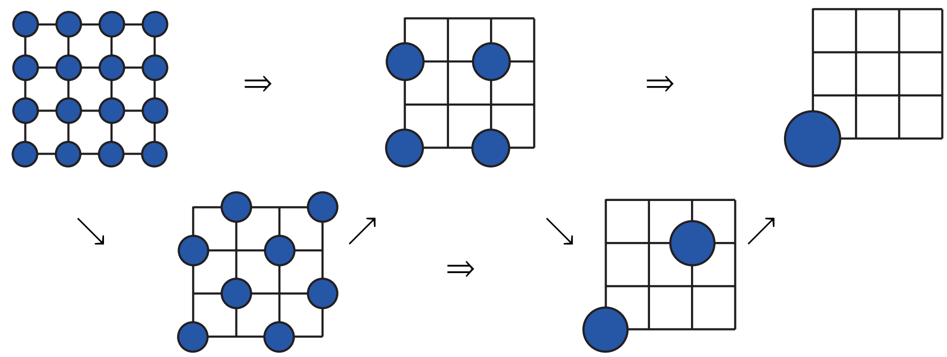
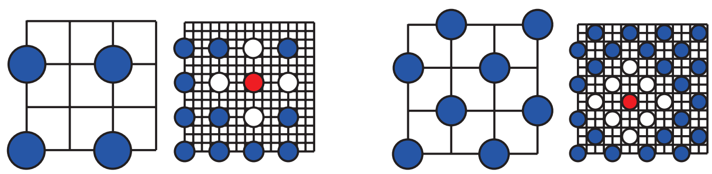


Figure 5: Spatial period doubling cascades for a lattice economy ($n = 4$); ($\Rightarrow$): full doubling; ($\searrow$) and ($\nearrow$): half doubling.



(a) Foursquare patterns (b) Oblique patterns

Figure 6: Foursquare and oblique spatial period doubling patterns.

For understanding the difference of a pair of spatial period doubling cascades (29) and (30), it is vital to classify spatial period doubling patterns into *foursquare patterns* and *oblique patterns*, as illustrated in Fig. 6. For foursquare patterns with a sufficiently large n (Fig. 6(a)), each first-level center (red circle) is surrounded by the four closest first-level centers (white circles). For oblique ones, each first-level center is surrounded by as many as eight closest first-level centers (Fig. 6(b)). In this sense, the first-level centers of the oblique ones are more densely distributed in comparison with those of the foursquare ones. Note that the first cascade in (29) occurs between foursquare ones, whereas the second cascade in (30) occurs between oblique ones. That classification is vital in the discussion of well-posedness of these patterns for the economic geography model (Section 8).

7. Break point initiating spatial agglomeration

Formulas for break points for the analytically solvable model (Section 3) are developed in this section. A break point is defined as the value of τ for the occurrence of a bifurcation that breaks uniformity. When investment in transportation infrastructure is committed, the break point indexes the functioning of this investment. Formulas for the lattice economy are newly developed and are presented in a synthetic manner to encompass the previous result for the racetrack economy (Ikeda, Akamatsu, and Kono, 2012 [15]).

The size n of the economy is chosen as 2 and $4m$ ($m = 1, 2, \ldots$). The total length $\mathcal{L}$ of the road on the racetrack is chosen as $\mathcal{L} = 1$, the spatial period of the lattice is also chosen as $\mathcal{L} = 1$, and neighboring places are connected by an inter-place road of the length $\tilde{L} = 1/n$.

7.1. Fundamentals for deriving the formulas for a break point

Breaking uniformity by bifurcation at the flat earth equilibrium $\boldsymbol{\lambda}^*$ is given by a zero eigenvalue of the Jacobian matrix $J(\boldsymbol{\lambda}^*)$. As worked out in (A.14)–(A.16), $J(\boldsymbol{\lambda}^*)$ is related to another Jacobian matrix $V(\boldsymbol{\lambda}^*) = \left(\partial v_i/\partial \lambda_j\right)(\boldsymbol{\lambda}^*)$ as

$$J(\boldsymbol{\lambda}^*) = \left(\frac{1}{K} I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right) V(\boldsymbol{\lambda}^*) - \frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top \tag{31}$$

with

$$V(\boldsymbol{\lambda}^*) = K\left[\kappa' \hat{D} + \left(I - \kappa\hat{D}\right)^{-1} \cdot \hat{D}\left(\kappa I - \hat{D}\right)\right], \tag{32}$$

where $\kappa = \frac{\mu}{\sigma}$, $\kappa' = \frac{\mu}{\sigma-1}$, and $\hat{D} = D/d$ is the normalized spatial discounting matrix. Here $D = (d_{ij})$ is defined by (4) and $d = d(r) = \sum_{j=1}^{K} d_{1j}$ with r being the trade freeness parameter introduced in (5). The spatial discounting matrices for the racetrack and the lattice economies are called D_{R} and D_{L}, respectively, and are given, for example, for $n = 2$ as

$$D_{\mathrm{R}} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}, \quad D_{\mathrm{L}} = D_{\mathrm{R}} \otimes D_{\mathrm{R}} = \begin{bmatrix} 1 & r & r & r^2 \\ r & 1 & r^2 & r \\ r & r^2 & 1 & r \\ r^2 & r & r & 1 \end{bmatrix}. \tag{33}$$

We have the relation $D_{\mathrm{L}} = D_{\mathrm{R}} \otimes D_{\mathrm{R}}$ that connects the two economies, while the matrices D_{R} for $n = 2^m$ ($m = 2, 3, 4$), for example, are given in Appendix D.1.

We present the following lemmas for the eigenproblems for the matrices $J(\boldsymbol{\lambda}^*)$, $V(\boldsymbol{\lambda}^*)$, and $\hat{D}$ (see Appendix D.2 for the proof).

Lemma 1. *The matrices $J(\boldsymbol{\lambda}^*)$, $V(\boldsymbol{\lambda}^*)$, and $\hat{D}$ have the common eigenvector*

$$\boldsymbol{\eta} = \begin{cases} \boldsymbol{\eta}_{\mathrm{Ra}} \text{ in (19)} & \text{for the racetrack economy,} \\ \boldsymbol{\eta}_{\mathrm{La}} \text{ in (24)} & \text{for the lattice economy (half doubling),} \\ \boldsymbol{\eta}_{\mathrm{Lb}} \text{ in (28)} & \text{for the lattice economy (full doubling).} \end{cases} \tag{34}$$

Lemma 2. *The eigenvalues β, γ, and ϵ of the matrices $J(\boldsymbol{\lambda}^*)$, $V(\boldsymbol{\lambda}^*)$, and $\hat{D}$, respectively, for the common eigenvector $\boldsymbol{\eta}$ in* (34) *are related as*

$$\gamma = K[\kappa'\epsilon + (1 - \kappa\epsilon)^{-1} \cdot \epsilon(\kappa - \epsilon)], \tag{35}$$

$$\beta = \Psi(\epsilon) = \frac{\epsilon\{\kappa + \kappa' - (\kappa\kappa' + 1)\epsilon\}}{1 - \kappa\epsilon}. \tag{36}$$

The break point τ^* can be determined as follows. First, $\epsilon = \epsilon^*$ for the break point[18] satisfying $(\beta =)\Psi(\epsilon^*) = 0$ is given by $\epsilon^* = (\kappa + \kappa')/(\kappa\kappa' + 1)$ and is rewrit-

[18] From (36), $\beta = 0$ is satisfied also by $\epsilon = 0$, which represents redispersion. This case, however, is not a major interest of this paper, and is excluded hereafter.

ten using $\kappa = \frac{\mu}{\sigma}$ and $\kappa' = \frac{\mu}{\sigma-1}$ as[19]

$$\epsilon^* = \frac{\mu(2\sigma - 1)}{\sigma(\sigma - 1) + \mu^2}. \tag{37}$$

Next, as shown in the sequel, the parameter for the remoteness r in (5) for the break point is given as a function of ϵ^* as $r^* = \Phi(\epsilon^*)$ with some function Φ. Last, the break point τ^* corresponding to $r = r^*$ can be determined from (5).

Remark 1. *The variable ϵ^* can be interpreted as an index for agglomeration as ϵ^* increases in association with an increase in μ or with a decrease of σ, both of which index a few large agglomerations.*

7.2. Formulas for break point: $n = 2$

As an illustration of basic ideas, formulas for break points are obtained for $n = 2$.[20] For the racetrack (two-place) economy with $D = D_{\mathrm{R}}$ in (33), we have

$$\hat{D}\boldsymbol{\eta} = \frac{D}{d}\boldsymbol{\eta} = \frac{1}{1+r}\begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1-r}{1+r}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \epsilon\boldsymbol{\eta}$$

with the eigenvalue $\epsilon = (1 - r)/(1 + r)$ and the eigenvector $\boldsymbol{\eta} = \boldsymbol{\eta}_{\mathrm{R}} = (1, -1)^\top$ for the spatial period doubling. Likewise, for the lattice economy, we have $\epsilon = (1 - r)^2/(1 + r)^2$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_{\mathrm{R}} \otimes \boldsymbol{\eta}_{\mathrm{R}} = (1, -1, -1, 1)^\top$. The relation between ϵ and r for the two economies can be expressed in a synthetic manner as

$$\epsilon = \left(\frac{1-r}{1+r}\right)^p, \quad \text{i.e.,} \quad r = \frac{1 - \epsilon^{1/p}}{1 + \epsilon^{1/p}} \tag{38}$$

[19] We have a no-black-hole condition $\frac{\mu}{\sigma-1} < 1$ (Forslid and Ottaviano, 2003 [11]) from (37) and $0 < \epsilon < 1$, which arises from (38) and (42) with $0 < r < 1$.

[20] The lattice economy with $n = 2$ is identical to the racetrack economy with $n = 4$.

using a variable p expressing the squared tensor product linkage as

$$p = \begin{cases} 1 & \text{for the racetrack economy and the lattice economy (full doubling),} \\ 2 & \text{for the lattice economy (half doubling).} \end{cases} \tag{39}$$

The break point for $n = 2$ is expressed as

$$\tau^* = \frac{2}{\mathcal{L}(\sigma - 1)} \log\left(\frac{1 + (\epsilon^*)^{1/p}}{1 - (\epsilon^*)^{1/p}}\right), \tag{40}$$

which gives the break point τ^* corresponding to $r = r^*$ with (5) and (38). Under a moderate assumption $\sigma \gg 1$, τ^* can be approximated as

$$\tau^* = \frac{2}{\mathcal{L}(\sigma - 1)} \log\left(\frac{1 + \epsilon^*}{1 - \epsilon^*}\right) \approx \frac{2}{\mathcal{L}(\sigma - 1)} 2\epsilon^* \approx \frac{8\mu}{\mathcal{L}(\sigma - 1)^2}. \tag{41}$$

7.3. Formulas for break point: $n = 4m$ $(m = 1, 2, \ldots)$

For $n = 4m$ $(m = 1, 2, \ldots)$, similarly to the case of $n = 2$, we can advance the relation between ϵ and r as

$$\epsilon = \left(\frac{1 - r}{1 + r}\right)^{2p}, \tag{42}$$

which encompasses both economies via the squared tensor product linkage (39).

Proposition 6. *The break point of the racetrack and the lattice economies for $n = 4m$ $(m = 1, 2, \ldots)$ can be formulated in a synthetic manner as*

$$\tau^* = \frac{n}{\mathcal{L}(\sigma - 1)} \log\left(\frac{1 + (\epsilon^*)^{1/2p}}{1 - (\epsilon^*)^{1/2p}}\right). \tag{43}$$

Proof. The relation (42) is solved for r as $r = \{1 + (\epsilon^*)^{1/2p}\}/\{1 - (\epsilon^*)^{1/2p}\}$ and is substituted into (5) to arrive at (43). □

Proposition 7. *As τ decreases from a large value for the lattice economy, the economic agglomeration is realized earlier for the half spatial doubling than for the full spatial doubling ($\tau^*_{\mathrm{Lb}} < \tau^*_{\mathrm{La}}$).*

Proof. For a given ϵ^*, (43) gives a larger τ^* for $p = 1$ than that for $p = 2$, which shows $\tau^*_{\mathrm{R}} = \tau^*_{\mathrm{Lb}} < \tau^*_{\mathrm{La}}$. □

Although the synthetic formula (43) is endowed with much desired independence from economic modeling, the influence of the parameter values σ and μ is contained implicitly in ϵ^* and is not transparent (Remark 1). As a remedy for this, we propose the following approximate formulas which clarify the influence of the values of these parameters on the break point τ^*.

Proposition 8. *Under an assumption $\sigma \gg 1$, the break point τ^* for $n = 4m$ ($m = 1, 2, \ldots$) is approximated by*

$$\tau^*_{\mathrm{R}} = \tau^*_{\mathrm{Lb}} \approx 2^{3/2} \frac{n}{\mathcal{L}} \frac{\mu^{1/2}}{(\sigma - 1)^{3/2}}, \quad \tau^*_{\mathrm{La}} \approx 2^{5/4} \frac{n}{\mathcal{L}} \frac{\mu^{1/4}}{(\sigma - 1)^{5/4}}. \tag{44}$$

Proof. The proof of these formulas is similar to the proof of (41) for $n = 2$. □

Remark 2. *The formulas for $n = 2$ presented in* (40) *have different forms than the formulas* (43) *for $n \geq 4$. Such a difference, which may be attributable to the influence of far places for $n \geq 4$, demonstrates the insufficiency of the two-place economy as a two-dimensional spatial platform for economic activities.*

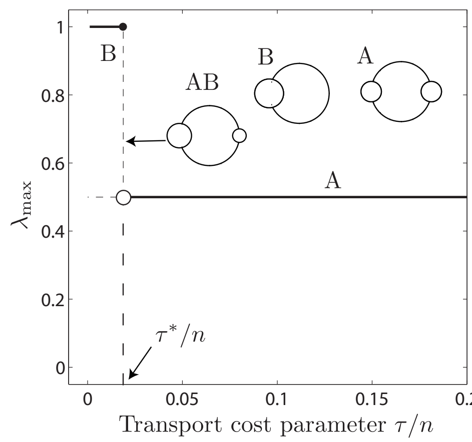
8. Progress of stable equilibria for an economic geography model

Spatial period doubling cascades of the two economies are studied in this section by a comparative static analysis with respect to the transport cost of the economic geography model (Section 3). The results of this analysis are examined in detail based on an ensemble of theoretical results in the previous sections: the theory of replicator dynamics (Section 5), the bifurcation mechanism of spatial period doubling (Section 6), and the formulas for the break point (Section 7).

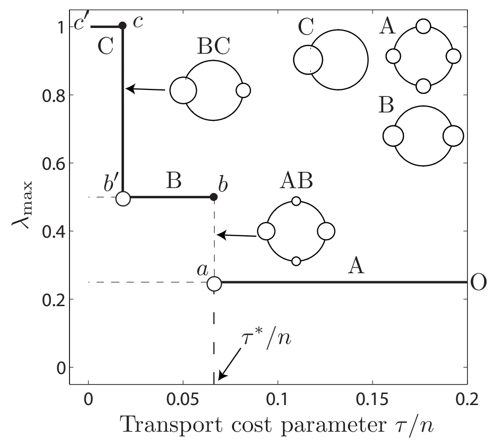
The size of the economies was chosen as $n = 2^m$ ($m = 1, 2, 3, 4$); note that the lattice economy with $n = 2$ is identical to the racetrack economy with $n = 4$. Parameter values were set as $\alpha = 1.0$ and $(\sigma, \mu) = (10.0, 0.4)$, which satisfy the no-black hole condition (Footnote 19).

8.1. Racetrack economy

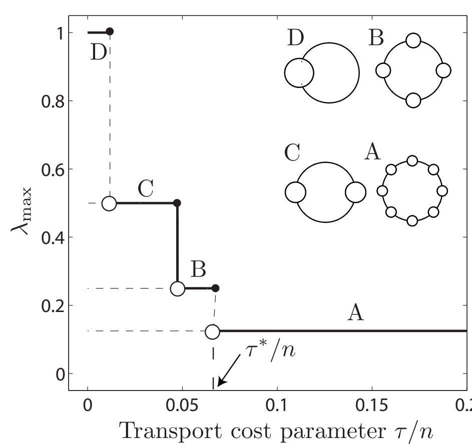
Curves of equilibria for the racetrack economy were computed and are plotted as a relation between $\lambda_{\max} = \max_{i=1}^{K} \lambda_i$ and the transport cost τ (Fig. 7). The horizontal lines A to E denote spatial period doubling trivial equilibria, whereas non-horizontal curves denote bifurcating equilibria. Stable and unstable equilibria are shown by solid and dashed lines, respectively. Every trivial solution was well-posed satisfying $\tau_{\rm B} < \tau_{\rm S}$ in (18) accommodating a range $\tau_{\rm B} < \tau < \tau_{\rm S}$ of stable equilibria, starting from a sustain point and ending with a break point as τ decreases. For example, for $n = 4$ (Fig. 7(b)), a spatial period doubling cascade between stable equilibria took place. There was a stable flat earth equilibrium for $\tau > \tau^*$ (state A). At the break bifurcation point a at $\tau = \tau^*$, there emerged an unstable transient state AB with two large places and two small places that connect



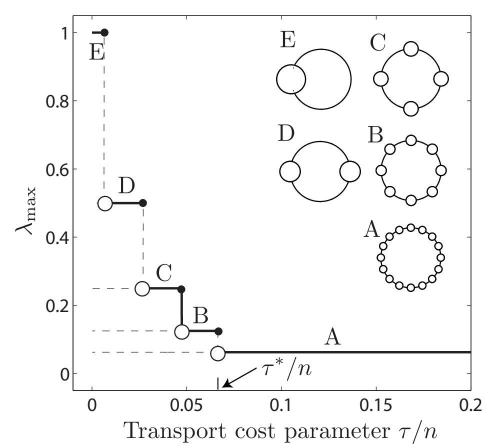
(a) $n = 2$



(b) $n = 4$



(c) $n = 8$



(d) $n = 16$

Figure 7: Curves of equilibria for the racetrack economy with $n = 2$, 4, 8, and 16 (solid lines denote stable equilibia and dashed curves denote unstable ones; (○): a simple break bifurcation point; (●): a sustain point; $\lambda_{\max} = \max_{i=1}^{n} \lambda_i$).

Table 1: Comparison of numerical, theoretical, and approximate break points (underlined values are approximate ones).

(a) Racetrack economy

	Number n of places	2	4	8	16
τ^*/n	Numerically computed	0.019	0.066	0.066	0.066
	Theoretical formula (40) or (43)	0.019	0.066	0.066	0.066
	Approximate formula (41) or (44)	$\underline{0.020}$	$\underline{0.066}$	$\underline{0.066}$	$\underline{0.066}$

(b) Lattice economy

	Number n of places	2	4	8	16
τ^*/n	Numerically computed	0.066	0.134	0.134	0.134
	Theoretical formula (40) or (43)	0.066	0.134	0.134	0.134
	Approximate formula (41) or (44)	$\underline{0.066}$	$\underline{0.121}$	$\underline{0.121}$	$\underline{0.121}$

the break point a and the sustain point b. This state regained stability at point b in the state B of two concentrated places and two extinguished places. Thereafter, at the break point b′, a stable transient state BC emerged en route to a stable atomic monocenter (state C starting from a sustain point c). As n increased to $n = 8$ and 16, there were cascades with more trivial equilibria. As τ decreased, stable equilibria shifted to fewer and larger agglomerations. Thus, the racetrack economy offers theoretically predicted idealistic agglomeration behavior (Section 6).

Normalized break points τ^*/n of the flat earth equilibrium A are listed in Table 1(a). Their numerically computed values were in complete agreement with the theoretical ones by (40) or (43) and were in good agreement with the approximate ones by (41) or or (44). Such an agreement is also seen in Table 1(b) for the lattice economy (Section 8.2), thereby ensuring the validity of these formulas.

8.2. Lattice economy

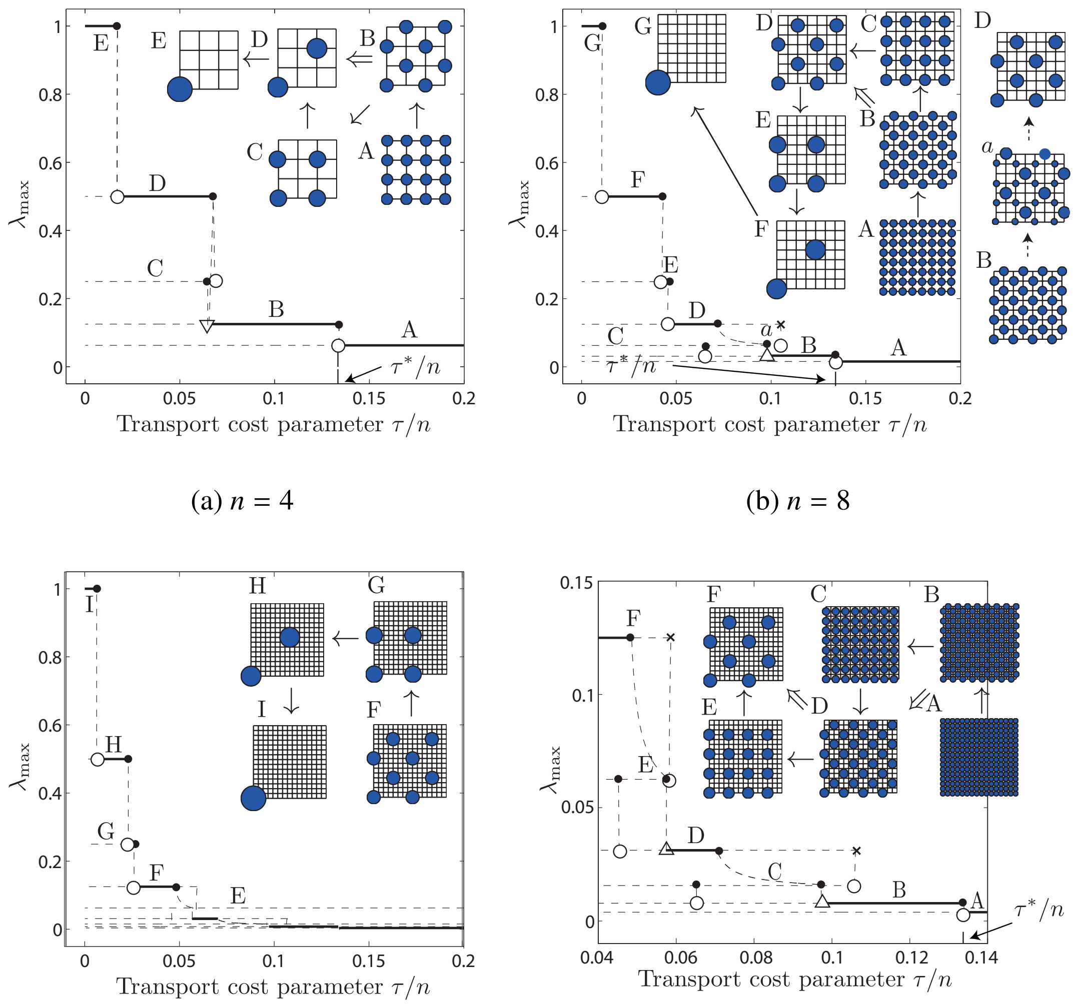
Curves of equilibria for the lattice economy (Fig. 8) displayed a spatial period doubling cascade between the trivial equilibria A to I. As τ decreased, stable equilibria shifted to fewer and larger agglomerations. Yet, unlike the racetrack economy, not all trivial equilibria were stable. All oblique patterns (cf., Fig. 6(b)) were well-posed satisfying $\tau_{\rm B} < \tau_{\rm S}$ in (16) and had stable equilibria. On the other hand, the foursquare patterns (cf., Fig. 6(a)) were either ill-posed solutions without stable equilibria (C for $n = 4$, 8 and 16 and E for $n = 16$) or well-posed but with very short durations of stable equilibria (E for $n = 8$ and G for $n = 16$).

The progress of agglomeration can be classified into three stages:[21] *dawn*, *intermediate*, and *mature stages*, as depicted in Fig. 9. In the dawn stage with a large transport cost, the underlying predominance of the market-crowding effect is weakened by an increase in the market-access effect that enlarges the agglomeration force, reorganizing firms into places with greater competition. Half spatial period doubling between two stable equilibria A and B took place for all cases ($n = 4,\ 8,\ 16$). The oblique pattern B engendered herein may be interpreted as a square lattice counterpart of a hexagon in central place theory.

In the intermediate stage, the market-crowding effect gradually decreases, whereas the market-access effect increases. In this stage, there were few stable equilibria unlike the other two stages. The equilibrium C was ill-posed and there were no stable equilibria for any cases. Full doubling[22] B⇒D took place bypass-

[21]This classification was introduced for the hexagonal lattice economy (Ikeda, Murota, and Takayama, 2017b [20])

[22]For $n = 4$, a break bifurcation in B led directly to D. For $n = 8$ and 16, a break bifurcation in



(a) $n = 4$

(b) $n = 8$

(c) $n = 16$

Figure 8: Curves of equilibria for the lattice economy with $n = 4$, 8, and 16 (solid lines denote stable equilibria and dashed ones denote unstable ones; (○): a simple break bifurcation point; (●): a sustain point; (△): a double bifurcation point; (▽): a triple bifurcation point; ×: a non-break point; $\lambda_{\max} = \max_{i=1}^{K} \lambda_i$).

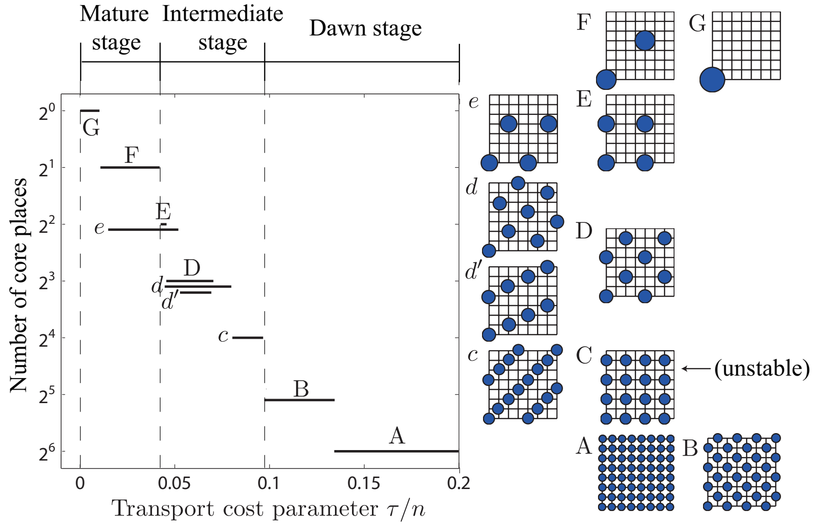


Figure 9: Durations of stable states for n = 8.

ing C and connecting stable equilibria B and D. For n = 16, another full doubling D⇒F took place bypassing an ill-posed equilibrium E and connecting stable equilibria D and F. Yet the transient states during the full doubling were all unstable.

In the mature stage, the transport cost became extremely low. Stability was recovered for all cases and the spatial period doubling cascade proceeded stably as

$$\begin{cases} \mathrm{D} \to \mathrm{E} & \text{for } n = 4, \\ \mathrm{F} \to \mathrm{G} & \text{for } n = 8, \\ \mathrm{F} \to \mathrm{G} \to \mathrm{H} \to \mathrm{I} & \text{for } n = 16. \end{cases}$$

Thus, a larger n entails more repeated occurrences of stable half doubling that are quite similar to the spatial period doubling cascade of the racetrack economy. Such similarity assesses the usefulness of the racetrack economy analogy.

There were several ranges of τ in which stable equlibria were absent in the intermediate stage for n = 8 and 16. To supplement such absence, the durations

B, followed by a non-break bifurcation, led to D.

of stable states were investigated for $n = 8$ encompassing other (non-doubling) equilibria that were obtained based on Proposition 2. Figure 9 depicts these durations in comparison with those of the spatial period doubling equilibria A to G. In the dawn stage, A and B were the only stable equilibria. In the intermediate stage and at the beginning of the mature stage, we encountered various kinds of stable trivial equilibria[23] c, d, d′, and e with stripe-like patterns, as well as the spatial period doubling ones D and E. At the end of the mature stage, a few large agglomerations, such as F, G, and e, were predominant. Thus, we have arrived at a more complete view on the transition of stable equilibria engendering fewer and larger agglomerations as τ decreases.

[23]Such emergence of various kinds of equilibria was also observed for a hexagonal lattice (Ikeda, Murota, and Takayama, 2017 [20]).

9. Conclusion

Agglomerations in a lattice economy were described by bifurcation theory with the aid of a racetrack economy analogy highlighting this economy as an idealized one-dimensional counterpart of two-dimensional economic agglomerations. A general methodology to find spatial patterns for trivial solutions in replicator dynamics was formulated. This methodology was applied to the racetrack economy and the lattice economy to set forth spatial period doubling patterns as important trivial solutions. Spatial period doubling cascades between these patterns were advanced as a theoretically possible course of the progress of agglomeration and was demonstrated to actually exist in both economies for an economic geography model. Knowledge of trivial solutions has turned out to be vital in understanding the mechanism of complicated agglomeration behavior of the lattice economy. It is to be emphasized that the proposed methodology is general and is readily applicable to other spatial platforms.

Progress of stable equilibria in association with decreasing transport cost τ in the lattice economy was observed for the economic geography model. In the dawn stage with large τ and in the mature stage with small τ, a spatial period doubling cascade between trivial equilibria was quite predominant. This demonstrates the usefulness of knowledge of trivial equilibria and the validity of the racetrack economy analogy. In the intermediate stage, however, equilibria of various kinds with stripe-like patterns were found to be stable. Such stage had also been previously observed for a hexagonal lattice (Ikeda, Murota, and Takayama, 2017b [20]) and may possibly be a general feature that is to be taken into consideration in the study of economic agglomerations.

As a quantitative measure of spatial agglomerations, analytical formulas for the break point in the lattice economy were newly developed for the economic geography model and were expressed in a synthetic manner to encompass the racetrack economy with the aid of the squared tensor product linkage. The validity of all these formulas has been ensured by comparative static analyses (Section 8).

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Appendix A. Details of the modeling of spatial economy

The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_{j=1}^{K} \int_0^{n_j} p_{ji}(\ell) q_{ji}(\ell) d\ell = Y_i, \tag{A.1}$$

where p_i^{A} is the price of A-sector goods in place i, $p_{ji}(\ell)$ is the price of a variety ℓ in place i produced in place j, and Y_i is the income of an individual in place i. The incomes (wages) of skilled workers and unskilled workers are represented, respectively, by w_i and w_i^{L}.

An individual in place i maximizes the utility in (1) subject to the budget constraint in (A.1). This yields the following demand functions of

$$C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \quad C_i^{\mathrm{M}} = \mu\frac{Y_i}{\rho_i}, \quad q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1} Y_i}{p_{ji}(\ell)^{\sigma}}, \tag{A.2}$$

where ρ_i denotes the price index of the differentiated products in place i, which is

$$\rho_i = \left(\sum_{j=1}^{K} \int_0^{n_j} p_{ji}(\ell)^{1-\sigma} d\ell \right)^{1/(1-\sigma)}. \tag{A.3}$$

Because the total income in place i is $w_i\lambda_i + w_i^{\mathrm{L}}$, the total demand $Q_{ji}(\ell)$ in place i for a variety ℓ produced in place j is given as

$$Q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1}}{p_{ji}(\ell)^{\sigma}}(w_i\lambda_i + w_i^{\mathrm{L}}). \tag{A.4}$$

The A-sector is perfectly competitive and produces homogeneous goods under constant-returns-to-scale technology, which requires one unit of unskilled labor per unit output. A-sector goods are transported without transportation cost and are chosen as the numéraire. In equilibrium, we have $p_i^{\mathrm{A}} = w_i^{\mathrm{L}} = 1$ for each i.

The M-sector output is produced under increasing-returns-to-scale technology and Dixit-Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. An M-sector firm located in place i chooses $(p_{ij}(\ell) \mid j = 1, \ldots, K)$ that maximizes its profit

$$\Pi_i(\ell) = \sum_{j=1}^{K} p_{ij}(\ell) Q_{ij}(\ell) - (\alpha w_i + \beta x_i(\ell)), \tag{A.5}$$

where $x_i(\ell)$ denotes the total supply of variety ℓ produced in place i and $(\alpha w_i + \beta x_i(\ell))$ signifies the cost function introduced by Flam and Helpman (1987).

With the use of the iceberg form of the transport cost, we have

$$x_i(\ell) = \sum_{j=1}^{K} T_{ij} Q_{ij}(\ell). \tag{A.6}$$

Then the profit function of an M-sector firm in place i, given in (A.5) above, can be rewritten as

$$\Pi_i(\ell) = \sum_{j=1}^{K} p_{ij}(\ell) Q_{ij}(\ell) - \left(\alpha w_i + \beta \sum_{j=1}^{K} T_{ij} Q_{ij}(\ell)\right), \tag{A.7}$$

which is maximized by the firm. The first-order condition for this profit maximization yields

$$p_{ij}(\ell) = \frac{\sigma\beta}{\sigma - 1} T_{ij}. \tag{A.8}$$

This implies that $p_{ij}(\ell)$, $Q_{ij}(\ell)$, and $x_i(\ell)$ are independent of ℓ. Therefore, argument ℓ is suppressed in the sequel.

In the short run, skilled workers are immobile between places, i.e., their spatial distribution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)$ is assumed to be given. The market equilibrium conditions consist of three conditions: the M-sector goods market clearing condition, the zero-profit condition attributable to the free entry and exit of firms, and the skilled labor market clearing condition. The first condition is written as (A.6) above. The second requires that the operating profit of a firm, given in (A.5), be absorbed entirely by the wage bill of its skilled workers. This gives

$$w_i = \frac{1}{\alpha}\left\{\sum_{j=1}^{K} p_{ij}Q_{ij} - \beta x_i\right\}. \tag{A.9}$$

The third condition is expressed as $\alpha n_i = \lambda_i$ and the price index ρ_i in (A.3) can be rewritten using (A.8) as

$$\rho_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha}\sum_{j=1}^{K}\lambda_j d_{ji}\right)^{1/(1-\sigma)}. \tag{A.10}$$

The market equilibrium wage w_i in (A.9) can be represented as

$$w_i = \frac{\mu}{\sigma}\sum_{j=1}^{K}\frac{d_{ij}}{\Delta_j}(w_j\lambda_j + 1) \tag{A.11}$$

using (4), (A.4), (A.6), (A.8), and (A.10). Here, $\Delta_j = \sum_{k=1}^{K} d_{kj}\lambda_k$. Equation (A.11) is solvable for w_i as follows. With the notation (7), (A.11) can be written as

$$\boldsymbol{w} = \frac{\mu}{\sigma}\, D\Delta^{-1}(\Lambda\boldsymbol{w} + \mathbf{1}), \tag{A.12}$$

which is solved for $\boldsymbol{w}$ as

$$\boldsymbol{w} = \frac{\mu}{\sigma}\left(I - \frac{\mu}{\sigma}\, D\Delta^{-1}\Lambda\right)^{-1} D\Delta^{-1}\mathbf{1}. \tag{A.13}$$

From the equilibrium equation $\boldsymbol{F}$ in (12) with (11), we have

$$\frac{\partial F_i}{\partial \lambda_j} = \left(v_i - \sum_{k=1}^{K}\lambda_k v_k\right)\delta_{ij} + \lambda_i\left(\frac{\partial v_i}{\partial \lambda_j} - v_j - \sum_{k=1}^{K}\lambda_k\frac{\partial v_k}{\partial \lambda_j}\right), \tag{A.14}$$

where δ_{ij} is the Kronecker delta. This shows that the Jacobian matrices $J(\boldsymbol{\lambda}) = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ and $V(\boldsymbol{\lambda}) = \partial \boldsymbol{v}/\partial \boldsymbol{\lambda}$ are related as

$$J(\boldsymbol{\lambda}) = \mathrm{diag}(v_1 - \bar{v}, \ldots, v_K - \bar{v}) + (\Lambda - \boldsymbol{\lambda}\boldsymbol{\lambda}^\top)V(\boldsymbol{\lambda}) - \boldsymbol{\lambda}\boldsymbol{v}^\top, \tag{A.15}$$

where $\bar{v} = \sum_{i=1}^{K} \lambda_i v_i$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_K)$, and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\lambda}, \tau) = (v_1(\boldsymbol{\lambda}, \tau), \ldots, v_K(\boldsymbol{\lambda}, \tau))^\top$.

At the flat earth equilibrium with $v_1 = \cdots = v_K = \bar{v}$, (A.15) gives

$$J(\boldsymbol{\lambda}^*) = \left(\frac{1}{K}I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right)V(\boldsymbol{\lambda}^*) - \frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top. \tag{A.16}$$

Appendix B. Details of the theory of trivial solutions in Section 5

We present details of Section 5. First, the proof of Proposition 2 is given as follows: Since the m places belonging to $\boldsymbol{\lambda}_+$ are assumed to permute each other by $T_+(g)$ ($g \in G$), we have $v_i = \bar{v}$ ($i = 1, \ldots, m$), thereby satisfying $\boldsymbol{F}_+(\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau) = \mathbf{0}$. For $K - m$ places with no population, we have $\lambda_j = 0$, thereby satisfying $\boldsymbol{F}_0(\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau) = \mathbf{0}$. This shows that $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) = (\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau)$ serves as a trivial solution.

Next, the proof of Corollary 1 reads: For an atomic monocenter for $m = 1$, Assumption 1 is satisfied by a group $G = \langle e \rangle$ and $T_+(e) = 1$. Then Proposition 2 guarantees that the corner solution of an atomic monocenter is a trivial solution. For twin places for $m = 2$, Assumption 1 is satisfied by a group $G = \langle h \rangle$ and

$$T_+(h) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where h denotes an exchange symmetry, i.e., $1 \leftrightarrow 2$. Then Proposition 2 guarantees that the corner solution for twin places is a trivial solution.

Last, the pattern in the left of Fig. 3(c), for example, is invariant to $\mathrm{D}_1 = \langle s \rangle$, i.e., the reflection $y \mapsto -y$. This invariance is expressed by the representation matrix

$$T_+(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which permutes places 2 and 3 but retains place 1 unchanged. Since there is no exchange symmetry between place 1 and other places, Assumption 1 is not satisfied. Hence that pattern is not a trivial solution in Proposition 2. The existence of a stationary point with this pattern is conditional on the value of τ.

Appendix C. Bifurcation of the lattice economy

After a brief introduction of group-theoretic bifurcation theory, bifurcation of the lattice economy is described.

Appendix C.1. Outline of group-theoretic bifurcation theory

The symmetry of the governing equation is formulated as the so-called equivariance condition[24]

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \tau), \quad g \in G \tag{C.1}$$

in terms of a $K \times K$ orthogonal matrix representation[25] T of the group G.

Consider a critical point $(\boldsymbol{\lambda}^*, \tau_c)$ on the flat earth equilibrium, which is said to have multiplicity $M\,(\geq 1)$ if the Jacobian matrix $J = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ of $\boldsymbol{F}$ at $(\boldsymbol{\lambda}^*, \tau_c)$ has M zero eigenvalues. Let $(\boldsymbol{\eta}_i \mid i = 1, \ldots, K)$ be an orthonormal basis of $\mathbb{R}^K$ such that

$$J\boldsymbol{\eta}_i = \boldsymbol{0}, \quad i = 1, \ldots, M. \tag{C.2}$$

We express the variable $\boldsymbol{\lambda}$ as $\boldsymbol{\lambda} = \boldsymbol{\lambda}^* + \sum_{i=1}^{M} \xi_i \boldsymbol{\eta}_i$ and τ as $\tau = \tau_c + \tilde{\tau}$, where $\tilde{\tau}$ denotes an increment of τ.

The full system of equations $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0}$ in (12) is reduced,[26] in a neighborhood of $(\boldsymbol{\lambda}^*, \tau_c)$, to a system of M equations (called bifurcation equations)

$$\tilde{\boldsymbol{F}}(\boldsymbol{\xi}, \tilde{\tau}) = \boldsymbol{0} \tag{C.3}$$

[24] This condition for the racetrack economy was proven in Ikeda, Akamatsu, and Kono (2012) [15]. The proof for the lattice economy can be achieved similarly.

[25] Matrix representation means that (i) for each element $g \in G$, $T(g)$ is a $K \times K$ matrix with $T(g)^\top T(g) = I$ (identity matrix), and (ii) $T(g)T(h) = T(gh)$ for all $g, h \in G$.

[26] This is a standard procedure called the *Liapunov–Schmidt reduction with symmetry* (Golubitsky, Stewart, and Schaeffer, 1988 [14]).

for some function $\tilde{\boldsymbol{F}}$ in $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_M) \in \mathbb{R}^M$ and $\tilde{\tau} \in \mathbb{R}$ defined above. In this reduction process, the symmetry condition (C.1) of the full system is inherited by the reduced system (C.3).

Appendix C.2. Half spatial period doubling

A simple break bifurcation point of the lattice is associated with the one-dimensional irreducible representation μ, which exists only when n is even and is given by

$$T^{\mu}(r) = 1, \quad T^{\mu}(s) = 1, \quad T^{\mu}(p_1) = -1, \quad T^{\mu}(p_2) = -1 \tag{C.4}$$

that satisfy the fundamental relations (Footnote 10). We assume that the variable $\boldsymbol{w} = w$ for the bifurcation equation (C.3) corresponds to the column vectors of

$$\begin{aligned}\boldsymbol{\eta} &= \{\cos(\pi(n_1 - n_2)) \mid n_1, n_2 = 1, \ldots, n\} \\ &= \{1, -1, \ldots, 1, -1; \ -1, 1, \ldots, -1, 1; \ \ldots \ ; \ -1, 1, \ldots, -1, 1\}.\end{aligned} \tag{C.5}$$

As stated in (24), when n is even, a bifurcating solution in the direction of $\boldsymbol{\eta}$ with the symmetry of $\Sigma = \langle r, s, p_1 p_2, p_1^{-1} p_2 \rangle$ arises from a critical point of multiplicity 1 associated with the irreducible representation μ. The proof of this statement is given below.

The fixed-point subspace of Σ for T^{μ} is given by

$$\mathrm{Fix}^{\mu}(\Sigma) = \{\boldsymbol{\xi} \in \mathbb{R}^M \mid T^{\mu}(g)\boldsymbol{\xi} = \boldsymbol{\xi} \text{ for all } g \in \Sigma\} = \{\xi \in \mathbb{R}\} \tag{C.6}$$

since $\boldsymbol{\xi} = \xi$ and

$$T^{\mu}(r)\xi = \xi, \quad T^{\mu}(s)\xi = \xi,$$

$$T^{\mu}(p_1p_2)\xi = (-1)(-1)\xi = \xi, \quad T^{\mu}(p_1^{-1}p_2)\xi = (-1)(-1)\xi = \xi$$

by (C.4). Thus, the fixed-point subspace $\mathrm{Fix}^{\mu}(\Sigma)$ of the targeted symmetry Σ is one-dimensional. The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ (see Chapter 8 of Ikeda and Murota, 2014 [16] for details of the equivariant branching lemma).

Secondary and further bifurcations for the lattice can be dealt with similarly. For example, for the secondary bifurcation, if we set $P_1 = p_1p_2$ and $P_2 = p_1^{-1}p_2$, we have the relations

$$\langle r, s, p_1p_2, p_1^{-1}p_2\rangle = \langle r, s, P_1, P_2\rangle, \quad \langle r, s, p_1^2, p_1^2\rangle = \langle r, s, P_1P_2, P_1^{-1}P_2\rangle.$$

Thus, the bifurcation analysis of the groups $\langle r, s, P_1, P_2\rangle$ and $\langle r, s, P_1P_2, P_1^{-1}P_2\rangle$ is identical to that of the groups $\langle r, s, p_1, p_2\rangle$ and $\langle r, s, p_1p_2, p_1^{-1}p_2\rangle$, respectively.

Appendix C.3. Full spatial period doubling

We consider a double bifurcation point that is associated with the two-dimensional irreducible representation μ, which exists only when n is even, and is given by

$$T^{\mu}(r) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \ T^{\mu}(s) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \ T^{\mu}(p_1) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \ T^{\mu}(p_2) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{C.7}$$

Let us assume that the variable $\boldsymbol{\xi} = (\xi_1, \xi_2)^{\top}$ for the bifurcation equation (C.3) corresponds to the vectors

$$\{\cos(\pi n_1) \mid n_1, n_2 = 1, \ldots, n\}, \ \{\cos(\pi n_2) \mid n_1, n_2 = 1, \ldots, n\}. \tag{C.8}$$

Bifurcating solutions associated with the irreducible representation μ exist in the direction of $\boldsymbol{q}_1 + \boldsymbol{q}_2$ with the symmetry of $\langle r, s, p_1^2, p_2^2 \rangle$, as shown below. Note

$$\mathrm{Fix}^{\mu}(\langle r, s, p_1^2, p_2^2 \rangle) = \mathrm{Fix}^{\mu}(\langle r \rangle) \ \cap \ \mathrm{Fix}^{\mu}(\langle s, p_1^2, p_2^2 \rangle).$$

Here we have $\mathrm{Fix}^{\mu}(\langle r \rangle) = \{c(1,1)^{\top} \mid c \in \mathbb{R}\}$ since $T^{\mu}(r)(\xi_1, \xi_2)^{\top} = (\xi_2, \xi_1)^{\top}$ by (C.7), whereas $\mathrm{Fix}^{\mu}(\langle s, p_1^2, p_2^2 \rangle) = \mathbb{R}^2$ since $T^{\mu}(s) = T^{\mu}(p_1^2) = T^{\mu}(p_2^2) = I$ by (C.7). Therefore,

$$\mathrm{Fix}^{\mu}(\Sigma) = \{c(1,1)^{\top} \mid c \in \mathbb{R}\},$$

that is, $\Sigma = \Sigma^{\mu}(\boldsymbol{\xi}_0)$ for $\boldsymbol{\xi}_0 = (1,1)^{\top}$. Thus, the targeted symmetry Σ is an isotropy subgroup with $\dim \mathrm{Fix}^{\mu}(\Sigma) = 1$. The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ.

Secondary and further bifurcations for full spatial period doubling can be dealt with similarly.

Appendix D. Details of derivation of formulas for break points

Details of derivation of formulas for break points in Section 7 are presented. In regard to $V(\boldsymbol{\lambda})$, we recall (8):

$$v_i = \frac{\mu}{\sigma - 1} \ln \Delta_i + \ln w_i \tag{D.1}$$

as well as (A.11):

$$w_i = \frac{\mu}{\sigma} \sum_k \frac{d_{ik}}{\Delta_k}(w_k \lambda_k + 1), \tag{D.2}$$

where

$$\Delta_k = \Delta_k(\boldsymbol{\lambda}, \tau) = \sum_{j=1}^{K} d_{jk} \lambda_j.$$

The differentiations of (D.1) and (D.2) with respect to λ_j yield, respectively,

$$\frac{\partial v_i}{\partial \lambda_j} = \kappa' \frac{d_{ji}}{\Delta_i} + \frac{1}{w_i} \frac{\partial w_i}{\partial \lambda_j}, \tag{D.3}$$

$$\frac{\partial w_i}{\partial \lambda_j} = \kappa \sum_{k=1}^{K} \frac{d_{ik}}{{\Delta_k}^2} \left[\left(\frac{\partial w_k}{\partial \lambda_j} \lambda_k + w_k \delta_{kj} \right) \Delta_k - (w_k \lambda_k + 1) d_{jk} \right], \tag{D.4}$$

where

$$\kappa = \frac{\mu}{\sigma}, \quad \kappa' = \frac{\mu}{\sigma - 1}. \tag{D.5}$$

We have $0 < \kappa < 1$ and $\kappa' > 0$ because $\sigma > 1$, $0 < \mu < 1$.

The matrix $V(\boldsymbol{\lambda}^*)$ in (31) can be evaluated as shown below. At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, we have

$$\Delta_j = \Delta_j(\boldsymbol{\lambda}^*, \tau) = \sum_{k=1}^{K} d_{kj} \lambda_k = \frac{d}{K}.$$

Because w_j is independent of j, we may write $w_j = w$; then (D.2) becomes

$$w = \kappa \sum_{j=1}^{K} \frac{K}{d} d_{ij} \left(\frac{w}{K} + 1 \right) = \kappa \left(w + K \right),$$

which yields

$$w = \frac{\kappa K}{1 - \kappa}. \tag{D.6}$$

At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, (D.4) becomes

$$\frac{\partial w_i}{\partial \lambda_j} = \kappa \sum_{k=1}^{K} \frac{K^2}{d^2} d_{ik} \left[\left(\frac{1}{K} \frac{\partial w_k}{\partial \lambda_j} + w\delta_{kj} \right) \frac{d}{K} - \left(\frac{w}{K} + 1 \right) d_{jk} \right],$$

which in matrix form reads as

$$W = \kappa \frac{K^2}{d^2} D \left[\frac{d}{K} \left(\frac{1}{K} W + wI \right) - \frac{w + K}{K} D \right]$$

with $W = (\partial w_i / \partial \lambda_j)$. With the use of (D.6), this equation can be rewritten as

$$\left(I - \kappa \frac{D}{d} \right) W = Kw \frac{D}{d} \left(\kappa I - \frac{D}{d} \right),$$

which can be further rewritten as

$$W = Kw \left(I - \kappa \frac{D}{d} \right)^{-1} \cdot \frac{D}{d} \left(\kappa I - \frac{D}{d} \right).$$

Then the partial derivatives in (D.3) can be evaluated in matrix form as

$$V(\boldsymbol{\lambda}^*) = K \left[\kappa' \frac{D}{d} + \left(I - \kappa \frac{D}{d} \right)^{-1} \cdot \frac{D}{d} \left(\kappa I - \frac{D}{d} \right) \right]. \quad \text{(D.7)}$$

Appendix D.1. Spatial discounting matrix

For the racetrack economy, the spatial discounting matrix D for $n = 4$ is given as

$$D_{\mathrm{R}} = \begin{bmatrix} 1 & r & r^2 & r \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r & r^2 & r & 1 \end{bmatrix}, \quad \text{(D.8)}$$

the matrix for $n = 8$ is given as

$$D_{\mathrm{R}} = R_8 = \begin{bmatrix} \tilde{R}_8 & \hat{R}_8 \\ \hat{R}_8 & \tilde{R}_8 \end{bmatrix} \quad \text{with} \quad \tilde{R}_8 = \begin{bmatrix} 1 & r & r^2 & r^3 \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r^3 & r^2 & r & 1 \end{bmatrix}, \quad \hat{R}_8 = r^4 \begin{bmatrix} 1 & r^{-1} & r^{-2} & r^{-3} \\ r^{-1} & 1 & r^{-1} & r^{-2} \\ r^{-2} & r^{-1} & 1 & r^{-1} \\ r^{-3} & r^{-2} & r^{-1} & 1 \end{bmatrix},$$

and that for $n = 16$ is given as

$$D_{\rm R} = R_{16} = \begin{bmatrix} \tilde{R}_8 & \hat{R}_{16} & r^4\hat{R}_8 & \hat{R}_{16}^\top \\ \hat{R}_{16}^\top & \tilde{R}_8 & \hat{R}_{16} & r^4\hat{R}_8 \\ r^4\hat{R}_8 & \hat{R}_{16}^\top & \tilde{R}_8 & \hat{R}_{16} \\ \hat{R}_{16} & r^4\hat{R}_8 & \hat{R}_{16}^\top & \tilde{R}_8 \end{bmatrix} \quad \text{with} \quad \hat{R}_{16} = \begin{bmatrix} r^4 & r^5 & r^6 & r^7 \\ r^3 & r^4 & r^5 & r^6 \\ r^2 & r^3 & r^4 & r^5 \\ r & r^2 & r^3 & r^4 \end{bmatrix}.$$

Appendix D.2. Proof of Lemmas 1 and 2

First, (C.1) gives a commutability $T(g)J(\boldsymbol{\lambda}^*) = J(\boldsymbol{\lambda}^*)T(g)$ $(g \in G)$ for the group G that labels the symmetry of each economy. Next, from (31), we have

$$T(g)\left(\frac{1}{K}I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right)V(\boldsymbol{\lambda}^*) - T(g)\frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top = \left(\frac{1}{K}I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right)V(\boldsymbol{\lambda}^*)T(g) - \frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top T(g),$$

which gives a commutability $T(g)V(\boldsymbol{\lambda}^*) = V(T(g)\boldsymbol{\lambda}^*)$ by $T(g)\mathbf{1}\mathbf{1}^\top = \mathbf{1}\mathbf{1}^\top T(g) = \mathbf{1}\mathbf{1}^\top$ and $\mathbf{1}\mathbf{1}^\top V(\boldsymbol{\lambda}^*) = V(\boldsymbol{\lambda}^*)\mathbf{1}\mathbf{1}^\top = \hat{V}\mathbf{1}\mathbf{1}^\top$, where $\hat{V}$ is the sum of the entries of a column of $V(\boldsymbol{\lambda}^*)$ that is identical for all the columns by the symmetry of the system. Last, from (32), we have a commutability $T(g)\hat{D} = \hat{D}T(g)$. These three commutabilities guarantee the existence of the common eigenvector $\boldsymbol{\eta}$ and a concrete form of $\boldsymbol{\eta}$ can be determined uniquely by adapting the method for the hexagonal lattice (Ikeda and Murota, 2014, Section 7.5 [16]).

Multiplying $V(\boldsymbol{\lambda}^*)$ in (32) by $\boldsymbol{\eta}$ from the right and using $\hat{D}\boldsymbol{\eta} = \epsilon\boldsymbol{\eta}$, we obtain $V(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\boldsymbol{\eta}$ with $\gamma = K[\kappa'\epsilon + (1 - \kappa\epsilon)^{-1} \cdot \epsilon(\kappa - \epsilon)]$. Multiplying (31) by $\boldsymbol{\eta}$ from the right and using $\mathbf{1}^\top\boldsymbol{\eta} = 0$ and $\mathbf{1}^\top V(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\mathbf{1}^\top\boldsymbol{\eta} = 0$, we obtain $J(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \frac{\gamma}{K}\boldsymbol{\eta}$. Then the eigenvalue β of the Jacobian matrix $J(\boldsymbol{\lambda}^*)$ for the eigenvector $\boldsymbol{\eta}$ is expressed in terms of ϵ as $\beta = \Psi(\epsilon)$ in (36).