Zipf's Law: A Microfoundation

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Abstract

Existing explanations of Zipf's law (Pareto exponent approximately equal to 1) in size distributions require strong assumptions on growth rates or the minimum size. I show that Zipf's law naturally arises in general equilibrium when individual units solve a homogeneous problem (*e.g.*, homothetic preferences, constant-returns-to-scale technology), the units enter/exit the economy at a small constant rate, and at least one production factor is in limited supply. My model explains why Zipf's law is empirically observed in the size distributions of cities and firms, which consist of people, but not in other quantities such as wealth, income, or consumption, which all have Pareto exponents well above 1.

Keywords: Gibrat's law, homogeneous problem, power law
JEL codes: D30, D52, D58, L11, R12

1 Introduction

Zipf's law is an empirical regularity that holds in the size distributions of cities and firms, stating that the frequency of observing a unit larger than the cutoff x is approximately inversely proportional to x:

$$P(X > x) \sim x^{-\zeta},$$

where the Pareto (power law) exponent ζ is slightly above 1. This relationship holds regardless of the choice of countries or time periods.[1] To get a sense of

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[1] Although Zipf's law is named after Zipf (1949), its discovery dates back at least to Auerbach (1913). For empirical studies documenting Zipf's law, see Rosen and Resnick (1980), Ioannides and Overman (2003), and Soo (2005) for cities (see Gabaix and Ioannides (2004) for a review and Nitsch (2005) for a meta analysis) and Axtell (2001), Fujiwara et al. (2004), di Giovanni et al. (2011), di Giovanni and Levchenko (2013), and Garicano et al. (2016) for firms, among many others. While most studies on cities use agglomerations and show that Zipf's law holds for the largest a few hundred agglomerations, Rozenfeld et al. (2011) define a "city" as a maximally connected cluster of populated sites (two adjacent sites are connected if the population densities exceed a cutoff value) and show that Zipf's law for population and area holds for a large range.

how the empirical size distribution looks like, Figure 1 shows a log-log plot of employment cutoffs and the number of firms larger than the cutoffs (essentially the ranks) using the 2011 U.S. Census Small Business Administration (SBA) data. Consistent with a power law, the figure shows a straight-line pattern up to small firms with as few as 10 employees. The Pareto exponent estimated by maximum likelihood is $\widehat{\zeta} = 1.0967$ with a standard error of 0.0020. We obtain similar patterns for all years from 1992 to 2011 for which data is available. Figure 2 shows the estimated Pareto exponent over the period 1992–2011, which is slightly above 1 in all years. As Krugman (1996) puts it, "there must be a compelling explanation of the astonishing empirical regularity."

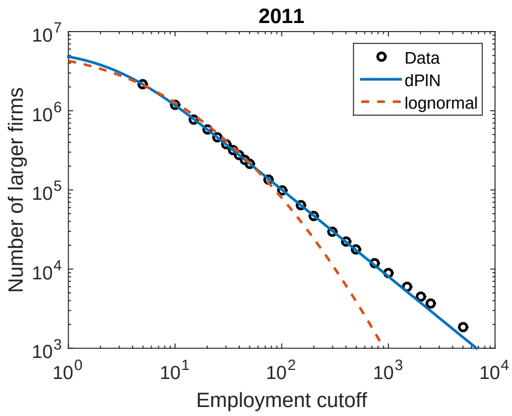

Figure 1: Log-log plot of firm size distribution.

Note: The figure plots employment cutoffs and the number of firms larger than the cutoffs (ranks). dPlN stands for *double Pareto-lognormal*, which is a distribution arising from the theoretical model in the paper. The straight-line pattern is consistent with a power law, with estimated exponent $\widehat{\zeta} = 1.0967$ and standard error 0.0020 using maximum likelihood with binned data (sample size $N = 5{,}684{,}424$; see Appendix D for details). Source: 2011 U.S. Census Small Business Administration data.

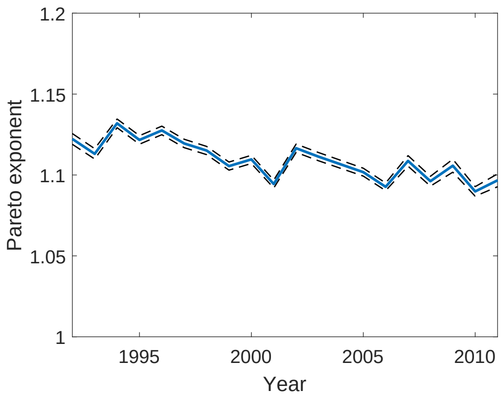

Figure 2: Time series of estimated Pareto exponent.

Note: The figure plots the estimated Pareto exponent $\widehat{\zeta}$ from 1992 to 2011. The two dashed curves indicate the 95% confidence interval. Source: U.S. Census Small Business Administration data.

In addition to its empirical regularity, Zipf's law is important because it may explain aggregate fluctuations from a micro level (Gabaix, 2011) and has distinct welfare implications of entry cost and trade barriers (di Giovanni and Levchenko, 2013). In a seminal paper, Gabaix (1999) has shown that Zipf's law arises when individual units follow Gibrat (1931)'s law of proportional growth and there is some small minimum size (relative to the average size) that the units must meet. His work has generated a large subsequent literature on power laws in economics and finance as well as models that attempt to explain Zipf's law. But where does this condition, which is equivalent to the condition that the expected growth rate of existing units is small in absolute value relative to the variance, come from? Given that Zipf's law is empirically so robust, an explanation of Zipf's law should not depend on a fine-tuning of particular parameters. Instead, there must be a mechanism that delivers the small growth condition endogenously. This paper proposes a new mechanism that endogenously delivers the small growth condition, and hence provides a microfoundation for Zipf's law.

My theory is surprisingly simple, and essentially relies on the following three elements: (i) Gibrat's law of proportional growth, (ii) individual units entering/exiting the economy with small probability ("rare disasters"), and (iii) existence of a production factor that is mobile but in limited supply. Conditions (i) and (ii) have already been known to be sufficient for generating Pareto tails (Reed, 2001), but Zipf's law (Pareto exponent close to 1) holds only in the knife-edge case in which the expected growth rate of units is small in absolute value. My contribution is thus in showing that condition (iii)—the existence of a production factor in limited supply, or to be precise, one production factor is bounded above by an exogenous process—limits aggregate growth, which *in equilibrium* also limits individual growth and delivers Zipf's law. The intuition is as follows. If individual units solve a homogeneous problem (*e.g.*, homothetic preferences, constant-returns-to-scale technology), the size of these units obeys Gibrat's law of proportional growth. But if one of the production factors is in limited supply, the aggregate economy exhibits decreasing returns to scale. Since the economy converges to the steady state (zero aggregate growth), and by accounting the aggregate is the sum of individuals, it follows that the individual growth rate endogenously becomes small. My theory explains why Zipf's law is empirically observed only for cities and firms, but not for other quantities such as wealth, income, or consumption, which all obey power laws but with exponents well above 1.[2] Cities and firms consist of people, which can be thought of as a production factor that is in limited supply (bounded by an exogenous process). On the other hand, there is no obvious exogenous process that bounds wealth, income, or consumption.

To illustrate these points in the simplest possible way, I first construct a stylized model of the population dynamics of cities (villages). In the model, there are a continuum of villages and households. The village authorities produce a single good ("potato") using a constant-returns-to-scale technology and hiring labor. Households migrate across villages freely without any cost. Villages are hit by two types of idiosyncratic shocks—technological shocks and rare disasters ("famine"). When a famine occurs, the potatoes in the village are wiped out, but the village authority receives deliveries of potatoes from other

[2]Reed (2003) and Toda (2012) find that the Pareto exponent for income is around 2. Toda and Walsh (2015) and Toda (2016) document a power law in consumption with an exponent around 4.

villages because they have a mutual insurance. This simple model has all the ingredients sufficient for generating Zipf's law: (i) with multiplicative technological shocks and constant-returns-to-scale technology, we obtain Gibrat's law for individual villages, (ii) famines are reset events and generate a stationary distribution with Pareto tails, and (iii) the inelastic labor supply endogenously forces the expected population growth rate in individual villages to be small in equilibrium, generating Zipf's law.

The intuition for this simple model carries over to more general models. Consider a dynamic general equilibrium model which consists of several agent types, and suppose that we are interested in the size distribution of an economic variable of a particular type (*e.g.*, firm size distribution measured by the number of employees). The main result of this paper, Theorem 3.5, shows that if agents of this type solve a homogeneous problem (*e.g.*, homothetic preferences, constant-returns-to-scale technology, proportional constraints), the agents enter/exit the economy at a constant rate $\eta > 0$, and at least one factor of production is in limited supply, then Zipf's law holds in the stationary equilibrium as $\eta \to 0$. This result holds in a wide variety of models, including those with elastic labor supply, balanced growth, random initial size, multiple types, and discrete time with non-Gaussian shocks.

Because the main theorem is an asymptotic result, the Pareto exponent need not be close to 1 for particular models or parameter configurations. To address the quantitative validity of my theory, I construct a model of entrepreneurship and firm size distribution. The economy is populated by entrepreneur-CEOs and household-workers. Each entrepreneur operates a firm using a constant-returns-to-scale technology and hiring labor, and makes consumption-saving-portfolio-hiring decisions optimally. Entrepreneurs are subject to idiosyncratic investment risk and bankruptcy. Workers supply labor inelastically but make consumption-saving decisions optimally. In this setting under mild conditions I prove that a unique stationary equilibrium exists and characterize the equilibrium in closed-form. I prove that the stationary firm size distribution obeys Zipf's law when the bankruptcy rate is small. I calibrate the model to the U.S. economy and find that the Pareto exponent is close to 1, consistent with Zipf's law.

Given the empirical robustness of Zipf's law, an "explanation" should not depend on a particular calibration of parameters. Hence to show its robustness, I generate random parameter configurations drawn from a uniform distribution with a large support (changing each parameter up to 5-fold independently), and for each case I compute the equilibrium Pareto exponent. For this particular model I find that the 95 percentile of the Pareto exponent is 1.13, so Zipf's law holds even for quite extreme (and unrealistic) parameter configurations, confirming its robustness.

1.1 Related literature

Pareto (1896) discovered that the size distribution of income obeys a power law. The idea of using random growth models[3] to explain power law distri-

[3] In this paper I focus on the random growth model because (i) it is the earliest model to explain power laws, and (ii) almost all existing explanations of Zipf's law rely on this mechanism one way or another. An exception is Geerolf (2016), who studies the production decision within an organization in a static setting. The Pareto exponent is exactly equal to 2 when there are two layers in the organization (*e.g.*, managers and workers). He also shows

butions dates back to Yule (1925), Champernowne (1953), Simon (1955), and Kesten (1973), among others. Because random *proportional* growth (Gibrat's law) alone does not lead to a stationary distribution (one would get a lognormal distribution, whose log variance increases linearly over time), one needs to introduce additional assumptions. Champernowne (1953) introduces a mean-reverting force and obtains the double Pareto distribution; Wold and Whittle (1957) consider random birth and death; Kesten (1973) considers both multiplicative and additive shocks. For reviews of generative mechanisms of the power law, see Mitzenmacher (2004), Gabaix (2009, 2016), and Benhabib and Bisin (2017). Although this early literature on power law used mechanical models (*i.e.*, they lacked optimizing behavior or general equilibrium analysis), more micro-founded models have been explored during the past decade.[4]

Since Zipf's law is a special case of power law (with Pareto exponent close to 1), one needs to introduce further assumptions to explain it. Gabaix (1999) considers the normalized size distribution of cities ("normalized" means dividing the size by the average size) and shows that we obtain Zipf's law if we assume that there is a small minimum size. As discussed in Section 2, this condition is equivalent to small expected growth relative to the variance, or $|g| \ll v^2$. (A similar condition is necessary in models with entry/exit, which Malevergne et al. (2013) call "balance condition.") In general, all existing explanations of Zipf's law require such a fine-tuning of parameters. For example, Simon and Bonini (1958) and Luttmer (2011) consider random growth models of firm size similar to Simon (1955) and show that Zipf's law obtains when the net growth attributed to new firms relative to that of existing firms approaches zero. Córdoba (2008) studies a model of city size distribution and shows that Zipf's law holds when the elasticity of substitution between goods is exactly 1.

Luttmer (2007, 2012) studies general equilibrium models of firms with entry/exit, where the entrant can pay an entry cost to sample at random from the population of incumbent firms. He shows that Zipf's law holds when the entry cost diverges to infinity. The mechanism is again similar since a large entry cost must be compensated by large profits, which imply a large average firm size that arises under small growth relative to the balanced growth path.[5]

Nirei and Aoki (2016) construct a heterogeneous-agent neoclassical growth model that accounts for the Pareto distributions of income and wealth in the upper tail. Because their model features constant-returns-to-scale at the individual level but decreasing returns at the aggregate level (due to the boundedness of labor), according to my theory their model (in Section 4.2) should generate Zipf's law. However, they do not discuss Zipf's law. Aoki and Nirei (2017) construct a neoclassical growth model that can simultaneously explain Zipf's law for the

that Zipf's law obtains as the number of layers tends to infinity.

[4]See, for example, Luttmer (2007), Nirei and Souma (2007), Benhabib et al. (2011, 2015, 2016), Toda (2014), Toda and Walsh (2015), Arkolakis (2016), Gabaix et al. (2016), Nirei and Aoki (2016), Aoki and Nirei (2017), Eisfeldt et al. (2017), and Jones and Kim (2017), among others.

[5]In Luttmer (2007), the equation that determines the Pareto exponent ζ is

$$\frac{\lambda_E}{\lambda_F} = \int_b^\infty V(s) f(s+\delta)\,\mathrm{d}s = \int_0^\infty \frac{1}{r-\kappa}\frac{\xi}{1+\xi}\left(\mathrm{e}^x - 1 - \frac{1-\mathrm{e}^{-\xi x}}{\xi}\right)\zeta^2 x \mathrm{e}^{-\zeta(x+\delta)}\,\mathrm{d}x,$$

which is equivalent to Equation (30) and can be derived by combining (11), (25), and (26). Since $r-\kappa$ and ξ are bounded away from 0 and the integrand grows like $\mathrm{e}^{-(\zeta-1)x}$, making the ratio between entry and fixed costs λ_E/λ_F large forces ζ to be close to 1.

firm size distribution and the evolution of the Pareto exponent in the income distribution. However, as in the existing literature they obtain Zipf's law by assuming a small minimum size.

2 Existing explanations and difficulties

In this section I review the existing explanations of Zipf's law based on random growth models and point out their difficulties.

2.1 Geometric Brownian motion with minimum size

Suppose that the size of individual units (*e.g.*, population of cities, number of employees in firms, etc.) satisfies Gibrat (1931)'s law of proportional growth: the growth rate of units is independent of their sizes.[6] The simplest of all such processes is the geometric Brownian motion (GBM)

$$\mathrm{d}X_t = gX_t\,\mathrm{d}t + vX_t\,\mathrm{d}B_t, \tag{2.1}$$

where X_t is the size of a typical unit,[7] g is the expected growth rate, $v > 0$ is the volatility, and B_t is a standard Brownian motion that is independent across units. As is well known, the geometric Brownian motion leads to the lognormal distribution whose log variance increases linearly over time, and hence does not admit a stationary distribution.

In order to obtain a stationary distribution, a common practice in the literature is to introduce a minimum size $x_{\min} > 0$ below which individual units cannot operate.[8] Mathematically, we are considering the geometric Brownian motion with a reflective barrier at $x_{\min}$. Assuming that the growth rate is negative ($g < 0$), it is well known (see Gabaix (1999) or Appendix A) that the system converges to the unique stationary distribution

$$P(X > x) = \left(\frac{x}{x_{\min}}\right)^{-\zeta}, \tag{2.2}$$

which is a Pareto distribution with minimum size $x_{\min}$ and Pareto exponent

$$\zeta = 1 - \frac{2g}{v^2} > 1. \tag{2.3}$$

Thus we obtain Zipf's law ($\zeta \approx 1$) when the growth rate is small in absolute value relative to the variance: $|g| \ll v^2$. Another way to formulate the condition for Zipf's law is to compare the minimum size $x_{\min}$ to the average size $\bar{x}$. Using the distribution function (2.2), the average size is

$$\bar{x} = \int_{x_{\min}}^{\infty} x\zeta x_{\min}^{\zeta} x^{-\zeta-1}\,\mathrm{d}x = \frac{\zeta}{\zeta - 1}x_{\min} \iff \zeta = \frac{1}{1 - x_{\min}/\bar{x}}. \tag{2.4}$$

[6] See Sutton (1997) for an early review of the empirical literature on Gibrat's law. More recent works include Ioannides and Overman (2003), Eeckhout (2004), and Giesen and Südekum (2011), among others.

[7] X_t is sometimes interpreted as the size of a typical unit relative to the cross-sectional average. In that case, g and v are the expected growth rate and volatility relative to the average, and $x_{\min}$ below is the minimum relative size.

[8] Such assumptions are made in Levy and Solomon (1996), Gabaix (1999), Malcai et al. (1999), Luttmer (2007), Córdoba (2008), and Aoki and Nirei (2017), among others.

Hence Zipf's law is also equivalent to $x_{\min} \ll \bar{x}$: the minimum size is small relative to the average. The intuition is that the minimum size is small relative to the average when the latter is large, which occurs precisely when the expected growth rate g is large, or when it is close to zero since it must be negative.

2.2 Geometric Brownian motion with entry/exit

Next, consider the same the geometric Brownian motion (2.1) but introduce entry and exit. Unlike in the previous example, there is no minimum size but new units constantly enter the economy at rate $\eta > 0$, with initial size x_0, and existing units exit at the same rate η as in the Yaari (1965)-Blanchard (1985) perpetual youth model.[9] It is well known (see Reed (2001) or Appendix A) that regardless of the parameter values, the size distribution of units always has a unique stationary distribution, with a density of the form

$$f(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} x_0^{\alpha} x^{-\alpha-1}, & (x \geq x_0) \\ \frac{\alpha\beta}{\alpha+\beta} x_0^{-\beta} x^{\beta-1}, & (0 < x < x_0) \end{cases} \tag{2.5}$$

which is known as *double Pareto*. The parameters $\alpha, \beta > 0$ are called Pareto (or power law) exponents. Given the parameters g, v, η of the stochastic process, the exponents $\zeta = \alpha, -\beta$ are the solutions to the quadratic equation

$$\frac{v^2}{2}\zeta^2 + \left(g - \frac{v^2}{2}\right)\zeta - \eta = 0. \tag{2.6}$$

Solving (2.6), we obtain the Pareto exponents

$$\alpha, \beta = \frac{1}{2}\left(\sqrt{\left(1 - \frac{2g}{v^2}\right)^2 + \frac{8\eta}{v^2}} \pm \left(1 - \frac{2g}{v^2}\right)\right). \tag{2.7}$$

As is clear from this formula, Zipf's law ($\alpha \approx 1$) arises when $|g|, \eta \ll v^2$, *i.e.*, when the growth and entry/exit rates are small compared to the variance.

2.3 Difficulties

Although the above models are purely mechanical, they underly the mechanism of generating Zipf's law in virtually all papers. Of course, in order to make it an economic model, one needs to provide mechanisms that generate Gibrat's law of proportional growth. However, this is not difficult if we assume that individual units solve a homogeneous problem (*e.g.*, homothetic preferences, constant-returns-to-scale production, proportional constraints).[10] The more difficult part

[9] Wold and Whittle (1957) is one of the earliest examples that shows that random entry/exit (birth/death) can generate Pareto tails. Working in continuous-time is convenient for tractability, though similar results hold in discrete time and in a Markov setting (Beare and Toda, 2016). For cities it may be unreasonable to assume that they exit at a constant rate. However, this assumption is not important because we obtain the exact same result if cities are infinitely lived, new cities are created at rate η, and the total population also grows at rate η. Also it is not important that the average size of cities is constant over time. If there is population growth, we obtain the same conclusion by considering the balanced growth path. See the discussion in Reed (2001) for details.

[10] See, for example, Saito (1998), Krebs (2003), Angeletos (2007), Benhabib et al. (2011, 2016), Toda (2014), and Toda and Walsh (2015), among others.

is to explain why there is a minimum size, and why the growth rate is small. These are the difficulties in existing explanations.

First, in many models a minimum size is often introduced as an ad hoc assumption. While a minimum size may be justified in some cases (*e.g.*, positive integer constraint, fixed cost of operation, borrowing constraints), in the presence of a minimum size, fully optimizing agents will typically behave differently depending on whether they are close to the lower boundary or not. Since Zipf's law is a statement about the upper tail, and large agents are likely not affected much by the lower boundary, it is reasonable to expect that the upper tail of the size distribution is similar in models where (i) agents behave rationally in the presence of an ex ante minimum size,[11] and (ii) agents ignore the minimum size but it is imposed ex post. Therefore the assumption of a minimum size is not really an issue, although characterizing the stationary distribution with fully optimizing agents in the presence of a minimum size is more challenging.

The second issue, which is more problematic, is the condition that the growth rate or the minimum size must be small in absolute value in order to obtain Zipf's law, which is a knife-edge case. Since the growth rate g is an endogenous variable in any fully specified economic model, there is no obvious reason why we should expect it to be close to zero. In order to obtain this condition, one usually needs to pick very particular parameter values.

To summarize, the explanation of Zipf's law remains incomplete until we provide a fully specified economic model with optimizing agents in which (i) there is no ad hoc minimum size, and (ii) the small growth condition emerges endogenously as an equilibrium outcome. I provide such models in the following sections.

3 Homogeneity and limited factor yield Zipf

In this section I show that whenever (i) individual units solve a dynamic optimization problem that is homogeneous in the state variable (size) as well as all control variables, (ii) individual units enter/exit the economy at a constant Poisson rate $\eta > 0$, and (iii) at least one production factor is in limited supply, we obtain Zipf's law in the limit $\eta \to 0$. This result does not depend on the details of the model and is thus robust. To illustrate the general result, as an example I provide a minimal model of population dynamics and city size distribution.

3.1 Example: a simple model of city size distribution

In this section I present a minimal stylized model of population dynamics and city size distribution in order to illustrate the main mechanism that generates Zipf's law. The general case is treated in Section 3.2.

Environment Consider an economy consisting of a continuum of villages and households. The mass of villages and households is normalized to 1 and N, re-

[11]For example, Benhabib et al. (2015) consider a Bewley model with capital income risk and show that the optimal consumption rule is asymptotically linear (*i.e.*, the lower boundary does not matter) as agents become rich. As a result, they show that the stationary wealth distribution exhibits a Pareto upper tail.

spectively. There is a single consumption good, which I call potato (the name does not matter: what matters is that the good can be used both as capital and consumption). For simplicity each household supplies 1 unit of labor inelastically and consumes the entire wage ("hand-to-mouth" behavior). Households migrate across villages freely without any moving costs; therefore in equilibrium, all villages must offer the same competitive wage. Each village authority ("dictator", or "landlord") uses its stock of potatoes and hires labor to produce new potatoes with a constant-returns-to-scale technology.

Each village is subject to two types of idiosyncratic shocks. First, the stock of potatoes is subject to a productivity shock coming from a Brownian motion. Second, each village is occasionally hit by a rare disaster—famine—which arrives at a (small) Poisson rate $\eta > 0$. When a famine hits a village, the entire stock of potatoes perishes. However, there is a mutual insurance agreement across villages: a village hit by a famine receives a delivery of potatoes from other villages and starts over at size $\kappa > 0$ times the aggregate stock of potatoes; this delivery is financed by contributions from other villages proportional to their stock of potatoes.

A stationary equilibrium is defined by a wage ω and size distributions of village population and stock of potatoes such that (i) profit maximization: given the wage and stock of potatoes, each village authority demands labor to maximize profits,[12] (ii) market clearing: for each village, population equals labor demand, and (iii) stationarity: the size distributions are invariant over time.

Population dynamics of individual villages Let ω be the equilibrium wage and x_t be the stock of potatoes in a typical village. Then the resource constraint when there is no famine is

$$\mathrm{d}x_t = (F(x_t, n_t) - \omega n_t)\,\mathrm{d}t - \eta\kappa x_t\,\mathrm{d}t + v x_t\,\mathrm{d}B_t, \tag{3.1}$$

where n_t is the labor input (population of the village in equilibrium), F is the production function (which is homogeneous of degree 1 since it exhibits constant-returns-to-scale), v is volatility, and B_t is a standard Brownian motion. $F(x_t, n_t) - \omega n_t$ is the amount of potatoes the village authority retains after paying the wage. The term $-\eta\kappa x_t$ reflects the delivery of potatoes to a village hit by a famine (in a short period of time Δt, there are $\eta\Delta t$ such villages, and each village gets κx_t, where $\kappa > 0$ is the constant of proportionality). The term $v x_t\,\mathrm{d}B_t$ is the technological shock to the stock of potatoes. The village authority maximizes the profit, so chooses n_t such that

$$n_t = \arg\max_n (F(x_t, n) - \omega n).$$

Let $f(x) = F(x, 1)$.[13] Since by assumption F is homogeneous of degree 1, we have $F(x, n) = n f(x/n)$. By the first-order condition, we obtain

$$\omega = f(y) - y f'(y), \tag{3.2}$$

[12] To keep the analysis as simple as possible, in this model I assume that the village authority maximizes profits point-by-point, without specifying fundamentals on the behavior (*e.g.*, utility function). One way to justify this behavior is to assume that the village authority (dictator) has an additive CRRA utility in the stock of potatoes (*i.e.*, gets utility from looking at potatoes) and the dictator gets replaced whenever a famine occurs.

[13] A typical example is the Cobb-Douglas production function $F(x, n) = A x^\alpha n^{1-\alpha} - \delta x$, so $f(x) = A x^\alpha - \delta x$, where δ is the depreciation rate.

where $y = x_t/n$ is the potato per capita. Hence given the wage ω and the stock of potatoes x_t, the labor demand is $n_t = x_t/y$, where y is determined by (3.2). The profit rate per unit of potato is then

$$\mu = \frac{F(x_t, n) - \omega n}{x} = \frac{1}{y}(f(y) - (f(y) - yf'(y))) = f'(y). \tag{3.3}$$

Substituting the profit (3.3) into the resource constraint (3.1), we obtain

$$\mathrm{d}x_t = (\mu - \eta\kappa)x_t \,\mathrm{d}t + vx_t \,\mathrm{d}B_t. \tag{3.4}$$

Therefore the stock of potatoes in each village evolves according to a geometric Brownian motion until a famine hits. Since $n_t = x_t/y$ is proportional to x_t, the village population n_t also obeys the same geometric Brownian motion (3.4).

Equilibrium To compute the equilibrium, we need to derive the dynamics of the aggregate stock of potatoes, X_t (which is constant in steady state). Consider what happens to the stock of potatoes in each village during a short period of time Δt. If the village does not experience a famine (which occurs with probability $1 - \eta\Delta t$), then by (3.4) the stock of potatoes grows at rate $\mu - \eta\kappa$ on average. If the village is hit by a famine (which occurs with probability $\eta\Delta t$), the potatoes are wiped out, and the village receives a delivery of κX_t from other villages according to the mutual agreement. Hence aggregating the stock of potatoes across villages and using the law of large numbers for the continuum (Uhlig, 1996; Sun, 2006), we obtain

$$\begin{aligned} X + \Delta X &= \underbrace{(1 - \eta\Delta t)(1 + (\mu - \eta\kappa)\Delta t)X}_{\text{Aggregate potatoes of non-famine villages}} + \underbrace{(\eta\Delta t)(\kappa X)}_{\text{Aggregate potatoes of famine villages}} \\ &= (1 + (\mu - \eta)\Delta t)X + \text{higher order terms.} \end{aligned}$$

Subtracting X from both sides and letting $\Delta t \to 0$, we obtain

$$\mathrm{d}X = (\mu - \eta)X \,\mathrm{d}t. \tag{3.5}$$

In steady state, since by definition the aggregate stock of potatoes is constant, we must have $\mathrm{d}X = 0$ and hence

$$\mu = \eta. \tag{3.6}$$

Combining (3.3) and (3.6), the equilibrium potato per capita y is determined by $f'(y) = \eta$. The equilibrium wage is then determined by (3.2). Substituting (3.6) into the equation of motion (3.4) of potatoes in each village (and hence the population), we obtain

$$\mathrm{d}x_t = \eta(1 - \kappa)x_t \,\mathrm{d}t + vx_t \,\mathrm{d}B_t. \tag{3.7}$$

The equation of motion (3.7) is identical to (2.1) with $g = \eta(1 - \kappa)$. Since η is small, we have $|g|, \eta \ll v^2$, so according to the formula for the Pareto exponent (2.7), we can expect that the upper tail exponent ζ is close to 1. In fact, as a special case of Theorem 3.5 below, we can show the bound (see (3.11))

$$1 < \zeta < 1 + \frac{2\eta\kappa}{v^2},$$

so we obtain Zipf's law $\zeta \to 1$ as $\eta \to 0$.

3.2 General theory

Next I consider the general setting.

3.2.1 Individual problem

Consider a dynamic optimization problem with one positive state variable (called "size") denoted by $x > 0$, finitely many control variables denoted by $y \in \mathbb{R}^{d_y}$, and finitely many parameters denoted by $\theta \in \Theta \subset \mathbb{R}^{d_\theta}$. Some parameters may be exogenous (*e.g.*, preference and technology parameters), while others are endogenous (*e.g.*, prices). Furthermore, the parameters may vary over time. Let $\Gamma(x;\theta) \subset \mathbb{R}^{d_y}$ be the constraint set of the control y given the state variable x and parameter θ, and $V(\{x_t, y_t; \theta_t\})$ be the objective function to be maximized. In this paper I introduce the following definition.

Definition 3.1 (Homogeneous problem)**.** The dynamic optimization problem is *homogeneous* if the followings hold:

1. for each parameter $\theta \in \Theta$, the constraint function $\Gamma(\cdot;\theta) : \mathbb{R}_+ \rightrightarrows \mathbb{R}^{d_y}$ is homogeneous of degree 1, so for all $\lambda > 0$ we have
$$y \in \Gamma(x;\theta) \implies \lambda y \in \Gamma(\lambda x;\theta),$$
2. the equation of motion for the state variable is a diffusion with homogeneous coefficients, so
$$\mathrm{d}x_t = g(x_t, y_t; \theta_t)\,\mathrm{d}t + v(x_t, y_t; \theta_t)\,\mathrm{d}B_t, \tag{3.8}$$
where B_t is a standard Brownian motion and g, v are drift and volatility, which are homogeneous of degree 1 in (x, y),
3. the objective function is homothetic, so for all $\lambda > 0$ and feasible $\{x_t, y_t\}_{t\ge 0}$ and $\{x'_t, y'_t\}_{t\ge 0}$, we have
$$V(\{x_t, y_t; \theta_t\}) \ge V(\{x'_t, y'_t; \theta_t\}) \implies V(\{\lambda x_t, \lambda y_t; \theta_t\}) \ge V(\{\lambda x'_t, \lambda y'_t; \theta_t\}).$$

Example 1. A typical example of a homogeneous problem is a Merton (1969)-type optimal consumption-portfolio problem. In this problem the investors maximize the expected utility

$$\mathrm{E}_0 \int_0^\infty \mathrm{e}^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma}\,\mathrm{d}t$$

subject to the budget constraint

$$\mathrm{d}x_t = (r x_t + (\mu - r)s_t - c_t)\,\mathrm{d}t + \sigma s_t\,\mathrm{d}B_t,$$

where x_t is total wealth, s_t is the amount of wealth invested in the risky asset (stock), c_t is consumption, r is the risk-free rate, μ is the expected return on stocks, and σ is volatility. In this case the control variable is $y = (c, s)$ and the parameter is $\theta = (\rho, \gamma, r, \mu, \sigma)$. Since consumption is nonnegative, the constraint set is $y = (c, s) \in \mathbb{R}_+ \times \mathbb{R} = \Gamma(x, \theta)$, which is homogeneous of degree 1. Clearly the objective function is homogeneous of degree $1 - \gamma$ in $\{c_t\}$, and the drift and volatility

$$g(x, y; \theta) = rx + (\mu - r)s - c, \quad v(x, y; \theta) = \sigma s$$

are homogeneous of degree 1 (in fact, linear) in (x, y).

As is well known, the solution to a homogeneous problem scales with the state variable.

Lemma 3.2. *If $\{y_t\}$ solves a homogeneous problem, then there exists a function $\alpha_t : \Theta \to \mathbb{R}^{d_y}$ such that $y_t = \alpha_t(\theta_t)x_t$.*

Proof. By homogeneity, if y is the optimal control given the state $x > 0$ and parameter θ, λy is the optimal control given the state λx and parameter θ. Letting $\lambda = 1/x$, y/x is the optimal control given the state 1 and parameter θ, which we can denote by $\alpha_t(\theta) \in \mathbb{R}^{d_y}$. Therefore $y = \alpha_t(\theta_t)x$. □

Next I study the general equilibrium in which at least one agent type solves a homogeneous problem.

3.2.2 Size distribution in general equilibrium

Consider the class of dynamic general equilibrium models that consist of one or several types of agents and feature only idiosyncratic risks. I define two notions of equilibria.

Definition 3.3. An *aggregate steady state* consists of endogenous parameters and decision rules of all agent types such that (i) agents optimize, (ii) markets clear, and (iii) all endogenous parameters and decision rules are time-invariant. If in addition the cross-sectional distributions of all agent types are time-invariant, the aggregate steady state is called a *stationary equilibrium*.

Suppose that in a dynamic general equilibrium model, a particular agent type solves a homogeneous problem. Since there is only idiosyncratic risks, the Brownian motion in (3.8) is i.i.d. across all agents.

The following lemma shows that if the aggregate supply of at least one positive control variable is bounded, then in a steady state the cross-sectional size distribution has a finite mean.

Lemma 3.4. *Suppose that a dynamic general equilibrium model has an aggregate steady state, and that one agent type solves a homogeneous problem. If the aggregate supply of at least one positive control variable is bounded, then the cross-sectional size distribution of that type has a finite mean. Furthermore, the size of individual units obeys some geometric Brownian motion*

$$\mathrm{d}x_t = g(\theta)x_t\,\mathrm{d}t + v(\theta)x_t\,\mathrm{d}B_t. \tag{3.9}$$

Using Lemmas 3.2 and 3.4, we can prove the main result: homogeneity, limited supply, and a (small) constant rate of Poisson entry/exit yield Zipf's law.

Theorem 3.5 (Zipf's law)**.** *Let everything be as in Lemma 3.4 and suppose that individual units of that particular type enter/exit the economy at a constant Poisson rate $\eta > 0$, and new units are drawn from some initial size distribution $x_0 \sim F(x;\theta,\eta)$ with finite mean. Assume that a stationary equilibrium exists and let $\theta(\eta) \in \Theta$ be all exogenous and endogenous parameters in stationary equilibrium given $\eta > 0$,*

$$\kappa(\eta) = \frac{\int_0^\infty xF(\mathrm{d}x;\theta(\eta),\eta)}{\mathrm{E}[x_t]} > 0$$

be the average initial size relative to the cross-sectional mean, and $v(\eta) := v(\theta(\eta)) > 0$ *be the volatility. Then the followings hold in equilibrium.*

1. *the size of individual units obeys the geometric Brownian motion*

$$\mathrm{d}x_t = \eta(1-\kappa(\eta))x_t \,\mathrm{d}t + v(\eta)x_t \,\mathrm{d}B_t, \tag{3.10}$$

   so $g(\theta(\eta)) = \eta(1-\kappa(\eta))$ *in* (3.9)*,*

2. *the cross-sectional size distribution has a Pareto upper tail with exponent* ζ *that satisfies*

$$1 < \zeta < 1 + \frac{2\eta\kappa(\eta)}{v(\eta)^2}. \tag{3.11}$$

   In particular, if

$$\lim_{\eta\to 0} \frac{\eta\kappa(\eta)}{v(\eta)^2} = 0, \tag{3.12}$$

   then $\zeta \to 1$ *as* $\eta \to 0$*, so we obtain Zipf's law.*

Theorem 3.5 is quite powerful since we obtain Zipf's law regardless of the details of the model ("detail-free"). All we need are that (i) individual units solve a homogeneous problem,[14] so the size variable obeys the geometric Brownian motion, (ii) individual units enter/exit at a constant Poisson rate, so the cross-sectional distribution is double Pareto, and (iii) there is a factor in the economy that is in limited supply, so in equilibrium all aggregate variables remain bounded, which forces the growth rate of GBM to be small in absolute value and makes the Pareto exponent close to 1.

Of course, Theorem 3.5 assumes that a stationary equilibrium exists and the technical condition (3.12) holds. In general, for a given model we need to verify these conditions on a case-by-case basis.

3.3 Robustness

In this section I show that the assumptions of Theorem 3.5 are satisfied in a wide variety of models and that the assumptions can be weakened further.

3.3.1 Elastic labor supply

In the city size example in Section 3.1, households supply labor inelastically. This assumption is inessential, since village authorities still solve a homogeneous problem regardless of whether labor supply is inelastic or not, and therefore the assumptions of Theorem 3.5 hold. Even if households make some labor-leisure choice, the conclusion of Theorem 3.5 remains valid because the total population is bounded and hence so is the total labor supply.

In other models, such as Angeletos and Panousi (2009, 2011), there is a single type of agents (entrepreneur-workers) that operates a constant-returns-to-scale technology while choosing labor supply and demand. In this case the individual problem is not homogeneous (according to Definition 3.1) because labor-leisure choice is bounded. However, after computing the present value of wage and

[14] Clearly, it is not necessary that *all* agent types solve homogeneous problems. All we need is that individual units of a particular type (whose distribution we are interested in) solve a homogeneous problem.

fixing the labor-leisure choice at the optimum, the remaining problem (optimal consumption-portfolio choice) becomes a homogeneous problem. Therefore Zipf's law still holds in this case.

3.3.2 Balanced growth equilibrium

In the city size example in Section 3.1, I assumed that the total population is constant at N, and hence bounded. Boundedness of some factor is sufficient for Zipf's law, but not necessary. Suppose, for example, that population grows (or shrink) at a constant rate ν, so $N_t = N_0 e^{\nu t}$. Since the equation of motion for the aggregate stock (3.5) still holds, we have a balanced growth equilibrium if and only if

$$\mu - \eta = \nu.$$

In this case the growth rate of individual cities relative to the mean is

$$g - \nu = (\mu - \eta\kappa) - \nu = \eta(1 - \kappa),$$

which is exactly the same as in the case with no population growth. Therefore in the balanced growth equilibrium, the mean of the cross-sectional distribution will grow at rate ν, but the upper tail Pareto exponent will still satisfy the bound (3.11). Hence we obtain Zipf's law as $\eta \to 0$.

3.3.3 Coexistence of Zipf and non-Zipf distributions

The simple model in Section 3.1 explains why Zipf's law for the city size distribution is possible. Is this theory consistent with the fact that empirically Zipf and non-Zipf distributions coexist? For example, while Zipf's law empirically holds for cities and firms, the Pareto exponent for household income is around 1.5–3 (Reed, 2003; Toda, 2012) and 4 for consumption (Toda and Walsh, 2015; Toda, 2016).

By slightly modifying the model, we can explain why Zipf's law holds for some size distributions but not for others. Instead of assuming that households are infinitely lived as in the above example, suppose that they enter/exit the labor market at a constant Poisson rate $\delta > 0$. Assume that new households have labor productivity normalized to 1, but the productivity evolves according to a geometric Brownian motion with growth rate $\mu < \delta$[15] and volatility $\sigma > 0$ over the life cycle. Letting H be the cross-sectional average labor productivity in steady state, by accounting we have

$$0 = \frac{\mathrm{d}H}{\mathrm{d}t} = (\mu - \delta)H + \delta N \iff H = \frac{\delta}{\delta - \mu} N > 0.$$

Suppose that a household with labor productivity h supplies h units of labor services inelastically. Since average productivity H is bounded, assuming that migration occurs independent of household income, by Theorem 3.5 the cross-sectional city size distribution obeys Zipf's law as $\eta \to 0$. Since the household labor productivity also satisfies a geometric Brownian motion (but with growth rate μ and volatility σ), the cross-sectional household income and consumption

[15] If $\mu \geq \delta$, the aggregate human capital grows indefinitely, and we need to consider the balanced growth equilibrium.

distributions will be double Pareto. By the discussion in Section 2.2, the upper tail exponent $\alpha > 0$ satisfies

$$\frac{\sigma^2}{2}\alpha^2 + \left(\mu - \frac{\sigma^2}{2}\right)\alpha - \delta = 0, \tag{3.13}$$

which corresponds to (2.6). However, since α is solely determined by household characteristics (μ, σ, δ), it need not be close to 1. To see this, substituting $\alpha \approx 1$ into (3.13), a necessary condition for Zipf's law is $\mu \approx \delta$. However, there is no reason to expect that the growth rate of individual labor productivity (μ) is close to the exit rate from the labor market (δ).

As a numerical illustration, Deaton and Paxson (1994, Table 1) report that within cohorts, the cross-sectional variance of household log consumption increases linearly over time (which is consistent with a geometric Brownian motion for consumption), and at a rate 0.0069 per annum in U.S. Toda and Walsh (2015) find that the entire cross-sectional distribution of household consumption has a Pareto exponent around 3–4. Hence setting $\mu = 0$ (cohort effects are controlled), $\sigma^2 = 0.0069$, and $\alpha = 3, 4$ in (3.13), the implied Poisson rate is $\delta = 0.0207, 0.0414$ (average $1/\delta = 48.3, 24.1$ years in the labor force), which is reasonable since typical households participate in the labor market for about 30–40 years. Thus a Zipf's law for firm size is entirely consistent with non-Zipf (but power law) distributions in income and consumption.

3.3.4 Random initial size

When the initial size of new units is constant, by the discussion in Section 2.2 and Theorem 3.5, the cross-sectional distribution is exactly double Pareto. Since the double Pareto distribution has a kink at the mode, it is unlikely to be observed in the data. Reed (2002) and Giesen et al. (2010) suggest that the entire size distribution of cities is closer to the *double Pareto-lognormal* (dPlN) distribution, which has two Pareto tails with a lognormal body (Reed, 2003). It is straightforward to obtain dPlN in my model: instead of assuming that the initial size after the reset event is constant, if the initial size distribution is lognormal, we obtain dPlN. Therefore my model can explain simultaneously why the size distribution of cities is close to dPlN and obeys Zipf's law. More generally, as long as the initial size distribution is thin-tailed, the initial size does not affect the upper tail of the cross-sectional distribution since the latter is governed by the distribution of relative size (*i.e.*, size divided by initial size), which is fat-tailed.

3.3.5 Multiple types

In the empirical literature on firm sizes, it is well known that Gibrat's law of proportional growth does not quite hold: small firms tend to grow faster but also exit at a higher rate (Mansfield, 1962; Evans, 1987a,b; Hall, 1987; Hart and Oulton, 1996). My theory is not necessarily inconsistent with these empirical facts. Suppose, for instance, that firms consist of several types, indexed by $j = 1, \ldots, J$. Suppose that all firm types solve (type-specific) homogeneous problems, and hence by Lemma 3.2, in a stationary equilibrium the size of type j firms evolve according to a geometric Brownian motion with growth rate g_j and volatility $v_j > 0$. Suppose also that type j firms either go bankrupt (exit from the economy) or transition to a different type at rate $\eta_j > 0$. Letting $\kappa_j > 0$ be the average initial size of new type j firms relative to the average existing type j firms, it follows from Theorem 3.5 that the cross-sectional size distribution of type j firms has a Pareto exponent ζ_j that satisfies

$$1 < \zeta_j < 1 + \frac{2\eta_j \kappa_j}{v_j^2}.$$

The entire cross-sectional distribution is some mixture of each component. Since tails are fatter the smaller the Pareto exponent is, the mixture of several distributions with Pareto upper tails has a Pareto tail with exponent equal to the minimum among its mixture components. Therefore the entire cross-sectional firm size distribution has a Pareto exponent ζ that satisfies

$$1 < \zeta = \min_j \zeta_j < 1 + \min_j \frac{2\eta_j \kappa_j}{v_j^2}.$$

Hence Zipf's law holds if $\eta_j \kappa_j / v_j^2$ is small for *at least one* type j.

Note that in this model the cross-sectional distributions are distinct across types. Hence if the firm type is imperfectly observed to the econometrician, the probability that a firm is of a particular type conditional on its size will generally depend on the size. The empirical fact that small firms tend to grow and exit faster need not be a violation of Gibrat's law but simply because firm types are imperfectly observed: a firm type that grows and exits fast may just happen to have a small average size.

3.3.6 Discrete-time model

So far I have considered a continuous-time model for tractability, but similar results obtain in a discrete-time model. As in Section 3.2, consider a dynamic general equilibrium model consisting of several agent types and featuring only idiosyncratic risk. We can define a homogeneous problem in a similar way to Definition 3.1: the only difference is that the equation of motion (3.8) is replaced by

$$x_{t+1} = G_{t+1}(x_t, y_t; \theta_t), \tag{3.14}$$

where $x_t > 0$ is the state variable, y_t is the control variable, θ_t is the parameter, and $G_{t+1}(x, y; \theta)$ is a positive random variable that is homogeneous of degree 1 in (x, y) and i.i.d. across agents and time, fixing x, y, θ.

By the same argument as in Lemma 3.4, in an aggregate steady state the equation of motion (3.14) becomes

$$x_{t+1} = G_{t+1}(\theta) x_t,$$

so the size of individual units grows at gross growth rate $G_t := G_t(\theta)$ between time $t - 1$ and t. To obtain a stationary distribution, assume that individual units enter/exit the economy with probability $0 < p < 1$ per period. The following theorem shows that under weak assumptions, the cross-sectional size distribution has Pareto upper tails.

Theorem 3.6 (Beare and Toda, 2016)**.** *Suppose that (i) $P(G > 1) > 0$, so existing units grow with positive probability, and (ii) there exists $\bar{s} > 0$ such that*

$$\frac{1}{1-p} < \mathrm{E}[G^{\bar{s}}] < \infty.$$

Then there exists a unique $\zeta \in (0, \bar{s})$ *such that*

$$(1-p)\,\mathrm{E}[G^{\zeta}] = 1. \tag{3.15}$$

In this case, the cross-sectional size distribution has a Pareto upper tail with exponent ζ*.*

A similar result holds for general (non-i.i.d.) Markov processes. Note that when G is lognormal, the condition (3.15) becomes equivalent to (2.6). To see this, let $\log G \sim N(g - v^2/2, v^2)$ and $p = 1 - \mathrm{e}^{-\eta}$, where g is expected growth rate, v is volatility, and η is the Poisson entry/exit rate per unit of time. Then (3.15) becomes

$$1 = (1-p)\,\mathrm{E}[G^{\zeta}] = \mathrm{e}^{-\eta}\mathrm{e}^{(g-v^2/2)\zeta + v^2\zeta^2/2} \iff \frac{v^2}{2}\zeta^2 + \left(g - \frac{v^2}{2}\right)\zeta - \eta = 0,$$

which is exactly (2.6).

In a general equilibrium model, the distribution of the growth rate G depends on exogenous parameters, and so does the Pareto exponent. Hence let $G^{(p)}, \zeta^{(p)}$ be growth rate and the Pareto exponent given the entry/exit rate p, fixing all other parameters. The following theorem shows that under additional assumptions, we obtain Zipf's law as $p \to 0$. Define the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} 0, & (0 \le x < 1) \\ x \log x - x + 1. & (x \ge 1) \end{cases}$$

Since $\phi(1) = 0$ and for $x > 1$ we have $\phi'(x) = \log x > 0$ and $\phi''(x) = 1/x > 0$, ϕ is continuous (in fact, differentiable), increasing, convex, and $\phi(x) > 0$ for $x > 1$.

Theorem 3.7. *Let everything be as in Theorem 3.6. Suppose that* $\mathrm{E}[G^{(p)}] < \frac{1}{1-p}$*, so the cross-sectional distribution has a finite mean.*[16] *Let* $\zeta^{(p)}$ *be the Pareto exponent determined by* (3.15)*. Then*

$$1 < \zeta^{(p)} < 1 + \frac{\frac{1}{1-p} - \mathrm{E}[G^{(p)}]}{\mathrm{E}[\phi(G^{(p)})]}. \tag{3.16}$$

In particular, if

$$\limsup_{p \to 0} \frac{\frac{1}{1-p} - \mathrm{E}[G^{(p)}]}{\mathrm{E}[\phi(G^{(p)})]} = 0,$$

*(*e.g., $\lim_{p\to 0} \mathrm{E}[G^{(p)}] = 1$ *and* $\liminf_{p \to 0} \mathrm{E}[\phi(G^{(p)})] > 0$*) then* $\lim_{p\to 0} \zeta^{(p)} = 1$*, so Zipf's law holds as* $p \to 0$*.*

4 A model of firm size distribution

Because Theorem 3.5 is an asymptotic result, the Pareto exponent need not be close to 1 for particular models or parameter configurations. To address the quantitative validity of my theory, in this section I construct a model of entrepreneurship and firm size distribution. The model builds on the continuous-time version of Angeletos (2007).

[16] Since existing units grow at rate $\mathrm{E}[G^{(p)}]$ on average and they remain in the economy with probability $1-p$, the growth rate of the economy is at least $(1-p)\,\mathrm{E}[G^{(p)}]$ (ignoring entry). Therefore $\mathrm{E}[G^{(p)}] < \frac{1}{1-p}$ is necessary for the cross-sectional distribution to have a finite mean.

4.1 Environment

Consider an economy populated by two types of agents, household-workers and entrepreneur-CEOs. There are a continuum of both types, and entrepreneurs and workers have mass 1 and N, respectively. There is a single consumption good produced by the firms operated by the entrepreneurs, which can also be used as capital.

Households are infinitely lived and supply 1 unit of labor inelastically in a perfectly competitive labor market. They are infinitely risk averse, so they only borrow or lend at the market risk-free rate up to the natural borrowing limit and make consumption-saving decisions optimally.

Entrepreneurs enter the economy and exit (go bankrupt) at Poisson rate $\eta > 0$. When an entrepreneur goes bust, her capital is wiped out and the firm disappears. Each new entrepreneur enters the economy with one "idea". Upon entry, she converts her "idea" to physical capital one-for-one[17] and starts to operate a constant-returns-to-scale technology with idiosyncratic investment risk. Entrepreneurs use their own physical capital and hire labor in a competitive market to carry out production. Markets are incomplete, so entrepreneurs may only invest in their own firms but can borrow or lend at the market risk-free rate.

A stationary equilibrium is defined by a wage ω, risk-free rate r, aggregate capital stock K, households' risk-free asset position X, households' consumption choice, entrepreneur's consumption-saving-portfolio-hiring choice, and size distributions of firms' capital and employment such that (i) households make optimal consumption-saving choice and entrepreneurs make optimal consumption-portfolio-saving-hiring choice, (ii) markets for labor and risk-free asset clear, and (iii) all aggregate variables and size distributions are invariant over time.

4.2 Individual decisions

Workers The utility function of a worker is

$$U_t = \int_0^\infty \mathrm{e}^{-\rho s} \frac{c_{t+s}^{1-1/\varepsilon}}{1-1/\varepsilon} \, \mathrm{d}s,$$

where $\rho > 0$ is the discount rate and $\varepsilon > 0$ is the elasticity of intertemporal substitution. Since workers hold only the risk-free asset, the budget constraint is

$$\mathrm{d}x_t = (rx_t + \omega_t - c_t) \, \mathrm{d}t,$$

where x_t is the financial wealth (which is entirely invested in the risk-free asset) and $\omega_t = \omega$ is the (constant) wage. Letting

$$h_t = \int_0^\infty \mathrm{e}^{-rs} \omega_{t+s} \, \mathrm{d}s = \frac{\omega}{r}$$

be the human wealth (present discounted value of future wages) and $w_t = x_t + h_t$ be the effective total wealth, we have

$$\mathrm{d}w_t = (rw_t - c_t) \, \mathrm{d}t. \tag{4.1}$$

[17] Since capital is wiped out when an entrepreneur goes bankrupt and entrepreneurs enter with one unit of capital, it is more appropriate to interpret capital as organization capital.

The problem thus reduces to a standard Merton (1969, 1971)-type optimal consumption-saving problem. A solution exists if and only if $\rho\varepsilon + (1-\varepsilon)r > 0$, in which case the optimal consumption rule is

$$c = (\rho\varepsilon + (1-\varepsilon)r)w = (\rho\varepsilon + (1-\varepsilon)r)(x + \omega/r). \tag{4.2}$$

Entrepreneurs Entrepreneurs have Epstein-Zin preferences with discount rate ρ, relative risk aversion γ, and elasticity of intertemporal substitution ε.

Let k_t be the physical capital, b_t be the risk-free asset position, and $x_t = k_t + b_t$ be the financial wealth (net worth) of a typical entrepreneur. The budget constraint is

$$\mathrm{d}x_t = (F(k_t, n_t) - \omega n_t + (r+\eta)b_t - c_t)\,\mathrm{d}t + \sigma k_t\,\mathrm{d}B_t, \tag{4.3}$$

where n_t is the labor input, c_t is consumption, F is a constant-returns-to-scale production function net of capital depreciation, $\sigma > 0$ is the volatility of the idiosyncratic shock, and B_t is a standard Brownian motion that is independent across entrepreneurs. Note that the effective risk-free rate faced by entrepreneurs is not r, but $r + \eta$, reflecting the fact that they go bankrupt at Poisson rate $\eta > 0$ and hence are charged an insurance premium $\eta > 0$ on their borrowing (they get annuities at the same rate if they are lending). η can also be interpreted as the spread of corporate bonds over the risk-free asset.

Because labor appears only in the budget constraint and can be chosen freely, letting $f(k) = F(k,1)$, as in (3.2) the capital-labor ratio $y = k_t/n_t$ satisfies $\omega = f(y) - yf'(y)$. The labor demand is $n_t = k_t/y$, and as in (3.3) the profit rate per unit of capital is $\mu = f'(y)$. Substituting into the budget constraint (4.3), we obtain

$$\mathrm{d}x_t = (r_e + (\mu - r_e)\theta - m)x_t\,\mathrm{d}t + \sigma\theta x_t\,\mathrm{d}B_t, \tag{4.4}$$

where $r_e = r+\eta$ is the effective risk-free rate faced by entrepreneurs, $\theta = k_t/x_t$ is the leverage (the fraction of wealth invested in the physical capital, so $k_t = \theta x_t$ and $b_t = (1-\theta)x_t$), and $m = c_t/x_t$ is the propensity to consume out of wealth. Therefore this problem also becomes a Merton (1971)-type optimal consumption-saving-portfolio problem. According to Svensson (1989), the solution for the case with Epstein-Zin utility is

$$\theta = \frac{\mu - r_e}{\gamma\sigma^2}, \tag{4.5a}$$

$$\begin{aligned} m &= (\rho+\eta)\varepsilon + (1-\varepsilon)\left(r_e + (\mu - r_e)\theta - \frac{1}{2}\gamma\sigma^2\theta^2\right) \\ &= (\rho+\eta)\varepsilon + (1-\varepsilon)\left(r_e + \frac{(\mu - r_e)^2}{2\gamma\sigma^2}\right), \end{aligned} \tag{4.5b}$$

provided that these θ, m are positive. Substituting these rules into the budget constraint (4.4), we obtain

$$\mathrm{d}x_t = g x_t\,\mathrm{d}t + v x_t\,\mathrm{d}B_t, \tag{4.6}$$

where the drift g and volatility v are given by

$$g = (r-\rho)\varepsilon + (1+\varepsilon)\frac{(\mu - r_e)^2}{2\gamma\sigma^2}, \tag{4.7a}$$

$$v = \sigma\theta = \frac{\mu - r_e}{\gamma\sigma}. \tag{4.7b}$$

4.3 Equilibrium

Next I characterize the equilibrium. So far I have implicitly assumed that the discount rate ρ and EIS ε are common across agent types, but this is not necessary. Hence let ρ_W, ε_W be the parameter values for the workers, and let the symbols without subscripts be those of the entrepreneurs. Throughout the rest of the paper I assume that the production function $f(x) = F(x, 1)$ satisfies the usual conditions $f(0) = 0$, $f' > 0$, $f'' < 0$, $f'(0) = \infty$, and $f'(\infty) \leq 0$.

Define $0 < y_0 < y_1 < y_2$ by

$$f'(y_0) = \rho_W + \eta + \gamma\sigma^2, \quad f'(y_1) = \rho_W + \eta, \quad f'(y_2) = \eta, \tag{4.8}$$

which uniquely exist by the Inada condition.

Depending on the discount rate of workers, in equilibrium workers may consume a positive amount or zero.[18] The following theorem characterizes the equilibrium.

Theorem 4.1. *A stationary equilibrium exists if and only if*

$$\left(1 - \frac{1}{y_2 N}\right)\eta > -\rho\varepsilon. \tag{4.9}$$

The equilibrium falls into exactly one of the following two categories.

1. *If*

$$\left(1 - \frac{1}{y_1 N}\right)\eta > (\rho_W - \rho)\varepsilon, \tag{4.10}$$

then the equilibrium is unique, the risk-free rate equals the discount rate of workers: $r = \rho_W$, and the capital-labor ratio $y = K/N$ is the unique solution in $(0, y_1)$ to

$$\left(1 - \frac{1}{yN}\right)\eta = (r - \rho)\varepsilon + (1 + \varepsilon)\frac{(f'(y) - r - \eta)^2}{2\gamma\sigma^2}. \tag{4.11}$$

In equilibrium workers consume a positive amount.

2. *If (4.10) fails, then the equilibrium capital-labor ratio y and risk-free rate r satisfy (4.11) and*

$$\frac{r}{r + f(y)/y - f'(y)} = \frac{f'(y) - r - \eta}{\gamma\sigma^2}. \tag{4.12}$$

In equilibrium workers consume zero. Furthermore, $y_0 < y < y_2$ and $0 < r < \rho_W$.

[18]Some readers may find it disturbing that in equilibrium workers consume a positive amount or zero depending on whether they are more or less patient than the entrepreneurs. In particular, if we assume positive consumption ($\rho_W < \rho$) and EIS is the same for the two types and is less than 1, then comparing the optimal consumption rules of workers (4.2) and entrepreneurs (4.5b), it follows that entrepreneurs (the rich) have a lower propensity to save. However, the empirical literature suggests that the rich save more (Lawrance, 1991; Dynan et al., 2004; Hurst and Lusardi, 2004). This parameter dependency arises only because I assume that workers are infinitely lived. If we consider a model in which workers also enter/exit the labor market at a constant rate, in equilibrium they consume a positive amount regardless of the parameter values. See Theorem C.4 in Appendix C for details.

In either case, the net worth x_t of individual entrepreneurs evolves according to the geometric Brownian motion (3.10)*, where $\kappa(\eta) = \frac{1}{K} = \frac{1}{yN}$ is the ratio between the initial and the steady state capital and $v(\eta) = \frac{f'(y)-r-\eta}{\gamma\sigma} > 0$ is volatility.*

It immediately follows that an equilibrium exists if η is sufficiently small.

Corollary 4.2. *An equilibrium exists if η is sufficiently small. If $\rho_W < \rho$ ($\rho_W \geq \rho$), then in equilibrium workers consume a positive (zero) amount.*

Proof. Since $0 < (f')^{-1}(\rho_W + \eta) \leq y_1 < y_2$, it follows that y_1, y_2 are bounded away from 0 as $\eta \to 0$. Therefore the left-hand sides of (4.9) and (4.10) converge to 0 as $\eta \to 0$. Since $\rho\varepsilon > 0$, for small enough $\eta > 0$ (4.9) holds, so by Theorem 4.1 a stationary equilibrium exists. If $\rho_W < \rho$, then for small enough $\eta > 0$ (4.10) holds, so in equilibrium workers consume a positive amount. Otherwise ($\rho_W \geq \rho$), workers consume zero. □

Since this model satisfies the assumptions of Theorem 3.5, the upper tail Pareto exponent ζ satisfies the bound (3.11). However, since κ, v are endogenous, it is not immediately clear whether Zipf's law holds as $\eta \to 0$. Nevertheless, we can show that the technical condition (3.12) holds, and so does Zipf's law.

Theorem 4.3 (Zipf's law)**.** *As $\eta \to 0$, we obtain Zipf's law $\zeta \to 1$.*

Theorem 4.3 is an asymptotic result, and hence for any given parameters the upper tail Pareto exponent need not be close to 1, although the bound (3.11) is always true. Whether ζ is close to 1 or not is therefore a quantitative question, which I address in the numerical example below.

4.4 Numerical example

In this section I compute a numerical example of the model of firm size distribution. For the production function, I assume the Cobb-Douglas form $F(k,n) = Ak^\alpha n^{1-\alpha} - \delta k$, where A is a constant (normalized to $A = 1$), α is the capital share, and δ is the capital depreciation rate.

4.4.1 Calibration

The model is completely specified by the parameters $(\rho_W, \rho, \gamma, \varepsilon, \alpha, \delta, \sigma, \eta, N)$.[19] I calibrate the model at the annual frequency. Following Angeletos (2007), I set $\rho = 0.04$, $\varepsilon = 1$, $\alpha = 0.36$, $\delta = 0.08$, and $\sigma = 0.2$, which are all relatively standard values. Since in steady state the risk-free rate r equals the discount rate of the workers ρ_W when they have positive consumption, I set $\rho_W = 0.01$ so that the risk-free rate is 1%, which is about the historical value in U.S. For N, which is the average number of workers per firm, according to 2011 *U.S. Census Small Business Administration* (SBA) data,[20] 5,684,424 firms employed 113,425,965 workers, which implies an average of 19.95 employees per firm. Therefore I set $N = 20$.

[19] Note that the elasticity of intertemporal substitution for the workers, ε_W, is irrelevant for the steady state, so there is no need to specify it.

[20] `https://www.sba.gov/advocacy/firm-size-data`

The parameters that may be controversial are the relative risk aversion γ and the bankruptcy rate η. Based on SBA data for 1988–2006, Luttmer (2010) reports that the average exit rate is 10.4% per annum for firms with fewer than 20 employees and 2.5% for firms with 500 or more employees. If we take the model literally, η is also the spread of (defaultable) corporate bond over the risk-free asset. Based on a monthly 1990–2008 sample of 899 publicly traded non-financial firms (mostly large firms) covered by the *Center for Research in Security Prices* (CRSP), Gilchrist et al. (2009) find that the mean spread of corporate bonds is 192 basis points (1.92%), which is comparable to the exit rate of large firms (2.5%). Since I am interested in the upper tail behavior (large firms), I set $\eta = 0.025$ or 2.5% spread, which implies an average lifespan of $1/\eta = 40$ years. However, since by Theorem 4.3 Zipf's law obtains when η is small, it is interesting to know the Pareto exponent under larger values of η, for which the bound (3.11) may not be so informative. Therefore I also consider the cases $\eta = 0.05$ (5% spread or 20 years lifespan) and $\eta = 0.1$ (10% spread or 10 years lifespan). One can think of the case $\eta = 0.025$ as a CEO operating a blue-chip firm, and the case $\eta = 0.05, 0.1$ as a young entrepreneur operating a start-up company.

For the relative risk aversion, it is reasonable to assume that the rich CEOs of large firms are not so risk averse, so I set $\gamma = 1$.[21] As a robustness check, I also consider the cases $\gamma = 0.5, 2$.

4.4.2 Results

By Theorem 4.1, computing the equilibrium with positive consumption reduces to solving a single nonlinear equation (4.11). If the existence condition (4.10) fails, we need to look for an equilibrium with zero consumption, in which case we need to solve a system of two nonlinear equations (4.11) and (4.12). Table 1 shows the results, which are all equilibria with positive consumption. The private equity premium, leverage (fraction of own physical capital to entrepreneur net worth), and volatility are all reasonable numbers, roughly in line with U.S. stock returns. In each case, the upper tail Pareto exponent ζ is close to 1, in agreement with Zipf's law.

As we make the environment riskier (larger γ or η), the private equity premium goes up, the capital-labor ratio goes down, which also suppresses the wage. However, the mechanism is very different depending on whether we increase risk aversion γ or the bankruptcy rate η. When γ increases, the entrepreneurs become less willing to invest capital, so they leverage less (portfolio effect). Since there is less investment in the high return capital, the aggregate capital goes down. On the other hand, when η increases, aggregate capital goes down just because there is more bankruptcy and hence destruction of capital (resource effect). Since capital is more scarce, the risk premium goes up, and entrepreneurs leverage more to take advantage.

It is not surprising that the upper tail Pareto exponent ζ is close to 1 regardless of the parameter specification. The reason is that, according to (3.11), we always have the bound

$$1 < \zeta < 1 + \frac{2\eta\kappa}{v^2}.$$

[21] Aoki and Nirei (2017) also assume $\gamma = 1$ (log utility), but the reason is for tractability for solving the entire transitional dynamics.

Table 1: Parameters and endogenous variables in steady state.

Quantity	Symbol	Values				
Risk aversion	γ	1	0.5	2	1	1
Bankruptcy rate (%)	η	2.5	2.5	2.5	5	10
Capital-labor ratio	y	3.49	4.01	2.93	2.58	1.65
Wage	ω	1.004	1.055	0.942	0.900	0.767
Private premium (%)	$\mu - r_e$	4.68	3.31	6.61	5.62	7.13
Equity premium (%)	$\mu - r$	7.18	5.81	9.11	10.62	17.13
Leverage	θ	1.17	1.65	0.83	1.41	1.78
Volatility (%)	v	23.4	33.1	16.5	28.1	35.6
Pareto exponent	ζ	1.007	1.004	1.011	1.011	1.019

Note: the table shows the values of endogenous variables in steady state. The capital-labor ratio is $y = K/N$, where K is the aggregate capital. The private premium is the expected return on capital in excess of the effective risk-free rate faced by entrepreneurs, $\mu - r_e$, where $\mu = f'(y)$ and $r_e = r + \eta = \rho_W + \eta$ is the effective risk-free rate (true risk-free rate plus spread). The equity premium is the expected return on capital in excess of the risk-free rate $r = \rho_W$ conditional on survival. The leverage $\theta = \frac{\mu - r_e}{\gamma\sigma^2}$ is the ratio between entrepreneur's own physical capital to net worth. $v = \sigma\theta$ is the volatility of entrepreneur's net worth (which is also the market capitalization of the firm). ζ is the upper tail Pareto exponent computed as in Theorem 3.5.

As a rough estimate, the bankruptcy rate η has order of magnitude about 10^{-1} or 10^{-2} and the volatility v has order of magnitude about 10^{-1}. Hence the upper bound of ζ is $1 + \frac{2\eta\kappa}{v^2} \approx 1 + \kappa$. Since κ is the ratio of the initial capital of new firms to that of the average firm, it is reasonable to expect that κ is quite small. Therefore ζ must be close to 1.

4.4.3 Sensitivity analysis

How robust is Zipf's law? In this section, I conduct two robustness checks.

First, I fix the parameter values $(\rho_W, \rho, \gamma, \varepsilon, \alpha, \delta, \sigma, \eta, N)$ at the baseline specification and vary one parameter at a time up to ten-fold increase or decrease. (For the capital share α, I consider all values in $(0, 1)$.) For example, since at the baseline we have $\gamma = 1$, I consider $\gamma \in [0.1, 10]$. Figure 3 shows the results. We can see that in all cases the Pareto exponent ζ is slightly above 1 regardless of the parameter values (which can be quite extreme), and in most cases below 1.1, consistent with Zipf's law.

In the second robustness check, I generate 10,000 random parameter configurations and compute the Pareto exponent for each simulation. For this experiment, I consider up to five-fold changes in the parameters, so in each simulation a parameter is 5^U times the baseline value, where U is uniformly drawn from $[-1, 1]$ independently across all parameters and simulations. (For the capital share α, it is uniformly drawn from $[0.1\alpha, 1.9\alpha]$.)

Figure 4 shows the histogram of the Pareto exponent ζ in the range $[1, 1.1]$. The mean, median, and the 95% percentile are 1.0312, 1.0089, and 1.1313, respectively. Again Zipf's law is quite robust.

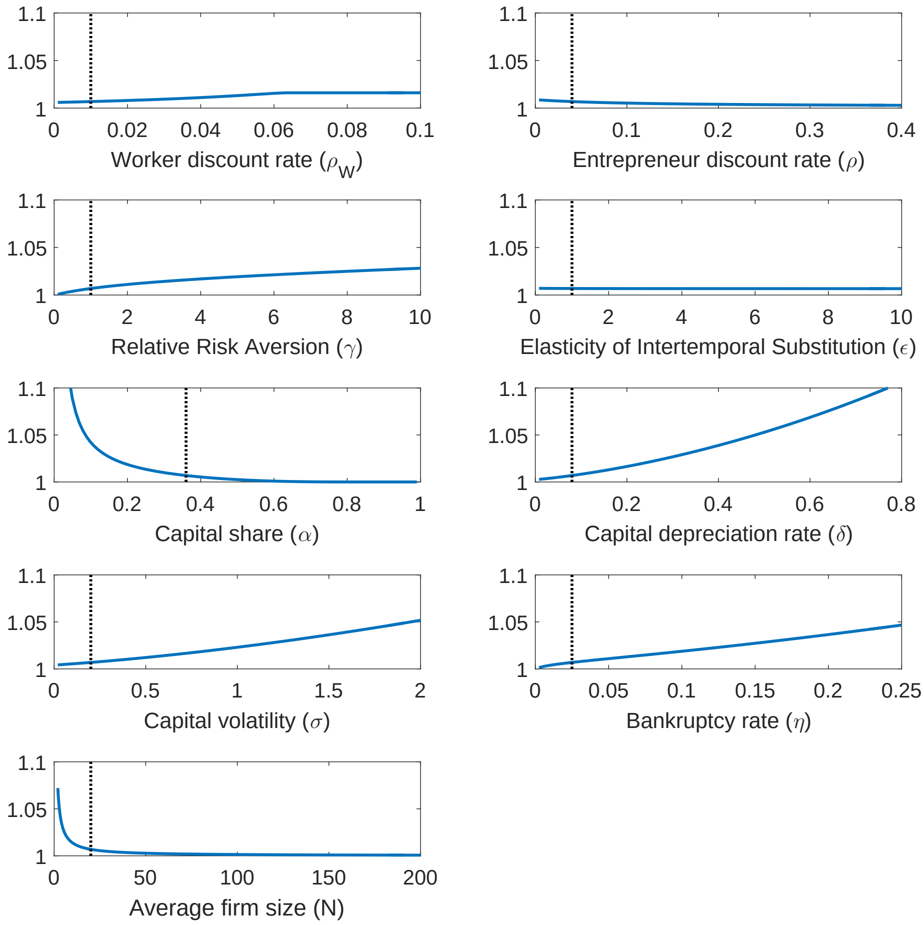

Figure 3: Sensitivity of Pareto exponent ζ with respect to model parameters.

Note: The figure plots the upper tail Pareto exponent ζ computed as in Theorem 3.5. The dotted vertical lines indicate the baseline parameter values.

5 Concluding remarks

This paper shows that Zipf's law (Pareto exponent slightly above 1) can be explained by embedding the standard random growth model into a general equilibrium model and introducing a factor of production that is mobile but in limited supply. Unlike existing explanations of Zipf's law, my theory does not require a fine-tuning of parameters.

Although my paper is theoretical, there are several anecdotal evidences that support my theory. First, Zipf's law is known to empirically hold for cities and firms, but not for other quantities such as wealth, income, and consumption, which all obey power laws but with Pareto exponents well above 1. What is special about cities and firms is that they consist of people, which can be thought of a production factor that is mobile but in limited supply. Second, using historical data, Dittmar (2011) documents that Zipf's law for cities in Europe emerged only after 1500. According to Dittmar, land entered city production as a quasi-fixed factor until 1500, but developments in trade, agricultural productivity, and

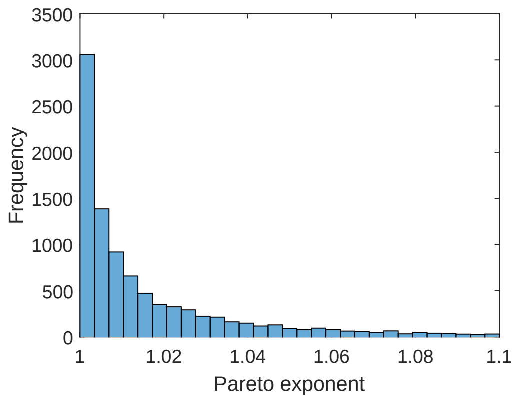

Figure 4: Histogram of Pareto exponent ζ with random parameter configurations.

knowledge-based activities relaxed this constraint thereafter. Since my theory requires that a factor of production is in limited supply but mobile, and land is clearly immobile, the failure of Zipf's law before 1500 is entirely consistent with my theory.

A Fokker-Planck equation

In this appendix, I explain the *Fokker-Planck equation*, also known as the *Kolmogorov forward equation*, which is useful in characterizing the cross-sectional distribution in general settings.

A.1 Fokker-Planck equation

Consider the diffusion

$$\mathrm{d}X_t = g(t, X_t)\,\mathrm{d}t + v(t, X_t)\,\mathrm{d}B_t, \tag{A.1}$$

where B_t is standard Brownian motion. Let $p(x, t)$ be the density of X_t at time t. Then

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(gp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(v^2 p), \tag{A.2}$$

which is known as the *Fokker-Planck (Kolmogorov forward) equation.*

The Fokker-Planck equation (A.2) holds if the diffusion (A.1) holds at all times. However, we can consider situations in which the process is occasionally reset. For example, if X_t in (A.1) describe individual wealth, since the individual will die eventually, we need to specify what happens when an individual dies. If there is influx $j_+(x, t)$ and outflux $j_-(x, t)$ per unit of time at location x at time t, then the Fokker-Planck equation (A.2) must be modified as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(gp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(v^2 p) + j_+ - j_-.$$

For example, if the units exit at constant probability η per unit of time (Poisson rate η) and enter at location x_0, then the FPE becomes

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(gp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(v^2 p) + \eta\delta(x - x_0) - \eta p,$$

where $\delta(x - x_0)$ is the Dirac delta function located at x_0.

A.2 Stationary density

If the diffusion has time-independent drift $g(x)$ and variance $v(x)$ and admits a stationary distribution $p(x)$, then we get

$$0 = -\frac{\mathrm{d}}{\mathrm{d}x}(gp) + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(v^2 p).$$

Integrating with respect to x and using the boundary condition $p(x), p'(x) \to 0$ as $x \to \pm\infty$, we get

$$0 = -g(x)p(x) + \frac{1}{2}(v(x)^2 p(x))'.$$

Letting $q(x) = v(x)^2 p(x)$ and solving the ODE, we get

$$\begin{aligned} q' = \frac{2g}{v^2} q \iff & \frac{q'}{q} = \frac{2g}{v^2} \\ \iff & \log q(x) = \int \frac{q'(x)}{q(x)}\,\mathrm{d}x = \int \frac{2g(x)}{v(x)^2}\,\mathrm{d}x \\ \iff & q(x) = \exp\left(\int \frac{2g(x)}{v(x)^2}\,\mathrm{d}x\right). \end{aligned}$$

Therefore the stationary density is

$$p(x) = \frac{q(x)}{v(x)^2} = \frac{1}{v(x)^2}\exp\left(\int \frac{2g(x)}{v(x)^2}\,\mathrm{d}x\right), \tag{A.3}$$

where the constant of integration is determined by the condition $\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1$ since $p(x)$ is a density.

If there is a constant probability of death η, the stationary density is the solution of the second-order ODE

$$0 = -\frac{\mathrm{d}}{\mathrm{d}x}(gp) + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(v^2 p) - \eta p,$$

which holds at every point except x_0.

A.2.1 Geometric Brownian motion with minimum size

As examples, consider the geometric Brownian motion with minimum size $x_{\min}$ or constant Poisson rate η of birth/death with reset size x_0. In the former case, setting $g(x) = gx$ (with $g < 0$) and $v(x) = vx$ in (A.3), the stationary density is

$$p(x) = \frac{1}{(vx)^2}\exp\left(\int \frac{2gx}{(vx)^2}\,\mathrm{d}x\right) = Cx^{\frac{2g}{v^2}-2}$$

for some constant $C > 0$. Since the minimum size is $x_{\min}$ and the probability must add up to 1, it follows that

$$1 = C \int_{x_{\min}}^{\infty} x^{\frac{2g}{v^2}-2} = \frac{C}{1-\frac{2g}{v^2}} x_{\min}^{-1+\frac{2g}{v^2}}.$$

Therefore

$$p(x) = \zeta x_{\min}^{\zeta} x^{-\zeta-1}$$

for $\zeta = 1 - 2g/v^2$, which is the probability density function of the Pareto distribution (2.2) with exponent $\zeta > 1$.

A.2.2 Geometric Brownian motion with Poisson entry/exit

Next, consider the geometric Brownian motion with entry/exit at Poisson rate $\eta > 0$ and initial size x_0. In this case, it is easier to solve in logs. Using Itô's lemma, $Y_t = \log X_t$ obeys the Brownian motion

$$\mathrm{d}Y_t = \left(g - \frac{1}{2}v^2\right)\mathrm{d}t + v\,\mathrm{d}B_t.$$

The Fokker-Planck equation in the steady state is

$$0 = -\left(g - \frac{1}{2}v^2\right)p'(y) + \frac{1}{2}v^2 p''(y) - \eta p(y)$$

except at $y_0 := \log x_0$, where I used the fact that g, v are constant. Since this is a linear second-order ODE with constant coefficients, the general solution is

$$p(y) = C_1 \mathrm{e}^{-\lambda_1 y} + C_2 \mathrm{e}^{-\lambda_2 y},$$

where $\lambda_1 > 0 > \lambda_2$ are solutions to the quadratic equation

$$\frac{1}{2}v^2\xi^2 + \left(g - \frac{1}{2}v^2\right)\xi - \eta = 0,$$

which is (2.6). Since the PDF must be continuous, $p(y) \to 0$ as $y \to \pm\infty$, and integrate to 1, letting $\alpha = \lambda_1 > 0$ and $\beta = -\lambda_2 > 0$, it follows that

$$p(y) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta}\mathrm{e}^{-\alpha|y-y_0|}, & (y \geq y_0) \\ \frac{\alpha\beta}{\alpha+\beta}\mathrm{e}^{-\beta|y-y_0|}, & (y \leq y_0) \end{cases}$$

which is the asymmetric Laplace distribution with mode y_0 and exponents α, β. Taking the exponential, we obtain the double Pareto distribution (2.5).

B Proofs

Proof of Lemma 3.4. Suppose that a positive control variable y_l has a finite aggregate supply $0 < e_l < \infty$. By lemma 3.2, the demand of a unit (of a particular type that solves a homogeneous problem) with size x_t is $y_{lt} = \alpha_l(\theta)x_t$. Since other types may also demand that variable, taking the cross-sectional expectation, by market clearing we have

$$\infty > e_l \geq \mathrm{E}[y_{lt}] = \mathrm{E}[\alpha_l(\theta)x_t] = \alpha_l(\theta)\,\mathrm{E}[x_t] > 0.$$

Since $\alpha_l(\theta) > 0$, we have $0 < \mathrm{E}[x_t] < \infty$.

Substituting the optimal control $y_t = \alpha(\theta)x_t$ into the equation of motion (3.8), we obtain

$$\begin{aligned} \mathrm{d}x_t &= g(x_t, \alpha(\theta)x_t; \theta)\,\mathrm{d}t + v(x_t, \alpha(\theta)x_t; \theta)\,\mathrm{d}B_t \\ &= g(1, \alpha(\theta); \theta)x_t\,\mathrm{d}t + v(1, \alpha(\theta); \theta)x_t\,\mathrm{d}B_t \\ &=: g(\theta)x_t\,\mathrm{d}t + v(\theta)x_t\,\mathrm{d}B_t, \end{aligned}$$

where I have used the homogeneity of g, v. $\square$

Prof of Theorem 3.5. By Lemma 3.4, the size of individual units evolves according to the geometric Brownian motion (3.9). Let $X = \mathrm{E}[x_t]$ be the cross-sectional average, which is positive and finite by Lemma 3.4 and constant over time by stationarity. Since individual units grow at rate $g(\theta)$, exits at rate η, and new units have average size $\kappa(\eta)X$, it follows that

$$0 = \frac{\mathrm{d}X}{\mathrm{d}t} = (g(\theta) - \eta)X + \eta\kappa(\eta)X \iff g(\theta) = \eta(1 - \kappa(\eta))$$

at $\theta = \theta(\eta)$. Substituting into (3.9), we obtain (3.10).

To show the bound (3.11), first assume that the initial size x_0 is deterministic. Letting $\kappa = \kappa(\eta)$ and $v = v(\eta)$, by (3.10) the expected growth rate is

$$g = g(\theta(\eta)) = \eta(1 - \kappa).$$

Therefore by (2.6), letting $c = 2\eta\kappa/v^2 > 0$ and $d = 2\eta/v^2 > 0$, the upper tail Pareto exponent is the positive root of the quadratic function

$$q(\zeta) = \zeta^2 - (1 + c - d)\zeta - d. \tag{B.1}$$

Since

$$\begin{aligned} q(1) &= 1 - (1 + c - d) - d = -c < 0, \\ q(1 + c) &= (1 + c)^2 - (1 + c)^2 + d(1 + c) - d = cd > 0, \end{aligned}$$

the positive root satisfies $1 < \zeta < 1 + c = 1 + \frac{2\eta\kappa}{v^2}$, which is (3.11).

By the results in Section 2.2, the cross-sectional size distribution relative to initial size x_0 is double Pareto with an upper tail exponent ζ that satisfies (3.11). The upper tail exponent of the (unconditional) cross-sectional size distribution also satisfies (3.11) since the initial size distribution $F(\cdot; \theta, \eta)$ either does not affect the tail (if F is thin-tailed) or makes the tail even fatter (if F is fat-tailed with exponent smaller than ζ). $\square$

Proof of Theorem 3.7. For notational simplicity, let $\zeta = \zeta^{(p)}$ and $G = G^{(p)}$.

Step 1. $\zeta > 1$.

The function $s \mapsto \mathrm{E}[G^s]$ is convex since $\mathrm{E}[G^s] = \mathrm{E}[\mathrm{e}^{s \log G}]$ and the exponential function is convex. Since by assumption $\mathrm{E}[G] < \frac{1}{1-p}$ and $\mathrm{E}[G^0] = 1 < \frac{1}{1-p}$, it follows that $\mathrm{E}[G^s] < \frac{1}{1-p}$ for all $0 \le s \le 1$. Therefore $\zeta > 1$.

Step 2. For any $x > 0$ we have

$$x^\zeta \ge 1 + \zeta(x - 1) + \zeta(\zeta - 1)\phi(x). \tag{B.2}$$

For any C^2 function, by Taylor's theorem in integral form, we have

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t) f''(t) \, dt.$$

Letting $f(x) = x^\zeta$ and $a = 1$, we obtain

$$\begin{aligned} x^\zeta &= 1 + \zeta(x-1) + \zeta(\zeta-1) \int_1^x (x-t) t^{\zeta-2} \, dt \\ &=: 1 + \zeta(x-1) + \zeta(\zeta-1) R(x;\zeta). \end{aligned}$$

Since $\zeta > 1$, to show (B.2) it suffices to show $R(x;\zeta) \geq \phi(x)$.

If $0 < x \leq 1$, then

$$R(x;\zeta) = \int_1^x (x-t) t^{\zeta-2} \, dt = \int_x^1 (t-x) t^{\zeta-2} \, dt \geq 0 = \phi(x).$$

If $x > 1$, since $\zeta > 1$, we obtain

$$\begin{aligned} R(x;\zeta) &= \int_1^x (x-t) t^{\zeta-2} \, dt \geq \int_1^x (x-t) t^{-1} \, dt \\ &= \int_1^x \left(\frac{x}{t} - 1 \right) dt = x \log x - x + 1 = \phi(x). \end{aligned}$$

Step 3. (3.16) *holds.*

Noting that $\mathrm{E}[G] < \frac{1}{1-p}$, $1 < \frac{1}{1-p}$, and $\mathrm{E}[\phi(G)] > 0$, define $c, d > 0$ by

$$c = \frac{\frac{1}{1-p} - \mathrm{E}[G]}{\mathrm{E}[\phi(G)]}, \quad d = \frac{\frac{1}{1-p} - 1}{\mathrm{E}[\phi(G)]}.$$

Letting $x = G$ in (B.2), taking expectations, and using (3.15), we obtain

$$\begin{aligned} & \frac{1}{1-p} = \mathrm{E}[G^\zeta] \geq 1 + \zeta(\mathrm{E}[G] - 1) + \zeta(\zeta-1) \, \mathrm{E}[\phi(G)] \\ \iff & \zeta^2 - (1 + c - d)\zeta - d \leq 0. \end{aligned} \tag{B.3}$$

Since the left-hand side of (B.3) is identical to (B.1), by the same argument as in the proof of Theorem 3.5, we obtain $\zeta < 1 + c$. ☐

Proof of Theorem 4.1. Note that since firms solve a homogeneous problem, if a stationary equilibrium exists, by Theorem 3.5 the firm size evolves according to the geometric Brownian motion (3.10).

Since the proof is long and tedious, I break it down into several steps.

Step 1. If a stationary equilibrium exists, then $r > 0$. *The propensity to consume out of wealth,* m *in* (4.5b)*, is positive. The volatility of entrepreneur's wealth is given by* $v = \frac{f'(y) - r - \eta}{\gamma\sigma} > 0$.

If $r \leq 0$, then the present value of a worker's wage $\int_0^\infty e^{-rt} \omega \, dt$ is infinite, so the utility maximization problem does not have a solution. Therefore if an equilibrium exists, it must be $r > 0$.

If an equilibrium exists, by (4.5a) the fraction of wealth invested in physical capital is

$$0 < \theta = \frac{\mu - r_e}{\gamma\sigma^2} = \frac{f'(y) - r - \eta}{\gamma\sigma^2},$$

where $\mu = f'(y)$ and $r_e = r + \eta$. By (4.7b), we have $v = \frac{f'(y)-r-\eta}{\gamma\sigma} > 0$. To show that the propensity to consume is positive, note that by (4.4), (4.5a), and (3.10), we have

$$g = r_e + \frac{(\mu - r_e)^2}{\gamma\sigma^2} - m = \eta(1-\kappa).$$

Since $r_e = r + \eta$, $\mu = f'(y)$, and $r, \kappa, \eta > 0$, it follows that

$$m = r + \eta\kappa + \frac{(f'(y) - r - \eta)^2}{\gamma\sigma^2} > 0.$$

Step 2. If a stationary equilibrium exists, the capital-labor ratio $y = K/N$ and risk-free rate r satisfy (4.11)*, and* (4.9) *must hold.*

By (3.10) and (4.7a), we must have

$$g = \eta(1-\kappa) = (r - \rho)\varepsilon + (1+\varepsilon)\frac{(\mu - r_e)^2}{2\gamma\sigma^2}.$$

Substituting $\kappa = \frac{1}{yN}$, $\mu = f'(y)$, and $r_e = r + \eta$, the equilibrium capital-labor ratio y must satisfy (4.11). To show (4.9), let

$$\phi(y, r) = \left(1 - \frac{1}{yN}\right)\eta - (r - \rho)\varepsilon - (1+\varepsilon)\frac{(f'(y) - r - \eta)^2}{2\gamma\sigma^2} \tag{B.4}$$

be the left-hand side minus the right-hand side of (4.11). If an equilibrium exists, since capital investment must be positive we have

$$\theta > 0 \iff f'(y) - r - \eta > 0 \iff y < (f')^{-1}(r + \eta).$$

If $y > 0$ satisfies this inequality, then

$$\frac{\partial\phi}{\partial y}(y, r) = \frac{\eta}{y^2 N} - (1+\varepsilon)\frac{f'(y) - r - \eta}{\gamma\sigma^2} f''(y) > 0,$$

so ϕ is strictly increasing in y. Since $\phi(0, r) = -\infty$ and ϕ is continuous, there exists $y \in (0, (f')^{-1}(r + \eta))$ such that (4.11) holds if and only if

$$\psi(r) := \phi((f')^{-1}(r + \eta), r) = \left(1 - \frac{1}{(f')^{-1}(r + \eta)N}\right)\eta - (r - \rho)\varepsilon > 0. \tag{B.5}$$

Since $f'' < 0$ and $\eta, \varepsilon > 0$, clearly $\psi(r)$ is strictly decreasing. Since ψ is continuous, there exists $r > 0$ such that $\psi(r) > 0$ if and only if

$$\psi(0) > 0 \iff \left(1 - \frac{1}{(f')^{-1}(\eta)N}\right)\eta + \rho\varepsilon > 0,$$

which is exactly (4.9).

Step 3. A stationary equilibrium in which workers consume a positive amount exists if and only if (4.10) *holds. In this case the equilibrium is unique and* $r = \rho_W$.

In steady state with positive consumption of workers, their wealth must be a positive constant. Setting $\mathrm{d}w/\mathrm{d}t = 0$ in (4.1), we have $c = rw$. Comparing to the optimal consumption rule (4.2), we obtain

$$r = \rho_W \varepsilon_W + (1 - \varepsilon_W) r \iff r = \rho_W. \tag{B.6}$$

In this case $(f')^{-1}(r + \eta) = (f')^{-1}(\rho_W + \eta) = y_1$, so by Step 2 an equilibrium exists if and only if

$$0 < \psi(\rho_W) = \phi(y_1, \rho_W) = \left(1 - \frac{1}{y_1 N}\right)\eta - (\rho_W - \rho)\varepsilon,$$

which is exactly (4.10). Since ϕ is strictly increasing in y, the capital-labor ratio $y = K/N$ is unique.

So far we have shown that (4.9) is necessary for equilibrium existence, and that (4.10) is necessary and sufficient for the existence of a stationary equilibrium in which workers consume a positive amount. Therefore it remains to show that if (4.9) holds but (4.10) fails, then there exists a stationary equilibrium in which workers consume zero.

Step 4. A stationary equilibrium in which workers consume zero exists if and only if there exist $y, r > 0$ *such that* (4.11) *and* (4.12) *hold.*

By Steps 1 and 2, $r > 0$ and (4.11) are necessary for equilibrium. Letting $y = K/N$ be the capital-labor ratio and $\theta = \frac{f'(y) - r - \eta}{\gamma\sigma^2} > 0$ be entrepreneurs' portfolio share of capital investment, their aggregate net worth is $K/\theta = \frac{yN}{\theta}$. Since they invest fraction $1 - \theta$ in the risk-free asset, its market capitalization is $B = \frac{1-\theta}{\theta} yN$. If workers consume zero in equilibrium, since they have zero net worth, all the wage must be used for interest payments on debt. Therefore the equilibrium condition is

$$rB = \omega N \iff r\frac{1 - \theta}{\theta} y = f(y) - yf'(y),$$

which is equivalent to (4.12). Conversely, if $y, r > 0$ satisfy (4.11) and (4.12), aggregate capital is constant and workers have zero net worth and consumption, so it is an equilibrium.

Step 5. If (4.9) *holds but* (4.10) *does not, then a stationary equilibrium in which workers consume zero exists. The capital-labor ratio* $y > 0$ *and risk-free rate* $r > 0$ *satisfy* (4.11) *and* (4.12)*. Furthermore,* $y_0 < y < y_2$ *and* $0 < r < \rho_W$.

Since (4.9) holds but (4.10) fails, we have

$$\psi(0) = \left(1 - \frac{1}{y_2 N}\right)\eta + \rho\varepsilon > 0 \geq \left(1 - \frac{1}{y_1 N}\right)\eta + (\rho_W - \rho)\varepsilon = \psi(\rho_W).$$

Since ψ is strictly decreasing, there exists a unique $\bar{r} \in (0, \rho_W]$ such that $\psi(\bar{r}) = 0$. For any $0 < r \leq \bar{r}$, we have $\psi(r) \geq 0$, so by the above argument there exists a unique $y \in (0, (f')^{-1}(r+\eta)]$ such that (4.11) holds. Denote this y by $y(r)$. By the definition of ψ and $\bar{r}$, we have $0 = \psi(\bar{r}) = \phi((f')^{-1}(\bar{r}+\eta), \bar{r})$, so

$$y(\bar{r}) = (f')^{-1}(\bar{r}+\eta) \iff f'(y(\bar{r})) - \bar{r} - \eta = 0. \tag{B.7}$$

Let

$$\varphi(r) = \frac{r}{r + f(y)/y - f'(y)} - \frac{f'(y) - r - \eta}{\gamma\sigma^2}$$

be the left-hand side minus the right-hand side of (4.12), where $y = y(r) > 0$. Note that φ is well-defined for all $0 \le r \le \bar{r}$. To see this, since f is strictly concave and $f(0) = 0$, for $y > 0$ we have

$$f(0) - f(y) < f'(y)(0 - y) \iff f(y)/y - f'(y) > 0,$$

so the denominator of the first term of φ is always positive. Since $f'(y(r)) - r - \eta > 0$, we have

$$\varphi(0) = -\frac{f'(y(0)) - 0 - \eta}{\gamma\sigma^2} < 0.$$

Furthermore, by (B.7) we have

$$\varphi(\bar{r}) = \frac{\bar{r}}{\bar{r} + f(y)/y - f'(y)} > 0,$$

where $y = y(\bar{r})$. Since φ is continuous, by the intermediate value theorem there exists $r \in (0, \bar{r})$ such that $\varphi(r) = 0$. Since $y = y(r)$ and $r > 0$ satisfy (4.11) and (4.12), an equilibrium exists. In this equilibrium $0 < r < \bar{r} \le \rho_W$.

It remains to show that $y_0 < y < y_2$. Since $0 < r < \rho_W$, $f(y)/y - f'(y) > 0$, and (4.12) holds, it follows that

$$\begin{aligned}
&\theta = \frac{f'(y) - r - \eta}{\gamma\sigma^2} > 0 \\
\implies &y < (f')^{-1}(r+\eta) < (f')^{-1}(\eta) = y_2, \\
&\theta = \frac{f'(y) - r - \eta}{\gamma\sigma^2} = \frac{r}{r + f(y)/y - f'(y)} < 1 \\
\implies &y > (f')^{-1}(r + \eta + \gamma\sigma^2) > (f')^{-1}(\rho_W + \eta + \gamma\sigma^2) = y_0.
\end{aligned}$$

□

Proof of Theorem 4.3. Since by Theorem 3.5 the bound (3.11) holds, in order to show $\zeta \to 1$ as $\eta \to 0$, it suffices to show that $\kappa > 0$ is bounded above and $v > 0$ is bounded away from 0.

Case 1: $\rho_W < \rho$. In this case (4.10) holds as $\eta \to 0$, so in equilibrium workers consume a positive amount and $r = \rho_W$.

Fix any $\underline{y} > 0$ such that

$$-(\rho_W - \rho)\varepsilon - (1+\varepsilon)\frac{(f'(\underline{y}) - \rho_W)^2}{2\gamma\sigma^2} < 0,$$

which exists by the Inada condition $f'(0) = \infty$. Let $\phi(y, \rho_W; \eta)$ be $\phi(y, r)$ in (B.4) with $r = \rho_W$, given $\eta > 0$. Then we have

$$\lim_{\eta \to 0} \phi(\underline{y}, \rho_W; \eta) = -(\rho_W - \rho)\varepsilon - (1+\varepsilon)\frac{(f'(\underline{y}) - \rho_W)^2}{2\gamma\sigma^2} < 0.$$

Since ϕ is strictly increasing in y and $\phi(y, \rho_W; \eta) = 0$ in equilibrium, it follows that for sufficiently small η we have $y > \underline{y}$. Therefore $\kappa = \frac{1}{yN} < \frac{1}{\underline{y}N}$ is bounded.

By Theorem 4.1, the equilibrium condition (4.11) is equivalent to

$$(1-\kappa)\eta = (\rho_W - \rho)\varepsilon + \frac{1+\varepsilon}{2}\gamma v^2 \iff v^2 = \frac{2\varepsilon(\rho - \rho_W) + 2(1-\kappa)\eta}{\gamma(1+\varepsilon)}.$$

Since κ is bounded and $\rho_W < \rho$, v is bounded away from 0 as $\eta \to 0$.

Case 2: $\boldsymbol{\rho_W \geq \rho}$. In this case (4.10) fails as $\eta \to 0$, so in equilibrium workers consume zero and $r < \rho_W$.

By Theorem 4.1, we have $0 < (f')^{-1}(\rho_W + \eta + \gamma\sigma^2) = y_0 < y$. Therefore as $\eta \to 0$ we have

$$\kappa = \frac{1}{yN} < \frac{1}{(f')^{-1}(\rho_W + \eta + \gamma\sigma^2)N} \to \frac{1}{(f')^{-1}(\rho_W + \gamma\sigma^2)N} < \infty,$$

so κ is bounded. To show that v is bounded away from 0, using (4.7b), (4.11), and (4.12), we have

$$\left(1 - \frac{1}{yN}\right)\eta - (r-\rho)\varepsilon - \frac{1+\varepsilon}{2}\gamma v^2 = 0, \tag{B.8a}$$

$$\frac{r}{r + f(y)/y - f'(y)} = \frac{v}{\sigma}. \tag{B.8b}$$

If $v \to 0$ as $\eta \to 0$, by (B.8b) we have $r \to 0$ since y is bounded away from 0. But letting $\eta \to 0$ (and hence $v, r \to 0$) in (B.8a), we obtain $\rho\varepsilon = 0$, which is a contradiction. Therefore v is bounded away from 0 as $\eta \to 0$. □

C Robustness of firm size model

In this appendix I consider different versions of the firm size model in Section 4 and show the existence of equilibrium and Zipf's law when (i) entrepreneurs and workers are in the same household, and (ii) workers also enter/exit the economy.

C.1 Entrepreneurs and workers in same household

In the model of firm size distribution in Section 4, I assumed that entrepreneurs and workers are different entities. This is because if they belong to the same household, then the size distribution of firms and consumption become identical up to rescaling, which is counter factual since empirically the Pareto exponent for consumption is 4 (Toda and Walsh, 2015), well above 1 for firms. In this appendix I study a continuous-time version of the Angeletos (2007) model and show that Zipf's law holds.

Consider the same model as in Section 4, except that entrepreneurs and workers belong to the same household. A household consists of 1 entrepreneur and N workers, which maintains a single budget. Letting x_t be the financial wealth and b_t be the risk-free asset position, the budget constraint is

$$\mathrm{d}x_t = (F(k_t, n_t) - \omega n_t + (r+\eta)b_t - c_t + \omega N)\,\mathrm{d}t + \sigma k_t\,\mathrm{d}B_t.$$

The difference from (4.3) is that there is labor income ωN. Letting $H = \int_0^\infty \mathrm{e}^{-(r+\eta)t}\omega N \, \mathrm{d}t$ be the human capital and $w_t = x_t + b_t + H$ be the total wealth, we obtain

$$\mathrm{d}w_t = (F(k_t, n_t) - \omega n_t + (r+\eta)b_t - c_t)\,\mathrm{d}t + \sigma k_t \,\mathrm{d}B_t,$$

which is identical to (4.3) except that x_t is now w_t. Letting $\theta = k_t/w_t$ be the fraction of wealth invested in capital, the optimal portfolio and consumption rule are given by (4.5a) and (4.5b). The following proposition characterizes the equilibrium.

Proposition C.1. *Let everything be as above. Then the stationary equilibrium is characterized by*

$$\theta = \frac{f'(y) - r - \eta}{\gamma\sigma^2} = \frac{y}{y + \frac{\omega}{r+\eta}}, \tag{C.1a}$$

$$\theta\eta = (r-\rho)\varepsilon + \frac{1}{2}(1+\varepsilon)\gamma\sigma^2\theta^2, \tag{C.1b}$$

where $\omega = f(y) - yf'(y) > 0$.

Proof. The left equality of (C.1a) follows from (4.5a). To show the right equality, let K, B, H, W be aggregate capital, risk-free asset, human capital, and wealth. By aggregating the individual budget constraint, we have $W = K + B + H$ and $K = \theta W$. By market clearing, $B = 0$. Therefore

$$\theta = \frac{K}{W} = \frac{K}{K+H} = \frac{yN}{yN + \frac{\omega N}{r+\eta}} = \frac{y}{y + \frac{\omega}{r+\eta}}.$$

Since the wage is $\omega = f(y) - yf'(y) > 0$, we obtain the right equality of (C.1a).

By the same argument as in the proof of Theorem 4.1, we obtain

$$\eta(1-\kappa) = (r-\rho)\varepsilon + \frac{1}{2}(1+\varepsilon)\frac{(f'(y) - r - \eta)^2}{2\gamma\sigma^2},$$

which is a consistency condition similar to (4.11) and κ is the ratio between initial household wealth and aggregate wealth. Since new households start with only human capital, we have

$$\kappa = \frac{H}{W} = \frac{W - K - B}{W} = \frac{W-K}{W} = 1 - \theta.$$

Therefore $1 - \kappa = \theta$, and using (C.1a), we obtain (C.1b). $\square$

Theorem C.2. *Suppose that entrepreneurs and workers belong to the same household. Then a stationary equilibrium always exists. Furthermore, the Pareto exponent ζ of the firm size distribution converges to 1 as $\eta \to 0$, so Zipf's law holds.*

Proof.

Step 1. A stationary equilibrium exists.

Since $f'' < 0$, $f'(0) = \infty$, and $f'(\infty) \le 0$, letting $y_3 = (f')^{-1}(0)$, we have $f'(y) > 0$ for $y \in (0, y_3)$. For $r > -\eta$, define

$$\phi_1(y, r) = \frac{f'(y) - r - \eta}{\gamma\sigma^2} - \frac{y}{y + \frac{f(y) - yf'(y)}{r + \eta}},$$

which is the difference between the middle and right expressions in (C.1a). Cleary ϕ is strictly decreasing in r. Furthermore, if $y \in (0, y_3)$, we have

$$\begin{aligned}\phi_1(y, -\eta) &= \frac{f'(y)}{\gamma\sigma^2} > 0,\\ \phi_1(y, \infty) &= -\infty < 0.\end{aligned}$$

Hence by the intermediate value theorem, there exists a unique $r(y) \in (-\eta, \infty)$ such that $\phi_1(y, r(y)) = 0$.

Clearly $r(y)$ is continuous in y. Since $\frac{y}{y + \frac{\omega}{r+\eta}} \in (0, 1)$, at $r = r(y)$ we have

$$0 < \frac{f'(y) - r - \eta}{\gamma\sigma^2} < 1 \iff f'(y) - \eta - \gamma\sigma^2 < r < f'(y) - \eta. \tag{C.2}$$

Since $r(y) > -\eta$, letting $y \to y_3$, from the left inequality in (C.2) we obtain $r(y_3) = -\eta$. Letting $y \to 0$ in (C.2), we obtain $r(0) = \infty$. Now let

$$\phi_2(y) = \frac{1}{2}(1 + \varepsilon)\gamma\sigma^2\theta(y)^2 - \theta(y)\eta + (r(y) - \rho)\varepsilon,$$

where $\theta(y) = \frac{f'(y) - r(y) - \eta}{\gamma\sigma^2}$. Since $f'(y_3) = 0$ and $r(y_3) = -\eta$, we get $\theta(y_3) = 0$, so $\phi_2(y_3) = -(\eta + \rho)\varepsilon < 0$. Since by (C.1a) we have $\theta(y) \in (0, 1)$, which is bounded, letting $y \to 0$ we obtain $\phi_2(0) = \infty$ since $r(0) = \infty$. Therefore by the intermediate value theorem there exists y such that $\phi_2(y) = 0$. Since this y and $r = r(y)$ satisfy (C.1), there exists an equilibrium.

Step 2. Zipf's law holds.

As in the proof of Theorem 4.3, it suffices to show that $\kappa = 1 - \theta$ is bounded above and $v = \frac{f'(y) - r - \eta}{\gamma\sigma} > 0$ is bounded away from zero. Since $\kappa = 1 - \theta \in (0, 1)$ in equilibrium, it is bounded above. Since $v = \sigma\theta$, it suffices to show that θ is bounded below. Suppose on the contrary that $\theta \to 0$ as $\eta \to 0$. Letting $\eta \to 0$ in (C.1b), we obtain $r \to \rho$. Since by (C.1a) we have $\theta = \frac{y}{y + \frac{\omega}{r+\eta}}$, where $\omega = f(y) - yf'(y) > 0$, in order for $\theta \to 0$, it is necessary that $y \to 0$. But then letting $\eta \to 0$ in (C.1a), we obtain

$$\theta = \frac{f'(y) - r - \eta}{\gamma\sigma^2} \to \frac{f'(0) - \rho}{\gamma\sigma^2} = \infty,$$

which is a contradiction. □

C.2 Workers enter/exit

In the model of firm size distribution in Section 4, to simplify the analysis I assumed that workers are infinitely lived. However, in such a model, in order for a stationary equilibrium (in which workers consume positive amounts) to exist, we need to assume that entrepreneurs are less patient than workers ($\rho > \rho_W$), which seems counterfactual (see Footnote 18). In this appendix I show that we do not need to take a stance on the relative magnitude of ρ, ρ_W if workers also enter/exit the economy.

Consider the same model as in Section 4, except that workers enter/exit the economy at constant Poisson rate η. Then the optimal consumption rule (4.2) remains true by replacing r, ρ with $r + \eta, \rho + \eta$. Substituting into the budget constraint, the total wealth (financial wealth plus human capital) of a typical worker evolves according to

$$\mathrm{d}w_t = \varepsilon(r - \rho)w_t \,\mathrm{d}t.$$

Letting W be the aggregate total wealth of workers, since there is a mass N of workers and the newborn are endowed with human capital alone (zero financial wealth), we obtain

$$W + \Delta W = (1 - \eta\Delta t)(W + \varepsilon(r - \rho)W\Delta t) + \eta\Delta t\frac{\omega N}{r + \eta}.$$

Subtracting W from both sides and letting $\Delta t \to 0$, we obtain

$$\frac{\mathrm{d}W}{\mathrm{d}t} = (\varepsilon(r - \rho) - \eta)W + \frac{\eta\omega N}{r + \eta}.$$

In the stationary equilibrium, we have $\mathrm{d}W/\,\mathrm{d}t = 0$, so

$$W = \frac{\eta\omega N}{(\eta - \varepsilon(r - \rho))(r + \eta)}.$$

Since consumption must be nonnegative, we need $\eta > \varepsilon(r - \rho)$. The aggregate financial wealth of workers is

$$X = W - \frac{\omega N}{r + \eta} = \frac{\varepsilon(r - \rho)\omega N}{(\eta - \varepsilon(r - \rho))(r + \eta)}.$$

So far I have omitted the W subscript for the workers, but since the parameters may differ from those of the entrepreneurs, ε, ρ, η should have W subscripts. Putting all the pieces together, we obtain the following result.

Proposition C.3. *Let everything be as above. Then the stationary equilibrium is characterized by* (4.11) *and*

$$\frac{1 - \theta}{\theta}y + \frac{\varepsilon_W(r - \rho_W)(f(y) - yf'(y))}{(\eta_W - \varepsilon_W(r - \rho_W))(r + \eta_W)} = 0, \tag{C.3}$$

where $\theta = \frac{f'(y) - r - \eta}{\gamma\sigma^2}$.

Proof. Since the equilibrium wage is $\omega = f(y) - yf'(y)$, the second term in (C.3) equals the risk-free asset holdings per worker. By Step 4 of the proof of Theorem 4.1, the first term in (C.3) equals the risk-free asset holdings of entrepreneurs divided by the number of workers. Since the risk-free asset is in zero net supply, by market clearing these two positions must cancel out in equilibrium. □

In this case we can show that a stationary equilibrium always exists and Zipf's law holds.

Theorem C.4. *Suppose that workers enter/exit the economy. Then a stationary equilibrium always exists. Furthermore, the Pareto exponent ζ of the firm size distribution converges to 1 as $\eta \to 0$, so Zipf's law holds.*

Proof.

Step 1. A stationary equilibrium exists.

Let $\psi(r)$ be as in (B.5), which is strictly decreasing. Since $f'(0) = \infty$ and $f'(\infty) \le 0$, we obtain $\psi(\infty) = -\infty$ and

$$\psi(-\eta) = \left(1 - \frac{1}{(f')^{-1}(0)N}\right)\eta - (-\eta - \rho)\varepsilon \ge \eta + (\eta + \rho)\varepsilon > 0,$$

so there exists a unique $\bar{r} > -\eta$ such that $\psi(\bar{r}) = 0$. For any $-\eta < r \le \bar{r}$, by the discussion in Step 5 of the proof of Theorem 4.1, there exists a unique $y \in (0, (f')^{-1}(r + \eta)]$ such that (4.11) holds. Denote this y by $y(r)$.

Letting $r \to -\eta$, $y(r)$ converges to a finite value. Since $\theta = \frac{f'(y)-r-\eta}{\gamma\sigma^2}$, θ also converges to a finite and positive value. Since $r - \rho_W \to -\eta - \rho_W < 0$, $f(y) - yf'(y) > 0$, and the denominator of the second term of (C.3) converges to 0, the left-hand side of (C.3) diverges to $-\infty$. Therefore in order to show the existence of equilibrium, it suffices to show that the left-hand side of (C.3) becomes positive for large r.

Case 1: $\boldsymbol{\bar{r} < \rho_W + \frac{\eta_W}{\varepsilon_W}}$. Letting $r \to \bar{r}$, by (B.7) and $\theta = \frac{f'(y)-r-\eta}{\gamma\sigma^2}$ we get $\theta \to 0$. Since the first term of (C.3) diverges to ∞ but the second term remains finite, the left-hand side of (C.3) diverges to ∞.

Case 2: $\boldsymbol{\bar{r} \ge \rho_W + \frac{\eta_W}{\varepsilon_W}}$. Since $\eta_W > \varepsilon_W(r - \rho_W) \iff r < \rho_W + \eta_W/\varepsilon_W$ is necessary for workers to consume a positive amount, in equilibrium it must be $r < \bar{r}$. Letting $r \to \rho_W + \eta_W/\varepsilon_W$, $y(r)$ converges to a finite positive value, and so does $\theta = \frac{f'(y)-r-\eta}{\gamma\sigma^2}$. Therefore the first term of (C.3) remains finite. Since $r - \rho_W \to \eta_W/\varepsilon_W > 0$, $f(y) - yf'(y) > 0$, and the denominator of the second term of (C.3) converges to 0, the left-hand side of (C.3) diverges to ∞.

Step 2. Zipf's law holds.

As in the proof of Theorem 4.3, it suffices to show that $\kappa = \frac{1}{yN}$ is bounded above and $v = \frac{f'(y)-r-\eta}{\gamma\sigma} > 0$ is bounded away from zero. By definition, $\bar{r}$ satisfies

$$0 = \psi(\bar{r}) = \left(1 - \frac{1}{(f')^{-1}(\bar{r} + \eta)N}\right)\eta - (\bar{r} - \rho)\varepsilon.$$

Since $\varepsilon > 0$, letting $\eta \to 0$ we obtain $\bar{r} \to \rho$.

Let us first show that κ is bounded above. Fix any $\underline{y} > 0$ such that $f'(\underline{y}) > \rho$ and

$$\rho\varepsilon - (1 + \varepsilon)\frac{(f'(\underline{y}) - \rho)^2}{2\gamma\sigma^2} < 0,$$

which exists by the Inada condition $f'(0) = \infty$. Let $r(\eta)$ be the equilibrium risk-free rate corresponding to η. Since $-\eta < r(\eta) < \bar{r}$, using the definition of

ϕ in (B.4), we obtain

$$\phi(\underline{y}, r(\eta)) = \left(1 - \frac{1}{\underline{y}N}\right)\eta - (r(\eta) - \rho)\varepsilon - (1+\varepsilon)\frac{(f'(\underline{y}) - r(\eta) - \eta)^2}{2\gamma\sigma^2}$$
$$\le \left(1 - \frac{1}{\underline{y}N}\right)\eta + (\eta - \rho)\varepsilon - (1+\varepsilon)\frac{(f'(\underline{y}) - r(\eta) - \eta)^2}{2\gamma\sigma^2}.$$

Letting $\eta \to 0$, since $\limsup r(\eta) \le \rho < f'(\underline{y})$, it follows that

$$\limsup_{\eta \to 0} \phi(\underline{y}, r(\eta)) \le \rho\varepsilon - (1+\varepsilon)\frac{(f'(\underline{y}) - \rho)^2}{2\gamma\sigma^2} < 0.$$

Since ϕ is strictly increasing in y and the equilibrium value $y(\eta)$ satisfies (4.11), or $\phi(y, r(\eta)) = 0$, it follows that $y(\eta) > \underline{y} > 0$ for sufficiently small η. Since y is bounded away from zero, $\kappa = \frac{1}{yN}$ is bounded above.

Finally, let us show that v is bounded away from zero. Suppose that $v \to 0$ as $\eta \to 0$. Since $v = \sigma\theta$, then $\theta \to 0$. Since by the above discussion $y(\eta) > \underline{y} > 0$ for small enough η, the first term in (C.3) diverges to ∞, while the second term remains finite, which is a contradiction. Therefore v is bounded away from zero. $\square$

D Estimating the firm size distribution

As discussed in Section 3.3.4, it is convenient to use the double Pareto-lognormal (dPlN) distribution to estimate the firm size distribution because it is obtained by introducing a random (lognormal) initial size to the random growth model with entry/exit in Sections 2.2 and 4. For comprehensive discussions of dPlN, see Reed (2003) and Reed and Jorgensen (2004). In this appendix I explain how to estimate the parameters of dPlN from binned data.

The logarithm of dPlN is called *normal-Laplace* (Reed and Jorgensen, 2004), which is the convolution of independent normal and asymmetric Laplace random variables. It has four parameters, the mean μ and standard deviation σ of the normal component and the exponents α, β of the Laplace component. The cumulative distribution function of normal-Laplace is

$$F(x;\theta) = \Phi(z) - \frac{b}{a+b}\mathrm{e}^{a^2/2 - az}\Phi^c(a - z) + \frac{a}{a+b}\mathrm{e}^{b^2/2+bz}\Phi^c(b+z), \quad \text{(D.1)}$$

where $z = \frac{x-\mu}{\sigma}$, $a = \alpha\sigma$, $b = \beta\sigma$, and $\Phi^c(z) = 1 - \Phi(z)$ is the counter cumulative distribution function of the standard normal distribution.[22]

If the data is binned, letting $(x_{k-1}, x_k]$ be the k-th bin ($k = 1, \dots, K$), the cell probability is

$$P_k(\theta) = F(x_k;\theta) - F(x_{k-1};\theta).$$

Therefore if the k-th bin contains n_k observations, the log likelihood function is

$$\log \mathcal{L}(\theta; \{n_k\}) = \sum_{k=1}^{K} n_k \log P_k(\theta).$$

[22] The CDF in the published version of Reed and Jorgensen (2004) contains a typographical error. The working paper version is correct.

The maximum likelihood estimate $\widehat{\theta}$ can be obtained by maximizing $\log \mathcal{L}$ over (μ, σ, a, b) and using $a = \alpha\sigma$, $b = \beta\sigma$. Letting θ_0 be the true parameter value and $N = \sum_{k=1}^{K} n_k$ the sample size, the asymptotic variance of $\sqrt{N}(\widehat{\theta} - \theta_0)$ is $I(\theta_0)^{-1}$, where the Fisher information matrix $I(\theta_0)$ can be estimated by

$$\widehat{I}(\widehat{\theta}) = \frac{1}{N}\sum_{k=1}^{K} \frac{n_k}{P_k(\widehat{\theta})^2} \nabla P_k(\widehat{\theta})(\nabla P_k(\widehat{\theta}))'.$$

To calculate this quantity it suffices to compute $\nabla_\theta F(x;\theta)$. Let

$$F(x;\mu,\sigma,\alpha,\beta) = \tilde{F}(x;\mu,\sigma,a,b),$$

where $a = \alpha\sigma$ and $b = \beta\sigma$. Letting $\phi(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}$ be the standard normal density, by simple algebra we obtain

$$\begin{aligned}
\tilde{F}_\mu =& -\frac{1}{\sigma}\left(\phi(z) - \frac{b}{a+b}\mathrm{e}^{a^2/2-az}(-a\Phi^c(a-z)+\phi(a-z))\right. \\
&\left. + \frac{a}{a+b}\mathrm{e}^{b^2/2+bz}(b\Phi^c(b+z)-\phi(b+z))\right), \\
\tilde{F}_\sigma =& z\tilde{F}_\mu, \\
\tilde{F}_a =& \frac{b}{a+b}\mathrm{e}^{a^2/2-az}\left(\left(\frac{1}{a+b}-(a-z)\right)\Phi^c(a-z)+\phi(a-z)\right) \\
&+ \frac{b}{(a+b)^2}\mathrm{e}^{b^2/2+bz}\Phi^c(b+z), \\
\tilde{F}_b =& -\frac{a}{(a+b)^2}\mathrm{e}^{a^2/2-az}\Phi^c(a-z) \\
&- \frac{a}{a+b}\mathrm{e}^{b^2/2+bz}\left(\left(\frac{1}{a+b}-(b+z)\right)\Phi^c(b+z)+\phi(b+z)\right).
\end{aligned}$$

Since $a = \alpha\sigma$ and $b = \beta\sigma$, by the chain rule we obtain

$$\begin{aligned}
F_\mu &= \tilde{F}_\mu, \\
F_\sigma &= \tilde{F}_\sigma + \alpha\tilde{F}_a + \beta\tilde{F}_b, \\
F_\alpha &= \sigma\tilde{F}_a, \\
F_\beta &= \sigma\tilde{F}_b.
\end{aligned}$$

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