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About the minimal magnitudes of measurement's forbidden zones. Version 1

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Suppose a random variable takes on values in an interval. The minimal distance from the expectation of the variable to the nearest boundary of the interval is considered here. One of the aims of the present article is also an analysis of the question when this minimal distance can be neglected with respect to the standard deviation. This minimal distance can determine the minimal magnitudes of forbidden zones caused by noise for results of measurements near the boundaries of the intervals. The most observed influence and problems of these forbidden zones are suffered in behavioral economics and decision sciences.

Contents

Introduction

The aims and the practical motivation of the article

Forbidden zones for results of measurements near the boundaries of the intervals, in particular, in behavioral economics and decision sciences, are considered in a number of works (see, e.g., [1] and [2]).

The works [3] and [4] were devoted to the well-known problems of utility and prospect theories. Such problems had been pointed out, e.g., in [5]. In [3] and [4] some examples of typical paradoxes were studied. Similar paradoxes may concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, etc.

The dispersion and noisiness of the initial data can lead to bounds (restrictions) on the expectations of these data. This should be taken into account when dealing with this kind of problems. The proposed bounds explained, at least partially, the analyzed examples of paradoxes.

The plenary report [1] presented the idea of these new general bounds (restrictions) on the expectations of random variables in the presence of a non-zero minimal variance.

The general aim of the present article is the consideration of the minimal distance from the nearest boundary of an interval to the expectation of a random variable that takes on values in this interval. This minimal distance is expressed in terms of the standard deviation of the random variable.

The consideration is concentrated on the normal and similar distributions. The calculations are given in details.

The first particular aim of the article is the determination of some typical reference points for considering of this minimal distance.

The second particular aim is the consideration of a question whether this minimal distance can be neglected with respect to the standard deviation of the random variable, especially when this standard deviation tends to zero.

The practical motivation of the present article is concerned with an idea of description of an influence of a noise on the expectations of random variables near the boundaries of intervals (see, e.g., [3]). This idea has explained, at least partially, some problems of behavioral economics, including the underweighting of high and the overweighting of low probabilities, risk aversion, etc. (see, e.g., [4]).

In [1] and [2], non-zero bounds on the expectation of a random variable, namely some symmetric forbidden zones for the expectation, under the condition of a non-zero noise were revealed near the boundaries of finite intervals. A non-zero noise was associated with the non-zero minimal variance of variables.

However, when the level of the noise and, hence, the non-zero minimal variance of variables tends to zero, then the ratio of the width of the forbidden zones to the standard deviation tends to zero as well. Therefore, in some cases these forbidden zones can be neglected at low level of the noise.

General definitions

For the purposes of the present article, let us define some terms:

The standard deviation is referred to as SD.

The **normal-like** distributions are defined as distributions that have symmetric probability density functions (PDF) with non-increasing sides.

The **contiguous situation** is defined as the situation when one side of the support of the PDF touches the boundary of the half-infinite or finite interval.

The **hypothetical reflection situation** is defined as the situation when the PDF f is transformed to the hypothetical function $f_{Refl}(x|x\geq 0) \equiv 2f(x_1\text{-}\boldsymbol{E}X_1|x_1\geq\boldsymbol{E}X_1)$.

In the **hypothetical adhesion** situation, the hypothetical function $f_{Adhes}(x|x\geq 0) \equiv f(x_1\text{-}\boldsymbol{E}X_1|x_1\geq\boldsymbol{E}X_1)$ and at $x \sim 0$ the integral of it equals $½$. In crude terms, a half of the reflected PDF is adhered to the point $x = 0$.

1. Normal distribution

The normal distribution is one of the most important distributions in the probability theory and statistics. Its PDF can be represented in a form of, e.g.,

$$f_X(x) \equiv N(0,\sigma^2) \equiv f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} .$$

1.1. Hypothetical situations

The standard deviation (SD) of the normal distribution equals σ.

One can calculate the expectation for the hypothetic situation of "reflection"

$$E(X) = 2\int_0^\infty xf(x)dx = 2\int_0^\infty x\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{2\sigma}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{x^2}{2\sigma^2}} d\left(\frac{x^2}{2\sigma^2}\right) =$$
$$= \frac{2\sigma}{\sqrt{2\pi}}\int_0^\infty e^{-y}dy = -\frac{2\sigma}{\sqrt{2\pi}} e^{-y}\Big|_0^\infty = \sigma\sqrt{\frac{2}{\pi}} .$$

The ratio $min(|E(X)\text{-}b|)/SD$ is equal to

$$\frac{\min(|E(X)-b|)}{SD} = \frac{E(X)}{SD} = \sqrt{\frac{2}{\pi}} \approx 0.789 \in \left(\frac{3}{4},\frac{4}{5}\right).$$

For the hypothetic situation of "adhesion" the ratio $min(|E(X)\text{-}b|)/SD$ is equal to $1/2$ of that of the hypothetic situation of "reflection" and is equal to

$$\frac{\min(|E(X)-b|)}{SD} = \frac{1}{\sqrt{2\pi}} \approx 0.399 \in \left(\frac{1}{3},\frac{1}{2}\right).$$

So, for the hypothetic situations of both "reflection" and "adhesion," the ratio $min(|E(X)\text{-}b|)/SD$ is not much less then unity and do not tend to zero when σ tends to zero.

2. A power test distribution with not compact support

Let us consider the power not compact "one-step" distribution. Its probability density function can be written as, e.g.,

$$f(x) = \alpha(|x-\mu|+\gamma)^{-\beta} = \frac{\alpha}{(\gamma+|x-\mu|)^{\beta}} = \frac{(\beta-1)\gamma^{\beta-1}}{2}\frac{1}{(\gamma+|x-\mu|)^{\beta}},$$

where $\gamma > 0$ and $\beta > 3$.

The parameter α can be calculated from the normalizing integration (under the simplifying condition $\mu = 0$) as

$$2\int_0^\infty f(x)dx = 2\int_0^\infty \alpha(x+\gamma)^{-\beta}dx = 2\alpha\int_0^\infty (x+\gamma)^{-\beta}dx =$$

$$= \left.\frac{-2\alpha}{(\beta-1)(x+\gamma)^{\beta-1}}\right|_0^\infty = \frac{2\alpha}{(\beta-1)\gamma^{\beta-1}} = 1$$

as

$$\alpha = \frac{(\beta-1)\gamma^{\beta-1}}{2}.$$

So

$$f(x) = \alpha(x+\gamma)^{-\beta} = \frac{(\beta-1)\gamma^{\beta-1}}{2}\frac{1}{(\gamma+|x-\mu|)^{\beta}}.$$

The variance can be calculated as

$$Var(X) = 2\int_0^\infty x^2 f(x)dx = 2\int_0^\infty x^2\alpha(x+\gamma)^{-\beta}dx = \alpha\int_0^\infty x^2(x+\gamma)^{-\beta}dx =$$

$$= -\alpha x^2\left.\frac{(x+\gamma)^{-\beta+1}}{\beta-1}\right|_0^\infty + \alpha\int_0^\infty 2x(x+\gamma)^{-\beta+1}dx = \alpha\int_0^\infty 2x(x+\gamma)^{-\beta-1}dx =$$

$$= -2\alpha x\left.\frac{(\gamma+x)^{-\beta-2}}{(\beta-1)(\beta-2)}\right|_0^\infty + \frac{2\alpha}{(\beta-1)(\beta-2)}\int_0^\infty(\gamma+x)^{-\beta-2}dx = \quad .$$

$$= \frac{2\alpha}{(\beta-1)(\beta-2)}\int_0^\infty(\gamma+x)^{-\beta-2}dx = -2\alpha\left.\frac{(\gamma+x)^{-\beta-3}}{(\beta-1)(\beta-2)(\beta-3)}\right|_0^\infty =$$

$$= \frac{2\alpha\gamma^{-\beta-3}}{(\beta-1)(\beta-2)(\beta-3)} = \frac{(\beta-1)\gamma^{\beta-1}}{2}\frac{2\gamma^{-\beta-3}}{(\beta-1)(\beta-2)(\beta-3)}$$

So, the standard deviation is

$$SD = \frac{\gamma}{\sqrt{(\beta-2)(\beta-3)}}.$$

2.1. Hypothetical situations

For the hypothetical reflection situation, the expectation can be calculated as

$$E(X)=2\int_0^\infty xf(x)dx=2\int_0^\infty x\alpha(x+\gamma)^{-\beta}dx=$$

$$=-2\alpha x\frac{(x+\gamma)^{-\beta+1}}{\beta-1}\Bigg|_0^\infty+2\alpha\int_0^\infty(x+\gamma)^{-\beta+1}dx=2\alpha\int_0^\infty(x+\gamma)^{-\beta+1}dx=$$

$$=-2\alpha x\frac{(\gamma+x)^{-\beta+2}}{(\beta-1)(\beta-2)}\Bigg|_0^\infty=2\alpha\frac{\gamma^{-\beta+2}}{(\beta-1)(\beta-2)}$$

or

$$E(X)=2\alpha\frac{\gamma^{-\beta+2}}{(\beta-1)(\beta-2)}=2\frac{(\beta-1)\gamma^{\beta-1}}{2}\frac{\gamma^{-\beta+2}}{(\beta-1)(\beta-2)}=\frac{\gamma}{(\beta-2)}$$

The ratio *min(|E(X)-b|)/SD* is equal to

$$\frac{\min(|E(X)-b|)}{SD}=\frac{E(X)}{SD}=\frac{\gamma}{\beta-2}\Bigg/\frac{\gamma}{\sqrt{(\beta-2)(\beta-3)}}=\sqrt{\frac{\beta-3}{\beta-2}}.$$

The variance can exist only if $\beta > 3$. Let $\beta = 3+\varepsilon > 3$, where $\varepsilon \to 0$, then

$$\frac{\min(|E(X)-b|)}{SD}=\sqrt{\frac{3+\varepsilon-3}{3+\varepsilon-2}}=\sqrt{\frac{\varepsilon}{1+\varepsilon}}\xrightarrow[\varepsilon\to 0]{}\sqrt{\varepsilon}\xrightarrow[\varepsilon\to 0]{}0.$$

So, if the power index tends down to three and is sufficiently close to three, then the ratio *min(|E(X)-b|)/SD* can be much less than one.

3. Test distributions with compact support

3.1. Power test distributions with compact support

Let us consider a sectionally continuous power test one-step distribution with compact support

$$f(x) = hx^{\beta}[\theta(x) - \theta(x-l)] + h(2l-x)^{\beta}[\theta(x-l) - \theta(x-2l)].$$

The parameter h can be calculated from the normalizing integration

$$2\int_0^l f(x)dx = 2\int_0^l hx^{\beta}dx = 2h\int_0^l x^{\beta}dx = 2h\frac{l^{\beta+1}}{\beta+1} = 1$$

and

$$h = \frac{\beta+1}{2l^{\beta+1}}$$

The contiguous situation

Due to the symmetry of the test distributions, $E(X) = l$. The variance equals

$$Var(X) = 2\int_0^l (x-\mu)^2 f(x)dx = 2\int_0^l (x-l)^2 hx^{\beta}dx = 2h\int_0^l (x-l)^2 x^{\beta}dx =$$

$$= 2h(x-l)^2 \frac{x^{\beta+1}}{\beta+1}\Bigg|_0^l - 2h\int_0^l \frac{2(x-l)x^{\beta+1}}{\beta+1}dx = -4h\int_0^l \frac{(x-l)x^{\beta+1}}{\beta+1}dx =$$

$$= -\frac{4h}{\beta+1}(x-l)\frac{x^{\beta+2}}{\beta+2}\Bigg|_0^l + \frac{4h}{\beta+1}\int_0^l \frac{x^{\beta+2}}{\beta+2}dx = \frac{4h}{\beta+1}\int_0^l \frac{x^{\beta+2}}{\beta+2}dx =$$

$$= \frac{4h}{(\beta+1)(\beta+2)}\frac{x^{\beta+3}}{\beta+3}\Bigg|_0^l = \frac{4hl^{\beta+3}}{(\beta+1)(\beta+2)(\beta+3)} =$$

$$= \frac{\beta+1}{2l^{\beta+1}}\frac{4l^{\beta+3}}{(\beta+1)(\beta+2)(\beta+3)} = \frac{2l^2}{(\beta+2)(\beta+3)}$$

So,

$$Var(X) = \frac{2l^2}{(\beta+2)(\beta+3)}$$

and the standard deviation is

$$SD = l\sqrt{\frac{2}{(\beta+2)(\beta+3)}}\,.$$

In particular, this expression gives the well-known formulae

$$SD_{Uniform} = \frac{l}{\sqrt{3}} \quad \text{and} \quad SD_{Triangle} = \frac{l}{\sqrt{6}}\,.$$

for the uniform ($\beta = 0$) and triangle ($\beta = 1$) distributions.

So, we have the ratio $min(|E(X)\text{-}b|)/SD$

$$\frac{\min(|E(X)-b|)}{SD} = \frac{E(X)}{SD} = \sqrt{\frac{(\beta+2)(\beta+3)}{2}}\,.$$

The minimal ratio $min(|E(X)\text{-}b|)/SD$ is reached at $\beta \rightarrow 0$ (the power distribution tends to the uniform one)

$$\frac{\min(|E(X)-b|)}{SD} = \sqrt{\frac{(\beta+2)(\beta+3)}{2}} \xrightarrow[\beta\to 0]{} \sqrt{3}\,.$$

So, the minimal ratio $min(|E(X)\text{-}b|)/SD$ is more, then unity for the contiguous situation of the one-step family.

The hypothetic situations

One can calculate the expectation for the hypothetic situation of "reflection"

$$E(X)=2\int_0^l xf(x)dx=2h\int_0^l x(l-x)^\beta dx=$$
$$=-2hx\frac{(l-x)^{\beta+1}}{\beta+1}\Bigg|_0^l+2h\int_0^l\frac{(l-x)^{\beta+1}}{\beta+1}dx=+2h\int_0^l\frac{(l-x)^{\beta+1}}{\beta+1}dx=$$
$$=-\frac{2h}{\beta+1}\frac{(l-x)^{\beta+2}}{\beta+2}\Bigg|_0^l=\frac{2h}{\beta+1}\frac{l^{\beta+2}}{\beta+2}=\frac{\beta+1}{2l^{\beta+1}}\frac{2}{\beta+1}\frac{l^{\beta+2}}{\beta+2}=$$
$$=\frac{l}{\beta+2}.$$

The ratio *min(|E(X)-b|)/SD* is equal to

$$\frac{\min(|E(X)-b|)}{SD}=\frac{E(X)}{SD}=\frac{l}{\beta+2}\frac{1}{l}\sqrt{\frac{(\beta+2)(\beta+3)}{2}}=$$
$$=\frac{1}{\sqrt{2}}\sqrt{\frac{\beta+3}{\beta+2}}.$$

The ratio *|E(X)-b|/SD* tends at $\beta \to 0$ (the power distribution tends to the uniform one) to

$$\frac{|E(X)-b|}{SD}=\frac{1}{\sqrt{2}}\sqrt{\frac{\beta+3}{\beta+2}}\xrightarrow[\beta\to 0]{}\frac{1}{\sqrt{2}}\sqrt{\frac{3}{2}}=\frac{\sqrt{3}}{2}\approx 0.87.$$

The minimal ratio *min(|E(X)-b|)/SD* is reached at $\beta \to \infty$

$$\frac{\min(|E(X)-b|)}{SD}=\frac{1}{\sqrt{2}}\sqrt{\frac{\beta+3}{\beta+2}}\xrightarrow[\beta\to\infty]{}\frac{1}{\sqrt{2}}\sqrt{\frac{\beta}{\beta}}=$$
$$=\frac{1}{\sqrt{2}}\approx 0.71\in\left(\frac{2}{3},\frac{4}{5}\right).$$

For the hypothetic situation of "adhesion" the ratio μ/σ is equal to 1/2 of that of the hypothetic situation of "reflection" and is equal to

$$\frac{\min(|E(X)-b|)}{SD}=\frac{1}{2\sqrt{2}}\approx 0.35\in\left(\frac{1}{3},\frac{2}{5}\right).$$

So, for the hypothetic situations of both "reflection" and "adhesion," the minimal ratio *min(|E(X)-b|)/SD* do not tend to zero when σ tend to zero.

3.2. The two-step stepwise test distribution with compact support

Let us consider the sectionally continuous test two-step stepwise distribution with compact support

$$f(x) = h_2[\theta(x) - \theta(x - l_2)] +$$
$$+ (h_2 + h_1)[\theta(x - l_2) - \theta(l_2 + 2l_1 - x)] + .$$
$$+ h_2[\theta(x - l_2 - 2l_1) - \theta(x - 2l_2 - 2l_1)]$$

The parameters h_2 and h_1, l_2 and l_1 are tied by the normalizing integration

$$1 = 2\int_0^{l_1+l_2} f(x)dx = 2\int_0^{l_1} (h_2 + h_1)dx + 2\int_{l_1}^{l_1+l_2} h_2 dx = .$$
$$= 2l_1(h_2 + h_1) + 2l_2h_2 = 2h_2(l_2 + l_1) + 2l_1h_1 = 2l_2h_2 + 2l_1h_2 + 2l_1h_1$$

or

$$2h_2(l_2 + l_1) + 2l_1h_1 = 1 .$$

For simplicity one can determine $E(X) = 0$.

For the test two-step stepwise distribution the variance equals

$$Var(X) = 2\int_0^{l_1+l_2} x^2 f(x)dx = 2\int_0^{l_1} x^2(h_2 + h_1)dx + 2\int_{l_1}^{l_1+l_2} x^2 h_2 dx =$$
$$= \frac{2(h_2 + h_1)}{3} l^3{}_1 + \frac{2h_2}{3}[(l_2 + l_1)^3 - l^3{}_1] = .$$
$$= \frac{2h_1}{3} l^3{}_1 + \frac{2h_2}{3}(l_2 + l_1)^3 = \frac{2}{3}\left[h_1 l^3{}_1 + h_2(l_2 + l_1)^3\right] =$$
$$= \frac{2}{3}\left[l^2{}_1 h_1 l_1 + (l_2 + l_1)^2 h_2(l_2 + l_1)\right]$$

So,

$$Var(X) = \frac{2}{3}\left[l^2{}_1 h_1 l_1 + (l_2 + l_1)^2 h_2(l_2 + l_1)\right]$$

and the standard deviation is

$$SD = \sqrt{\frac{2}{3}}\sqrt{h_1 l^3{}_1 + h_2(l_2 + l_1)^3} .$$

Note, for the uniform distribution we have: for $h_2 = 0$ the variance equals

$$Var(X) = \frac{2}{3}h_1 l^3{}_1 = \frac{2}{3}l^2{}_1 h_1 l_1 = \frac{2}{3}l^2{}_1 \frac{1}{2} = \frac{1}{3}l^2{}_1 .$$

For $h_1 = 0$ it equals

$$Var(X) = \frac{2}{3}h_2(l_2 + l_1)^3 = \frac{2}{3}(l_2 + l_1)^2 h_2(l_2 + l_1) = \frac{2}{3}(l_2 + l_1)^2 \frac{1}{2} = $$
$$= \frac{1}{3}(l_2 + l_1)^2 .$$

Due to the normalizing equality

$$2h_2(l_2 + l_1) + 2l_1 h_1 = 1$$

none of these parameters can be changed independently. Using

$$h_1 = \frac{1 - 2h_2(l_2 + l_1)}{2l_1}$$

the variance can be rewritten in terms of h_2 as

$$Var(X) = \frac{2}{3}\left[h_1 l^3{}_1 + h_2(l_2 + l_1)^3\right] =$$
$$= \frac{2}{3}\left[\frac{1 - 2h_2(l_2 + l_1)}{2l_1} l^3{}_1 + h_2(l_2 + l_1)^3\right] =$$
$$= \frac{1}{3}\left[[1 - 2h_2(l_2 + l_1)]l^2{}_1 + 2h_2(l_2 + l_1)^3\right] =$$
$$= \frac{1}{3}\left[l^2{}_1 - 2h_2(l_2 + l_1)l^2{}_1 + 2h_2(l_2 + l_1)^3\right] = .$$
$$= \frac{1}{3}\left[l^2{}_1 + 2h_2(l_2 + l_1)[(l_2 + l_1)^2 - l^2{}_1]\right] =$$
$$= \frac{1}{3}\left[l^2{}_1 + 2h_2(l_2 + l_1)(l^2{}_2 + 2l_2 l_1)\right] =$$
$$= \frac{1}{3}\left[l^2{}_1 + 2h_2 l_2(l_2 + l_1)(l_2 + 2l_1)\right]$$

The derivative of the variance with respect to h_2 is

$$\frac{\partial Var(X)}{\partial h_2} = \frac{2}{3}l_2(l_2 + l_1)(l_2 + 2l_1) > 0 .$$

The variance increases when h_2 increases.

Using

$$h_2 = \frac{1-2l_1h_1}{2(l_2+l_1)}$$

the variance can be rewritten in terms of h_1 as

$$Var(X) = \frac{2h_1}{3}l^3{}_1 + \frac{2h_2}{3}(l_2+l_1)^3 =$$

$$= \frac{2h_1}{3}l^3{}_1 + \frac{2}{3}\frac{1-2l_1h_1}{2(l_2+l_1)}(l_2+l_1)^3 = \frac{2h_1}{3}l^3{}_1 + \frac{1-2l_1h_1}{3}(l_2+l_1)^2 =$$

$$= \frac{2h_1}{3}l^3{}_1 + \frac{1}{3}(l_2+l_1)^2 - \frac{2l_1h_1}{3}(l_2+l_1)^2 =$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2l_1h_1\left[(l_2+l_1)^2 - l^2{}_1\right]\right\} = \qquad .$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2l_1h_1\left[l^2{}_2 + 2l_2l_1 + l_1^{\ 2} - l^2{}_1\right]\right\} =$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2l_1h_1\left[l^2{}_2 + 2l_2l_1\right]\right\} =$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2h_1l_2l_1\left[l_2 + 2l_1\right]\right\}$$

The derivative with respect to h_1 is

$$\frac{\partial Var(X)}{\partial h_1} = \frac{2}{3}\left\{-l_2l_1\left[l_2+2l_1\right]\right\} < 0.$$

The variance increases when h_1 decreases.

So, the derivative of the variance with respect to h_2 is positive but the derivative with respect to h_1 is negative. Remember, when h_1 increases then h_2 decreases.

Therefore the variance is maximal at the condition $h_1 = 0$ and equals

$$Var(X) = \frac{2}{3}\left[l^2{}_1 h_1 l_1 + (l_2 + l_1)^2 h_2 (l_2 + l_1)\right] =$$

$$= \frac{2}{3}(l_2 + l_1)^2 h_2 (l_2 + l_1) = \frac{2}{3}(l_2 + l_1)^2 \frac{1}{2} = \frac{(l_2 + l_1)^2}{3} \equiv$$

$$\equiv \frac{2}{3} l^2 hl = \frac{2}{3} l^2 \frac{1}{2} = \frac{l^2}{3}$$

The standard deviation is

$$SD = \frac{l_2 + l_1}{\sqrt{3}}.$$

The contiguous case

For the contiguous case, due to the symmetry of the PDF, the expectation is

$$\min(|E(X) - b|) = \min(E(X)) = \frac{l_2 + l_1}{2}.$$

So,

$$\frac{\min(|E(X) - b|)}{SD} = \frac{\sqrt{3}}{2} \frac{l_2 + l_1}{l_2 + l_1} = \frac{\sqrt{3}}{2}.$$

So, the minimal ratio *min(|E(X)-b|)/SD* for the two-step compact test stepwise distribution with compact support for the contiguous case is finite.

The hypothetic situations

For the two-step compact test stepwise distribution with compact support the expectation equals

$$E(X)=2\int_0^{l_1+l_2} xf(x)dx = 2\int_0^{l_1} x(h_2+h_1)dx + 2\int_{l_1}^{l_1+l_2} xh_2 dx =$$

$$=\frac{2(h_2+h_1)}{2}l^2{}_1+\frac{2h_2}{2}[(l_2+l_1)^2-l^2{}_1]= \qquad .$$

$$=h_2l^2{}_1+h_1l^2{}_1+h_2(l_2+l_1)^2-h_2l^2{}_1=h_1l^2{}_1+h_2(l_2+l_1)^2$$

So,

$$E(X)=h_1l^2{}_1+h_2(l_2+l_1)^2.$$

Remember that

$$SD=\sqrt{\frac{2}{3}}\sqrt{h_1l^3{}_1+h_2(l_2+l_1)^3}\,.$$

So,

$$\frac{\min(|E(X)-b|)}{SD}=\sqrt{\frac{3}{2}}\frac{h_1l^2{}_1+h_2(l_2+l_1)^2}{\sqrt{h_1l^3{}_1+h_2(l_2+l_1)^3}}\,.$$

The ratio depends on four parameters. The form of the ratio and preliminary calculations show that the full analysis of it is rather complicated. In addition, such an analysis is not a goal of this article.

We can see for the ratio

$$\frac{\min(|E(X)-b|)}{SD}=\sqrt{\frac{3}{2}}\frac{h_1l^2{}_1+h_2(l_2+l_1)^2}{\sqrt{h_1l^3{}_1+h_2(l_2+l_1)^3}}=$$

$$=\sqrt{\frac{3}{2}}\frac{h_1}{\sqrt{h_1}}\frac{l^2{}_1+\frac{h_2}{h_1}(l_2+l_1)^2}{\sqrt{l^3{}_1+\frac{h_2}{h_1}(l_2+l_1)^3}}=\sqrt{\frac{3}{2}}\sqrt{h_1}\frac{l^2{}_1+\frac{h_2}{h_1}(l_2+l_1)^2}{\sqrt{l^3{}_1+\frac{h_2}{h_1}(l_2+l_1)^3}}=$$

$$=\sqrt{\frac{3}{2}}\sqrt{h_1}\frac{l^2{}_2}{l^{3/2}{}_2}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}}=$$

$$=\sqrt{\frac{3}{2}}\sqrt{h_1l_2}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}} \qquad .$$

So, the ratio can be rewritten as

$$\frac{\min(|E(X)-b|)}{SD}=\sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\,\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}}.$$

One can search its minimum at

$$\frac{l_1}{l_2}\to 0 \quad \text{and} \quad \frac{h_2}{h_1}\to 0.$$

If $l_1/l_2 \to 0$, then the ratio $min(|E(X)-b|)/SD$ is

$$\frac{\min(|E(X)-b|)}{SD}=\sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\,\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}}\xrightarrow[\frac{l_1}{l_2}\to 0]{}.$$

$$\xrightarrow[\frac{l_1}{l_2}\to 0]{}\sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\,\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}}}$$

One can consider the two mutually excluding cases

$$\left(\frac{l_1}{l_2}\right)^2>>\frac{h_2}{h_1}. \quad \text{or} \quad \frac{h_2}{h_1}>>\left(\frac{l_1}{l_2}\right)^2.$$

If

$$\left(\frac{l_1}{l_2}\right)^2 >> \frac{h_2}{h_1}$$

then

$$\frac{\min(|E(X)-b|)}{SD} = \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\frac{\left(\frac{l_1}{l_2}\right)^2 + \frac{h_2}{h_1}}{\sqrt{\left(\frac{l_1}{l_2}\right)^3 + \frac{h_2}{h_1}}} \xrightarrow[\frac{h_2}{h_1} << \left(\frac{l_1}{l_2}\right)^2 << 1]{}$$

$$\xrightarrow[\frac{h_2}{h_1} << \left(\frac{l_1}{l_2}\right)^2 << 1]{} \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\frac{\left(\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3}} = \sqrt{\frac{3}{2}}\sqrt{h_1 l_2 \frac{l_1}{l_2}} = .$$

$$= \sqrt{\frac{3}{2}}\sqrt{h_1 l_1}$$

Due to

$$\frac{h_2}{h_1} << \left(\frac{l_1}{l_2}\right)^2$$

it is true that

$$\frac{h_2}{h_1} << \frac{l_1}{l_2}$$

and

$$h_2 l_2 << h_1 l_1$$

and remembering

$$h_2 l_2 + h_1 l_1 = \frac{1}{2}$$

we have

$$h_2 l_2 \approx \frac{1}{2}.$$

So, at

$$\left(\frac{l_1}{l_2}\right)^2 >> \frac{h_2}{h_1}$$

the ratio is finite and equals

$$\frac{\min(|E(X)-b|)}{SD} \xrightarrow[\frac{h_2}{h_1} << \left(\frac{l_1}{l_2}\right)^2 << 1]{} \sqrt{\frac{3}{2}}\sqrt{h_1 l_1} \xrightarrow[\frac{h_2}{h_1} << \left(\frac{l_1}{l_2}\right)^2 << 1]{} \frac{\sqrt{3}}{2}.$$

So, under these conditions the ratio is finite.

If

$$\frac{h_2}{h_1} >> \left(\frac{l_1}{l_2}\right)^2$$

then

$$\frac{\min(|E(X)-b|)}{SD} = \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\frac{\left(\frac{l_1}{l_2}\right)^2 + \frac{h_2}{h_1}}{\sqrt{\left(\frac{l_1}{l_2}\right)^3 + \frac{h_2}{h_1}}} \xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} \to 0]{}$$

$$\xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} \to 0]{} \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\frac{\frac{h_2}{h_1}}{\sqrt{\frac{h_2}{h_1}}} = \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\sqrt{\frac{h_2}{h_1}} = \quad .$$

$$= \sqrt{\frac{3}{2}}\sqrt{h_2 l_2}$$

If

$$\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} << 1$$

then

$$\frac{h_2}{h_1}\frac{l_2}{l_1} << 1$$

and it follows

$$h_2 l_2 = h_1 l_1 \frac{h_2}{h_1}\frac{l_2}{l_1} < \frac{1}{2}\frac{h_2}{h_1}\frac{l_2}{l_1} << 1$$

and

$$\frac{\min(|E(X)-b|)}{SD} \xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} \to 0]{} \sqrt{\frac{3}{2}}\sqrt{h_2 l_2} \xrightarrow[\frac{h_2}{h_1} << \frac{l_1}{l_2} \to 0]{} 0 .$$

For example, the condition

$$\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} << 1 .$$

can be true if

$$\frac{l_1}{l_2} = 10^6 \quad \text{and} \quad \frac{h_2}{h_1} = 10^9 .$$

So, it has been proven that for the hypothetic situations the minimal ratio *min(|E(X)-b|)/SD* for the two-step compact test stepwise distribution with compact support can be much less than unity.

Conclusions

The minimal distance from the expectation of a random variable to the nearest boundary of the interval has been considered in the present article. The distance has been expressed in terms of the standard deviation (SD).

The question whether this minimal distance can be neglected with respect to the SD at low SD has been particularly analyzed.

This minimal distance can determine the minimal magnitudes of forbidden zones caused by noise for results of measurements near the boundaries of the intervals (see, e.g., [1] and [2]). The most observed influence and problems of these forbidden zones are suffered in behavioral economics and decision sciences, in utility and prospect theories.

For the purposes of this article, the following definitions have been given:

The **normal-like** distributions are defined as distributions that have symmetric probability density functions (PDF) with non-increasing sides.

The **contiguous situation** is defined as the situation when one side of the support of the PDF touches the boundary of the half-infinite or finite interval.

The **hypothetical reflection situation** is defined as the situation when the PDF f is transformed to the hypothetical function $f_{Refl}(x|x\geq 0) \equiv 2f(x_1\text{-}\boldsymbol{E}X_1|x_1\geq\boldsymbol{E}X_1)$.

In the **hypothetical adhesion** situation, the hypothetical function $f_{Adhes}(x|x\geq 0) \equiv f(x_1\text{-}\boldsymbol{E}X_1|x_1\geq\boldsymbol{E}X_1)$ and at $x \sim 0$ the integral of it equals $½$. In crude terms, a half of the reflected PDF is adhered to the point $x = 0$.

The hypothetical reflection situation and the corresponding adhesion situation have been analyzed for the normal distribution.

The hypothetical reflection situation of the minimal expectation and the corresponding adhesion situation have been analyzed for the test distribution having sectionally continuous probability density function with not compact support.

The contiguous and hypothetical situations have been analyzed for "normal-like" test distribution having sectionally continuous PDF with compact support.

In this preliminary version the calculations are given as detailed as possible to be the verification for a following journal article.

The following deductions have been drawn for the normal and tested "normal-like" distributions having sectionally continuous PDFs with not compact support:

1) The normal distribution exhibit finite ratio $min(|E(X)\text{-}b|)/SD = \sqrt{2/\pi}$ for the hypothetical reflection situation (and $min(|E(X)\text{-}b|)/SD = 2\sqrt{2/\pi}$ for the corresponding adhesion situation).

2) The existence of a distribution that exhibits the ratio $min(|E(X)\text{-}b|)/SD$ that can be much less than unity has been proven for "normal-like" distributions with not compact support. This is done by means of the tested "normal-like" power one-step probability density function with not compact support.

The following deductions have been drawn for the tested "normal-like" distributions having sectionally continuous PDFs with compact support:

1) For the contiguous situation, the tested "normal-like" distributions with compact support exhibit finite ratio $min(|E(X)\text{-}b|)/SD$ that is not less than $\sqrt{3}$ for the tested sectionally continuous PDFs.

2) The existence of a distribution that exhibits the ratio $min(|E(X)\text{-}b|)/SD$ that can be much less than unity has been proven for "normal-like" distributions with compact support. This is done by means of the tested "normal-like" two-step stepwise probability density function with compact support.

The final resume. The obtained proofs determine the need of further research of the minimal distance from the expectation of a random variable to the nearest boundary of the interval to refine the conditions of finite ratios $min(|E(X)\text{-}b|)/SD$.

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