Assets with possibly negative dividends

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Abstract

The paper introduces assets whose dividends can take any value (positive, negative or zero) in a dynamic general equilibrium model with financial market imperfections. We investigate the interplay between the asset markets and the production sector. The behavior of asset price and value is also studied.

Keywords: Infinite-horizon, general equilibrium, productivity, asset price, negative dividend
JEL Classification Numbers: D5, D90, E44, G12.

1 Introduction

The standard literature of asset pricing (Lucas, 1978; Ljungqvist and Sargent, 2012; Asparouhova et al., 2016) considers that dividends of assets are positive. However, recently some central banks and governments issue assets with negative nominal interest rates (see Figures 1, 2 below). Such these assets may have interpretation: once we buy an asset (money, for example), we will (1) be able to resell it, and (2) have to pay an amount (instead of receiving an amount as in the case of positive dividend). Motivated by this fact, our paper investigates the behavior of prices and values of assets whose dividends (or yields) may take any value (negative, positive or zero), and the interplay between the asset markets and the production sector.

To do so, we build an infinite-horizon general equilibrium model with a production sector and an imperfect financial market. There are a finite number of heterogeneous consumers and one representative firm (without market power). Consumers have two choices for investing: buy physical capital and/or buy a long-lived asset (whose initial supply is exogenous and positive) which brings dividends in the future (similar to the Lucas tree). The novelty is that asset dividend at each period may be positive, negative or zero.

Without the positivity of asset dividends, it is not trivial that asset prices are positive because it is possible that nobody buys this asset; in such a case, issuing

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Figure 1: Policy rate on excess reserves, percent

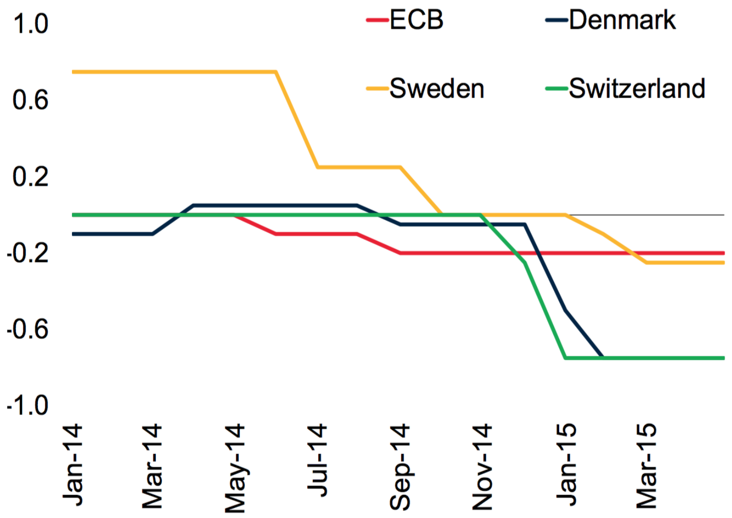
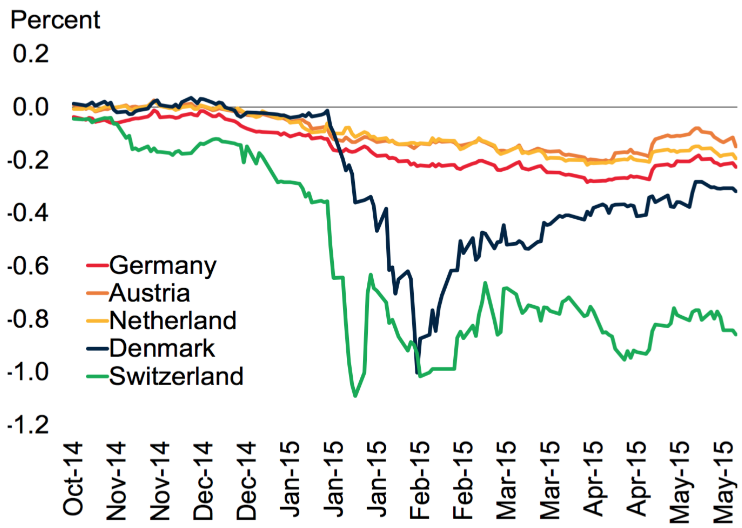


Figure 2: Two-year government bond yields



Source: The World Bank, Global Economic Prospects, June 2015.

assets with negative dividend has no effect on the economy. Hence, we may interpret that one can run negative dividend policy if there exists an equilibrium where asset prices are positive at all dates.

The first set of our contribution is to find out conditions under which we can run negative dividend policy. We show that negative dividend policies cannot be sustained without a strong production sector or high endowments. The idea behind is the following. If one agent buys asset whose future dividends are negative, she will be able to resell this asset but have to pay an amount at the same time. If this amount is so high (i.e., negative dividend is so low), her income including that from capital may be not enough to cover this amount; this can happen if the production sector is weak or endowments are low. In this case, no one wants to buy assets with negative dividends, which implies in turn that asset prices must be zero.

We also prove that when asset dividends are negative at any period, there is no equilibrium with positive asset prices and borrowing constraints are not binding. We then provide examples where agents cannot borrow and asset dividends are negative at any date but asset prices are positive. Let us explain the intuition of this example where we assume that there is a fluctuation on endowments, which in turn creates a fluctuation on agents' income. Consider a date: Agents, who have low endowment at the next date and cannot borrow, have to transfer their wealth from the present date to the next period, hence they accept to buy financial asset in the present date with positive prices even this asset brings negative dividends in the future. The same argument is applied for other dates and agents. Therefore, asset prices are positive at all periods. This result is stronger than the existence of fiat money, i.e., asset without dividend has positive price (Bewley, 1980; Tirole, 1985; Pascoa et al., 2011), and the existence of rational bubbles, i.e., the asset price is higher than the present value of future dividends (Santos and Woodford, 1997; Le Van and Pham, 2016), in the general equilibrium context.

Our second contribution is to identify conditions under which we should run negative dividend policy, i.e., we study the optimal distribution of dividends, given that the objective is the welfare. We find out that, when the productivity in the future is high enough, the government should issue an asset in the present, which will have negative dividend in the future. This action will provide investment for production sector, which will bring a high return because the productivity in the future is high. This suggests that while a central bank encourages people to invest by reducing interest rates (ECB, 2014), it is important for the economy to stimulate investments in R&D in order to improve the productivity in the future. Moreover, our analyses indicate that when the aggregate resource of the economy is low today, dividend at this date should be positive because asset dividend can provide financial support for the purchase of the physical capital, and then increase investment.

The last set of our contribution concerns the behavior of asset price and value. Let us denote q_t and ξ_t the equilibrium asset price (in terms of consumption good) and asset dividend at date t. Given that the asset supply is positive at any date, we have, as in Santos and Woodford (1997), so-called *no-arbitrage condition*

$$q_t = \gamma_{t+1}(q_{t+1} + \xi_{t+1}) \tag{1}$$

where γ_{t+1} is the endogenous discount factor of the economy from date t to $t+1$. By iterating (1), we get the following decomposition

$$q_0 = \Big(\sum_{t=1}^{T} Q_t \xi_t\Big) + Q_T q_T \tag{2}$$

where $Q_t \equiv \gamma_1 \cdots \gamma_t$ is the endogenous discount factor of the economy from date 0 to t.

In the standard theory,[1] ξ_t is assumed to be positive for any t. So, $\sum_{t=1}^{T} Q_t \xi_t$ is increasing in T, which implies that the discounted asset value $Q_T q_T$ decreasingly converges to some value. When $Q_T q_T$ converges to zero, we can compute the asset

[1] See Tirole (1982), Kocherlakota (1992), Santos and Woodford (1997), Le Van and Pham (2016) among others.

price by $q_0 = \sum_{t=1}^{\infty} Q_t \xi_t$; this kind of equilibrium is referred to no-bubble equilibrium (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997; Le Van and Pham, 2016).

We point out, by some examples, that when asset dividends (ξ_t) may be negative, the sum $\sum_{t=1}^{T} Q_t \xi_t$ may diverge, and the discounted asset value $Q_T q_T$ may diverge or converge to any value (even converge to infinity). Our examples are not trivial because $\sum_{t=1}^{T} Q_t \xi_t$ converges to q_0 if intertemporal marginal rates of substitution of agents are the same or borrowing constraints are never binding (or without financial frictions). We also show that asset prices (q_t) may fluctuate over time. Interestingly, there are some cases where asset prices are zero at infinitely many dates and positive at other dates. These findings show how hard is the searching for a robust result on prices and values of assets whose dividends may be negative.

The remainder of the paper is organized as follows. Section 2 introduces our framework and presents some basic properties of equilibria. Section 3 provides analyses of equilibrium with positive asset prices by studying the interaction between asset market and production sector. In Section 4, we investigate asset valuation and provide some examples illustrating and complementing our theoretical results. Section 5 concludes. Technical proofs are gathered in Appendix A.

2 Framework

Our model is based on Lucas (1978), Santos and Woodford (1997) and Le Van and Pham (2016). The novelty is that we do not require the positivity of dividends. In additional, different from Lucas (1978), Santos and Woodford (1997), we introduce capital accumulation. However, for simplicity, we assume that consumers are prevented from borrowing.

Time is discrete and runs from 0 to ∞. There are a finite number of households. Let us denote $I \equiv \{1, 2, \cdots, m\}$ the set of households.

Consumption good. There is a single consumption good at each date. At period t, the price of consumption good is denoted by p_t and agent i consumes $c_{i,t}$ units of consumption good.

Physical capital. Let us denote r_t the capital return at date t and δ the depreciation rate, $k_{i,t+1}$ the quantity of physical capital bought by agent i at date t.

Financial asset. At period t, if agent i buys $a_{i,t}$ units of financial asset with price q_t, she will receive ξ_{t+1} units of consumption good as dividend and she will be able to resell $a_{i,t}$ units of financial asset with price q_{t+1}. Note that ξ_t may take any value (negative, positive or zero). When $\xi_t = 0$ for any t, the asset becomes fiat money as in Bewley (1980), Pascoa et al. (2011) or pure bubble asset as in Tirole (1985), Hirano and Yanagawa (2017). When $\xi_t > 0$ for any t, we recover the Lucas' tree in Lucas (1978), or security in Santos and Woodford (1997) or stock in Kocherlakota (1992).

Each household i takes the sequence of prices $(p, q, r) = (p_t, q_t, r_t)_{t=0}^{\infty}$ as given and chooses allocation sequences $(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^{\infty}$ to maximize her intertemporal utility. The utility maximization problem of agent i is the following:

$$(P_i(p, q, r)) : \max_{(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \right] \tag{3}$$

subject to

$$k_{i,t+1} \geq 0, \quad a_{i,t} \geq 0, \tag{4}$$

$$p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} \leq r_t k_{i,t} + (q_t + p_t\xi_t)a_{i,t-1} + p_t e_{i,t} + \theta_t^i \pi_t, \tag{5}$$

where $e_i \equiv (e_{i,t})$ is the sequence of endowment of agent i while π_t is the profit of the firm at date t (see below). $(\theta_t^i)_{i=1}^m$ is the share of profit at date t. $\theta_i \equiv (\theta_t^i)_t$ is exogenous, $\theta_t^i \geq 0$ for all i and $\sum_{i=1}^m \theta_t^i = 1$.

For each period t, there is a representative firm which takes prices (p_t, r_t) as given and maximizes its profit by choosing physical capital amount K_t.

$$(P(p_t, r_t)) : \qquad \pi_t \equiv \max_{K_t \geq 0} \Big[p_t F(K_t) - r_t K_t\Big] \tag{6}$$

Denote $\mathcal{E}$ the economy which is characterized by a list

$$\Big((u_i, \beta_i, e_i, k_{i,0}, a_{i,-1}, \theta^i)_{i=1}^m, F, \delta, (\xi_t)_{t=0}^\infty\Big).$$

Definition 1. *Consider the economy $\mathcal{E}$. A sequence of prices and quantities $\Big(p_t, q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1}^m, K_t\Big)_{t=0}^\infty$ is an equilibrium of the economy $\mathcal{E}$ if the following conditions are satisfied:*

(i) Price positivity: $p_t > 0, r_t > 0$ and $q_t \geq 0$ for $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\sum_{i \in I}(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) = e_t + F(K_t) + \xi_t \sum_{i \in I} a_{i,t-1} \tag{7}$$

$$K_t = \sum_{i \in I} k_{i,t}, \tag{8}$$

$$\sum_{i \in I}(a_{i,t} - a_{i,t-1}) \leq 0, \quad q_t \sum_{i \in I}(a_{i,t} - a_{i,t-1}) = 0 \tag{9}$$

where $e_t \equiv \sum_{i \in I} e_{i,t}$ is the aggregate endowment of the economy.

(iii) Optimal allocation plan: for each i, $(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^\infty$ is a solution of the problem $(P_i(p, q, r))$.

(iv) Optimal production plan: for each $t \geq 0$, K_t is a solution of the problem $(P(p_t, r_t))$.

Comments. In this definition, we do not require that $q_t > 0$ for any t. The asset's market clearing condition (9) is in the spirit of Arrow and Debreu (1954), and $\sum_{i \in I}(a_{i,t} - a_{i,t-1}) = 0$ if price $q_t > 0$. As we will mention below, in some cases where asset dividends are not positive, it is not easy to find out an equilibrium with $q_t > 0$ for any t. In condition (7), the term $\xi_t \sum_{i \in I} a_{i,t-1}$ will be ξ_t if $\sum_{i \in I} a_{i,t-1} = 1$. However, when nobody buys asset, we have $\sum_{i \in I} a_{i,t-1} = 0$.

Standard assumptions are required in our paper.

Assumption (H1): u_i is in C^1, $u_i'(0) = +\infty$, and u_i is strictly increasing, concave.

Assumption (H2): $F(\cdot)$ is in C^1, strictly increasing, concave, $F(0) = 0$.

Assumption (H3): At initial period 0, $k_{i,0}, a_{i,-1} \geq 0$, and $(k_{i,0}, a_{i,-1}) \neq (0, 0)$ for $i = 1, \ldots, m$. Moreover, we assume that $\sum_{i=1}^m a_{i,-1} = 1$ and $K_0 \equiv \sum_{i=1}^m k_{i,0} > 0$.

Definition 2. *Given (ξ_t), we say that a positive sequence of consumption and capital (C_t, K_t) is feasible if $C_t + K_{t+1} \leq e_t + F(K_t) + (1-\delta)K_t + \xi_t$ for any t.*

Let (D_t) be defined by

$$\begin{aligned} D_0 &\equiv e_0 + F(K_0) + (1-\delta)K_0 + max(0, \xi_0), &(10)\\ D_t &\equiv e_t + F(D_{t-1}) + (1-\delta)D_{t-1} + max(0, \xi_t) \quad \forall t \geq 0. &(11)\end{aligned}$$

We see that D_t is exogenous and depends on the function F and $K_0, \delta, \xi_1, \ldots, \xi_t$. Moreover, $C_t + K_{t+1} \leq D_t$ for every $t \geq 0$. This leads to the following result.

Lemma 1 (the boundedness of consumption and capital stocks)**.** *Consider a feasible path (C_t, K_t). We have*

1. *Capital and consumption are in a compact set for the product topology.*
2. *Moreover, they are uniformly bounded if $(e_t)_t$ and $(\xi_t)_t$ are uniformly bounded from above and there exists x_0 such that $F(x) + (1-\delta)x + \sup_t(e_t + \xi_t) \leq x$ for every $x \geq x_0$.*

One can prove that conditions in point 2 are satisfied if $\sup_t(e_t + \xi_t) < \infty$ and $F'(\infty) < \delta$.

The following assumption ensures that utility of each agent is finite.
Assumption (H4): For each agent i,

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t(F, \delta, K_0, \xi_0, \ldots, \xi_t)) < \infty. \tag{12}$$

Price normalization. Since the utility function u_i is strictly increasing, at any equilibrium (if it exists), p_t must be positive for any t. So, without loss of generality, we can normalize by setting $p_t = 1$ for any t. In this case, we also call $\Big(q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1}^m, K_t\Big)_t$ equilibrium.

2.1 Basis properties

We provide a necessary and sufficient condition to verify that a list of prices and allocations is an equilibrium.

Lemma 2. *$\Big(q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1}^m, K_t\Big)_t$ is an equilibrium if and only if there exist sequences $(\sigma_{i,t}, \nu_{i,t})_{i,t}$ such that the following conditions are satisfied, for any i and for any t,*

(i) $c_{i,t} > 0, k_{i,t+1} \geqslant 0,\ a_{i,t} \geqslant 0,\ \sigma_{i,t} \geqslant 0, \nu_{i,t} \geqslant 0,\ K_t \geqslant 0, q_t \geq 0, r_t > 0.$

(ii) First order conditions:

$$\begin{aligned} \frac{1}{r_{t+1} + 1 - \delta} &= \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \sigma_{i,t}, \quad \sigma_{i,t}k_{i,t+1} = 0\\ \frac{q_t}{q_{t+1} + \xi_{t+1}} &= \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \nu_{i,t}, \quad \nu_{i,t}a_{i,t} = 0.\end{aligned}$$

(iii) Transversality condition:

$$\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t})(k_{i,t+1} + q_t a_{i,t}) = 0. \tag{13}$$

(iv) $F(K_t) - r_t K_t = \max\{F(k) - r_t k : k \geqslant 0\}$.

(v) $c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} + q_t a_{i,t} = r_t k_{i,t} + (q_t + \xi_t)a_{i,t-1} + \theta_t^i \pi_t + e_{i,t}$ *where* $\pi_t = F(K_t) - r_t K_t$.

(vi) $K_t = \sum_{i\in I} k_{i,t}$

(vii) $\sum\limits_{i\in I}(a_{i,t} - a_{i,t-1}) \leq 0$, *and* $\sum\limits_{i\in I}(a_{i,t} - a_{i,t-1}) = 0$ *if* $q_t > 0$.

Transversality condition (13) which is not trivial can be proved by adapting the argument in the proof of Theorem 1 in Kamihigashi (2002). The proof of Lemma 2 is left to the reader. The readers are referred to Araujo et al. (2002), Pascoa et al. (2011), Bosi et al. (2017) for similar conditions in economies with uncertainty, incomplete markets and collateral constraints.

Remark 1. *Consider a finite T-period economy. If $\xi_t \leq 0$ for any $t \leq T$, there does not exist an equilibrium with $q_t > 0$ for any $t \leq T-1$.*

Let us denote, for each $t \geq 0$, $\gamma_{i,t+1}$ (respectively, $Q_{i,t}$) the agent i's discount factor from date t to date $t+1$ (respectively, from initial date to date t) as follows.

$$\gamma_{i,t+1} \equiv \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_{i,0} = 1, \quad Q_{i,t} \equiv \gamma_{i,t} \cdots \gamma_{it}. \tag{14}$$

We also define γ_{t+1} the discount factor of the economy from date t to $t+1$ and Q_t the discount factor of the economy from date 0 to t

$$\gamma_{t+1} \equiv \max_i \left\{ \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \right\}, \quad Q_0 = 1, \quad Q_t \equiv \gamma_1 \cdots \gamma_t. \tag{15}$$

According to point (iii) of Lemma 2, we have so-called non-arbitrage inequalities.

Lemma 3. *At equilibrium, we have, for each t,*

$$q_t \geq \gamma_{t+1}(q_{t+1} + \xi_{t+1}) \text{ with equality if } \sum_i a_{i,t} > 0 \tag{16}$$

$$1 \geq \gamma_{t+1}(r_{t+1} + 1 - \delta) \text{ with equality if } K_{t+1} > 0 \tag{17}$$

Note that $Q_t k_{i,t+1} = (1 - \delta + r_{t+1})Q_{t+1}k_{i,t+1}$ for any t and for any i.

In the remainder of the paper, we will focus on equilibria where all prices are positive, i.e. $q_t > 0$ for any t. In this case, we have $\sum_i a_{i,t} = 1$ for any t, and therefore

$$q_t = \gamma_{t+1}(q_{t+1} + \xi_{t+1}). \tag{18}$$

3 Negative dividend and production

The asset in our framework can be interpreted as an asset issued by the government who can choose negative dividends at some or all dates. However, such an action has no effect on the economy if the asset price is zero or nobody buys the asset. This motivates us to introduce the following notion.

Definition 3. *We say that the government can run negative dividend policy if there exists an equilibrium with $q_t > 0$ for any t.*

The aim of this section is to find out conditions under which the government can run negative dividend policy.

3.1 Can we run negative dividend policy?

First, we consider the case where asset dividend at only one date may be negative. We have the following result.

Proposition 1. *Assume that Assumptions (H1)-(H4) hold and $u_i(0) = 0$ for any i.*

Consider a date $s^* \geq 0$. Assume that $\xi_t \geq 0$ for any $t \neq s^*$, and there is an infinite sequence $(\xi_{t_n})_n$ such that $\xi_{t_n} > 0$ for any t. Then, there exists $\bar{\xi} > 0$ such that: for any $\xi_{s^*} \geq -\bar{\xi}$, there exists an equilibrium with $q_t > 0$ for any t.

Proof. See Appendix A.1. □

According to this result, the existence of equilibrium is ensured if asset dividend at some date is negative but not far from zero (in the sense that $B_{i,t} > 0$). In particular, we recover the existence result in Le Van and Pham (2016) for the case $\xi_t > 0\ \forall t$.

In what follows, we will consider more general cases where dividends at any date may be negative. We start by pointing out the behavior of asset price and value in the very long run.

Lemma 4. *Assume that $0 < \liminf_{t\to\infty} \xi_t \leq \limsup_{t\to\infty} \xi_t < +\infty$,*[2] *and conditions in point 2 of Lemma 1 hold. Then, for any equilibrium, we have $\lim\limits_{t\to\infty} Q_t q_t = 0$ and $q_s = \sum_{t=s+1}^{\infty} Q_t \xi_t / Q_s$ for each $s \geq 0$. Consequently, $q_t > 0$ when t is high enough.*

Proof. See Appendix A.2. □

Lemma 4 provides a sufficient condition under which the present value $\sum_{t=1}^{\infty} Q_t \xi_t$ converges. Moreover, the equilibrium price at any date is equal to the present value of future dividends, which is equivalent to the fact that the discounted value of asset will converge to zero, i.e. $\lim_{t\to\infty} Q_t q_t = 0$. Lemma 4 also gives a sufficient condition under which asset prices are positive in the very long run. Notice that under assumptions in Lemma 4, aggregate consumption and capital stocks are uniformly bounded from above.

Some interesting consequences of Lemma 4 should be mentioned.

[2] Note that there may be some t such that $\xi_t < 0$.

Corollary 1. *Assume that* $0 < \liminf_{t\to\infty} \xi_t \leq \limsup_{t\to\infty} \xi_t < +\infty$*, and conditions in point 2 of Lemma 1 hold. Let us consider a date* $s \geq 0$ *such that* $\xi_s < 0$*. Consider an equilibrium.*

1. *If* $q_{s-1} > 0$*, then* $\sum_{t=s+1}^{\infty} Q_t\xi_t > Q_s|\xi_s| > 0$.

2. *If* $K_t > 0$ *for any* $t \geq 0$*, then we have* $\sum_{t=s+1}^{\infty}(F'(0)+1-\delta)^{t-s}|\xi_t| \geq |\xi_s| > 0$.

Point 1 in Corollary 1 indicates that when dividend at some date, say s, is negative, the asset price at date $(s-1)$ is positive only if the present value of dividends at date s is strictly higher than the absolute discounted value of asset at this date. Note that when conditions $0 < \liminf_{t\to\infty} \xi_t \leq \limsup_{t\to\infty} \xi_t < +\infty$ are violated, $\sum_{t\geq s+1}^{\infty} Q_t\xi_t$ may be lower than $Q_s|\xi_s|$; this property will be readdressed in Section 4.2. Point 2 in Corollary 1 complements Proposition 1 by providing an upper bound of $-\xi_s$ when $\xi_s < 0$. This upper bound depends on productivity and future dividends.

Using transversality condition (13) in Lemma 2, we have the following result showing the role of intertemporal marginal rates of substitutions $\gamma_{i,t+1} \equiv \beta_i u_i'(c_{i,t+1})/u_i'(c_{i,t})$.

Proposition 2 (role of agents' heterogeneity)**.**

1. *If there is a date* t_0 *such that* $\gamma_{i,t} = \gamma_t$ *for any* $t > t_0$ *and for any* i*, then* $\lim_{t\to\infty} Q_tq_t = 0$ *and* $Q_tq_t = \sum_{s=t+1}^{\infty} Q_s\xi_s$

2. *Consequently, if* $\xi_t \leq 0$ *for any* t*, then there is no equilibrium with positive prices such that* $\gamma_{i,t} = \gamma_t$ *for any* t, i *or* $a_{i,t} > 0$ *for any* t, i.

Proof. See Appendix A.3. □

According to Proposition 2, when all dividends are negative, there is no equilibrium with positive prices in which the intertemporal marginal rates of substitutions are the same at any period (this happens if agents are identical). The intuition is the following: when the intertemporal marginal rates of substitutions are the same, agents' investment behavior are similar; in such a case, nobody buys assets with negative dividends. So, asset prices are zero at any date.

If borrowing constraint $a_{i,t} \geq 0$ is not binding for any i, we have $\gamma_{i,t+1} = \gamma_{t+1}$. By the way, point 2 of Proposition 2 indicates the role of borrowing constraints: When all dividends are negative, at each equilibrium with positive prices, there exist an agent i and an infinite sequence $(t_n)_{n\geq 1}$ such that $a_{i,t_n} = 0$ for any $n \geq 1$. Section 4.2 will provide some examples where $\xi_t \leq 0$ for any t, asset prices are positive, and borrowing constraints are frequently binding.

We now analyze the role of productivity. We prepare our exposition by an intermediate step.

Lemma 5. *If there exists an equilibrium with* $q_t > 0\ \forall t$*, then* ξ_t *is bounded from below by an exogenous parameter:* $\xi_t \geq -e_t - F(D_{t-1}) - (1-\delta)D_{t-1}$ *for any* t*, where the sequence* $(D_t)_t$ *is defined by (10) and (11).*

Proof. If an equilibrium exists, we have $0 \leq C_t + K_{t+1} \leq F(K_t) + (1-\delta)K_t + \xi_t$. By definition of (D_t), we see that $\xi_t \geq -F(D_{t-1}) - (1-\delta)D_{t-1}$. □

Lemma 5 indicates that the existence of equilibrium with positive prices ($q_t > 0$ for any t) requires that asset dividends must be bounded from below by exogenous parameters. This leads to the following result.

Proposition 3 (role of productivity)**.** *Assume that $e_t = 0$ for any t.*

1. *Assume that there exists d such that $\xi_t \leq -d < 0$ for any t. If $F'(0) < \delta$ and $F(0) = 0$, then there is no equilibrium with $q_t > 0$ for any t.*

2. *(collapse). Assume that $\xi_t \leq 0$ for any t, $F'(0) < \delta$ and $F(0) = 0$. If there exists an equilibrium with $q_t > 0$ for any t, then $\lim_{t\to\infty} \xi_t = 0$ and $\lim_{t\to\infty} K_t = 0$.*

Proof. See Appendix A.4. □

The first point shows that when dividends are negative and bounded above by a negative constant, there is no equilibrium with positive prices if the productivity is low. Point 2 of Proposition 3 indicates that when dividends are negative and productivity is low, an equilibrium exists only if dividends tend to zero and in this case the economy will collapse (aggregate consumption stocks converge to zero).

Let us explain the economic intuition of our result. When asset prices are positive at any date, there are always some agents who buy this asset. At any date, if one agent buys asset whose future dividends are negative, she will be able to resell this asset but have to pay an amount at the same time. In the aggregate level, the economy has to finance an amount (corresponding to negative dividends) at any date, which is bounded away from zero ($-\xi_t > d > 0$). However, when productivity is very low ($F'(0) < \delta$), the production level decreases in time and tends to zero, the economy collapses. By consequence, there will be some period when the resource of the economy will not be enough to cover negative dividends. Therefore, asset prices cannot be positive.

Propositions 1 and 3 suggest that negative dividend policies may be sustained only if (1) the production sector is strong enough (high productivity) and (2) dividends are not so low.

3.2 Should we run negative dividend policy? Optimal dividend distribution

In this section, we wonder whether we should run negative dividend policies or not. It is reasonable to assume that the government chooses dividends in order to maximize the welfare of agents in the decentralized economy.

Since we are interested in the role of productivity, we allow for non-stationary production functions: the production function at date t is given by $F_t(K) = A_t F(K)$, where F is strictly increasing, strictly concave, $F(0) = 0, F'(0) = \infty$, $F(\infty) = \infty$, and $A_t \geq 0$ represents the total-factor productivity (TFP) of the economy..

For the sake of tractability, we assume that there is one representative household with instantaneous utility function u, the rate of time preference $\beta \in (0, 1)$, and non-negative endowments (e_t). The agent's allocation is denoted by $(c_t, k_{t+1}, a_t)_{t\geq 0}$. In this case, according to the definition of equilibrium, we have $a_t = 1$ and $r_t = F_t'(k_t)$, and therefore the welfare function which depends on (ξ_t) is given by

$$W((\xi_t)_{t\geq 0}) \equiv \max_{(c_t,k_{t+1})_{t\geq 0}} \left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{19}$$

$$\text{subject to: } k_{t+1} \geq 0, \quad c_t + k_{t+1} \leq G_t(k_t) + e_t + \xi_t \tag{20}$$

where $G_t(k) = (1-\delta)k + F_t(k)$.

By the concavity of u and G_t, one can prove that the function $W((\xi)_t)$ is concave in (ξ_t).

Assume that the government's problem is to maximize the welfare function by choosing the sequence of dividend (ξ_t) subject to

$$\xi_t \geq -b_t \ \forall t \geq 0, \text{ and } \sum_{t=0}^{\infty} \xi_t \leq B, \tag{21}$$

where $B \geq 0, b_t > 0$ for any t. Within this setup, the government, having an endowment $B \geq 0$ units of consumption good, has to distribute dividends across periods. The government can choose negative dividend at each date but there is a lower bound b_t.

To find out the properties of the government's optimal choice (ξ_t), we will study the following problem.

$$(PW): \max_{(c_t,k_{t+},\xi_t)_t} \left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{22}$$

$$\text{subject to: } k_{t+1} \geq 0, \tag{23}$$

$$c_t + k_{t+1} \leq G_t(k_t) + e_t + \xi_t \tag{24}$$

$$\xi_t \geq -b_t \tag{25}$$

$$\sum_t \xi_t \leq B. \tag{26}$$

We assume that (b_t) is not so high so that the set of choices of the problem (PW) is not empty. Therefore, the problem (PW) has a solution.

Notice that the non-standard optimal growth problem (PW) is non-stationary and has no closed-form solution. Moreover, it is not easy to find out global property of the solution. Here, we can provide some qualitative analysis. The following result shows the role of productivity.

Proposition 4. *Let assumptions in this section be satisfied and $u(\infty) = \infty, u'(\infty) = 0$. Fix a date t and all parameters, except A_t. Then, there exists $\bar{A}_t$ such that each solution of the problem (PW) satisfies: $\xi_t = -b_t < 0$ for any $A_t \geq \bar{A}_t$.*

Proof. See Appendix A.5. □

The intuition of Proposition 4 is the following. Let us interpret date $(t-1)$ as the present and date t as the future. When the productivity in the future is high enough, the government should issue an asset in the present which will have negative dividend in the future. This action provides investment for the production sector, which will bring a high return because the productivity in the future is high.

Proposition 4 has an interesting consequence.

Corollary 2. *Let assumptions in this section be satisfied and $u(\infty) = \infty, u'(\infty) = 0$. Assume that the government can only choose ξ_0, ξ_1 such that $\xi_0 \geq -b_0, \xi_1 \geq -b_1$ and $\xi_0 + \xi_1 \leq D$ where $D > 0$.*

Fix all parameters, except A_1. There exists $\bar{A}_1$ such that at optimal, $\xi_1 = -b_1 < 0$ and $\xi_0 = D + b_1 > 0$ for any $A_1 \geq \bar{A}_1$.

This result complements Proposition 4. It indicates the moments when the government should choose positive and negative dividends. Dividend should be positive today if the productivity in the future is high enough because in this case asset dividend can provide financial support for the purchase of the physical capital, and then increase investment.

3.2.1 A closed-form solution

In order to see more clearly economic intuitions, this section provides an example where we can find a closed-form solution. Consider a two-period model. Assume that $u(c) = ln(c)$. Assume also that $\delta = 1$, $F_t(k) = A_t k^{1/2}$ for $t = 0, 1$. The welfare function, which depends on ξ_0, ξ_1, is defined by

$$\begin{aligned} W(\xi_0, \xi_1) \equiv \max_{(c_0,k_1,c_1)} & \Big[ln(c_0) + \beta ln(c_1) \Big] \\ & \text{subject to: } k_1 \geq 0, \\ & c_0 + k_1 \leq A_0 k_0^{1/2} + e_0 + \xi_0 \\ & c_1 \leq A_1 k_1^{1/2} + e_1 + \xi_1. \end{aligned}$$

The standard Euler equation $u'(c_0) = \beta u'(c_1) \frac{1}{2} A_1 k_1^{-1/2}$ implies that

$$(1 + \beta \frac{1}{2}) A_1 k_1 + (e_1 + D - \xi_0) k_1^{1/2} - \beta \frac{1}{2} A_1 (A_0 k_0^{1/2} + e_0 + \xi_0) = 0.$$

Some comments should be mentioned here.

- It is easy to see that when ξ_0 increases, k_1 increases.
- Since $\xi_0 + \xi_1$ is constant, if we decrease ξ_1, then ξ_0 and so k_1 increases, therefore the interest rate on the capital market $r_1 = \frac{1}{2} A_1 k_1^{-1/2}$ decreases. This corresponds to what the banks wanted to do: decrease the interest rate to enhance investment (see for example ECB (2014)).

When ξ_0 increases, the aggregate at initial date $(A_0 k_0^{1/2} + e_0 + \xi_0)$ and production at next date $(A_1 k_1^{1/2})$ increase but ξ_1 decreases. So, the consumption at next date $(A_1 k_1^{1/2} + e_1 + \xi_1)$ may decrease, and therefore it is not trivial that the welfare is an increasing function of ξ_0. The following example gives a closed-form solution for the optimal level of dividend distribution (ξ_0, ξ_1).

Example 1 (A closed-form solution)**.** *Assume that the government chooses ξ_0, ξ_1 such that $\xi_0 \geq -b_0, \xi_1 \geq -b_1$ and $\xi_0 + \xi_1 \leq D$ where $D \geq 0$ in order to maximize $W(\xi_0, \xi_1)$.*

Then, the government's solution is given by

$$\xi_0 = \bar{\xi_0} \equiv \frac{(2+\beta)A_1^2 + 4\Big(e_1 + B - \beta(A_0 k_0^{1/2} + e_0)\Big)}{4(1+\beta)} \tag{27}$$

$$\xi_0 + \xi_1 = B \tag{28}$$

Here, we explicitly assume that $\bar{\xi_0} \in [-b_0, B]$*. We also need* $\bar{\xi_1} \equiv B - \bar{\xi_0} > 0$ *in order to ensure that* $q_0 > 0$*.*[3]

Proof. See Appendix A.6. □

According to (27), the optimal level of ξ_0 may be negative or positive. It depends positively on A_1, e_1 but negatively on A_0, k_0, e_0. Moreover, it depends positively on the total amount B of dividends. This result complements the findings in Proposition 4 and Corollary 2.

It is easy to see that the optimal level of ξ_0 is negative if the endowment at the initial period $A_0 k_0^{1/2} + e_0$ is high enough. It means that the country is rich enough (high k_0 and e_0) or/and has high productivity (high A_0). This finding is totally consistent with empirical data mentioned in the introduction: only rich countries experience negative nominal interest rates. Moreover, our analyses in this example and Proposition 4 suggest that while a central bank encourages people to invest by reducing interest rates (ECB, 2014), it is important for the economy to stimulate investments in R&D in order to improve the productivity in the future.

The optimal level of ξ_0 is positive if

- the productivity at the second period A_1 is high enough, as in this situation, the higher the investment at the initial period, the higher the level of output at the second period. This is consistent with Corollary 2;
- or/and endowment at the second period e_1 is high enough. Indeed, when the economy has enough endowment at the second period, positive dividend at the initial period may benefit consumption at this period and then the total welfare.

4 Asset valuation

In this section, we investigate the asset valuation by developing the standard asset pricing for the case where dividends may be negative. By iterating (18) we have the following decomposition

$$q_0 = \Big(\sum_{t=1}^{T} Q_t \xi_t\Big) + Q_T q_T \ \forall T \geq 1. \tag{29}$$

The price q_0, the value of one unit of asset at date 0, equals the sum of two terms: The first $FV_0^T \equiv \sum_{t=1}^{T} Q_t \xi_t$ is the sum of discounted values of dividends until date T,

[3] As mentioned in Remark 1, in this two-period model, no agent buys the asset today if they will not be able to resell this asset and have to pay an amount tomorrow.

and the second one $Q_T q_T$, called *re-sold term*, is the discounted value of one unit of asset at date T. We also have a similar decomposition for the asset price at date t.

$$Q_t q_t = \Big(\sum_{s=t+1}^{T} Q_s \xi_s\Big) + Q_T q_T \ \forall T > t. \tag{30}$$

Consider the Lucas tree with the sequence of positive dividends (ξ_t). The standard literature of asset pricing in infinite-horizon models (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997; Le Van and Pham, 2016) defines the fundamental value of this asset by $FV \equiv \sum_{t=1}^{\infty} Q_t \xi_t$ and the bubble of this asset as the difference between the equilibrium price and the fundamental value $q_0 - FV$; in some mild situations, there is no bubble and we can compute the asset price by $q_0 = FV$.

This standard approach is suitable for assets with positive dividends because $\sum_{t=1}^{T} Q_t \xi_t$ is increasing in T and so converges if $\xi_t \geq 0$ for any t. However when we consider assets whose dividends may be negative, there is a room for the divergence of the series $\sum_{t\geq 1} Q_t \xi_t$. Hence, the standard approach cannot be applied.

4.1 Asset value at infinity

A natural question concerns the behavior of the discounted value of one unit of the asset in the long run, i.e., $\lim_{t\to} Q_t q_t$. This makes sense because in the case where $\lim_{t\to} Q_t q_t = 0$, we can compute the asset price by $q_0 = \sum_{t=1}^{\infty} Q_t \xi_t$. When dividends are positive, thanks to the decomposition (29), $Q_t q_t$ is bounded and decreasingly converges to some value (which is referred to the bubble of asset price bubble). However, if dividends may be negative, the story becomes more complicated. Let us start by considering two particular cases.

Proposition 5 (Value and price of asset)**.**

1. *(Montrucchio, 2004; Le Van and Pham, 2014). Assume that $\xi_t > 0$ for any t. At any equilibrium, both $\sum_{s=t+1}^{T} Q_s \xi_s$ and $Q_T q_T$ converge, and*

$$Q_s q_t = \Big(\sum_{s=t+1}^{\infty} Q_s \xi_s\Big) + \lim_{T\to\infty} Q_T q_T.$$

Moreover, we have (i) $(Q_T q_T)$ is decreasing in time T, and (ii) $\lim_{T\to\infty} Q_T q_T > 0$ if and only if $\sum_{t=1}^{\infty} (\xi_t / q_t) < \infty$.

2. *Assume that $\xi_t \leq 0$ for any t. At any equilibrium with $q_t > 0$ for any t, both $\sum_{s=t+1}^{T} Q_s \xi_s$ and $Q_T q_T$ converge, and*

$$Q_t q_t = \Big(\sum_{s=t+1}^{\infty} Q_s \xi_s\Big) + \lim_{T\to\infty} Q_T q_T.$$

Moreover, we have

(i) $(Q_T q_T)_T$ is increasing in time T,

(ii) $\lim\limits_{T\to\infty} Q_T q_T < \infty$ if and only if $\lim\limits_{T\to\infty} \prod_{t=1}^{T} (1 + \frac{\xi_t}{q_t}) > 0$, which is equivalent to $\sum\limits_{t=1}^{\infty} \dfrac{-\xi_t}{q_t + \xi_t} < +\infty$, and this implies that $\sum\limits_{t=1}^{\infty} \frac{-\xi_t}{q_t} < +\infty$.

Proof. See Appendix A.7. $\square$

In Proposition 5, we see that $Q_t q_t$ converges because either $\xi_t \geq 0\ \forall t$ or $\xi_t \leq 0\ \forall t$. However, in more general cases, $Q_t q_t$ may diverge. This issue will be addressed in the next section.

To understand the meaningful of Proposition 5's point 2, let us observe the budget constraint of agent i at date t

$$c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} + q_t a_{i,t} \leq r_t k_{i,t} + (q_t + \xi_t)a_{i,t-1} + e_{i,t} + \theta_t^i \pi_t.$$

We see that one unit of asset bought at date $t-1$ will give one unit of the same asset and ξ_t units of consumption good (i.e., $(q_t + \xi_t)$ units of consumption good) at date t. When $\xi_t < 0$, the ratio $\frac{-\xi_t}{q_t+\xi_t}$ can be interpreted as the interest-to-value ratio (proportion of interest to asset value) of asset at date t. By the way, point 2.ii shows that asset value at infinity is finite if and only if the sum (over time) of interest-value ratios is finite. This also implies that interest rate (in terms of asset), $-\xi_t/q_t$, must converge to zero.

4.2 Example: Positive asset prices with negative dividends

This section provides an example where the price of asset may be positive even its dividends are negative at all dates.

Fundamentals of the economy. In this section, we will work under the following setup. There are 2 consumers H and F with the same utility function and rate of time preference: $u_i(c) = ln(c), \beta_i = \beta \in (0,1) \quad \forall i = \{H, F\}$. Their initial endowments are respectively $k_{H,0} = 0,\ a_{H,-1} = 0,\ k_{F,0} > 0,\ a_{F,-1} = 1$. Their endowments are given by

$$(e_{2t}^H, e_2^F) = (e_t, 0), \quad (e_{2t+1}^H, e_{2t+1}^F) = (0, e_{t+1}).$$

Assume that the production functions are given by $F(K) = a_t K$ where $a_t \geq 0$ and $\beta(1-\delta+a_t) \leq 1$ for any t. Note that $\pi_t = 0$ for any t.

We also need $\sum_{s=1}^{\infty} \beta^t ln(e_t) < \infty$ to ensure that consumers' utilities are finite.

Computing equilibria. With the above setup, equilibria can be computed as follows. Allocations of the consumer H are given by

$$\begin{aligned}
&k_{H,2t} = 0, \quad a_{H,2t-1} = 0, \quad k_{H,2t+1} = K_{2t+1}, \quad a_{H,2t} = 1\\
&c_{H,2t-1} = (1-\delta+r_{2t-1})K_{2t-1} + q_{2t-1} + \xi_{2t-1}, \quad c_{H,2t} = e_{2t} - K_{2t+1} - q_{2t}
\end{aligned}$$

while allocations of the consumer F are

$$\begin{aligned}
&k_{F,2t} = K_{2t}, \quad a_{F,2t} = 1, \quad k_{F,2t+1} = 0, \quad a_{F,2t} = 0\\
&c_{F,2t-1} = e_{2t-1} - K_{2t} - q_{2t-1}, \quad c_{F,2t} = (1-\delta+r_{2t})K_{2t} + q_{2t} + \xi_{2t}.
\end{aligned}$$

Prices and the aggregate capital are given by the following system: for any t, $p_t = 1, r_t = a_t$, and

$$K_{t+1} + q_t = \frac{\beta}{1+\beta} e_t \tag{31}$$

$$q_{t+1} + \xi_{t+1} = q_t(a_{t+1} + 1 - \delta) \tag{32}$$

$$q_t \geq 0, \quad K_t > 0. \tag{33}$$

By using Lemma 2, we can verify that this sequence of allocations and prices is an equilibrium. For short, we also call $(K_{t+1}, q_t)_{t\geq 0}$ equilibrium. It is easy to see that

$$Q_t = \frac{1}{(1-\delta+a_1)\cdots(1-\delta+a_t)}; \quad Q_t q_t = q_0 - \sum_{s=1}^{t} Q_s \xi_s \tag{34}$$

$$q_0 \in \Big(0, \frac{\beta e_0}{1+\beta}\Big); \quad 0 \leq q_0 - \sum_{s=1}^{t} \frac{\xi_s}{(1-\delta+a_1)\cdots(1-\delta+a_s)} < \frac{\beta e_t}{1+\beta} \quad \forall t \tag{35}$$

Example 2 (Multiple equilibrium prices). *Assume $a_t = \delta$ and $e_t = e$ for any t, then $\gamma_t = 1$ and $Q_t = 1$ for any t. For each q_0 such that*

$$q_0 \in \Big(0, \frac{\beta e}{1+\beta}\Big), \quad 0 < q_0 - \sum_{s=1}^{t} \xi_s < \frac{\beta e}{1+\beta} \quad \forall t, \tag{36}$$

we determine

$$q_t \equiv q_0 - \sum_{s=1}^{t} \xi_s, \quad K_{t+1} \equiv \frac{\beta e}{1+\beta} - q_t > 0. \tag{37}$$

It is easy to see that $(K_{t+1}, q_t)_{t\geq 0}$ is an equilibrium and $q_t > 0$ for any t.

In Example 2, we see that even when ξ_t is negative for any t, all asset prices are positive. Let us explain the intuition of this fact. A fluctuation on wealth (or endowments) creates a fluctuation on agents' income. In the odd periods $(2t+1)$, agent H has no endowment. She wants to smooth consumption over time but she cannot transfer her wealth from the future back to this date because of borrowing constraint. By consequence, she needs to transfer her wealth from date $2t$ to date $2t+1$, hence she accepts to buy financial asset at date $2t$ with positive prices even this asset brings negative dividends in the future. The same argument is applied for the even periods and agent F. Therefore, asset prices are positive at any date.

This result is stronger than the existence of fiat money or of rational bubbles in the general equilibrium context. Indeed, Bewley (1980), Tirole (1985), Kocherlakota (1992), Pascoa et al. (2011), Hirano and Yanagawa (2017) point out that the price of an asset without dividends (i.e., fiat money) may be positive.[4] Santos and Woodford (1997), Le Van and Pham (2016), Bosi et al. (2017) provide some examples where the asset price is higher than the sum of discounted values of asset dividends (which are always positive).

Remark 2. *Let $s \geq 0$. Take $\xi_t = 0$ for any $t \geq s$ and $\xi_s < 0$. In this case $\sum\limits_{t=s+1}^{\infty} Q_t \xi_t = 0 < -\xi_s < Q_s q_s$. This suggests that condition $\liminf\limits_{t\to\infty} \xi_t > 0$ in Corollary 1 is essential in order to ensure that $\sum_{t=s+1}^{\infty} Q_t \xi_t > Q_s|\xi_s|$.*

[4] This may happen in an OLG model (Tirole, 1985) if the real interest rate of the economy without bubble asset is lower than the population growth rate

4.3 Fluctuations of asset price and (discounted) value

Given an equilibrium, the conventional view[5] is that the discounted value of one unit of the asset (i.e., $(Q_t q_t)_t$) is bounded from above and converges. This property holds because the existing literature only considers the case where dividends are always positive. In this section, we will investigate the behavior of $(Q_t q_t)$ to know whether it can diverge or converge when dividends may be negative.

Example 3. *Consider again the example in Section 4.2 but we only require $a_t = \delta$ for any t. It is easy to see that $(K_{t+1}, q_t)_{t\geq 0}$ determined by (31), (32), and (33), constitutes an equilibrium if*

$$q_0 \in \Big(0, \frac{\beta e_0}{1+\beta}\Big), \quad 0 \leq q_0 - \sum_{s=1}^{t} \xi_s < \frac{\beta e_t}{1+\beta} \quad \forall t \tag{38}$$

Notice that under these conditions, we have $Q_t = 1$ and $Q_t q_t = q_t = q_0 - \sum_{s=1}^t \xi_s$.

Let us point out some consequences of Example 3.

1. *Fluctuations of asset price and value.* When we choose (ξ_t) such that $(\sum_{s=1}^t \xi_s)_t$ diverges, then the sequence of asset prices (q_t) diverges and so does $(Q_t q_t)$.

   In particular, we can choose $q_0 \in \Big(0, \frac{\beta e_0}{1+\beta}\Big)$ and (ξ_t) such that $q_0 - \sum_{s=1}^t \xi_s > 0$ for any even t and $q_0 - \sum_{s=1}^t \xi_s = 0$ for any odd t. Therefore, in general case, asset price q_t may be zero at infinitely many dates and it may also be positive at infinitely many other dates.

2. *Asset value converges to infinity.* When we take (e_t) such that $\lim_{t\to\infty} e_t = \infty$ and $\frac{\beta e_t}{1+\beta} > \frac{\beta b_0}{1+\beta} - \sum_{s=1}^{t} \xi_s$,[6] we have that: $Q_t q_t$ tends to infinity if and only if $\sum_{s=1}^\infty \xi_s = -\infty$. According to point 2 of Proposition 5, this is equivalent to $\sum_{t\geq 1} \frac{\xi_t}{q_0 - (\xi_1 + \cdots + \xi_{t-1})} = -\infty$.

5 Conclusion

When dividends may be negative, asset prices are positive only when the production sector is strong enough and dividends are not so low. Our analysis suggests that when the productivity in the future is high, issuing an asset in the present having negative dividends in the future may increase the welfare.

It is hard to find out robust behaviors of asset prices and values when dividends may be negative. The discounted value of one unit of asset $Q_t q_t$ may fluctuate over time. It may also converge to any positive value, even to infinity. Interestingly, asset prices may be positive even dividends are negative at all dates.

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[5] See Tirole (1982, 1985), Santos and Woodford (1997), Le Van and Pham (2016) among others.

[6] For example, take $\xi_t = \xi < 0$ for any t and e_t such that $\frac{\beta e_t}{1+\beta} > \frac{\beta e_0}{1+\beta} + t\xi$.

A Appendix: Formal proofs

A.1 Proof of Proposition 1

Let $(B_{i,t})$ be defined by

$$B_{i,0} \equiv (1-\delta)k_{i,0} + \xi_0 a_{i,-1}, \quad B_{i,t} \equiv (1-\delta)B_{i,t-1} + \xi_t a_{i,t-1}.$$

We will show that: if $B_{i,t} > 0$ for any i, t then there exists an equilibrium with $q_t \geq 0$ and $p_t > 0$ for any t. This can be done by adapting the argument in Le Van and Pham (2016).

We have $q_t > 0$ because there is an infinite sequence $(\xi_{t_n})_n$ such that $\xi_{t_n} > 0$ for any t

A.2 Proof of Lemma 4

By iterating (18) we have the following decomposition

$$q_0 = \Big(\sum_{t=1}^{T} Q_t \xi_t\Big) + Q_T q_T \ \forall T \geq 1 \tag{A.1}$$

$$Q_t q_t = \Big(\sum_{s=t+1}^{T} Q_s \xi_s\Big) + Q_T q_T \ \forall T > t. \tag{A.2}$$

We see that there exists $\xi > 0$ and $t_0 > s$ such that $\xi_t \geq \xi$ for any $t \geq t_0$. So, when T is high enough, the sequence $(FV_0^T \equiv \sum_{t=1}^{T} Q_t \xi_t)_{T > t_0}$ is increasing in T. Moreover, $FV_0^T \leq q_0$ for any T. By consequence, FV_0^T converges to $FV_0 \equiv \sum_{t=1}^{\infty} Q_t \xi_t < \infty$ and hence $Q_t q_t$ converge. Since $\liminf_{t\to\infty} \xi_t > 0$, we get that $\sum_{t=1}^{\infty} Q_t < \infty$.

According to point 2 of Lemma 1, $e_t + F(K_t)$ is uniformly bounded from above. As a result, we obtain that $\lim_{T\to\infty} Q_T k_{i,T+1} = 0$ for any i, and so $\sum_{t=1}^{\infty}(e_t + F(K_t)Q_t) < \infty$ because e_t is also uniformly bounded from above.

For each agent i, we rewrite her/his budget constraint at date t as follows

$$Q_t c_{i,t} + Q_t k_{i,t+1} + Q_t q_t a_{i,t} \quad = \quad Q_t(r_t + 1 - \delta)k_{i,t} + Q_t(q_t + \xi_t)a_{i,t-1} + (e_{i,t} + \theta^i \pi_t)Q_t.$$

By summing the budget constraints from t equals 0 to t, and use (17), (18), we get that

$$\Big(\sum_{t=0}^{T} Q_t c_{i,t}\Big) + Q_T k_{i,T+1} + Q_T q_T a_{i,T}$$
$$= (r_0 + 1 - \delta)k_{i,0} + (q_0 + \xi_0)a_{i,-1} + \sum_{t=0}^{T}(e_{i,t} + \theta^i \pi_t)Q_t < +\infty$$

where the last inequality is from the fact that $\sum_{t=1}^{\infty}\Big(e_t + F(K_t)\Big)Q_t < \infty$.

We have $Q_T q_T a_{i,T} + Q_T k_{i,T+1} \geq 0$, hence $\sum_{t=0}^{\infty} Q_t c_{i,t} < +\infty$, and then $(Q_T k_{i,T+1} + Q_T q_T a_{i,T})_T$ converges where T tends to infinity. Since $\lim_{T\to+\infty} Q_T k_{i,T+1} = 0$, the sequence $(Q_T q_T a_{i,T})_T$ will converge.

If $\lim_{T\to+\infty} Q_T q_T > 0$, then $a_{i,T}$ converges for any i. By consequence, there exists i such that $\lim\limits_{t\to+\infty} a_{i,t} > 0$. For such an agent, there exists T such that the $a_{i,t} > 0$ for any $t \geq T$. Thus, $\frac{Q_t}{Q_T} = \frac{Q_{i,t}}{Q_{i,T}}$ for any $t \geq T$. According to condition (13) in Lemma 2, we have

$$\lim_{t\to+\infty} Q_t q_t a_{i,t} = \frac{Q_T}{Q_{i,T}} \lim_{t\to\infty} Q_{i,t} q_t a_{i,t} = 0$$

which is a contradiction. We conclude that $Q_T q_T$ converges to 0. Therefore, it is easy to see that $q_t = \sum_{s=t+1}^{\infty} Q_s \xi_s / Q_t > 0$ for any $t > t_0$.

A.3 Proof of Proposition 2

Point 1. Let an equilibrium such that $\gamma_{i,t} = \gamma_t$ for any $t > t_0$ and for any i. According to point (iii) of Lemma 2, we have $\lim_{t\to\infty} Q_t \big(q_t a_{i,t} + k_{i,t+1} \big) = 0$ for any i. This implies that $\lim_{t\to\infty} Q_t q_t = \lim_{t\to\infty} Q_t K_{t+1} = 0$. By combining this with (A.2), we obtain $Q_t q_t = \sum\limits_{s=t+1}^{\infty} Q_s \xi_s$.

Point 2 is a direct consequence of point 1.

A.4 Proof of Proposition 3

Proof of point (1). According to Lemma 5, we have $F(D_{t-1}) + D_{t-1} \geq \xi_t \geq d > 0$ for any t. So, D_t is bounded away from zero.

By definition, we have

$$\begin{aligned} D_t &= e_t + F(D_{t-1}) + (1-\delta) D_{t-1} + \max(0, \xi_t) \\ &= F(D_{t-1}) + (1-\delta) D_{t-1} < (F'(0) + 1 - \delta) D_{t-1}. \end{aligned}$$

Since $F'(0) < \delta$, we obtain that D_t converges to zero, a contradiction.

Proof of point (2). By definition, we have $D_t \geq C_t + K_{t+1}$, so both C_t and K_{t+1} converge to zero.

If ξ_t does not converge to zero, there exist $\xi > 0$ and an infinite sequence $(t_n)_n$ such that $\xi_{t_n} \leq -\xi$ for any n. Hence, $F(K_{t_n}) + (1-\delta) K_{t_n} \geq -\xi_{t_n} \geq \xi > 0$. So, K_{t_n} is bounded away from zero, a contradiction.

A.5 Proof of Proposition 4

By Lemma 1 and Assumption H4, the problem (PW) has a solution. Let $(c_t, k_{t+}, \xi_t)_t$ be a solution of this problem. We have first order conditions

$$\lambda_t = \beta^t u'(c_t) \tag{A.3}$$
$$\lambda = \lambda_t + \mu_t, \quad \mu_t(\xi_t + b_t) = 0 \tag{A.4}$$
$$\lambda_t = \lambda_{t+1} G'_{t+1}(k_{t+1}) \tag{A.5}$$

for any t, where λ_t, μ_t, λ are non-negative multipliers associated to constraints (24), (25), (26) respectively.

We fix a date t.

Suppose that $\xi_t > -b_t$, then $\mu_t = 0$. We have

$$1 = \frac{\lambda}{\lambda} = \frac{\lambda_t}{\lambda_{t+1} + \mu_{t+1}} \leq \frac{\lambda_t}{\lambda_{t+1}} = G'_{t+1}(k_{t+1}). \tag{A.6}$$

We will claim that $G_t(K_t)$ tends to infinity when A_t tends to infinity. Indeed, suppose $G_t(K_t)$ is bounded, then the sequence $(c_t)_t$ is bounded and so is the welfare. However, it is easy to see the the welfare tends to infinity when A_t tends to infinity because $u(\infty) = F(\infty) = \infty$.

We now prove that K_{t+1} tends to infinity when A_t tends to infinity. Suppose that K_{t+1} is bounded, then $\lim_{A_t \to \infty} c_t = \infty$ (because $c_t + k_{t+1} = G_t(k_t) + e_t + \xi_t$). We see that (c_{t+1}) is bounded because K_{t+1} is bounded. Hence $u'(c_{t+1})$ and $G'_{t+1}(k_{t+1})$ are bounded away from zero.

By FOCs, we get that

$$1 = \frac{\beta u'(c_{t+1})}{u'(c_t)} G'_{t+1}(k_{t+1}).$$

Hence, $u'(c_t)$ is also bounded away from zero. We have a contradiction because $\lim_{A_t \to \infty} c_t = \infty$ and $u'(\infty) = 0$.

Therefore, we have proved that K_{t+1} tends to infinity when A_t tends to infinity. This implies that $\lim\limits_{A_t \to \infty} G'_{t+1}(k_{t+1}) = 1 - \delta < 1$, a contradiction to (A.6). Finally, we get $\xi_t = -b_t < 0$.

A.6 Proof of Example 1

It is easy to see that $W(\xi_0, D - \xi_0)$ is differentiable. We have, by noting that $u'(c_0) = \beta u'(c_1) G'_1(k_1)$,

$$\frac{\partial W}{\partial \xi_0} = u'(c_0) - \frac{\partial k_1}{\partial \xi_0} u'(c_0) - \beta u'(c_1) + \beta u'(c_1) G'_1(k_1) \frac{\partial k_1}{\partial \xi_0} \tag{A.7}$$

$$= u'(c_0) - \beta u'(c_1) = \beta u'(c_1)(G'_1(k_1) - 1). \tag{A.8}$$

We now compute $G'_1(k_1)$. Euler condition $u'(c_0) = \beta u'(c_1) G'_1(k_1)$ implies that

$$A_1 k_1^{1/2} + e_1 + \xi_1 = \beta \frac{1}{2} A_1 k_1^{-1/2} (A_0 k_0^{1/2} + e_0 + \xi_0 - k_1).$$

Hence, $(1 + \beta \frac{1}{2}) A_1 k_1 + (e_1 + D - \xi_0) k_1^{1/2} - \beta \frac{1}{2} A_1 (A_0 k_0^{1/2} + e_0 + \xi_0)$. So, we can find that

$$k_1^{1/2} = \frac{-(e_1 + D - \xi_0) + \sqrt{(e_1 + D - \xi_0)^2 + \beta(2+\beta) A_1^2 (A_0 k_0^{1/2} + e_0 + \xi_0)}}{(2+\beta) A_1}. \tag{A.9}$$

Thus, condition $G'_1(k_1) \geq 1$ becomes

$$\frac{\frac{1}{2} A_1 (2+\beta) A_1}{-(e_1 + D - \xi_0) + \sqrt{(e_1 + D - \xi_0)^2 + \beta(2+\beta) A_1^2 (A_0 k_0^{1/2} + e_0 + \xi_0)}} \geq 1$$

$$\Leftrightarrow \frac{\frac{1}{2} A_1 (2+\beta) A_1 \Big(e_1 + D - \xi_0 + \sqrt{(e_1 + D - \xi_0)^2 + \beta(2+\beta) A_1^2 (A_0 k_0^{1/2} + e_0 + \xi_0)} \Big)}{\beta(2+\beta) A_1^2 (A_0 k_0^{1/2} + e_0 + \xi_0)} \geq 1.$$

This inequality is equivalent to

$$\sqrt{(e_1+D-\xi_0)^2+\beta(2+\beta)A_1^2(A_0k_0^{1/2}+e_0+\xi_0)} \geq 2\beta(A_0k_0^{1/2}+e_0+\xi_0)-(e_1+D-\xi_0). \tag{A.10}$$

When the right hand side of (A.10) is positive, this inequality becomes

$$\beta(2+\beta)A_1^2(A_0k_0^{1/2}+e_0+\xi_0) \geq 4\beta^2(A_0k_0^{1/2}+e_0+\xi_0)^2-4\beta(A_0k_0^{1/2}+e_0+\xi_0)(e_1+D-\xi_0)$$
$$\Leftrightarrow (2+\beta)A_1^2 \geq 4\beta(A_0k_0^{1/2}+e_0+\xi_0)-4(e_1+D-\xi_0).$$

So, we find that: $\frac{\partial W}{\partial \xi_0}=0$ if and only if $\xi_0=\bar{\xi}_0$. We also check that the right hand side of (A.10) is positive when $\xi_0=\bar{\xi}_0$.

The solution is unique because the welfare function W is concave.

A.7 Proof of Proposition 5

According to (18), we have $Q_tq_t = Q_{t+1}q_{t+1}(1+\frac{\xi_{t+1}}{q_{t+1}})$ for any t, so

$$\begin{aligned} q_0 &= (1+\frac{\xi_1}{q_1})q_1Q_1 = (1+\frac{\xi_1}{q_1})(1+\frac{\xi_2}{q_2})q_2Q_2 \\ &= \ldots = (1+\frac{\xi_1}{q_1})\cdots(1+\frac{\xi_T}{q_T})q_TQ_T. \end{aligned}$$

Point (1). $\lim_{T\to\infty} Q_Tq_T > 0$ if and only if $\lim_{T\to\infty}(1+\frac{\xi_1}{q_1})\cdots(1+\frac{\xi_T}{q_T}) < \infty$, which is equivalent to $\sum_{t=1}^{\infty}(\xi_t/q_t) < \infty$.

Point (2). We see that $\lim_{T\to\infty} Q_Tq_T < \infty$ if and only if $\lim_{T\to\infty}(1+\frac{\xi_1}{q_1})\cdots(1+\frac{\xi_T}{q_T}) > 0$.

Denote $d_t = -\xi_t \geq 0$. We observe that

$$1+\frac{\xi_t}{q_t} = 1-\frac{d_t}{q_t} = \frac{1}{1+\frac{d_t}{q_t-d_t}}.$$

Therefore, $\lim_{T\to\infty}\prod_{t=1}^{T}(1-\frac{d_t}{q_t}) > 0$ if and only if $\lim_{T\to\infty}\prod_{t=1}^{T}(1+\frac{d_t}{q_t-d_t}) < \infty$ which is equivalent to $\sum_{t=1}^{\infty}\frac{d_t}{q_t-d_t} < \infty$.

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