



Munich Personal RePEc Archive

Huggett Economies with Multiple Stationary Equilibria

Alexis Akira Toda

University of California, San Diego

13 March 2017

Online at <https://mpa.ub.uni-muenchen.de/78984/>
MPRA Paper No. 78984, posted 8 May 2017 02:57 UTC

Huggett Economies with Multiple Stationary Equilibria

Alexis Akira Toda*

April 23, 2017

Abstract

I obtain a closed-form solution to a Huggett economy with CARA utility when the vector of individual state variables follows a VAR(1) process with an arbitrary shock distribution. The stationary equilibrium is unique if the income process is AR(1), but not necessarily so otherwise. With Gaussian shocks, I provide general sufficient conditions for the existence of at least three equilibria when the income process is either ARMA(1,1), AR(2), or has a persistent-transitory (PT) representation with negatively correlated shocks. The possibility of multiple equilibria calls for caution in comparative statics exercises and policy analyses using heterogeneous-agent models.

Keywords: CARA utility, income fluctuation problem, persistent-transitory representation.

JEL codes: C62, D52, D58, E21.

1 Introduction

General equilibrium models with agents that are subject to uninsurable idiosyncratic income risk (the so-called Bewley (1983)-Huggett (1993)-Aiyagari (1994) models) are one of the workhorses of modern macroeconomics. Since such heterogeneous-agent models rarely admit closed-form solutions, they are typically solved numerically. Based on such numerical solutions, researchers often conduct comparative statics exercises and policy analyses. If the equilibrium is unique, then the result of such exercises are unambiguous. However, if there exist multiple equilibria, the conclusion may depend on the choice of the equilibrium, both quantitatively (*i.e.*, in terms of magnitude) and qualitatively (*i.e.*, in terms of direction) (Kehoe, 1985, 1991). While it is known that multiple (stationary) equilibria are possible in a variety of economies,¹ examples of multiple stationary equilibria in canonical Bewley-Huggett-Aiyagari models do not seem to be known.

*Department of Economics, University of California San Diego. Email: atoda@ucsd.edu.

¹For examples of multiple equilibria, see Kehoe (1985), Gjerstad (1996), and Toda and Walsh (2017) for Edgeworth box economies, Galí (1996) for a growth model with monopolistic competition, Bodenstein (2010) for an international business cycle model, Kubler and Schmedders (2010a) for a static incomplete market model, and Kubler and Schmedders (2010c) for overlapping generations models.

In this paper, exploiting constant absolute risk aversion (CARA) preferences and the VAR(1) dynamics of the state variables, I obtain a new closed-form solution (up to a single nonlinear equation that determines the equilibrium risk-free rate) to a canonical Huggett (1993) economy, where agents are subject to uninsurable idiosyncratic income risk and trade a risk-free asset in zero net supply. As an application, I explore whether or not such heterogeneous-agent economies admit a unique stationary equilibrium. I show that (i) the stationary equilibrium is unique if the income process is AR(1) with an arbitrary shock distribution, but (ii) multiple stationary equilibria are possible if the income process is more general. In particular, I show how to construct economies with at least three stationary equilibria when the income process is either ARMA(1,1), AR(2), or has a persistent-transitory (PT) representation that is the sum of an AR(1) process and a transitory shock. The possibility of multiple equilibria in CARA Huggett (incomplete market) economies may be surprising since it is well known that CARA preferences are sufficient for equilibrium uniqueness under complete markets (Hens and Loeffler, 1995).

The case with the persistent-transitory (PT) income process is especially relevant because it is widely used in empirical and quantitative works. The PT model has the general form

$$y_t = x_{1t} + x_{2t}, \quad (1.1)$$

where x_{1t}, x_{2t} are the persistent and transitory components of y_t with various specifications.² In this paper I focus on the case in which the persistent component x_{1t} is AR(1) and the transitory component x_{2t} is white noise, which is by far the most widely used in the literature.³ When the shocks are restricted to be Gaussian, I prove that the stationary equilibrium is unique when the correlation between the two shocks is nonnegative, but multiple equilibria are possible when the correlation is negative. Although this result may seem restrictive at first glance since many papers assume zero correlation, this is not the case: I show that multiple equilibria are possible even with correlations that are arbitrarily close to zero, and also the recent evidence in Hryshko (2014) suggests that the correlation between the shocks to the persistent and transitory components is actually negative.

Taken together, the possibility of multiple stationary equilibria calls for caution in comparative statics exercises and policy analyses using incomplete market heterogeneous-agent models.

1.1 Related literature

The closed-form solution obtained in this paper is similar to Calvet (2001) and Wang (2003).⁴ Using an incomplete market CARA economy in an i.i.d. Gaussian setting, Calvet (2001) shows that a continuum of *nonstationary* equilibria

²In Moffitt and Gottschalk (2002) and Jappelli and Pistaferri (2010), x_{1t} is a random walk and x_{2t} is ARMA(1,1). In Meghir and Pistaferri (2004) and Blundell et al. (2008), x_{1t} is a random walk and x_{2t} is MA(q).

³See, for example, Topel and Ward (1992), Hubbard et al. (1995), Storesletten et al. (2004), Guvenen (2007), Ejrnaes and Browning (2014), Guvenen et al. (2014), and Hryshko (2014), among many others.

⁴Other papers that exploit the tractability of CARA preferences with additive shocks include Caballero (1990, 1991), Wang (2004, 2007), and Angeletos and Calvet (2005, 2006), among others.

may exist, but the stationary equilibrium is unique. Since macroeconomists are typically concerned with stationary equilibria, my examples of multiple stationary equilibria may be surprising. Wang (2003) obtains a closed-form solution to a Huggett economy (up to a single nonlinear equation) with CARA preferences and AR(1) income processes in the context of the permanent income hypothesis. My model generalizes his result to the VAR(1) case, and I prove the uniqueness of equilibrium in the special case of AR(1) studied in Wang (2003).

In a series of papers, Kubler and Schmedders (2010a,b,c) show how to compute *all* equilibria using the Gröbner basis in algebraic geometry when the economy is “semi-algebraic”, *i.e.*, the equilibrium conditions reduce to a system of finitely many polynomial equations after a suitable transformation. This is the case when agents are finitely lived and have constant relative risk aversion (CRRA) preferences with rational risk aversion coefficients. Kubler and Schmedders (2010c) apply the Gröbner basis approach to OLG economies and find that multiplicity becomes less likely as the life span of agents increases. My examples show that multiplicity is possible in economies with infinitely lived agents and robust ranges of parameters, possibly because the CARA Huggett economy is not semi-algebraic.

This paper is also related to papers that obtain closed-form solutions to dynamic general equilibrium models, such as Labadie (1989), Burnside (1998), Tsonas (2003), and de Groot (2015), among others. Most of these papers provide solutions to asset pricing models, which have been applied to evaluate the solution accuracy of numerical methods (Collard and Juillard, 2001; Schmitt-Grohé and Uribe, 2004; Farmer and Toda, 2016). My solution to the income fluctuation problem may be useful for evaluating the accuracy of numerical algorithms to solve heterogeneous-agent models.

2 Huggett economy with closed-form solution

In this section, exploiting constant absolute risk aversion (CARA) preferences and VAR(1) dynamics, I show how to obtain exact solutions to Huggett (1993) economies, where there are a continuum of independent agents that are subject to uninsurable idiosyncratic income risk and trade a risk-free asset in zero net supply. I first solve an income fluctuation problem (Schechtman and Escudero, 1977), which I subsequently embed into a general equilibrium model.

2.1 CARA-VAR(1) income fluctuation problem

Consider an agent with additive CARA utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad (2.1)$$

where $0 < \beta < 1$ is the discount factor and $u(c) = -e^{-\gamma c}/\gamma$ has constant absolute risk aversion $\gamma > 0$. The agent can borrow or save at an exogenous gross risk-free rate $R > 1$ and is subject to uninsurable idiosyncratic income risk. Let y_t be the income in period t , which evolves according to some Markov process, and w_t be the financial wealth at the beginning of time t excluding current income. The timing is as follows. At the beginning of period t , the

agent sees his financial wealth w_t and current income y_t . The agent chooses consumption c_t and saves the rest $w_t + y_t - c_t$. Hence the budget constraint is

$$w_{t+1} = R(w_t - c_t + y_t). \quad (2.2)$$

The Bellman equation is

$$V(w, x) = \max_c \{u(c) + \beta \mathbf{E}[V(R(w - c + y), x') | w, x]\}, \quad (2.3)$$

where x denotes the vector of state variables that include income but not wealth. Since the CARA utility is defined on the entire real line, as is usual I assume that consumption can be negative. Under an AR(1) specification for the income process, Wang (2003) obtains a closed-form solution to the utility maximization problem. Below, I show that a similar solution exists for a VAR(1) income process with an arbitrary shock distribution.

Suppose that the vector of state variables x_t follows the VAR(1)

$$x_t = Ax_{t-1} + \eta_t, \quad (2.4)$$

where A is a square matrix with spectral radius less than 1 and η_t has a general shock distribution that is i.i.d. over time.⁵ Suppose that the income in period t is a linear function of current state variables,

$$y_t = f'x_t,$$

where f is a vector of loadings of x . For example, if income follows an AR(p) process, we can rewrite the AR(p) process as a p -dimensional VAR(1) process using $x_t = (y_t, \dots, y_{t-p+1})'$, and can pick $f = (1, 0, \dots, 0)'$. Even under this generality, the optimization problem still admits a closed-form solution.

Proposition 2.1. *Suppose that the moment generating function of η is finite. Then the value function and the optimal consumption rule of the CARA-VAR(1) income fluctuation problem are given by*

$$V(w, x) = -\frac{1}{\gamma a} e^{-\gamma(aw + b + d'x)}, \quad (2.5a)$$

$$c(w, x) = aw + b + d'x, \quad (2.5b)$$

where

$$a = 1 - 1/R > 0,$$

$$b = \frac{1}{\gamma(1-R)} \log \beta R \mathbf{E}[e^{-\gamma(R-1)f'(RI-A)^{-1}\eta}],$$

$$d = (R-1)(RI-A')^{-1}f.$$

As is clear from the the proof of Proposition 2.1, the model is tractable as long as all expressions are exponential-affine (things like $e^{\kappa_0 + \kappa_1'x}$). For example, the discount factor β can be an exponential-affine function of current consumption c , and the distribution of η can depend on current state variable x as long as its moment generating function is exponential-affine in x . Wang (2007) considers such a case in the context of the wealth distribution.

⁵Without loss of generality, we may assume that the VAR(1) process (2.4) does not contain a constant term. This is because we have put no structure on the distribution of η , so if there is a constant term we can always shift the distribution of η so that the constant term is 0.

2.2 General equilibrium

Next I embed the income fluctuation problem into a general equilibrium model. I consider a Huggett (1993) economy, where there are a continuum of independent agents that trade a risk-free asset in zero net supply.⁶ The equilibrium concept I use is the stationary equilibrium, which consists of a constant gross risk-free rate $R > 1$ such that (i) agents solve the income fluctuation problem, and (ii) the risk-free asset market clears.

Using the budget constraint (2.2), the consumption rule (2.5b), and $a = 1 - 1/R$, individual wealth evolves according to

$$\begin{aligned} w' &= R(w - (aw + b + d'x) + f'x) \\ &= w + R(-b + (f - d)'x). \end{aligned} \quad (2.6)$$

Since wealth is a random walk in levels, with infinitely lived agents there is no stationary wealth distribution. In order to obtain a stationary distribution, I assume that agents enter/exit the economy at constant probability p as in Yaari (1965) and Blanchard (1985). Because agents survive each period with probability $1 - p$, the effective discount factor is $\tilde{\beta} = \beta(1 - p)$. Suppose that there are perfectly competitive insurance companies that offer annuities and life insurances. Let \tilde{R} be the effective risk-free rate that agents face. If an agent saves or borrows 1, the position grows to \tilde{R} next period if the agent survives, and 0 if he dies (an agent who dies with positive assets surrender to the insurance company; the debt of an agent who dies with negative assets is covered by life insurance). Hence by accounting we obtain

$$R = (1 - p)\tilde{R} + p0 \iff \tilde{R} = \frac{R}{1 - p}.$$

The following theorem shows the existence of equilibrium.

Theorem 2.2. *There exists a stationary equilibrium in the CARA-VAR(1) Huggett economy. The effective risk-free rate \tilde{R} solves*

$$\mathbb{E}[v(\tilde{R})'\eta] - \log \tilde{\beta} \tilde{R} \mathbb{E}[e^{v(\tilde{R})'\eta}] = 0, \quad (2.7)$$

where $v(\tilde{R}) = \gamma(1 - \tilde{R})(\tilde{R}I - A')^{-1}f$. The gross risk-free rate $R = \tilde{R}(1 - p)$ satisfies $1 - p < R < 1/\beta$.

In light of the subsequent applications, it is convenient to consider Gaussian VAR(1) processes. In this case the equilibrium condition (2.7) simplifies as follows. For notational simplicity, assume there is no death ($p = 0$), so $\tilde{\beta} = \beta$ and $\tilde{R} = R$. The case $p > 0$ is completely analogous.

Corollary 2.3. *Suppose that the income process is Gaussian VAR(1), so $\eta \sim N(\mu_\eta, \Sigma_\eta)$ in (2.4). Then the equilibrium condition (2.7) is equivalent to*

$$\log \beta R + \frac{1}{2}v(R)'\Sigma_\eta v(R) = 0. \quad (2.8)$$

⁶It is straightforward to generalize the results to Aiyagari (1994) models with capital accumulation. I do not consider this case since the point of the paper is to show that multiple equilibria are possible even in simple endowment economies.

Proof. Let $v = v(R)$. Since $\eta \sim N(\mu_\eta, \Sigma_\eta)$, we have $E[v'\eta] = v'\mu_\eta$ and the moment generating function of η is

$$M_\eta(v) = E[e^{v'\eta}] = e^{v'\mu_\eta + \frac{1}{2}v'\Sigma_\eta v}.$$

Therefore the equilibrium condition (2.7) is equivalent to

$$0 = -E[v'\eta] + \log \beta R + \log E[e^{v'\eta}] = \log \beta R + \frac{1}{2}v'\Sigma_\eta v. \quad \square$$

While $v(R)$ is a rational function of R , because (2.8) contains $\log R$, the CARA-Gaussian VAR(1) Huggett economy is not semi-algebraic. Consequently, the Gröbner basis method in Kubler and Schmedders (2010a) is not applicable.

If there exist multiple equilibria, it is interesting to rank the equilibria in terms of welfare. To compute the equilibrium welfare, assume that newborn agents are endowed with zero financial wealth and the initial state variable is drawn from the stationary distribution. Then the ex ante welfare is $V = E[V(0, x)]$, where $V(w, x)$ is given by (2.5a) and the expectation is taken over the stationary distribution of x . To convert this quantity into consumption equivalent, suppose that an agent consumes a constant amount c in every period, with associated utility equal to V . Then we have

$$V = -\frac{1}{\gamma}e^{-\gamma c} \sum_{t=0}^{\infty} \beta^t = -\frac{1}{\gamma(1-\beta)}e^{-\gamma c} \iff c = -\frac{1}{\gamma} \log(-\gamma(1-\beta)V).$$

Letting $v = \gamma(1-R)(RI - A')^{-1}f$, (2.5a) with $w = 0$ becomes

$$V(0, x) = -\frac{1}{\gamma a}e^{-\gamma b + v'x},$$

where $a = 1 - 1/R$. If the VAR is Gaussian, then $x \sim N(\mu_x, \Sigma_x)$, so using the moment generating function of the Gaussian distribution, we can compute V as

$$V = E[V(0, x)] = -\frac{1}{\gamma a}e^{-\gamma b + v'\mu_x + \frac{1}{2}v'\Sigma_x v}.$$

Therefore the welfare in consumption equivalent is

$$c = -\frac{1}{\gamma} \left(\log \frac{1-\beta}{1-1/R} - \gamma b + v'\mu_x + \frac{1}{2}v'\Sigma_x v \right). \quad (2.9)$$

Σ_x can be computed using $\text{vec}(\Sigma_x) = (I - A \otimes A)^{-1} \text{vec}(\Sigma_\eta)$. Since $\eta \sim N(\mu_\eta, \Sigma_\eta)$ with $\mu_\eta = (I - A)\mu_x$, we can compute b using Proposition 2.1.

3 Uniqueness and multiplicity of equilibria

Theorem 2.2 shows that a Huggett economy with CARA preference and VAR(1) income process always has an equilibrium, whose interest rate is the solution to the single nonlinear equation (2.7). In this section, I show that the equilibrium is unique if the income process is AR(1), but not necessarily so otherwise.

Proposition 3.1. *Suppose that the income process is AR(1) with an arbitrary shock distribution, so $A = \phi$ with $|\phi| < 1$ and $f = 1$ in the VAR(1) model. Then the equilibrium is unique.*

The CARA-AR(1) Huggett economy has been studied in the context of the permanent income hypothesis by Wang (2003), who proved the existence of equilibrium. The uniqueness result in Proposition 3.1 seems to be new. Once we move away from the AR(1) specification, multiple equilibria are possible. Below, I consider several such cases.

3.1 ARMA(1,1) income process

First, as the income process I consider the ARMA(1,1):

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}, \quad \varepsilon_t \sim N(\mu_\varepsilon, \sigma^2). \quad (3.1)$$

This case is of particular interest because (i) it is a direct generalization of AR(1), (ii) it is fairly widely used (MaCurdy, 1982; Moffitt and Gottschalk, 2002; Jappelli and Pistaferri, 2010), and (iii) the sufficient condition for the existence of multiple equilibria is especially simple. Letting

$$x_t = \begin{bmatrix} y_t \\ \varepsilon_t \end{bmatrix}, \quad \eta_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_t \end{bmatrix}, \quad A = \begin{bmatrix} \phi & \theta \\ 0 & 0 \end{bmatrix}, \quad \mu_\eta = \begin{bmatrix} \mu_\varepsilon \\ \mu_\varepsilon \end{bmatrix}, \quad \text{and } \Sigma_\eta = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

we have $x_t = Ax_{t-1} + \eta_t$ with $\eta_t \sim N(\mu_\eta, \Sigma_\eta)$, so we can think of the ARMA(1,1) economy as a VAR(1) economy with $f = (1, 0)'$. Since

$$(RI - A')^{-1} = \begin{bmatrix} R - \phi & 0 \\ -\theta & R \end{bmatrix}^{-1} = \frac{1}{R(R - \phi)} \begin{bmatrix} R & 0 \\ \theta & R - \phi \end{bmatrix},$$

the equilibrium condition (2.8) becomes

$$\log \beta R + \frac{1}{2} \gamma^2 \sigma^2 \left(\frac{(R-1)(R+\theta)}{R(R-\phi)} \right)^2 = 0. \quad (3.2)$$

Let $g(R)$ be the left-hand side of (3.2). Since $g(1) = \log \beta < 0$ and $g(1/\beta) > 0$, $g(R) = 0$ has a solution in $(1, 1/\beta)$. To show that g can have multiple zeros, following the idea of Toda and Walsh (2017), it suffices to construct an example with $g(R_2) = 0$ with $g'(R_2) < 0$. In fact, if such an R_2 exists, since $g(1) < 0$ and $g(R) > 0$ for sufficiently large $R < R_2$, by the intermediate value theorem there exists $R_1 \in (1, R_2)$ such that $g(R_1) = 0$. Similarly there exists $R_3 \in (R_2, 1/\beta)$ such that $g(R_3) = 0$, so there are at least three equilibria.

Since

$$g'(R) = \frac{1}{R} + \gamma^2 \sigma^2 \frac{(R-1)(R+\theta)}{(R(R-\phi))^3} ((1-\phi-\theta)R^2 + 2\theta R - \phi\theta),$$

if γ, σ are chosen to satisfy (3.2), after some algebra we obtain

$$g'(R) < 0 \iff 1 + \kappa \frac{(1-\phi-\theta)R^2 + 2\theta R - \phi\theta}{(R+\theta)(R-\phi)} < 0, \quad (3.3)$$

where $\kappa = -\frac{2 \log \beta R}{R-1} > 0$ since $1 < R < 1/\beta$. Since the ARMA(1,1) process is stationary if and only if $|\phi| < 1$, we can choose θ freely. Therefore constructing an example with multiple equilibria is quite simple.

Proposition 3.2. Fix any $R > 1$, $0 < \beta < 1/R$, $\gamma > 0$, and $\mu_\varepsilon \in \mathbb{R}$. If θ satisfies (3.3), then the CARA-ARMA(1,1) Huggett economy with the ARMA(1,1) income process (3.1) has at least three equilibria, with one equilibrium risk-free rate being R . In particular, (3.3) is satisfied by either

1. taking $\theta < -R$ and θ sufficiently close to $-R$, or
2. taking $\beta < \frac{1}{R}e^{\frac{1}{2(1-R)}}$, $1 + \frac{1}{\kappa} - (R-1)^2 < \phi < 1$, and $\theta > 0$ sufficiently large.

Proof. If (3.3) holds, then $g'(R) < 0$ by choosing $\sigma > 0$ to satisfy (3.2), or $g(R) = 0$. Therefore there exist at least three equilibria.

Case 1: $\theta < -R$ and θ is sufficiently close to $-R$. In this case, the denominator of the second term in (3.3) is negative and close to zero. Since $\kappa > 0$, to show (3.3), it suffices to show that the numerator is positive when θ is close to $-R$. Letting $\theta \rightarrow -R$, the numerator becomes

$$\begin{aligned} (1 - \phi - \theta)R^2 + 2\theta R - \phi\theta &\rightarrow (1 - \phi + R)R^2 - 2R^2 + \phi R \\ &= R(R-1)(R-\phi) > 0. \end{aligned}$$

Case 2: $\beta < \frac{1}{R}e^{\frac{1}{2(1-R)}}$, $1 + \frac{1}{\kappa} - (R-1)^2 < \phi < 1$, and $\theta > 0$ is sufficiently large. Letting $\theta \rightarrow \infty$, (3.3) becomes

$$1 + \kappa(-R^2 + 2R - \phi) < 0 \iff \phi > 1 + \frac{1}{\kappa} - (R-1)^2.$$

Since $\phi < 1$ for stationarity, if $1 > 1 + \frac{1}{\kappa} - (R-1)^2$, then we can take ϕ in this interval and construct three equilibria by taking $\theta > 0$ sufficiently large. The condition for such a ϕ to exist is

$$1 > 1 + \frac{1}{\kappa} - (R-1)^2 \iff (R-1)^2 > -\frac{R-1}{2\log \beta R} \iff \beta < \frac{1}{R}e^{\frac{1}{2(1-R)}}. \quad \square$$

Since the upper bound of β in the second case, $\frac{1}{R}e^{\frac{1}{2(1-R)}}$, attains the maximum $\frac{1}{2}e^{-\frac{1}{2}} = 0.3033$ when $R = 2$, the second case is not useful for constructing realistic examples (with β close to 1). Therefore I consider the first case, and take $R = 1.03$ (one of the equilibrium risk-free rates is 3%), $\beta = 0.95$, $\phi = 0.6$, and $\theta = -1.04$, which satisfy (3.3). (3.2) is then satisfied by taking $\gamma = 10$ and $\sigma = 30.7804$. Figure 1 shows the graph of g (left-hand side of the equilibrium condition (3.2)). Since the graph crosses the horizontal axis three times, there exist three equilibria. The equilibrium risk-free rates are $R_1 = 1.0146$, $R_2 = 1.03$, and $R_3 = 1.0443$. Assuming $\mu_y = 100$, the welfare in consumption equivalent computed by (2.9) is $c_1 = 98.0962$, $c_2 = 92.9687$, and $c_3 = 85.7410$, so the equilibrium with the lowest interest rate is the most efficient. (Under complete markets, since agents consume 100 in each period, the welfare is 100. Thus with incomplete markets the welfare declines by 1.9%, 7.0%, and 14.3%, respectively.) The intuition is that in Huggett economies it is the poor agents that borrow, so making the interest rate low alleviates the burden of poor agents, who have high marginal utility.

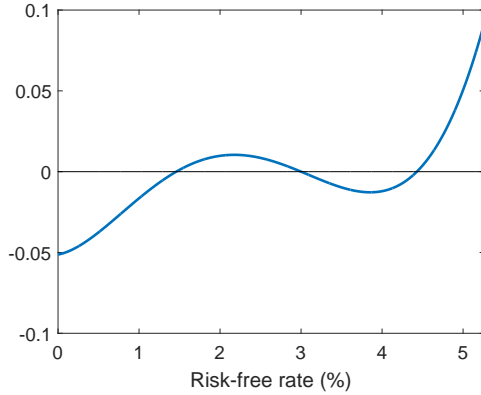


Figure 1. Graph of the equilibrium condition (3.2) for $R = 1.03$, $\beta = 0.95$, $\gamma = 10$, $\phi = 0.6$, $\theta = -1.04$, and $\sigma = 30.7804$ in the ARMA(1,1) economy.

3.2 Persistent-transitory (PT) income process

Next, I consider the following income process with a persistent-transitory (PT) representation with AR(1) and white noise components:

$$\begin{aligned} y_t &= x_{1t} + x_{2t}, \\ x_{1t} &= \phi x_{1,t-1} + \eta_{1t}, \\ x_{2t} &= \eta_{2t}, \end{aligned}$$

where $\eta_t = (\eta_{1t}, \eta_{2t})' \sim N(\mu_\eta, \Sigma_\eta)$. The PT process is by far the most widely used in the literature.⁷ We can rewrite this process as VAR(1) by setting $A = \begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix}$ and $f = (1, 1)'$. Since

$$(RI - A')^{-1} = \begin{bmatrix} R - \phi & 0 \\ 0 & R \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{R - \phi} & 0 \\ 0 & \frac{1}{R} \end{bmatrix},$$

the equilibrium condition (2.8) becomes

$$\log \beta R + \frac{1}{2} \gamma^2 (R - 1)^2 \left(\frac{\sigma_1^2}{(R - \phi)^2} + \frac{2\rho\sigma_1\sigma_2}{R(R - \phi)} + \frac{\sigma_2^2}{R^2} \right) = 0, \quad (3.4)$$

where $\Sigma_\eta = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$. Letting $g(R)$ be the left-hand side of (3.4), we obtain

$$g'(R) = \frac{1}{R} + \gamma^2 (R - 1) \left(\frac{\sigma_1^2(1 - \phi)}{(R - \phi)^3} + \frac{\rho\sigma_1\sigma_2(2R - \phi(1 + R))}{R^2(R - \phi)^2} + \frac{\sigma_2^2}{R^3} \right). \quad (3.5)$$

In this case we have the following proposition.

Proposition 3.3. *If $\rho \geq 0$, the CARA-PT Huggett economy has a unique equilibrium. Otherwise, multiple equilibria are possible. In particular, suppose that the parameter values are chosen as follows:*

⁷See, for example, Topel and Ward (1992), Hubbard et al. (1995), Storesletten et al. (2004), Guvenen (2007), Ejrnaes and Browning (2014), Guvenen et al. (2014), and Hryshko (2014), among many others.

1. Take any $R > 1$.
2. Take any γ, σ_1 such that $\gamma\sigma_1 > 2\sqrt{R-1}$.
3. Take any ρ such that $-1 < \rho < -\frac{2\sqrt{R-1}}{\gamma\sigma_1}$.
4. Take any $\sigma_2 > 0$ such that $R^2 + \gamma^2(\rho\sigma_1\sigma_2R + \sigma_2^2(R-1)) < 0$.
5. Take ϕ sufficiently close to 1 such that $g'(R) < 0$.
6. Take $\beta > 0$ such that $g(R) = 0$.

Then there exist at least three equilibria, with one equilibrium risk-free rate being R .

Proof. Suppose that $\rho \geq 0$. Since $R > 1$ and $|\phi| < 1$, we have $1 - \phi > 0$ and $2R - \phi(1 + R) = R(1 - \phi) + (R - \phi) > 0$, so all terms in (3.5) are positive. Since $g'(R) > 0$, (3.4) has at most one solution, so the equilibrium is unique.

To construct an example with multiple equilibria, it suffices to choose parameters such that $g(R) = 0$ and $g'(R) < 0$. Letting $\phi \rightarrow 1$ in (3.5), we have $g'(R) < 0$ if and only if

$$\begin{aligned} \frac{1}{R} + \gamma^2(R-1) \left(\frac{\rho\sigma_1\sigma_2(R-1)}{R^2(R-1)^2} + \frac{\sigma_2^2}{R^3} \right) < 0 \\ \iff R^2 + \gamma^2(\rho\sigma_1\sigma_2R + \sigma_2^2(R-1)) < 0. \end{aligned} \quad (3.6)$$

Therefore if the parameters satisfy this last inequality, we can choose ϕ sufficiently close to 1 such that $g'(R) < 0$. Regarding the last expression in (3.6) as a quadratic function of σ_2 , we can choose $\sigma_2 > 0$ to make (3.6) hold if $\rho < 0$ and

$$(\gamma^2\rho\sigma_2R)^2 - 4\gamma^2R^2(R-1) > 0 \iff \rho < -\frac{2\sqrt{R-1}}{\gamma\sigma_1}.$$

Since $|\rho| < 1$, we can choose ρ to make the inequality hold if

$$-1 < -\frac{2\sqrt{R-1}}{\gamma\sigma_1} \iff \gamma\sigma_1 > 2\sqrt{R-1}.$$

Therefore if γ, σ_1 satisfy this inequality, we can choose parameters such that $g'(R) < 0$. Since β does not enter $g'(R)$, we can choose β such that $g(R) = 0$, namely

$$0 < \beta = \frac{1}{R} \exp \left(-\frac{1}{2}\gamma^2(R-1)^2 \left(\frac{\sigma_1^2}{(R-\phi)^2} + \frac{2\rho\sigma_1\sigma_2}{R(R-\phi)} + \frac{\sigma_2^2}{R^2} \right) \right) < \frac{1}{R}. \quad \square$$

As a concrete example, take $R = 1.01$ (one of the equilibrium risk-free rates is 1%), $\gamma = 10$, $\sigma_1 = 0.25$, and $\rho = \phi = -0.99$. Since (3.5) is quadratic in σ_2 , to give it the best chance to be negative, take the minimizer

$$\sigma_2 = -\frac{\rho\sigma_1R(2R - \phi(1 + R))}{2(R - \phi)^2} = 9.4053.$$

Then (3.5) is equal to -6.7432 , so we have $g'(R) < 0$. Finally, take β to satisfy $g(R) = 0$, which implies $\beta = 0.9302$. Figure 2 shows the graph of g (left-hand

side of the equilibrium condition (3.4)). Since the graph crosses the horizontal axis three times, there exist three equilibria. The equilibrium risk-free rates are $R_1 = 1.0047$, $R_2 = 1.01$, and $R_3 = 1.0204$. Assuming $\mu_y = 100$, the welfare in consumption equivalent (2.9) is $c_1 = 98.1648$, $c_2 = 95.9506$, and $c_3 = 92.9373$, so again the equilibrium with the lowest interest rate is the most efficient.

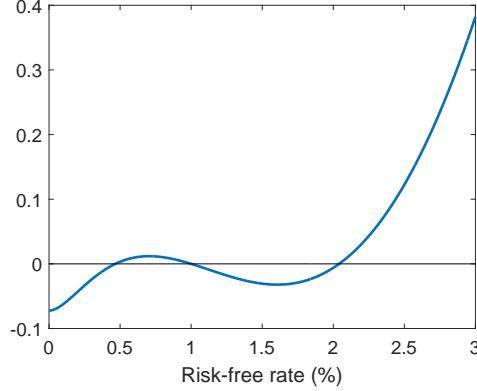


Figure 2. Graph of the equilibrium condition (3.4) for $R = 1.01$, $\beta = 0.9302$, $\gamma = 10$, $\rho = \phi = -0.99$, $\sigma_1 = 0.25$, and $\sigma_2 = 9.4053$ in the PT economy.

3.3 AR(2) income process

Finally, I consider the AR(2) income process:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim N(\mu_\varepsilon, \sigma^2). \quad (3.7)$$

Letting

$$x_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}, \quad \eta_t = \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}, \quad \mu_\eta = \begin{bmatrix} \mu_\varepsilon \\ 0 \end{bmatrix}, \quad \text{and } \Sigma_\eta = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix},$$

we have $x_t = Ax_{t-1} + \eta_t$ with $\eta_t \sim N(\mu_\eta, \Sigma_\eta)$, so we can think of the AR(2) economy as a VAR(1) economy with $f = (1, 0)'$. Since

$$(RI - A')^{-1} = \begin{bmatrix} R - \phi_1 & -1 \\ -\phi_2 & R \end{bmatrix}^{-1} = \frac{1}{R^2 - \phi_1 R - \phi_2} \begin{bmatrix} R & 1 \\ \phi_2 & R - \phi_1 \end{bmatrix},$$

the equilibrium condition (2.8) becomes

$$\log \beta R + \frac{1}{2} \gamma^2 \sigma^2 \left(\frac{R(R-1)}{R^2 - \phi_1 R - \phi_2} \right)^2 = 0. \quad (3.8)$$

Let $g(R)$ be the left-hand side of (3.8). Since the spectral radius of A is less than 1, clearly we have $R^2 - \phi_1 R - \phi_2 > 0$ for $R \geq 1$. Therefore g is continuous in $R \geq 1$. By some algebra, we have

$$g'(R) = \frac{1}{R} + \gamma^2 \sigma^2 \frac{R(R-1)}{(R^2 - \phi_1 R - \phi_2)^3} ((1 - \phi_1)R^2 - 2\phi_2 R + \phi_2).$$

Therefore if γ, σ are chosen to satisfy (3.8), we obtain

$$g'(R) < 0 \iff R(1 + R\kappa)\phi_1 + (1 + (2R - 1)\kappa)\phi_2 > R^2(1 + \kappa), \quad (3.9)$$

where $\kappa = -\frac{2\log\beta R}{R-1} > 0$ since $1 < R < 1/\beta$. Putting all the pieces together, we obtain the following proposition.

Proposition 3.4. *Fix any $R > 1$, $\gamma > 0$, and $\mu_\varepsilon \in \mathbb{R}$. For any discount factor β such that $0 < \beta < \frac{1}{R}e^{\frac{1-R}{2}}$, there exists a Gaussian AR(2) income process (3.7) such that the CARA-AR(2) Huggett economy has at least three equilibria, with one equilibrium risk-free rate being R . In particular, letting $\kappa = -\frac{2\log\beta R}{R-1} > 1$, we can construct such an example by taking*

$$\frac{R + 1 + (R - 1)\kappa}{1 + (R - 1)\kappa} < \phi_1 < 2, \quad (3.10a)$$

$$\max \left\{ -1, \frac{R^2(1 + \kappa) - R(1 + R\kappa)\phi_1}{1 + (2R - 1)\kappa} \right\} < \phi_2 < 1 - \phi_1. \quad (3.10b)$$

Proof. If $0 < \beta < \frac{1}{R}e^{\frac{1-R}{2}}$, then $\kappa = -\frac{2\log\beta R}{R-1} > 1$. Hence the denominators in the left-hand sides of (3.10) are positive.

Let us show that there exist at least three equilibria if $\phi_1 > 0$ and ϕ_2 satisfies (3.10b). If (3.10b) holds, in particular $-1 < \phi_2 < 1 - |\phi_1|$, so by Lemma A.1 the spectral radius of A is less than 1. Again by (3.10b), we have

$$\frac{R^2(1 + \kappa) - R(1 + R\kappa)\phi_1}{1 + (2R - 1)\kappa} < \phi_2,$$

which is equivalent to (3.9). Letting g be the left-hand side of (3.8), we have $g'(R) < 0$. Since $0 < \beta R < 1$, for any parameter values we can choose $\sigma > 0$ to satisfy (3.8), or $g(R) = 0$. Therefore there exist at least three equilibria.

Finally, let us show that there exists ϕ_2 satisfying (3.10b) if ϕ_1 satisfies (3.10a). Since $R > 1$ and $\kappa > 1$, we have

$$\frac{R + 1 + (R - 1)\kappa}{1 + (R - 1)\kappa} < 2 \iff (R - 1)(\kappa - 1) > 0,$$

so there exists ϕ_1 that satisfies (3.10a). In this case $0 < \phi_1 < 2$. For any such ϕ_1 , since $-1 < 1 - \phi_1$ and

$$\frac{R^2(1 + \kappa) - R(1 + R\kappa)\phi_1}{1 + (2R - 1)\kappa} < 1 - \phi_1 \iff \phi_1 > \frac{R + 1 + (R - 1)\kappa}{1 + (R - 1)\kappa},$$

we can take ϕ_2 that satisfies (3.10b). \square

As a concrete example, take $R = 1.05$ (one of the equilibrium risk-free rates is 5%) and $\gamma = 1$. Since $\frac{1}{R}e^{\frac{1-R}{2}} = 0.9289$, take $\beta = 0.9$ to satisfy the assumption of Proposition 3.4. Then in order to satisfy (3.8), it must be $\sigma = 0.0243$. With these numbers (3.10a) becomes $1.9433 < \phi_1 < 2$, so take $\phi_1 = 1.98$. Then (3.10b) becomes $-0.9806 < \phi_2 < -0.98$, so take $\phi_2 = -0.9803$. Figure 3 shows the graph of g (left-hand side of the equilibrium condition (3.8)). Since the graph crosses the horizontal axis three times, there exist three equilibria. The equilibrium risk-free rates are $R_1 = 1.0120$, $R_2 = 1.05$, and $R_3 = 1.0679$. Assuming $\mu_y = 100$, the welfare in consumption equivalent (2.9) is $c_1 = 87.7905$, $c_2 = 76.8134$, and $c_3 = 76.0348$, so again the equilibrium with the lowest interest rate is the most efficient.

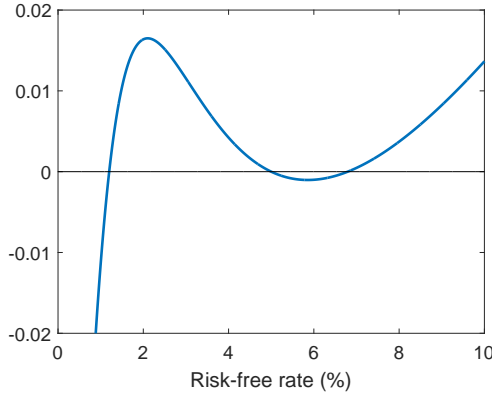


Figure 3. Graph of the equilibrium condition (3.8) for $R = 1.05$, $\beta = 0.9$, $\gamma = 1$, $\phi_1 = 1.98$, $\phi_2 = -0.9803$, and $\sigma = 0.0243$ in the AR(2) economy.

4 Concluding remarks

The paper generalizes the closed-form solution to the CARA-AR(1) Huggett economy in Wang (2003) to the case in which the vector of individual state variables follows a VAR(1) process with an arbitrary shock distribution. Although the AR(1) case has a unique stationary equilibrium, the economy with VAR(1) dynamics may have multiple equilibria. In particular, when the income process is either ARMA(1,1), AR(2), or has a persistent-transitory (PT) representation that consists of AR(1) and white noise, I provide general sufficient conditions for the existence of at least three equilibria. With multiplicity of equilibria, the quantitative implications such as comparative statics with respect to parameter values may depend on the choice of the equilibrium. Applied researchers should be aware of this possibility even in simple Huggett economies.

A Proofs

Proof of Proposition 2.1. I prove by guess-and-verify. Substituting (2.5a) into the Bellman equation, we obtain

$$\begin{aligned}
 & -\frac{1}{\gamma a} e^{-\gamma(aw+b+d'x)} \\
 & = \max_c \left\{ -\frac{1}{\gamma} e^{-\gamma c} - \frac{\beta}{\gamma a} \mathbf{E} \left[e^{-\gamma(aR(w-c+f'x)+b+d'x')} \mid w, x \right] \right\}. \quad (\text{A.1})
 \end{aligned}$$

The first-order condition with respect to c is

$$e^{-\gamma c} - \beta R \mathbf{E} \left[e^{-\gamma(aR(w-c+f'x)+b+d'x')} \mid w, x \right] = 0. \quad (\text{A.2})$$

Substituting (A.2) into (A.1), we obtain

$$-\frac{1}{\gamma a} e^{-\gamma(aw+b+d'x)} = -\frac{1}{\gamma a} \left(a + \frac{1}{R} \right) e^{-\gamma c}. \quad (\text{A.3})$$

Comparing the coefficients, (A.3) trivially holds if $a = 1 - 1/R$ and $c = aw + b + d'x$. In this case,

$$aR(w - c + f'x) = aw + (1 - R)b + (1 - R)(d - f)'x,$$

so (A.2) becomes

$$\begin{aligned} e^{-\gamma(aw+b+d'x)} &= \beta R \mathbf{E} \left[e^{-\gamma(aw+(1-R)b+(1-R)(d-f)'x+b+d'x')} \mid w, x \right] \\ \iff e^{-\gamma d'x} &= \beta R \mathbf{E} \left[e^{-\gamma((1-R)b+(1-R)(d-f)'x+d'(Ax+\eta))} \mid x \right]. \end{aligned} \quad (\text{A.4})$$

Since (A.4) is an identity, comparing the coefficients of y , we obtain

$$d = (1 - R)(d - f) + A'd \iff d = (R - 1)(RI - A')^{-1}f. \quad (\text{A.5})$$

(Note that since the spectral radius of A is less than 1 and $R > 1$, the matrix $RI - A'$ is regular.) Substituting (A.5) into (A.4), we obtain

$$\begin{aligned} 1 &= \beta R \mathbf{E} \left[e^{-\gamma((1-R)b+((R-1)(RI-A')^{-1}f)'\eta)} \right] \\ \iff b &= \frac{1}{\gamma(1-R)} \log \beta R \mathbf{E} [e^{-\gamma(R-1)f'(RI-A)^{-1}\eta}]. \end{aligned}$$

To show that this is the solution, it remains to show the transversality condition. Dividing (A.2) by $-\gamma a$ and using (2.5a), (2.5b), we obtain

$$V(w, x) = \beta R \mathbf{E} [V(w', x') \mid w, x] \iff V(w_t, x_t) = \beta R \mathbf{E}_t [V(w_{t+1}, x_{t+1})].$$

Iterating this equation we obtain

$$\beta^t \mathbf{E}_0 [V(w_t, x_t)] = \frac{1}{R^t} V(w_0, x_0) \rightarrow 0$$

as $t \rightarrow \infty$ since $R > 1$. Therefore the transversality condition holds. \square

Proof of Theorem 2.2. Let C, W be the aggregate consumption and wealth. By the optimal consumption rule (2.5b), we have $C = aW + b + d' \mathbf{E}[x]$, where $\mathbf{E}[x]$ is the unconditional mean of the VAR(1) (2.4). Since the risk-free asset is in zero net supply, which is the only saving vehicle, we have $W = 0$. By market clearing, aggregate consumption must equal aggregate income, so $C = f' \mathbf{E}[x]$. Combining these three equations, the equilibrium condition is

$$b + d' \mathbf{E}[x] = f' \mathbf{E}[x] \iff b = (f - d)' \mathbf{E}[x]. \quad (\text{A.6})$$

For the rest of the proof, to simplify the notation assume that $p = 0$, so $\tilde{R} = R$ and $\tilde{\beta} = \beta$.

Step 1. If an equilibrium exists, then

$$\gamma(1 - R)f'(RI - A)^{-1} \mathbf{E}[\eta] - \log \beta R \mathbf{E} [e^{\gamma(1-R)f'(RI-A)^{-1}\eta}] = 0. \quad (\text{A.7})$$

By (A.5) we obtain

$$\begin{aligned} f - d &= (I - (R - 1)(RI - A')^{-1})f \\ &= (RI - A' - (R - 1)I)(RI - A')^{-1}f \\ &= (I - A')(RI - A')^{-1}f. \end{aligned} \quad (\text{A.8})$$

Taking the unconditional expectation of (2.4), we obtain

$$\mathbb{E}[x] = A \mathbb{E}[x] + \mathbb{E}[\eta] \iff \mathbb{E}[x] = (I - A)^{-1} \mathbb{E}[\eta]. \quad (\text{A.9})$$

Combining (A.6), (A.8), and (A.9), it follows that

$$b = f'((RI - A')^{-1})(I - A)(I - A)^{-1} \mathbb{E}[\eta] = f'(RI - A)^{-1} \mathbb{E}[\eta].$$

(A.7) then follows from this equation and Proposition 2.1.

Step 2. If an equilibrium exists, then $\beta R < 1$.

Suppose that an equilibrium exists, and let $X = \gamma(1 - R)f'(RI - A)^{-1}\eta$. Since $\log(\cdot)$ is concave, by Jensen's inequality and (A.7), we obtain

$$0 = \mathbb{E}[X] - \log \mathbb{E}[\beta R e^X] < \mathbb{E}[X] - \mathbb{E}[\log(\beta R e^X)] = -\log \beta R.$$

Therefore $\beta R < 1$.

Step 3. An equilibrium exists. The gross risk-free rate satisfies $1 - p < R < 1/\beta$.

Let $g(R) = \mathbb{E}[v(R)'\eta] - \log \mathbb{E}[\beta R e^{v(R)'\eta}]$, where $v(R) = \gamma(1 - R)(RI - A')^{-1}f$. Since $v(1) = 0$ and $\beta < 1$, we obtain $g(1) = -\log \beta > 0$. By the previous step, we obtain $g(1/\beta) < -\log(\beta/\beta) = 0$. By the intermediate value theorem, there exists $R \in (1, 1/\beta)$ such that $g(R) = 0$. If $p > 0$, by the same argument as above we obtain $1 < \tilde{R} < 1/\tilde{\beta} \iff 1 - p < R < 1/\beta$. \square

Proof of Proposition 3.1. Using the equilibrium condition (A.7), we obtain

$$\gamma \frac{1 - R}{R - \phi} \mathbb{E}[\varepsilon] - \log \beta R \mathbb{E}[e^{\gamma \frac{1 - R}{R - \phi} \varepsilon}] = 0. \quad (\text{A.10})$$

Since $R > 1 > \phi$, let $x = \gamma \frac{1 - R}{R - \phi} \in (-\gamma, 0)$. Solving for R , we get

$$R = \frac{\gamma + \phi x}{\gamma + x} = \phi + \frac{\gamma(1 - \phi)}{\gamma + x},$$

which is decreasing in x . Substituting this R into (A.10), we obtain

$$\mathbb{E}[\varepsilon]x - \log \left(\beta \frac{\gamma + \phi x}{\gamma + x} \right) - \log \mathbb{E}[e^{\varepsilon x}] = 0. \quad (\text{A.11})$$

Let $h(x)$ be the left-hand side of (A.11). To show the uniqueness of equilibrium, it suffices to show that $h(x) = 0$ has at most one solution $x \in (-\gamma, 0)$. Let us show that h is strictly increasing on $(-\gamma, 0)$, which completes the proof.

For any $x \in (-\gamma, 0)$, letting $m = \mathbb{E}[\varepsilon e^{\varepsilon x}] / \mathbb{E}[e^{\varepsilon x}]$, we obtain

$$\begin{aligned} h'(x) &= \mathbb{E}[\varepsilon] - \frac{\phi}{\gamma + \phi x} + \frac{1}{\gamma + x} - \frac{\mathbb{E}[\varepsilon e^{\varepsilon x}]}{\mathbb{E}[e^{\varepsilon x}]}, \\ h''(x) &= \frac{\phi^2}{(\gamma + \phi x)^2} - \frac{1}{(\gamma + x)^2} - \frac{\mathbb{E}[\varepsilon^2 e^{\varepsilon x}] \mathbb{E}[e^{\varepsilon x}] - (\mathbb{E}[\varepsilon e^{\varepsilon x}])^2}{(\mathbb{E}[e^{\varepsilon x}])^2} \\ &= -\frac{\gamma(1 - \phi)(\phi(\gamma + x) + \gamma + \phi x)}{(\gamma + \phi x)^2(\gamma + x)^2} - \frac{\mathbb{E}[(\varepsilon - m)^2 e^{\varepsilon x}]}{\mathbb{E}[e^{\varepsilon x}]} < 0 \end{aligned}$$

since $\phi < 1$ and $x > -\gamma$. Therefore h is strictly concave on $(-\gamma, 0)$. Substituting $x = 0$, we have $h'(0) = \frac{1 - \phi}{\gamma} > 0$, so $h'(x) > 0$ for all $x \in (-\gamma, 0)$ because $h''(x) < 0$. Therefore h is strictly increasing. \square

Lemma A.1. Let $A = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}$, where $\phi_1, \phi_2 \in \mathbb{R}$. Then the spectral radius of A is less than 1 if and only if $-1 < \phi_2 < 1 - |\phi_1|$.

Proof. Let $\Phi_A(z) = |zI - A| = z^2 - \phi_1 z - \phi_2$ be the characteristic polynomial of A and $D = \phi_1^2 + 4\phi_2$ its discriminant. If $D \geq 0 \iff \phi_2 \geq -\phi_1^2/4$, then A has two real eigenvalues. In this case the spectral radius is less than 1 if and only if $-1 < \phi_1/2 < 1$ and $\Phi_A(\pm 1) > 0$, or $|\phi_1| < 2$ and $-\phi_1^2/4 \leq \phi_2 < 1 - |\phi_1|$. Note that since

$$1 - |\phi_1| + \frac{\phi_1^2}{4} = \frac{1}{4}(2 - |\phi_1|)^2 > 0$$

if $|\phi_1| < 2$, we have $-\phi_1^2/4 < 1 - |\phi_1|$. If $D < 0 \iff \phi_2 < -\phi_1^2/4 \leq 0$, then A has two complex eigenvalues that are conjugate of each other. Letting $\zeta, \bar{\zeta}$ be the roots of Φ_A , we have $-\phi_2 = \zeta\bar{\zeta} = |\zeta|^2$, so the spectral radius is less than 1 if and only if $-\phi_2 < 1 \iff \phi_2 > -1$. Combining the two cases together, the spectral radius of A is less than 1 if and only if $-1 < \phi_2 < 1 - |\phi_1|$. \square

References

- S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics*, 109(3):659–684, 1994. doi:10.2307/2118417.
- George-Marios Angeletos and Laurent-Emmanuel Calvet. Incomplete-market dynamics in a neoclassical production economy. *Journal of Mathematical Economics*, 41(4-5):407–438, August 2005. doi:10.1016/j.jmateco.2004.09.005.
- George-Marios Angeletos and Laurent-Emmanuel Calvet. Idiosyncratic production risk, growth and the business cycle. *Journal of Monetary Economics*, 53:1095–1115, 2006. doi:10.1016/j.jmoneco.2005.05.016.
- Truman F. Bewley. A difficulty with the optimum quantity of money. *Econometrica*, 51(5):1485–1504, September 1983. doi:10.2307/1912286.
- Olivier J. Blanchard. Debt, deficits, and finite horizons. *Journal of Political Economy*, 93(2):223–247, April 1985. doi:10.1086/261297.
- Richard Blundell, Luigi Pistaferri, and Ian Preston. Consumption inequality and partial insurance. *American Economic Review*, 98(5):1887–1921, December 2008. doi:10.1257/aer.98.5.1887.
- Martin Bodenstein. Trade elasticity of substitution and equilibrium dynamics. *Journal of Economic Theory*, 145(3):1033–1059, May 2010. doi:10.1016/j.jet.2010.01.008.
- Craig Burnside. Solving asset pricing models with Gaussian shocks. *Journal of Economic Dynamics and Control*, 22(3):329–340, March 1998. doi:10.1016/S0165-1889(97)00075-4.
- Ricardo J. Caballero. Consumption puzzles and precautionary savings. *Journal of Monetary Economics*, 25(1):113–136, January 1990. doi:10.1016/0304-3932(90)90048-9.

- Ricardo J. Caballero. Earnings uncertainty and aggregate wealth accumulation. *American Economic Review*, 81(4):859–871, September 1991.
- Laurent-Emmanuel Calvet. Incomplete markets and volatility. *Journal of Economic Theory*, 98(2):295–338, June 2001. doi:10.1006/jeth.2000.2720.
- Fabrice Collard and Michel Juillard. Accuracy of stochastic perturbation methods: The case of asset pricing models. *Journal of Economic Dynamics and Control*, 25(6-7):979–999, June 2001. doi:10.1016/S0165-1889(00)00064-6.
- Oliver de Groot. Solving asset pricing models with stochastic volatility. *Journal of Economic Dynamics and Control*, 52:308–321, March 2015. doi:10.1016/j.jedc.2015.01.001.
- Mette Ejrnæs and Martin Browning. The persistent-transitory representation for earnings processes. *Quantitative Economics*, 5(3):555–581, November 2014. doi:10.3982/QE239.
- Leland E. Farmer and Alexis Akira Toda. Discretizing nonlinear, non-Gaussian Markov processes with exact conditional moments. *Quantitative Economics*, 2016. URL <http://ssrn.com/abstract=2585859>. Forthcoming.
- Jordi Galí. Multiple equilibria in a growth model with monopolistic competition. *Economic Theory*, 8(2):251–266, 1996. doi:10.1007/BF01211817.
- Steven Gjerstad. Multiple equilibria in exchange economies with homothetic, nearly identical preferences. 1996.
- Fatih Guvenen. Learning your earning: Are labor income shocks really very persistent? *American Economic Review*, 97(3):687–712, June 2007. doi:10.1257/aer.97.3.687.
- Fatih Guvenen, Serdar Ozkan, and Jae Song. The nature of countercyclical income risk. *Journal of Political Economy*, 122(3):621–660, June 2014. doi:10.1086/675535.
- Thorsten Hens and Andrés Loeffler. Gross substitution in financial markets. *Economics Letters*, 49(1):39–43, July 1995. doi:10.1016/0165-1765(94)00647-K.
- Dmytro Hryshko. Correlated income shocks and excess smoothness of consumption. *Journal of Economic Dynamics and Control*, 48:41–62, November 2014. doi:10.1016/j.jedc.2014.08.022.
- R. Glenn Hubbard, Jonathan Skinner, and Stephen P. Zeldes. Precautionary saving and social insurance. *Journal of Political Economy*, 102(2):360–399, April 1995. doi:10.1086/261987.
- Mark Huggett. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969, September–November 1993. doi:10.1016/0165-1889(93)90024-M.
- Tullio Jappelli and Luigi Pistaferri. The consumption response to income changes. *Annual Review of Economics*, 2:249–506, 2010. doi:10.1146/annurev.economics.050708.142933.

- Timothy J. Kehoe. Multiplicity of equilibria and comparative statics. *Quarterly Journal of Economics*, 100(1):119–147, February 1985. doi:10.2307/1885738.
- Timothy J. Kehoe. Computation and multiplicity of equilibria. In Kenneth J. Arrow and Michael D. Intriligator, editors, *Handbook of Mathematical Economics*, volume 4, chapter 38, pages 2049–2144. North Holland, 1991. doi:10.1016/S1573-4382(05)80013-X.
- Felix Kubler and Karl Schmedders. Competitive equilibria in semi-algebraic economies. *Journal of Economic Theory*, 145(1):301–330, January 2010a. doi:10.1016/j.jet.2009.10.004.
- Felix Kubler and Karl Schmedders. Tackling multiplicity of equilibria with Gröbner bases. *Operations Research*, 58(4):1037–1050, 2010b. doi:10.1287/opre.1100.0819.
- Felix Kubler and Karl Schmedders. Uniqueness of steady states in models with overlapping generations. *Journal of the European Economic Association*, 8(2-3):635–644, April-May 2010c. doi:10.1111/j.1542-4774.2010.tb00533.x.
- Pamela Labadie. Stochastic inflation and the equity premium. *Journal of Monetary Economics*, 24(2):277–298, September 1989. doi:10.1016/0304-3932(89)90007-X.
- Thomas E. MaCurdy. The use of time series processes to model the error structure of earnings in a longitudinal data analysis. *Journal of Econometrics*, 18(1):83–114, January 1982. doi:10.1016/0304-4076(82)90096-3.
- Costas Meghir and Luigi Pistaferri. Income variance dynamics and heterogeneity. *Econometrica*, 72(1):1–32, January 2004. doi:10.1111/j.1468-0262.2004.00476.x.
- Robert A. Moffitt and Peter Gottschalk. Trends in the transitory variance of earnings in the United States. *Economic Journal*, 112(478):C68–C73, March 2002. doi:10.1111/1468-0297.00025.
- Jack Schechtman and Vera L. S. Escudero. Some results on “an income fluctuation problem”. *Journal of Economic Theory*, 16(2):151–166, December 1977. doi:10.1016/0022-0531(77)90003-5.
- Stephanie Schmitt-Grohé and Martín Uribe. Solving dynamic general equilibrium models using a second-order approximation to the policy function. *Journal of Economic Dynamics and Control*, 28(4):755–775, January 2004. doi:10.1016/S0165-1889(03)00043-5.
- Kjetil Storesletten, Chris I. Telmer, and Amir Yaron. Cyclical dynamics in idiosyncratic labor market risk. *Journal of Political Economy*, 112(3):695–717, June 2004. doi:10.1086/383105.
- Alexis Akira Toda and Kieran James Walsh. Edgeworth box economies with multiple equilibria. *Economic Theory Bulletin*, 5(1):65–80, April 2017. doi:10.1007/s40505-016-0102-3.

- Robert H. Topel and Michael P. Ward. Job mobility and the careers of young men. *Quarterly Journal of Economics*, 107(2):439–479, May 1992. doi:10.2307/2118478.
- Efthymios G. Tsionas. Exact solution of asset pricing models with arbitrary shock distributions. *Journal of Economic Dynamics and Control*, 27(5):843–851, March 2003. doi:10.1016/S0165-1889(02)00017-9.
- Neng Wang. Caballero meets Bewley: The permanent-income hypothesis in general equilibrium. *American Economic Review*, 93(3):927–936, June 2003. doi:10.1257/000282803322157179.
- Neng Wang. Precautionary saving and partially observed income. *Journal of Monetary Economics*, 51(8):1645–1681, November 2004. doi:10.1016/j.jmoneco.2004.06.001.
- Neng Wang. An equilibrium model of wealth distribution. *Journal of Monetary Economics*, 54(7):1882–1904, October 2007. doi:10.1016/j.jmoneco.2006.11.005.
- Menahem E. Yaari. Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies*, 32(2):137–150, April 1965. doi:10.2307/2296058.