Conditionally Additive Utility Representations

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Abstract

Advances in behavioral economics have made decision theoretic models increasingly complex. Utility models incorporating insights from psychology often lack additive separability, a major obstacle for decision theoretic axiomatizations. We address this challenge by providing representation theorems which yield utility functions of the form $u(x, y, z) = f(x, z) + g(y, z)$. We call these representations conditionally separable as they are additively separable only once holding fixed z. Our representation theorems have a wide range of applications. For example, extensions to finitely many dimensions yield both consumption preferences with reference points $\sum_i u_i(x_i, r)$, as well as consumption preferences over time with dependence across time periods $\sum_t u_t(x_t, x_{t-1})$.

1 Introduction

In an important contribution to utility theory, Debreu (1954) characterized what is known as additively separable preferences. He showed that certain assumptions on the preferences of a consumer hold if and only if these preferences can be represented by an additive utility function. For example, if preferences are defined on a product space $\prod_{i \in I} X_i$ of commodities $x_i \in X_i$, then $\sum_{i \in I} f_i(x_i)$ is an additive utility function. A wide class of problems can be addressed with such utility functions. In preferences over time, we often assume that the consumption in one time period has no effect on the desirability of consumption in another period. Constant elasticity of substitution preferences over goods spaces have an additive representation. In economic policy evaluation, utilitarian policy makers have additively separable preferences across individuals.

However, in the more recent literature, economic models have introduced more nuanced preferences in many of these cases. Consumption preferences may depend on reference points. In the case of preferences over time, the marginal utility of consumption in one period may depend on the consumption in the previous period. Policy makers who are not utilitarian may care about inequality, diversity, or the freedom of individuals, which usually lead to preferences which are not additively separable.

In the present paper, we generalize the idea of additively separable preferences to what we call conditionally separable preferences. Consider the example

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of preferences over consumption x_t in three periods of time t. Additively separable preferences would yield a utility representation such as $f_1(x_1) + f_2(x_2) + f_3(x_3)$. If the marginal utility of consumption depends on the previous period's consumption, we may instead have a conditionally additive utility representation $f_2(x_1, x_2) + f_2(x_2, x_3)$. In this representation we say that x_1 and x_3 are additively separable conditionally on x_2. We provide an axiomatization for such conditionally additive utility representations. Maintaining the usual continuity and order assumptions, our axiomatization differs from axiomatizations of additive utility functions in two ways.

Firstly, we weaken the usual independence assumptions such that we require only x_1 and x_3 to be independent of each other for fixed x_2. Additive utility functions over all three components would require x_1 to be independent of (x_2, x_3) and x_2 to be independent of (x_1, x_3).

Secondly, we sharpen the so-called Reidemeister condition. The Reidemeister condition is a necessary condition for additive representations of the kind $f(x_1) + f_2(x_2)$. Usually, in additive representations with at least three dimensions the Reidemeister condition is implied by the two independence conditions and continuity. However, even though our representation contains three dimensions, we only have one (conditional) independence assumption, requiring the use of the Reidemeister condition. The usual Reidemeister condition on x_1, x_3 would however only yield representations of the type $f_1(f_2(x_1, x_2) + f_3(x_2, x_3), x_2)$. Our generalization of the Reidemeister condition ensures that the additive utility functions across each value of x_2 are cardinally comparable.

We generalize our results in two important ways. Firstly, we extend our results to finitely many dimensions. Unlike additive representations, conditionally additive representations have more than one natural extension to higher dimensions. We consider the representations $\sum_i u_i(x_i, x_1)$ and $\sum_i u_i(x_i, x_{i-1})$ and provide axiomatizations. The former has a natural interpretation as a utility function with a reference point x_1. The latter utility function can be used to characterize preferences over time where an agent may be satiated from the consumption in the previous period.

Secondly, our representation theorem holds for subsets of product spaces with nonempty interiors. An important corollary of the latter generalization is a representation theorem for *additively* separable utility functions on probability spaces. To our knowledge, this is the first representation theorem of additively separable utility functions on a space with an *empty* interior in the product topology. Special cases of our representation are von Neumann-Morgenstern preferences with arbitrary probability distortions.

The applications of our results are not limited to utility theory. As an interesting example application to game theory, we provide a representation theorem for ordinal beliefs in a Bayesian game. We consider an agent who can only comprehend i) the structure of the game and ii) whether any state s of the world is more likely than another state s'. However, the agent cannot judge whether iii) a state is twice (half, three times, etc.) as likely than another state. Using our representation theorems, we provide a surprisingly simple representation for the beliefs of this agent.

The paper continues as follows. First, we will introduce some basic notation and definitions (Section 2). Next, we prove the representation theorem for the basic case in Section 3. The following Section 4 covers the finite dimensional case while Section 5 covers the case of subsets of a product space.

2 Model and Notation

Let $S = X \times Y \times Z$ be a product space where X, Y, Z are connected and separable spaces. $\succsim$ is a relation on S, i.e., a subset of $S \times S$. We say that s is weakly preferred to s' if $s \succsim s'$. We assume throughout the paper that $\succsim$ is complete and transitive and then call it a preference relation.[1] Let $\succ$ be the strict part of $\succsim$ and $\sim$ the symmetric part. X is independent (of Y) given Z if for all $x, x' \in X$, $y, y' \in Y$, and $z \in Z$, we have:

$$\begin{aligned} & (x, y, z) \succsim (x', y, z) \\ \Leftrightarrow \quad & (x, y', z) \succsim (x', y', z) \end{aligned} \tag{1}$$

$\succsim$ is independent with respect to X, Y given Z if X and Y are independent given Z. $\succsim$ is continuous if the sets $\overline{S}(t) = \{s \in S : s \succsim t\}$ and $\underline{S}(t) = \{s \in S : t \succsim s\}$ are closed with respect to the product topology for all t. X is essential if for all $x \in X$ there exist $(y, z) \in Y \times Z$ and $(y', z') \in Y \times Z$ such that $(x, y, z) \succ (x, y', z')$. X is essential given Z if for all $x \in X$ and all $z \in Z$ there exist $y \in Y$ and $y' \in Y$ such that $(x, y, z) \succ (x, y', z)$. $\succsim$ is essential if X, Y, Z are essential. $\succsim$ is essential given Z if X and Y are essential given Z.

Definition 1. $\succsim$ fulfills the generalized Reidemeister condition with respect to X given Z if for all $z, \bar{z} \in Z$ and all $x, x', \bar{x}, \bar{x}' \in X$ and all $y, y', \bar{y}, \bar{y}' \in Y$ such that the following points exist, we have:

$$\begin{aligned} & (x, y, z) \sim (\bar{x}', \bar{y}', \bar{z}) \\ \wedge \quad & (x', y, z) \sim (\bar{x}, \bar{y}', \bar{z}) \\ \wedge \quad & (x, y', z) \sim (\bar{x}', \bar{y}, \bar{z}) \\ \Rightarrow \quad & (x', y', z) \sim (\bar{x}, \bar{y}, \bar{z}) \end{aligned}$$

Definition 2. To simplify notation, in the following we will write $X \perp Y \mid Z$ if $\succsim$ is independent with respect to X, Y given Z, $\succsim$ fulfills the generalized Reidemeister condition with respect to X given Z and X and Y are essential given Z. Our borrowing of notation from the statistical literature will be justified when we relate our axioms to statistical independence.

We say that $\succsim$ fulfills restricted solvability given Z if for all $y \in Y, z \in Z, s \in S$: If $(x, y, z) \succsim s \succsim (x', y, z)$ then there exists x'' such that $(x'', y, z) \sim s$. If $(x, y, z) \succsim s \succsim (x, y', z)$ then there exists y'' such that $(x, y'', z) \sim s$

Lemma 1. *Suppose $\succsim$ is a continuous preference relation on S. Then $\succsim$ satisfies restricted solvability given Z.*

Proof. See Wakker (1989) Lemma III.3.3. □

3 Representation theorem for 3 dimensions

In this section, we will state our representation theorems for three dimensions and prove a lemma from which the main intuition of our result follows. The three dimensional case is the key building block for higher dimensional cases.

The main representation result for the case of three dimensions is as follows.

[1] The results can possibly be generalized by dropping the completeness assumption. Vind (1991) gives a representation theorem for additive utility functions.

Theorem 1. *Let $\succsim$ be a continuous preference relation on $S = X \times Y \times Z$ where X, Y, Z are connected and separable topological spaces. Then $\succsim$ fulfills $X \perp Y \mid Z$ if and only if:*
a) there exists a representation $v(x, y, z) = f(x, z) + g(y, z) + h(z)$ such that for some x_0, y_0, $f(x_0, z) = 0$ and $g(y_0, z) = 0$ and all functions f, g, h are continuous, and
b) the representation is unique up to affine transformations, i.e. if $\bar{v}(x, y, z) = \bar{f}(x, z) + \bar{g}(y, z) + \bar{h}(z)$ is another representation fulfilling a) for some choice of $\bar{x}_0, \bar{y}_0$, then $\bar{v} = \alpha v + \beta_f + \beta_g + \beta_h$, $\bar{f} = \alpha f + \beta_f$, $\bar{g} = \alpha g + \beta_g$, and $\bar{h} = \alpha h + \beta_h$ with $\alpha > 0, \beta_f, \beta_g, \beta_h \in \mathbb{R}$.

The reader may be intrigued by our use of the function h and the requirement that $f(x_0, z) = 0$ and $g(y_0, z) = 0$. While h is certainly superfluous in part a) of the theorem, it is crucial for the uniqueness result b). To see this, let $h_f(z) + h_g(z) = h(z)$ and $\bar{f} = f + h_f$ and $\bar{g} = g + h_g$. While it holds that $v = f + g + h = \bar{v} = \bar{f} + \bar{g}$, it is for example not necessarily true that $f = \alpha\bar{f} + \beta$. Intuitively, we may understand the function h as the separable part of the preference of Z. Thus, once we fix some point x_0, y_0, h is the utility function over the points (x_0, y_0, z). In the following, when we state the uniqueness of a representation, it is always meant in the above way.

Due to its length, we delegate the proof of the representation theorem to the appendix with the exception of a Lemma which provides the main intuition behind the result and the proof of which links well with proofs of additive representations, in particular the one in Wakker (1989).

Lemma 2. *Let $\succsim$ be a continuous preference relation on $S = X \times Y \times Z$ where X, Y, Z are connected and separable topological spaces. Let $\succsim$ fulfill $X \perp Y \mid Z$. Then:*
a) For any pair $z', z'' \in Z$ with some (x', y', z'), $(\underline{x}', \underline{y}', z')$, (x'', y'', z''), and $(\underline{x}'', \underline{y}'', z'')$ such that $(\underline{x}'', \underline{y}'', z'') \sim (\underline{x}', \underline{y}', z') \prec (x'', y'', z'') \sim (x', y', z')$ there exists a utility representation $u(x, y, z) = f(x, z) + g(y, z) + h(z)$ on $X \times Y \times \{z', z''\}$.
b) The representation is unique up to affine transformations $v' = \alpha v + \beta$, $f' = f\alpha + \beta_f$, $g' = g\alpha + \beta_g$, $h' = \alpha h + \beta - \beta_f - \beta_g$.

We included Lemma 2 and its proof into the main text for two reasons. First, it gives an insight into the utility construction process and how this process differs from the procedure for additive representations. Second, Lemma 2 is of independent interest. Our main representation theorem assumes Z to be connected and separable. The lemma shows that our result would also hold for finite Z with overlapping utility ranges.

Proof. We start out by constructing a utility function on $X \times Y \times \{z'\}$. We use the same utility construction process as in Wakker (1989). Essentiality given Z guarantees that there exist x_0,$x_1 \in X$ and y_0 $y_1 \in Y$ such that

$$\begin{aligned}(x_0, y_0, z') &\prec (x_1, y_0, z')\\(x_0, y_0, z') &\prec (x_0, y_1, z')\\(x_1, y_0, z') &\sim (x_0, y_1, z')\end{aligned} \tag{2}$$

Next, since the generalized Reidemeister condition implies the Reidemeister condition on each z-layer $X \times Y \times \{z\}$, we can construct an order grid on the

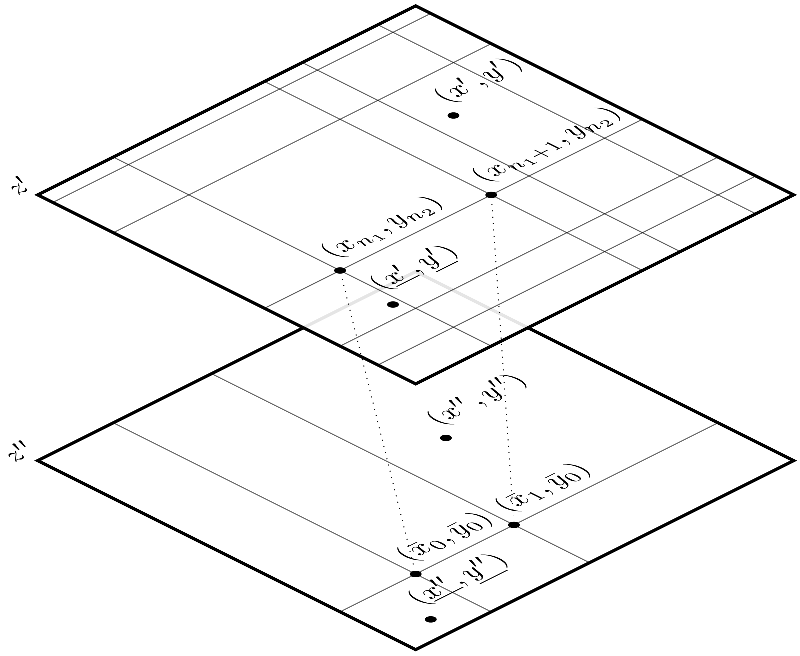

Figure 1: Extension of the utility grid from the z' layer to the z'' layer.

z'-layer such that for any rational numbers n, n', m, m' we have $(x_n, y_m, z') \sim (x_{n'}, y_{m'}, z') \Leftrightarrow n + m = n' + m'$. For details of how to construct this grid, see Wakker (1989).

We next extend this representation to the z''-layer. Since our grid is dense in the z'-layer, we can find a grid point (x_{n_1}, y_{n_2}, z') on the z'-layer such that $(\underline{x}', \underline{y}', z') \prec (x_{n_1}, y_{n_2}, z') \prec (x', y', z')$. Therefore, by restricted solvability on the z''-layer, we can find a point $(\bar{x}, \bar{y}, z'') \sim (x_{n_1}, y_{n_2}, z')$. Next, we construct the grid on both z-layers in the following way. We use the point $(\bar{x}_0, \bar{y}_0, z'')$ on the z''-layer satisfying $(\bar{x}_0, \bar{y}_0, z'') \sim (x_{n_1}, y_{n_2}, z')$ as the center on the z''-layer and construct the grid with an initial point $\bar{x}_1, \bar{y}_0$ satisfying $(\bar{x}_1, \bar{y}_0, z'') \sim (x_{n_1+1}, y_{n_2}, z')$. These points exist by restricted solvability and by the fact that we can choose our initial points (x_0, y_0, z') and (x_1, y_0, z') to be arbitrarily close to each other.

We now show that the grid points are indeed consistent on both layers. That is, we want to show that

$$
\begin{aligned}
(x_{n+1}, y_m, z') &\sim (x_n, y_{m+1}, z') \\
(\bar{x}_{n+1}, \bar{y}_m, z'') &\sim (\bar{x}_n, \bar{y}_{m+1}, z'') \\
(x_n, y_m, z') &\sim (\bar{x}_{n_1+n}, \bar{y}_{n_2+m}, z'')
\end{aligned} \tag{3}
$$

for all n, m.

Similar to the argument of Wakker (1989), we use induction on our subcripts. For $n + m = 0$, the result directly follows from $(\bar{x}, \bar{y}, z'') \sim (x_{n_1}, y_{n_2}, z')$. For $n+m = 1$ the condition follows from the construction of the grid. For $n+m \geq 2$,

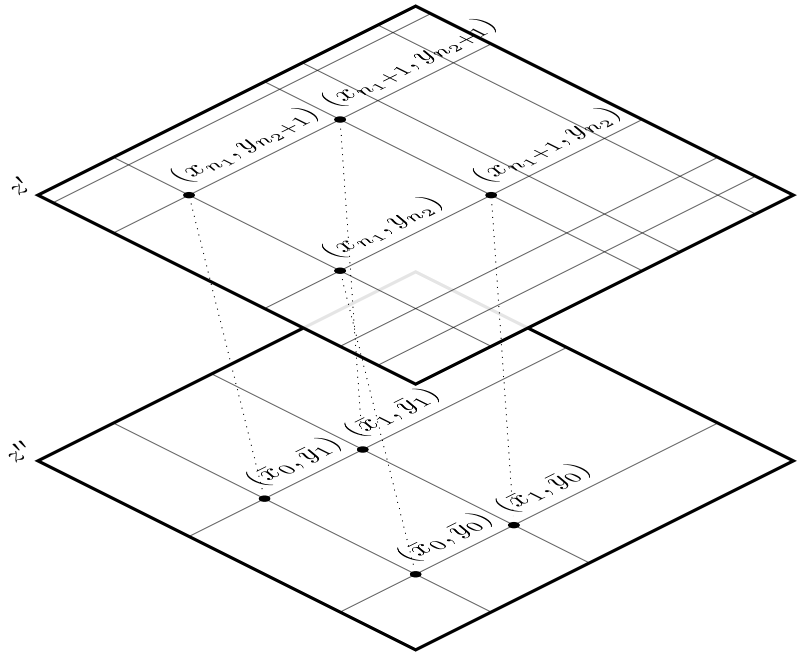

Figure 2: Consistency of the utility grid between the z' layer and the z'' layer.

we simply notice that the generalized Reidemeister condition given Z implies

$$
\begin{aligned}
(x_{n_1+n-2}, y_{n_2}, z') &\sim (\bar{x}_{n-2}, \bar{y}_0, z'') \\
(x_{n_1+n-1}, y_{n_2}, z') &\sim (\bar{x}_{n-1}, \bar{y}_0, z'') \\
(x_{n_1+n-2}, y_{n_2+1}, z') &\sim (\bar{x}_{n-2}, \bar{y}_1, z'') \\
&\text{and therefore} \\
(x_{n_1+n-1}, y_{n_2+1}, z') &\sim (\bar{x}_{n-1}, \bar{y}_1, z''). \qquad (4)
\end{aligned}
$$

We can extend the integer-valued grid on the z''-layer to a rational-valued grid by the same method as in Wakker (1989). Via transitivity and the fact that for any $x_n, n \in \mathbb{Q}$ we can find y_m such that there exist $x_{n'}, n' \in \mathbb{Z}$ and $y_{m'}, m' \in \mathbb{Z}$ such that $n + m = n' + m'$ and thus $(x_n, y_m, z'') \sim (x_{n'}, y_{m'}, z'')$, the extended grid on the rationals is also consistent.

Next, we define the functions

$$
\begin{aligned}
f(x_n, z') &:= n \\
f(\bar{x}_n, z'') &:= n \\
g(y_m, z') &:= m \\
g(\bar{y}_m, z'') &:= m \\
h(z'') &:= n_1 + n_2 \\
h(z') &:= 0 \qquad (5)
\end{aligned}
$$

Since our grid is dense in the z' and z''-layers, due to continuity we can extend the utility functions on the entire z' and z''-layers by taking the limit to obtain a continuous additive utility representation $u(x, y, z) = f(x, z) + g(y, z) + h(z)$ on both layers.

b) The uniqueness of the representation on the z' layer follows from the arbitrary choices of $u(x_0, y_0, z) = 0, u(x_1, y_0, z) = 1$ the choice of $f(x_0, z') = 0$, and $h(z') = 0$ when constructing the grid on the z' layer. Given the representation on the z' layer, the extension to the z'' layer is unique. $\square$

In summary, the construction of the utility representation on a single layer follows the construction by Wakker (1989). In this step, our generalized Reidemeister condition fulfills the same role as the Reidemeister condition in Wakker (1989): if a preference relation over a product space $X \times Y$ is continuous and independent, the Reidemeister condition is required to ensure that an additive representation exists.

However, when extending the representation to the second layer, the generalized Reidemeister condition fulfills an additional role: it makes the additive representations on both layers consistent with each other. Assuming only the Reidemeister condition without our generalization, we could obtain an additive representation on each layer. But notice that for example the preference induced by the utility function $(f(x) + g(y))^{h(z)}$ has an additive representation on each layer z, but does not have a utility representation of the desired form. The generalized Reidemeister condition excludes such preferences.

3.1 Example: Ordinal belief systems

We now present an interesting connection of our axioms to the theory of probability. Suppose our relation $s \succsim s'$ on a state space $S = X \times Y \times Z$ can be interpreted as s being at least as likely as s'. Then it turns out that $X \perp Y \mid Z$ holds if and only if there exists a probability measure p on S fulfilling conditional independence in the statistical sense.

Corollary 1. *Let $\succsim$ be a continuous preference relation on $S = X \times Y \times Z$ where X, Y, Z are connected, separable, and compact topological spaces. Let $X \perp Y \mid Z$. Then for all finite measures $\mu = \mu_x \mu_y \mu_z$ there exists a representation of $\succsim$ of the form*

$$p(x, y, z) = p_{X|Z}(x, z) \cdot p_{Y|Z}(y, z) \cdot p_Z(z) \tag{6}$$

where

$$\int_Z p_Z(z) d\mu_z = 1 \tag{7}$$

$$\int_{X \times Z} p_{X|Z}(x, z) p_Z(z) d\mu_x d\mu_z = 1 \tag{8}$$

$$\int_{Y \times Z} p_{Y|Z}(y, z) p_Z(z) d\mu_y d\mu_z = 1 \tag{9}$$

$$\int_S p(x, y, z) d\mu = 1 \tag{10}$$

The probability representation is not unique, exponentiating all functions by a common exponent (and renormalizing) yields another representation of $\succsim$. This is not surprising since $\succsim$ provides only ordinal information. In order to fix a unique probability representation, in addition we would need a statement such as "(x, y, z) is twice as likely as (x', y', z')".

In many cases the assumption of boundedly rational agents having beliefs consistent with the laws of probability is implausible. Nonetheless, boundedly rational agent may have well defined ordinal beliefs whether state s or s' is more likely.

Consider a game where Alice, Bob, and Charlie observe a signal from Nature. Next, Alice gets to make a hidden choice followed by a hidden choice by Bob. Finally, it is Charlie's turn to make a choice and he ponders what Alice and Bob have previously chosen. Charlie may have an idea of the structure of the game (Bob and Alice's actions are independent given the public signal) and may have an ordinal ranking of what is more or less likely (given a certain signal z from Nature, Alice is more likely to play x than x'). However, Charlie may be unable to state precisely how much more likely it is that Bob will play y instead of y'. This type of boundedly rational belief system is made precise by our axiomatization: Charlie's beliefs can be described by a set of probability density functions $p(x, y, z)$ which are all independent in Alice's and Bob's strategies x and y given the signal z and which are exponential transformations of each other.

Charlie's decision problem can therefore be written as a decision under Knightian uncertainty, where the set of probability distributions is given by all exponential transformations of a single joint probability distribution over the actions of Nature, Alice and Bob.

4 Representation theorems for higher dimensions

In the following, we will extend our representation result to product spaces of higher dimensions. Notice that as soon as there are more than three dimensions, different extensions are possible. In terms of utility functions, we may for example be interested in the conditions which yield a representation of the kind $f(x, w) + g(y, w) + h(z, w)$ or $f(x, y) + g(y, z) + h(z, w)$. In the following, we will consider what we believe are the most interesting cases. We hope that our treatment of these cases is instructive for the cases we omit.

In this section we therefore assume that our space is a product space $S = \prod_{i \in I} X_i$ with finite index set $I = \{1, 2.., n\}$. We first note that our definition of $X = \prod_{i \in I_X} X_i$ being independent of $Y = \prod_{i \in I_Y} X_i$ given $Z = \prod_{j \in I - I_X - I_Y}$ continues to work well in higher dimensions if $I_X \cap I_Y = \emptyset$ and $I_X, I_Y \subseteq I$. The same holds for our generalized Reidemeister condition with respect to X given Z.

Theorem 2. *Let $\succsim$ be a continuous preference relation on $S = \prod_{i=1}^{n} X_i$, $3 \leq i < \infty$ where all X_i are connected and separable topological spaces. Then $\succsim$ fulfills $X_i \perp \prod_{j \neq i,1} X_j \mid X_1$ for all $i \neq 1$ if and only if:*
a) $\succsim$ can be represented by $v(s) = \sum_{i=2}^{n} f_i(x_i, x_1) + h(x_1)$ where each f_i is continuous, h is continuous, and $f_i(x_i^0, x_1) = 0$ for all i, and all x_1.
b) the representation is unique up to affine transformations.

A natural application of this representation theorem are preferences which are reference dependent (Sugden (2003)). x_1 can be interpreted as being the reference point of the utility of an $n-1$-dimensional consumption space. For example, the reference point x_1 of a consumer may be the consumption bundle of the Jones's. It is straightforward to add further structure onto the utility representation. If we impose that the elasticity of substitution is constant between any two goods, we get the representation $v(s) = \sum_{i=2}^{n}(x_i)^{\theta} w_i(x_1) + h(x_1)$. We therefore have CES preferences where the Jones's consumption influences the weight attached to each consumption good. Note that deriving a representation theorem for these preferences from scratch would be a nontrivial task. Most likely, the above result would need to be derived somewhere along the way.

The above theorem covers the case where all additive component functions f_i share one argument z. Another interesting case arises if none of the functions f_i share an argument:

Theorem 3. *Let $\succsim$ be a continuous preference relation on $S = \prod_{i=1}^{n} X_i$, $3 \leq i < \infty$ where all X_i are connected and separable topological spaces. Then $\succsim$ fulfills $\prod_{j=1}^{i-1} X_j \perp \prod_{k=i+1}^{n} X_k \mid X_i$ for all $i = 2, \ldots, n-1$ if and only if: $\succsim$ can be represented by $v(s) = \sum_{i=2}^{n} f_i(x_i, x_{i-1})$ where each f_i is continuous.*

A natural application for this representation theorem are preferences over time. Preferences over consumption streams need not be additively separable if individuals experience satiation, addiction, or form consumption habits (Rozen (2010)). In this case, preferences over consumption periods sufficiently distant in time may be additively separable when holding fixed the consumption in between. The above representation theorem captures the case where the marginal utility of consumption depends on the previous period's consumption. The overlapping number of dimensions can of course be increased by a corresponding change in the independence conditions.

5 Subsets of a Product Space

So far we have only discussed the case of product spaces. However, from the proofs of Lemma 2 it is obvious that our analysis extends to any axiomatic structure which yields an additive representation on each z-layer (or x_1-layer, x_i-layer in Theorems 2 and 3, respectively). We continue our analysis by considering an important generalization in which our set S is no longer a product space. Instead, we allow S to be a subset of a product space. More precisely, we allow $S \subset \prod_{i=1}^{n} X_i \times Z$ and assume that for each $z^* \in Z$, the set $\{(x, z) \in S\} : z = z^*$ has a nonempty interior in the product topology of $\prod_{i=1}^{n} X_i$.

Definition 3. $S \subset \prod_{i=1}^{n} X_i \times Z$, ($X_i, Z$ connected, separable sets) is well behaved given Z if for all $z^* \in Z$
i) S is connected, $int(S)$ is connected and nonempty
ii) for all i, x_i^*, z^* $\{(x, z) \in S : x_i = x_i^*, z = z^*\}$ is connected
ii)b) for all z^* $\{(x, z) \in S : z = z^*\}$ is connected
iii) all equivalence classes in $int(\{(x, z) \in S : z = z^*\})$ are connected
iv) all boundary points of S are limit points of interior points.

These assumptions correspond to the structural assumptions of Wakker and Chateauneuf (1993) and guarantee the existence of additive representations on each z-layer.

Lemma 3. *Suppose $\succsim$ is a continuous preference relation on a well behaved space $S \subseteq \prod_{i=1}^{n} X_i \times Z$ and $X_i \perp \prod_{j \neq i} X_i$ for all i. Suppose for some, z' and $z'' \in Z$ there exist $s_{z'} \in int(S_{z'}) = int(\{(x,z) \in S : z = z'\})$ and $s_{z''} \in int(S_{z''}) = int(\{(x,z) \in S : z = z''\})$ such that $s_{z'} \sim s_{z''}$. Then there exists a utility representation*

$$u(x,y,z) := \sum_{i=1}^{n} u_i(x_i,z) + h(z)$$

on $S_{z'} \cup S_{z''}$.

Proof. Since $S_{z'}$ is connected, the projections of $S_{z'}$ on each coordinate are connected. The open connect subset of topological space induced by the order topology would be a preference interval. Therefore, we can write each projection $\pi_i(S(z'))$ on coordinate i as $(x_i^{z'}, \bar{x_i^{z'}})$. By Wakker and Chateauneuf (1993), there exists additive representation on $S_{z'}$ and $S_{z''}$. That means that we can uniquely define the utility level on open cubes $O_{z'} := \prod_{i=1}^{N}(x_i^{z'}, \bar{x_i^{z'}}) \times \{z'\}$ and $O_{z''} := \prod_{i=1}^{N}(x_i^{z''}, \bar{x_i^{z''}}) \times \{z''\}$. Note that $S_{z'} \subset O_{z'}$ and $S_{z''} \subset O_{z''}$. We can use exactly the same argument as in Lemma 2 to construct a utility representation on $O_{z'} \cup O_{z''}$. □

Theorem 4. *Suppose $\succsim$ is a preference relation on a well behaved space $S \subseteq \prod_{i=1}^{n} X_i \times Z$ satisfying continuity and $X_i \perp \prod_{j \neq i} X_i$ for all i. Then there exists a representation $u(x,z) = \sum_i u_i(x_i,z)$ on S.*

Proof. The proof is almost literally identical to the proof of Theorem 1 with Lemma 2 replaced by Lemma 3. □

We can use the above theorem to provide an interesting new result on additive representations. So far, additive representations have only been axiomatized for sets with nonempty interiors in the product topology. However, an important space in economics which does not fulfill this requirement is the lottery space. The reason for this is that the classical independence axiom is not well defined for lottery spaces. Take a statement such as $(x,y) \succsim (x',y) \Rightarrow (x,y') \succsim (x',y')$ where x is the probability of the first state and y the probability of the second state. Then by the laws of probability $x = x'$, $y = y'$ and the axiom is not meaningful.

In order to provide a meaningful account of independence in probability spaces we need our conditional independence axiom. Let us first expand the probability space to 4 dimensions with p_i being the probability of state i. Next, let $x = (p_1, p_2)$ and $y = (p_3, p_4)$. Now a statement such as $(x,y) \succsim (x',y) \Rightarrow (x,y') \succsim (x',y')$ is meaningful as long as $p_1 + p_2 = p_1' + p_2'$.

An interesting corollary of the above result is the following additive representation on a probability space.

Corollary 2. *Suppose $S = \{x \in \prod_{i=1}^{n} X_i : x_n = 1 - \sum_{i=1}^{n-1} x_i\}$ and each $X_i = [0,1]$. Let $\succsim$ be a continuous preference relation fulfilling for all i,j:*

- $(X_i, X_n) \perp \prod_{k \neq i,n} X_k$ *given* $\sum_{k \neq i,n} X_k$,

- *comeasurability of* (X_i, X_n) *and* (X_j, X_n)*:*

$$\begin{aligned}
&(\ldots, x_i + \alpha, x_j, x_k, x_n - \alpha)\\
\sim&(\ldots, x_i, x_j + \beta, x_k - \beta, x_n)\\
\sim&(\ldots, x_i, x_j + \gamma, x_k, x_n - \gamma)\\
\sim&(\ldots, x_i + \delta, x_j, x_k - \delta, x_n)\\
\Rightarrow&\\
&(\ldots, x_i + \alpha, x_j + \beta, x_k - \beta, x_n - \alpha)\\
\sim&(\ldots, x_i + \delta, x_j + \gamma, x_k - \delta, x_n - \gamma) \qquad (11)
\end{aligned}$$

for all $x_i, x_j, x_k, x_n, \alpha, \beta, \gamma, \delta$ *for which the above elements of* S *are defined.*

Then $\succsim$ *can be represented by:*

$$u(x) = \sum_{i=1}^{n} u_i(x_i) \qquad (12)$$

To our knowledge, so far no representation theorems for additively separable utility function have been provided for probability spaces. We attribute this to the fact that probabilities can never be fully independent, but only independent conditional on some subset of the probability space, keeping the sum of certain probabilities constant. Our novel axiom system allows us to bypass this challenge.

This representation theorem can be further sharpened to give more structure to the functions u_i. For example, $u_i(x) = \phi(x)v_i$ would yield an expected utility model with vNM utilities v_i and a probability distortion function ϕ in the spirit of loss aversion models (Kahneman and Tversky (1979)).

6 Future Research

As a final remark, we note that the generalized Reidemeister condition can be used to extend any additively separable representation to other layers. Thus, we see potential extensions of our representation theorems to mixture spaces from Herstein and Milnor (1953). Moreover, some applications may have more complex conditioning structures than a single variable. We are currently working on a general framework to combine conditional independence assumptions and generalized Reidemeister conditions across multiple variables. We believe that such results will be a valuable toolbox for economic theorists.

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A Proof of Theorem 1

We first state and proof some further intermediate results before turning to the proof of the representation theorem.

Lemma 4. *Let $\succsim$ be a continuous preference relation on $S = X \times Y \times Z$ where X, Y, Z are connected and separable topological spaces. Let $u : X \times Y \times Z \to \mathbb{R}$ be a continuous representation of $\succsim$. Then for every $u^* \in int(u(X \times Y \times Z))$ there exists an open interval around u^* and $\hat{Z} \subseteq Z$ such that there exists a continuous representation $v : X \times Y \times \hat{Z} \to \mathbb{R}$ with $v(x, y, z) = f(x, z) + g(y, z) + h(z)$ and $f(x_0, z) = 0$ and $g(y_0, z) = 0$ covering the indifference classes of the interval.*

Proof. We claim that there exists a layer z' such that

$$\inf_{a \in X, b \in Y} u(a, b, z') < u^* < \sup_{a \in X, b \in Y} u(a, b, z'). \tag{13}$$

Since u^* is in the interior of $u(X \times Y \times Z)$, there must exist a z' such that $u^* < \sup_{a \in X, b \in Y} u(a, b, z')$. By continuity of u, $sup_{x,y} u(x, y, z)$ is lower semicontinuous. Thus, the set $Z' = \{z' | \sup_{a \in X, b \in Y} u(a, b, z') > u^*\}$ of all such z' is open. Since Z is connected, the boundary of Z' is nonempty (unless $Z' = Z$ in which case u^* is not in the interior). Let z^* be a boundary point of Z'. Clearly, $\sup_{x,y} u(x, y, z^*) = u^*$.

Since z^* is a boundary point of Z', the 2.14 Theorem of Aliprantis and Border (2006) guarantees that there exists a net $\{z_\alpha\}_{\succsim^*} \to z^*$ with $z_\alpha \in Z'$. By essentiality, there exists $u(x', y', z^*) < \sup_{a \in X, b \in Y} u(a, b, z^*)$.

Since u is continuous, we have $\{u(x', y', z_\alpha)\} \to u(x', y', z^*)$. Therefore, there exists for some $\epsilon < u^* - u(x', y', z)$ an α_0 such that $u(x', y', z_\alpha) < u(x', y', z^*) + \epsilon < u^*$ for all $\alpha \succsim^* \alpha_0$. Thus, there is a $z' \in Z'$ such that $inf_{a \in X, b \in Y} u(a, b, z') < u^* < \sup_{a \in X, b \in Y} u(a, b, z')$.

By the existence of a layer z' such that Equation (13) holds and the fact that all assumptions of Lemma 2 are fulfilled for layers z', z^*, we can construct the utility function v on $\hat{Z} = \{z', z^*\}$ to obtain the result. $\square$

Lemma 5. *Let $\succsim$ be a continuous preference relation on $S = X \times Y \times Z$ where X, Y, Z are connected and separable topological spaces. Moreover, $S' \subseteq S$ such that S' covers all indifference classes. Then for all $z \in Z$, there exists $(x', y', z') \in S'$ such that there exist points fulfilling $(x, y, z) \succsim (x', y', z') \succ (x'', y'', z) \succsim (x''', y''', z')$.*

Proof. Let u be a continuous representation on S. If $u(x, y, z) = \sup_{a \in X, b \in Y} u(a, b, z)$ then we claim that there exists another layer z' and x'', y'' such that $u(x'', y'', z') < u(x, y, z) < \sup_{a \in X, b \in Y} u(a, b, z')$. Since u^* is in the interior of $u(X \times Y \times Z)$, there must exist a z' such that $u(x, y, z) = u^* < \sup_{a \in X, b \in Y} u(a, b, z')$. Since sup is lower semicontinuous, the set $Z' = \{z' | \sup_{a \in X, b \in Y} u(a, b, z') > u^*\}$ of all such z' is open and there exists a boundary point z^* of the set Z' with $u^* = \sup_{a \in X, b \in Y} u(a, b, z^*)$. Therefore, by the 2.14 Theorem of Aliprantis and Border (2006) there exists a net $\{z_\alpha\} \to z^*$ with $z_\alpha \in Z'$. By continuity of u, $sup_{x,y} u(x, y, z)$ is lower continuous. By the 2.42 Lemma of Aliprantis and Border (2006) $\lim_\alpha inf_\alpha sup_{x,y} u(x, y, z) \geq u^*$. Since u is continuous and there exists $u(x', y', z^*) < u(x, y, z^*)$, we have $\{u(x', y', z_\alpha)\} \to u(x', y', z^*)$. Therefore, there exists for some $\epsilon < u(x, y, z^*) - u(x', y', z^*)$ an α_0 such that

$u(x', y', z_\alpha) < u(x', y', z^*) + \epsilon$ for all $\alpha \geq \alpha_0$. Thus, there is a $z' \in Z'$ such that $inf_{x,y} u(x, y, z) < u^* < \sup_{x,y} u(x, y, z')$. □

We now turn to the proof of the main representation theorem.

Proof. Let $u : X \times Y \times Z \to \mathbb{R}$ be a continuous representation of $\succsim$ which exists by continuity. From Essentiality given Z we have that the interior of $u(X, Y, Z)$ is nonempty. From Lemma 4 there exists a representation v covering the indifference classes of an open interval containing $u^* \in int(u(X, Y, Z))$. We claim that we can extend this representation to all indifference classes in $int(u(X, Y, Z))$. Let $\hat{U}$ be the set of utility values which we cannot extend our representation to. We will show that we can extend our representation to an open set around $inf \hat{U} \cap \{u \in \mathbb{R} \geq u^*\}$. By Lemma 4 there exists a representation v' on a layer z' covering the indifference classes of an open interval containing $inf \hat{U} \cap \{u \in \mathbb{R} \geq u^*\}$. We can extend v as follows. Take some layer z which overlaps in its utility with z'. Then by the uniqueness of the representations v and v' we can use an affine transformation A to obtain $A[v(s)] = v'(s')$ for points s on the z layer and s' on the z' layer with $s \sim s'$. Without loss of generality, assume A to be the identity transformation. We know that for all points such that $s \sim s'$, $s \in dom(v), s' \in dom(v')$, we have $T(v(s)) = T(v'(s'))$. Moreover, for every z, we can obtain a representation on layers z, z', which we may name v'''. v''' must be both an affine transformation of v and v'. Thus, T is locally affine everywhere and thus the identity transformation.

We can extend $v(x, y, z)$ defined on $X \times Y \times Z'$ to $X \times Y \times Z$ as follows: Let $z \in Z - Z'$. Since v covers all indifference classes of $\succsim$, every $(x, y, z) \sim (x', y', z')$ such that $z' \in Z'$. We then choose $v(x, y, z) = v(x', y', z')$. Moreover, define $h(z) = v(x_0, y_0, z)$, $f(x, z) = v(x, y_0, z)$, and $g(y, z) = v(x_0, y, z)$. By Lemma 5 for every z there exists a layer $z' \in Z'$ such that their indifference classes overlap. Notice that a continuous representation u on the entire space $X \times Y \times Z$ exists. Moreover, by strict essentiality, $\inf u(X, Y, z) < \sup u(X, Y, z)$. Take now some z' such that $u(x, y, z') \in (\inf u(X, Y, z) < \sup u(X, Y, z))$. Then by strict essentiality and continuity of u, there exist points such that $u(x', y', z') < \sup u(X, Y, z)$. By Lemma 2 a) there then exists a representation v' on z, z' layers fulfilling

$$v'(x, y, z) = f'(x, z) + g'(y, z) + h'(z) \tag{14}$$

with $f'(x_0) = 0$ and $g'(y_0) = 0$. By Lemma 2 b) we can assume that $v' = v$.

We will now show that v is continuous. To do so, we will show that each additive component is continuous.

We claim $h(z)$ is continuous. Let $\{z_k\} \to z$ be a sequence in Z. Then, there exists $z' \in Z'$ with points $(x', y', z') \succ (x'', y'', z') \sim (x_0, y_0, z) \succ (x''', y''', z')$. Since the representation on z' is continuous, there exists some $m \in \mathbb{N}$ such that for all $k \geq m$ there exist x_k, z_k such that $(x_0, y_0, z_k) \sim (x_k, y_k, z')$. Since v is continuous in x and y in each layer $\{v(x_0, y_0, z_k)\} = \{v(x_k, y_k, z')\} \to v(x'', y'', z') = v(x_0, y_0, z)$.

Next, we need to show continuity of $f(x, z)$ and $g(y, z)$. Consider any sequence $\{x_k, z_k\} \to x, z$. There exists some z' such that $(x', y', z') \sim (x, y, z)$ is neither minimal nor maximal in the z' layer. Since the representation on z' is continuous, there exists some $m \in \mathbb{N}$ such that for all $k \geq m$ there exist x'_k, y'_k such that $(x_k, y, z_k) \sim (x'_k, y'_k, z')$. Since v is continuous in x and y in each layer

$\{f(x_k, z_k)\} = \{v(x_k, y, z_k) - h(z) - g(y, z_k)\} = \{v(x'_k, y'_k, z')\} \to v(x'', y'', z') = v(x_0, y_0, z)$. The continuity of g follows by a similar argument.

According to our construction of the grid, we arbitrarily set $f(x_0, z^*) = 0$, $g(y_0, z^*) = 0$, and $f(x_1, z^*) = 1$. Moreover, we arbitrarily set $h(z^*) = 0$. The remainder of the construction has been unique. Thus, suppose we set $f(x_0, z^*) = b_1$, $g(y_0, z^*) = b_2$, and $f(x_1, z^*) = a_1 + b_1$, and $h(z^*) = b_3 - b_1 - b_2$. Then the remainder of the construction of the utility function is unique. Moreover, the constructed functions f_1, g_1, h_1 fulfill $f_1 = a_1 f + b_1$, $g_1 = a_1 g + b_2$, and $h_1 = a_1 h + b_3$.

We note that any preference relation induced by a continuous utility function of the form in Theorem 1 a) will also fulfill our axioms. Thus, we have indeed provided both necessary and sufficient conditions for our representation.

□

B Proof of Corollary 1

Proof. By Theorem 1 we have a representation $u(x, y, z) = f(x, z) + g(y, z) + h(z)$. If u represents $\succsim$ then so does e^u. Since e^u is a continuous function on a compact measurable space, it is Lebesgue measurable with upper bound $\mu(S) \max_{(x,y,z)\in S} e^{u(x,y,z)}$. We define

$$p_Z(z) = \frac{e^{h(z)c}}{\int_Z e^{h(z)c} d\mu_z} \tag{15}$$

$$p_{X|Z}(x, z) = \frac{e^{f(x,z)c}}{\int_{X\times Z} e^{f(x,z)c} p_Z(z) d\mu_x d\mu_z} \tag{16}$$

$$p_{Y|Z}(y, z) = \frac{e^{g(y,z)c}}{\int_{Y\times Z} e^{g(y,z)c} p_Z(z) d\mu_y d\mu_z} \tag{17}$$

$$p(x, y, z) = \frac{e^{u(x,y,z)c}}{\int_S e^{u(x,y,z)c} d\mu}. \tag{18}$$

Since $p(x, y, z) = p_{X|Z}(x, z) \cdot p_{Y|Z}(y, z) \cdot p_Z(z)$ is the result of a monotone transformation of $u(x, y, z)$, it also represents $\succsim$. Note that $p(x, y, z)$ is unique up to changes in the constant c, i.e. up to exponential transformations. □

C Proof of Theorem 2

Proof. We prove this by induction on the number n of elements of I. By Theorem 1, we have the case where $n = 3$. Suppose the result holds for $n - 1$. Then for each $x_n^* \in X_n$, we can obtain a continuous representation $v_{x_n^*}(s) = \sum_{i=2}^{n-1} f_i(x_i, x_1) + h(x_1)$ over $s \in \prod_{i=2}^{n-1} X_i \times \{x_n^*\}$. Since $\prod_{i=2}^{n-1} X_i$ is a connected and separable set, by independence given X_1, the generalized Reidemeister condition with respect to X_n given X_1, and essentiality given X_1, we have a representation $v(s) = f_n(x_n, x_1) + g((x_2, \ldots, x_{n-1}), x_1) + \hat{h}(x_1)$ on S. Without loss of generality, assume $f_1(x_n^*, z) = 0$, $g((x_2^0, \ldots, x_{n-1}^0), x_1) = 0$, $\hat{h}(x_1^0) = h(x_1^0) = 0$, and $\hat{h}(x_1^1) = h(x_1^1) = 1$. Then $T(g((x_2, \ldots, x_{n-1}), x_1) + \hat{h}(x_1)) = \sum_{i=2}^{n} f_i(x_i, z) + h(x_1)$ for some increasing function T. From this, we can derive $T(\hat{h}(x_1)) = h(x_1)$ and $\sum_{i=2}^{n} f_i(x_i, z) = T(g((x_2, \ldots, x_{n-1}), x_1))$.

By Cauchy's functional equation[2] T is linear and in fact $T(1) = 1$. Thus, $v(s) = \sum_{i=1}^{n} f_i(x_i, z) + h(x_1)$. $\square$

D Proof of Theorem 3

Proof. Again the proof goes by induction on n. For $n = 3$ the result holds in virtue of Theorem 1.

Suppose our result holds for $n = k$. We have $X_1 \times, \ldots, X_{k-1} \perp X_{k+1} \mid X_k$. Thus, we have a representation $u(x) = f(x_1, \ldots, x_k) + g(x_k, x_{k+1})$ by the case of $n = 3$. Since our result holds for $n = k$,

$$\begin{aligned} T[f(x_1, \ldots, x_k) + g(x_k, x_{k+1})] = &\sum_{i=2}^{k-1} f_i(x_i, x_{i-1}) \\ &+ f_k((x_k, x_{k+1}), x_{k-1}) \end{aligned} \tag{19}$$

for some montone transformation T. Fixing $x_l = x_l^0$ for all $l \neq k+1, k-2$, we have:

$$\begin{aligned} T[f((x_l^0)_{l=1}^{k-3}, x_{k-2}, x_{k-1}^0, x_k^0) + g(x_k^0, x_{k+1})] = &\sum_{i=2}^{k-3} f_i(x_i^0, x_{i-1}^0) \\ &+ f_{k-2}(x_{k-2}, x_{k-3}^0) \\ &+ f_{k-1}(x_{k-1}^0, x_{k-2}) \\ &+ f_k((x_k^0, x_{k+1}), x_{k-1}^0) \end{aligned} \tag{20}$$

Noticing that both the term inside $T[\ldots]$ and the RHS are additive representations on the $X_{k-2} \times X_{k+1} \times \prod_{l \neq k-2, k+1} \{x_l^0\}$ space, by the uniqueness of additive representations follows that T is affine. We may assume without loss of generality that $T[f] = f$. Thus,

$$\begin{aligned} f(x_1, \ldots, x_k) + g(x_k, x_{k+1}) = &\sum_{i=2}^{k-1} f_i(x_i, x_{i-1}) \\ &+ f_k((x_k, x_{k+1}), x_{k-1}) \end{aligned} \tag{21}$$

From which follows

$$\begin{aligned} f(x_1^0, \ldots, x_{k-2}^0, x_{k-1}, x_k) + g(x_k, x_{k+1}) = &\sum_{i=2}^{k-2} f_i(x_i^0, x_{i-1}^0) \\ &+ f_{k-1}(x_{k-1}, x_{k-2}^0) \\ &+ f_k((x_k, x_{k+1}), x_{k-1}) \end{aligned} \tag{22}$$

Thus, we can write f_k in the form: $f_k((x_k, x_{k+1}), x_{k-1}) = g_k(x_k, x_{k-1}) + g_{k+1}(x_{k+1}, x_k)$ which concludes the proof. $\square$

[2] We actually need Cauchy's functional equation for an interval of $\mathbb{R}$. The derivation of this is straightforward and not very insightful, and thus omitted.

E Proof of Theorem 4

We provide a proof for the three dimensional case. Future versions of the paper will include the more general case of a subset of a finite dimensional product space. We again state our intermediate results (this time for subsets of the product space) before turning to the proof of the representation theorem.

Lemma 6. *Let $\succsim$ be a continuous preference relation on a well-behaved space $S \subseteq X \times Y \times Z$ where X, Y, Z are connected and separable topological spaces. Let $u : S \to \mathbb{R}$ be a continuous representation of $\succsim$. Then for every $u^* \in int(u(X \times Y \times Z))$ there exists an open interval around u^* and $\hat{Z} \subseteq Z$ such that there exists a continuous representation $v : X \times Y \times \hat{Z} \to \mathbb{R}$ with $v(x, y, z) = f(x, z)+g(y, z)+h(z)$ and $f(x_0, z) = 0$ and $g(y_0, z) = 0$ covering the indifference classes of the interval.*

Proof. We claim that there exists a layer z' such that

$$\inf_{a\in X,b\in Y} u(a, b, z') < u^* < \sup_{a\in X,b\in Y} u(a, b, z'). \tag{23}$$

Since u^* is in the interior of $u(S)$, there must exist a z' such that $u^* < \sup_{a\in X,b\in Y:(a,b,z')\in S} u(a, b, z')$. By continuity of u, $sup_{x,y}u(x, y, z)$ is lower semicontinuous. Thus, the set $Z' = \{z' | \sup_{a\in X,b\in Y:(a,b,z')\in S} u(a, b, z') > u^*\}$ of all such z' is open. Since Z is connected, the boundary of Z' is nonempty (unless $Z' = Z$ in which case u^* is not in the interior). Let z^* be a boundary point of Z'. Clearly, $\sup_{a,b:(a,b,z')\in S} u(a, b, z^*) = u^*$.

Suppose $(x', y', z^*) \in int(S)$. Then there exists some neighborhood $N = N_{x'} \times N_{y'} \times N_{z^*}$ around (x', y', z^*) in S where N_X, N_Y, N_Z are neighborhoods of x', y', z^* in X, Y, Z, respectively. Let $Z'' = Z' \cap N_{z^*}$ where N_{z^*}. Since z^* is a boundary point of Z', one can show that it is also a boundary point of Z''. Since z^* is a boundary point of Z'', the 2.14 Theorem of Aliprantis and Border (2006) guarantees that there exists a net $\{z_\alpha\}_{\succsim^*} \to z^*$ with $z_\alpha \in Z''$. By essentiality, there exists $u(x', y', z^*) < \sup_{a\in X,b\in Y} u(a, b, z^*)$.

Since u is continuous, we have $\{u(x', y', z_\alpha)\} \to u(x', y', z^*)$. Therefore, there exists for some $\epsilon < u^* - u(x', y', z)$ an α_0 such that $u(x', y', z_\alpha) < u(x', y', z^*) + \epsilon < u^*$ for all $\alpha \succsim^* \alpha_0$. Thus, there is a $z' \in Z''$ such that $inf_{a\in X,b\in Y} u(a, b, z') < u^* < \sup_{a\in X,b\in Y} u(a, b, z')$.

By the existence of a layer z' such that Equation (23) holds and the fact that all assumptions of Lemma 2 are fulfilled for layers z', z^*, we can construct the utility function v on $\hat{Z} = \{z', z^*\}$ to obtain the result. □

Lemma 7. *Let $\succsim$ be a continuous preference relation on a well behaved space $S \subseteq X \times Y \times Z$ where X, Y, Z are connected and separable topological spaces. Moreover, $S' \subseteq S$ such that S' covers all indifference classes. Then for all $z \in Z$, there exists $(x', y', z') \in S'$ such that there exist points fulfilling $(x, y, z) \succsim (x', y', z') \succ (x'', y'', z) \succsim (x''', y''', z')$.*

Proof. Let u be a continuous representation on S. If $u(x, y, z) = \sup_{a\in X,b\in Y} u(a, b, z)$ then we claim that there exists another layer z' and x'', y'' such that $u(x'', y'', z') < u(x, y, z) < \sup_{a\in X,b\in Y} u(a, b, z')$. Since u^* is in the interior of $u(X \times Y \times Z)$, there must exist a z' such that $u(x, y, z) = u^* < \sup_{a\in X,b\in Y} u(a, b, z')$. Since sup is lower semicontinuous, the set $Z' = \{z' | \sup_{a\in X,b\in Y} u(a, b, z') > u^*\}$ of all such z' is open and there exists a boundary point z^* of the set Z' with $u^* = \sup_{a\in X, b\in Y} u(a,b,z^*)$. Therefore, by the 2.14 Theorem of Aliprantis and Border (2006) there exists a net $\{z_\alpha\} \to z^*$ with $z_\alpha \in Z'$. Without loss of generality, we will assume that the net lies entirely in an open set N_Z where $N_X \times N_Y \times N_Z$ is an open set in S. By continuity of u, $sup_{x,y}u(x,y,z)$ is lower continuous. By the 2.42 Lemma of Aliprantis and Border (2006) $\lim_\alpha inf_\alpha sup_{x,y} u(x,y,z) \geq u^*$. Since u is continuous and there exists $u(x',y',z^*) < u(x,y,z^*)$, we have $\{u(x',y',z_\alpha)\} \to u(x',y',z^*)$. Therefore, there exists for some $\epsilon < u(x,y,z^*) - u(x',y',z^*)$ an α_0 such that $u(x',y',z_\alpha) < u(x',y',z^*) + \epsilon$ for all $\alpha \geq \alpha_0$. Thus, there is a $z' \in Z'$ such that $inf_{x,y}u(x,y,z) < u^* < \sup_{x,y} u(x,y,z')$. □

For the proof of the representation theorem, we note that we previously only used the topological assumption of S being a product space when invoking Lemma 4 and Lemma 5. Lemma 6 and Lemma 7 acting as replacements for these two lemmas, the proof of Theorem 1 also holds for Theorem 4.

F Proof of Corollary 2

Proof. The assumptions of Definition 3 are satisfied for $\hat{X}_1(z) = \{(x_i, x_n) \in X_i \times X_n : x_i + x_n = z\}$, $\hat{X}_2(z) = \{x_{-i} \in \prod_{j\neq i,n} X_j : \sum_{j\neq i,n} = 1 - z\}$, $Z = [0,1]$. From Theorem 4 we have the following representations for all $i < n$:

$$U_i(x) = f_i(x_i, \sum_{k\neq i,n} x_k) + g_i((x_k)_{k\neq i,n}) \tag{24}$$

Using comeasurability, we can ensure during the utility construction process of U_i, U_j that

$$U_i(x) = U_j(x) = U(x). \tag{25}$$

for all $i, j < n$. Therefore,

$$f_i(x_i, \sum_{m\neq i,n} x_m) + g_i((x_m)_{m\neq i,n}) = \tag{26}$$

$$f_j(x_j, \sum_{m\neq i,n} x_m) + g_j((x_m)_{m\neq i,n}). \tag{27}$$

Setting $x_m = 0$ for all $m \neq i, j, k$, we obtain:

$$f_i(x_i, x_j + x_k) + g_i(0, \dots, 0, x_j, x_k) = \tag{28}$$

$$f_j(x_j, x_i + x_k) + g_j(0, \dots, 0, x_i, x_k). \tag{29}$$

By Lemma 8, this functional equation has the solution $f_i(x_i, x_j + x_k) = u_i(x_i) + \bar{u}_i(x_i + x_j + x_k) + \hat{u}_i(x_j + x_k)$ and $f_j(x_j, x_i + x_k) = u_j(x_j) + \bar{u}_j(x_i + x_j + x_k) + \hat{u}_j(x_i + x_k)$. We therefore have for all i

$$\begin{aligned} U(x) &= u_i(x_i) + \bar{u}_i(\sum_{m\neq n} x_m) + \hat{g}_i((x_m)_{m\neq i,n}) \\ &= u_j(x_j) + \bar{u}_j(\sum_{m\neq n} x_m) + \hat{g}_j((x_m)_{m\neq j,n}) \end{aligned} \tag{30}$$

where $\hat{g}_i((x_m)_{m\neq i,n}) = g_i((x_m)_{m\neq i,n}) - \hat{u}(\sum_{m\neq i,n} x_m)$ and $\hat{g}_j((x_m)_{m\neq j,n}) = g_j((x_m)_{m\neq j,n}) - \hat{u}(\sum_{m\neq j,n} x_m)$. Setting $u_n(x_n) = \bar{u}(1 - x_n)$ we obtain:

$$U(x) = u_i(x_i) + u_n(x_n) + \hat{g}_i((x_m)_{m\neq i,n}) \tag{31}$$

Our initial choice of n was arbitrary. We have thus shown that for any i, n, the utility representation is additively separable. Thus,

$$U(x) = \sum_{i=1}^{n} u_i(x_i) \tag{32}$$

□

Lemma 8. *Let $\mathbb{S}, +$ be a cancellative abelian monoid and let $\bar{f}$, $\bar{g}$, f and g be real valued functions defined on $\mathbb{S}^2$ and satisfy the relation*

$$\bar{f}(x_3, x_1 + x_2) + \bar{g}(x_1, x_2) = f(x_2, x_1 + x_3) + g(x_1, x_3)$$

for all x_1, x_2, x_3 in $\mathbb{S}$. Then $f(x_2, x_1 + x_3) + g(x_1, x_3) = v_{123}(x_1 + x_2 + x_3) + v_1(x_1) + v_2(x_2) + v_3(x_3)$. In particular, $f(a, b) = a_1(a) + a_2(b) + a_3(a + b)$.

Proof. The functional equation to be solved is[3]

$$\bar{g}(x_1, x_2) = f(x_2, x_1 + x_3) + g(x_1, x_3) - \bar{f}(x_3, x_1 + x_2). \tag{33}$$

We set $x_3 = 0$ and define $\bar{u}_1(x_1) = g(x_1, 0)$ and $\bar{u}(x_1) = \bar{f}(0, x_1)$ to obtain:

$$\bar{g}(x_1, x_2) = f(x_2, x_1) + \bar{u}_1(x_1) + \bar{u}_3(x_1 + x_2) \tag{34}$$

By a symmetric argument with $x_2 = 0$, we have

$$g(x_1, x_3) = \bar{f}(x_3, x_1) + u_1(x_1) + u_3(x_1 + x_3). \tag{35}$$

Inserting the above result into Equation (33), we have

$$\begin{aligned} &f(x_2, x_1 + x_3) + \bar{f}(x_3, x_1) + u_1(x_1) + u_3(x_1 + x_3) \\ &= \bar{f}(x_3, x_1 + x_2) + f(x_2, x_1) + \bar{u}_1(x_1) + \bar{u}_3(x_1 + x_2) \end{aligned} \tag{36}$$

Let $x_1 = 0$ in Equation (36). Then we get the following relation between $\bar{f}$ and f

$$\bar{f}(x_3, x_2) = f(x_2, x_3) + A_1(x_2) + A_2(x_3)$$

for some suitably defined functions A_1, A_2. Inserting this result into Equation (36) we get

$$\begin{aligned} &f(x_1 + x_2, x_3) + f(x_2, x_1) + \bar{U}_1(x_1) + \bar{U}_2(x_2) + \bar{U}_3(x_1 + x_2) \\ &= f(x_2, x_1 + x_3) + f(x_1, x_3) + U_1(x_1) + U_2(x_3) + U_3(x_1 + x_3). \end{aligned} \tag{37}$$

We want to characterize the function f, for any $(x_1, x_2) \in \mathbb{S}^2$. Gathering terms, we have

$$\begin{aligned} f(x_1, x_2) &= f(x_1, x_2 + x_3) + f(x_2, x_3) - f(x_1 + x_2, x_3) \\ &+ v_1(x_2) + v_2(x_1) + v_3(x_3) + v_{12}(x_1 + x_2) + v_{13}(x_2 + x_3) \end{aligned} \tag{38}$$

Our goal is to prove $f(x, x_2) = a_1(x) + a_2(x_2) + a_3(x + x_2)$. To achieve this, we provide the following Lemma.

[3] In the remainder of the proof, we will omit stating that equations such as (33) hold for all x_1, x_2, x_3. It will be clear from the context whether a variable is a free variable or not.

Lemma 9. *Let $g : \mathbb{S}^2 \to \mathbb{R}$. Then $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$ if and only if $g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = 0 \; \forall \; x_1, x_2$.*

Proof. If $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$ then, $g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = g_1(x_1') + g_2(x_2') - g_1(x_1') - g_2(0) - g_1(0) - g_2(x_2') + g_1(0) + g_2(0) = 0$.

On the other hand, suppose $g(x_1, x_2)$ satisfies the condition $g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = 0$. Then we define the $g_1(x_1) := g(x_1, 0)$ and $g_2(x_2) := g(0, x_2) - g(0, 0)$. Then, by the condition, $g(x_1, x_2) = g(0, x_2) + g(x_1, 0) - g(0, 0) = g_1(x_1) + g_2(x_2)$. □

By Lemma 9, $f(x_1, x_2) = a_1(x_1) + a_2(x_2) + a_3(x_1 + x_2)$ if and only if $f(x_1, x_2) - f(x_1, 0) - f(0, x_2) - f(0, 0) = a_3(x_1 + x_2) - a_3(x_1) - a_3(x_2) + a_3(0)$. Therefore, we define

$$G(x_1, x_2) \equiv f(x_1, x_2) - f(x_1, 0) - f(0, x_2) - f(0, 0). \tag{39}$$

Substituting Equation (38) for $f(x_1, x_2)$, we get

$$\begin{aligned} G(x_1, x_2) = f(x_1, x_2 + x_3) + f(x_2, x_3) - f(x_1 + x_2, x_3) - f(0, x_2 + x_3) \\ +(v_{12}(x_1 + x_2) - v_{12}(x_1) - v_{12}(x_2) + v_{12}(0)) \end{aligned} \tag{40}$$

Thus f has the desired functional form if and only if

$$N(x_1, x_2) \equiv f(x_1, x_2 + x_3) + f(x_2, x_3) - f(x_1 + x_2, x_3) - f(0, x_2 + x_3) \tag{41}$$
$$= a(x_1 + x_2) - a(x_1) - a(x_2) + a(0) \tag{42}$$

for some real-valued function a. To show that this is the case, notice that

$$\begin{aligned} N(x_1 + x_2, x_3) &= -[f(x_1 + x_2 + x_3, c) - f(x_3, c)] + [f(x_1 + x_2, x_3 + c) - f(0, x_3 + c)] \\ N(x_1, x_2) &= -[f(x_1 + x_2, c) - f(x_2, c)] + [f(x_1, x_2 + c) - f(0, x_2 + c)] \\ N(x_1, x_2 + x_3) &= -[f(x_1 + x_2 + x_3, c) - f(x_2 + x_3, c)] + [f(x_1, x_2 + x_3 + c) - f(0, x_2 + x_3 + c)] \\ N(x_2, x_3) &= -[f(x_2 + x_3, c) - f(x_3, c)] + [f(x_2, x_3 + c) - f(0, x_3 + c)] \end{aligned} \tag{43}$$

We choose $c = 0$ in $N(x_1 + x_2, x_3)$, $N(x_1, x_2 + x_3)$ and $N(x_2, x_3)$, and $c = x_3$ in $N(x_1, x_2)$ to obtain $N(x_1 + x_2, x_3) + N(x_1, x_2) = N(x_1, x_2 + x_3) + N(x_2, x_3)$. By M.Hosszu (1971), $N(x_1, x_2) = B(x_1, x_2) + a(x_1 + x_2) - a(x_1) - a(x_2)$ where $B(x_1, x_2)$ is a skew-symmetric biadditive function. Since $N(0, 0) = N(x_1, 0) = N(0, x_2) = 0$, $B(x, x_2) = B(0, 0) = a(0) = 0$. Thus, the function f has the functional form

$$f(a, b) = a_1(a) + a_2(b) + a_3(a + b). \tag{44}$$

To show that $f(x_2, x_1 + x_3) + g(x_1, x_3)$ has the desired functional form, we substitute Equation (44) in Equation (37). Then we obtain

$$\begin{aligned} &U_1(x_1) + U_2(x_2) + U_3(x_3) + U_4(x_1 + x_2) \\ =&\bar{U}_1(x_1) + \bar{U}_2(x_2) + \bar{U}_3(x_3) + \bar{U}_4(x_1 + x_3). \end{aligned} \tag{45}$$

Letting $x_3 = 0$, we obtain that $U_4(x_1 + x_2)$ is additively separable in variables x_1 and x_2. Similarly, letting $x_2 = 0$, $\bar{U}_4$ is additively separable in x_1 and x_3. The desired result follows. □