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Abueg, Luisito

De La Salle University

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Luisito C Abueg[1]
School of Economics
De La Salle University Manila

Abstract

A mathematical-historical revisit of the controversy of GFA L'Hospital and J Bernoulli, and related developments in the history of calculus in the seventeenth and eighteenth centuries. This paper presents some of the indeterminate forms occurring in problems in mathematics and statistics relevant to economic analysis.

Keywords: calculus, L'Hospital's rules, mathematical history, mean value theorem, logarithmic mean, mathematical statistics, recreational mathematics

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[1] Assistant Professor, and Angel Caloia chair in international economics, School of Economics, DLSU Manila. Email: luisito.abueg@dlsu.edu.ph. The author would like also to thank Dr Yoshihiro Nakijima and Dr Kohei Wakimura and students of Osaka City University (Japan), and the Economics Department for the comments obtained during the DLSU School of Economics brown Bag Presentation on 23 September 2015, Board Room, St La Salle Building. The author is much indebted to queries raised by Dr Jesus C Dumagan Jr in coming up with this mathematical note.

A historical walkthrough[2] with L'Hospital, from indeterminates to applied problems in mathematics

Luisito C Abueg
School of Economics
De La Salle University Manila

There are infinite numbers between 0 and 1. There's 0.1 and 0.12 and 0.112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between zero and two, or between zero and a million. Some infinities are bigger than other infinities. … . But, Gus, my love, I cannot tell you how thankful I am for our little infinity. I wouldn't trade it for the world. You gave me a forever within the numbered days, and I'm grateful. ---Hazel Grace Lancaster, from the movie *The Fault In Our Stars* [2014]

Some preliminaries from elementary calculus

Recall from elementary algebra that the set of rational numbers is the smallest field of quotients of nonzero integers, i.e.,

$$\mathbf{Q} = \left\{ \frac{a}{b} : a,b \in \mathbf{Z} \wedge b \neq 0 \right\}$$

In particular, we use the notation "∞" to denote a number whose numerator is nonzero but whose denominator is [possibly approaching to] zero, i.e.,

$$\infty = \frac{a}{b}, \quad a \neq 0, \; b \to 0$$

It is for this reason that $\infty - \infty \neq 0$ (in fact, one indeterminate form). In the language of limits, some would not even consider this as valid: limits must be finite for them to exist. We first recall a result in elementary calculus, stated in the following theorem:

Theorem 1. Let $A \subseteq \mathbf{R} \wedge f,g : A \to \mathbf{R}$ and suppose $a \in \mathbf{R}, (a,\infty) \subseteq A$ such that $g(x) > 0, \forall x > a$. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L \in \mathbf{R}^*$, then

i. $L > 0 \Rightarrow \left(\lim\limits_{x\to\infty} f(x) = \infty \Leftrightarrow \lim\limits_{x\to\infty} g(x) = \infty \right)$

[2] This "walkthrough" is a historical revisit of materials in calculus and literature in mathematical analysis, done in two Good Fridays, 2015 and 2016.

ii. $L < 0 \Rightarrow \left(\lim_{x\to\infty} f(x) = -\infty \Leftrightarrow \lim_{x\to\infty} g(x) = \infty \right)$

Proof. Leithold [1995]. ■

We ask ourselves (a natural question in mathematics): is the converse true? That is, can we find the limit of some quotient whenever it already takes the form "∞/∞", called an indeterminate form? A partial solution to answer if the converse of the previous theorem, is provided by the next result (under an important condition: differentiability). This is the "controversial" L'Hospital's rules, stated in the next theorem:

Theorem 2. [G.F.A. L'Hospital (1696)][3] Let $f(x)$ and $g(x)$ have continuous n^{th}-order derivatives on some open interval (a,b), and let c be in that interval. If $f(c) = g(c) = 0$ (or if $f(c) = g(c) = \infty$), then

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \;\Rightarrow\; \lim_{x\to c} \frac{f(x)}{g(x)} = L$$

Proof. Bartle and Shebert [2011]. ■

L'Hospital started with the problem of obtaining this limit:

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}, \quad aa = a^2$$

Although this result tends to save us from the troubles of indeterminate forms, a lot of noted and counterexamples have been provided in support of what classes of functions do these rules apply. For example, successive application of the L'Hospital's rule does not always lead to a limit that is finite or infinite, as in the example below

$$\lim_{x\to\infty} \frac{\sqrt{ax+1}}{\sqrt{x+1}}, \quad a > 0$$

(apart from a counterexample given by Sydsæter and Hammond [2008]). The idea is that the problem arises "since it holds only in the implicitly understood case that $g'(x)$ does not change sign infinitely often in a neighborhood of infinity" (Boas [1986], Rickert [1968], cited in Wolfram Mathematica). Often cited examples are the transcendental functions: sine and cosine functions.

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[3] It was known in mathematical history that L'Hospital forged a deal with Johann Bernoulli: L'Hospital paid Bernoulli a regular salary of 300 francs per year (beginning 1691) to tell him of his discoveries, which L'Hospital described in the introduction of the book. The first calculus textbook, *l'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* [1696], which had numerous editions during the whole of the eighteenth century: founded on infinitesimal calculus. After L'Hospital's death, Bernoulli cried plagiarism, for it is him who had first discovered the rule, and L'Hospital placed that rule on his book.

We now turn to a next result—in a similar historical context with L'Hospital—the approximation of [almost all] differentiable functions of some order, due to Brook Taylor. We state this result formally as follows:

Theorem 3. [B. Taylor (1715)] Let $f(x)$ have continuous derivatives up to the $(n+1)^{th}$ order on an open interval (a,b). For every pair of points x, x_0 in (a,b), there is a p between x, x_0 such that

$$f(x) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + R_{n+1}$$

where

$$R_{n+1} = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}$$

is the remainder, which is provided by J.L. Lagrange [1765].

Proof. Bartle and Shebert [2011]. ∎

It is also known that Colin Maclaurin[4] worked on these approximation methods that made his name be credited for a special case (where the center of approximation $x_0 = 0$)[5]. In fact, an immediate consequence of these results is the following problem:[6] let I be an open interval, and let f be twice continuously differentiable and real-valued. If $f''(a)$ exists at a point a in I, then

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Although it is natural to ask: why is there a theorem on approximation of functions? To answer the above question, recall that every polynomial is of the form

$$\sum_{k=0}^{n} a_k x^k, \quad a_n \neq 0, \; a_k \in \mathbf{R}$$

We can interpret this sum with every summand as a term in the sequence $\left(a_k x^k\right)_{k=0}^{n}$. We call $\sum_{k=0}^{n} a_k x^k$ a series. A generalization of polynomials are series of the form

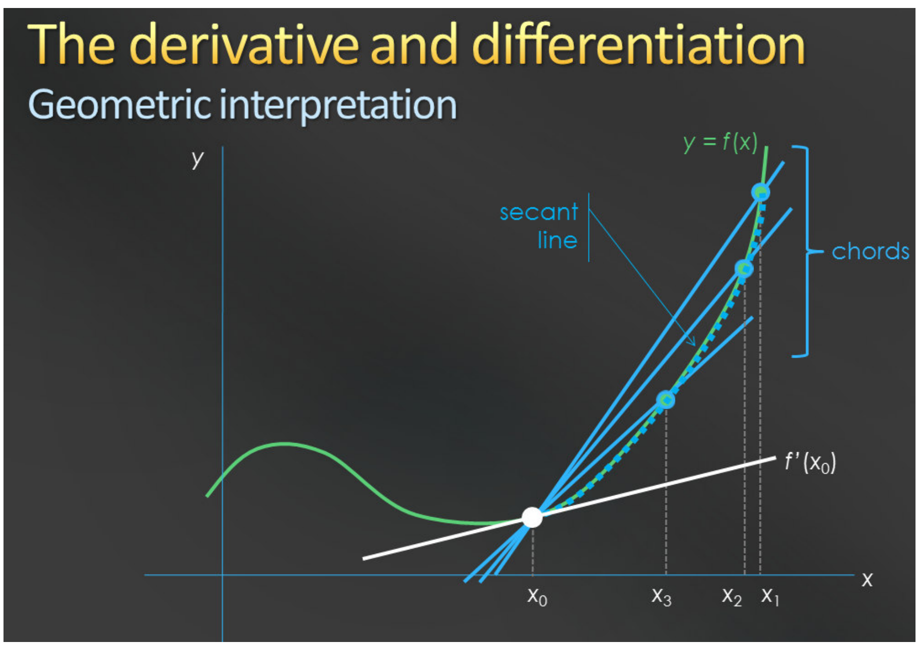
[4] Maclaurin held the record being the "youngest professor" for three hundred years, until 2008. The current record is now held by a woman of age 18, in South Korea.
[5] Theory of fluxions (1742).
[6] In *Berkeley problems in Mathematics* (2003).

$\sum_{k=0}^{\infty} a_k (x - x_0)^k$ is called a power series centered at x_0. Calculus was invented with the tacit assumption that power series provided a unified function theory; i.e., every function can be approximated by a power series (Wade [2010]). However, A.L. Cauchy (1823) has shown that there is a function that contains one point in its domain that will not correspond to any Taylor approximation or a Taylor series expansion (in particular, at the origin):

$$f(x) = \begin{cases} \exp(-x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We now turn to a replete of indeterminates in the history of calculus, as depicted in contemporary sources. We only provide a few examples, although a lot may have not been covered in this discussion.

The replete of indeterminates in calculus and approximation of differentiable functions



1. **"Consistency" of the definition of the derivative**

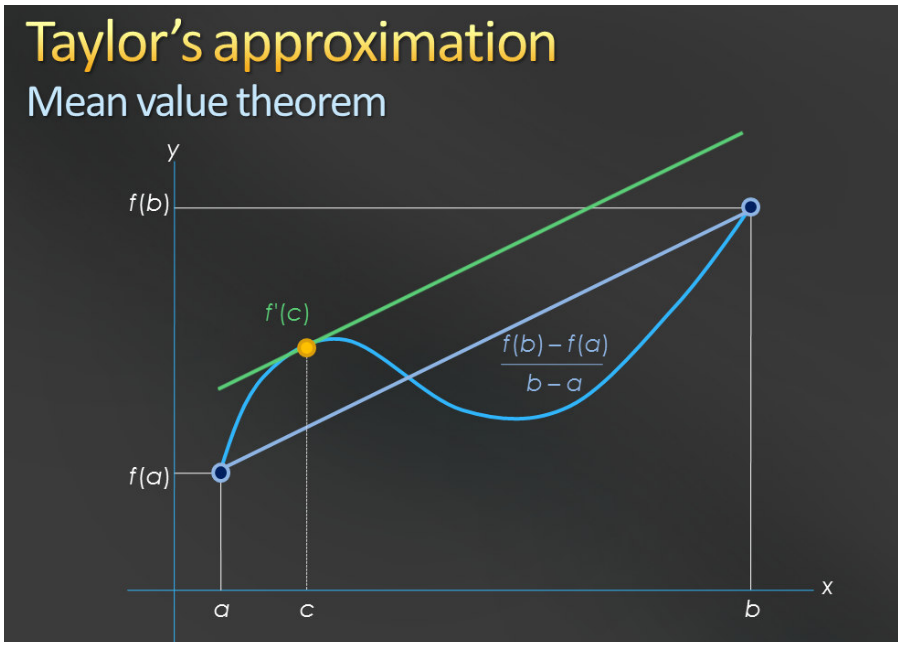
To begin with, we look more closely into the standard definition of a differentiable function $y = f(x)$: Let f be a function defined on an open interval (a,b) and let $x \in (a,b)$. The derivative of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists. We then say that f is differentiable at x, and this process is called differentiation. Observe carefully that carrying out the right hand side of this definition, we get an indeterminate form "∞/∞". Thus, appealing to L'Hospital, we obtain

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{d}{dh}\left[f(x+h)-f(x)\right]}{\frac{d}{dh}h}=\lim_{h\to 0}f'(x+h)=f'(x)$$

Thus, this definition is "consistent" with the result of L'Hospital.



2. Mean value theorems for differentiation

Next, we investigate a special case of Taylor's approximation, when n = 0: the mean value theorem [for differentiation]. Formally, Taylor's approximation reduces to the following:

$$f(x)=f(x_0)+f'(p)(x-x_0) \quad\Rightarrow\quad \frac{f(x)-f(x_0)}{x-x_0}=f'(p)$$

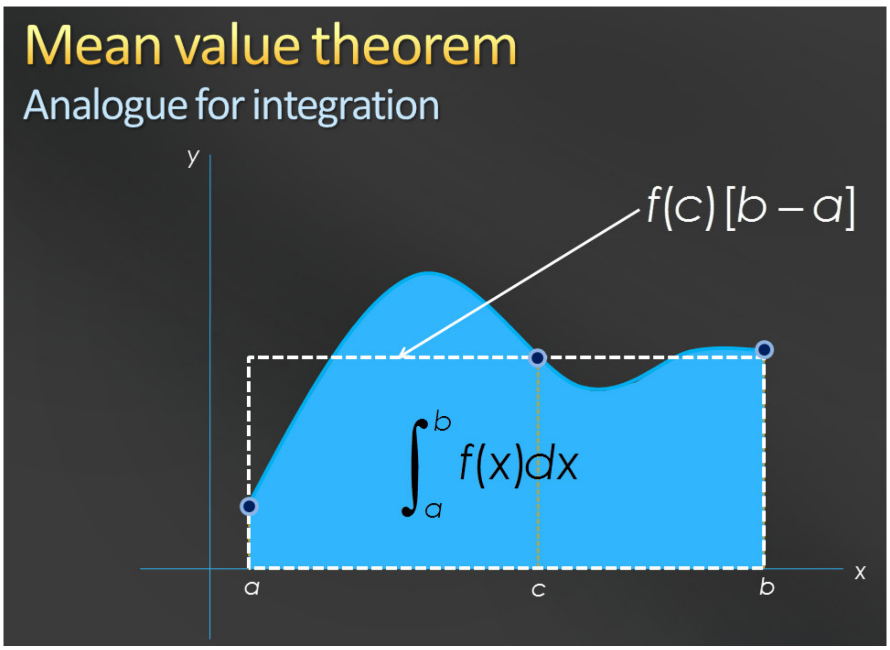
Note that the expression $\frac{f(x)-f(x_0)}{x-x_0}$ is just the slope of the line connecting the points $(x,f(x))$, and $(x_0,f(x_0))$, which is often referred to as the "rise-over-run" in elementary algebra. Clearly, as the "run" (more formally, the change in x) approaches zero, this slope approaches the slope of the tangent line (which is $f'(x)$) at x. Clearly, appealing to L'Hospital, we again obtain a "consistent" result.[7]

[7] Although noteworthy, the hypothesis of continuity cannot be relaxed, as provided by an example in Danao [2001].

A similar argument can also be applied in the more general case of the mean value theorem for integration (referred to as Cauchy's[8] mean value theorem), given two functions *f* and *g* under the same hypotheses as in the mean value theorem for integration: there is a *p* between x and x_0 such that

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(p)}{g'(p)}$$

The usual issue arises: as $x \to x_0$, the left hand side of the above approaches an indeterminate form "0/0", which appeals for L'Hospital.[9]



3. Mean value theorems for integration

We also look at the "other case" of a mean value theorem, this time for integration. Formally, we state the result as follows: let $f:[a,b]\to\mathbf{R}$ be continuous and let $g:[a,b]\to\mathbf{R}_+$ be integrable, then, there is a *p* between *a* and *b* such that

$$\int_a^b f(x)g(x)dx = f(p)\int_a^b g(x)dx$$

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[8] Note that by setting *g*(x) = x, we obtain the mean value theorem for differentiation.

[9] Wade [2010] mentioned that this generalized version of the mean value theorem is necessary for comparison of functions, say *f* and *g*, and may also be related with other properties that may arise due to differentiation (e.g., indeterminacy of the ratio *f*/*g*).

We call this p an average value of f on $[a,b]$. in particular, if g is the constant function 1, the above reduces to

$$\int_a^b f(x)dx = f(p)[b-a] \quad \Rightarrow \quad f(p) = \frac{\int_a^b f(x)dx}{b-a}$$

It is also natural to ask, what happens [to the above] when b approaches a? Formally, the said question translates to the problem

$$\lim_{b \to a} f(p) = \lim_{b \to a} \frac{\int_a^b f(x)dx}{b-a}$$

Note that the right hand side above is of the form "0/0", thus we apply L'Hospital's rule and obtain

$$\lim_{b \to a} f(p) = \lim_{b \to a} \frac{\int_a^b f(x)dx}{b-a} = \lim_{b \to a} \frac{\frac{d}{db}\int_a^b f(x)dx}{\frac{d}{db}(b-a)}$$

Observe that the numerator $\frac{d}{db}\int_a^b f(x)dx$ is the first form of the fundamental theorem of calculus, which is precisely $f(b)$. Hence,

$$\lim_{b \to a} \frac{\frac{d}{db}\int_a^b f(x)dx}{\frac{d}{db}(b-a)} = \lim_{b \to a} \frac{f(b)}{1} = f(a)$$

The appeal to graphs seems to be apparent: as b approaches a, the point p in between b and a is ultimately "pulled" towards a.

4. A "convergence lemma": from a constant-elasticity-substitution production function to a Cobb-Douglas production function.

Consider a constant-elasticity-substitution (CES) function

$$f(x,y) = A\left[ax^{-p} + (1-a)y^{-p}\right]^{-1/p}, \quad A > 0, \quad a \in (0,1), \quad p \in \mathbf{R}^*$$

Taking the natural logarithm of f, we obtain

$$\ln f(x,y) = \ln A + \left(\frac{-1}{p}\right)\ln\left[ax^{-p} + (1-a)y^{-p}\right]$$

Define

$$z = \left(\frac{-1}{p}\right)\ln\left[ax^{-p} + (1-a)y^{-p}\right]$$

Note that as p approaches zero, z takes an indeterminate form "0/0". Applying L'Hospital's rule, we obtain

$$\begin{aligned}\lim_{p\to 0} z &= -\lim_{p\to 0}\frac{\ln\left[ax^{-p}+(1-a)y^{-p}\right]}{p}\\ &= -\lim_{p\to 0}\frac{\dfrac{ax^{-p}(-1)\ln x+(1-a)y^{-p}(-1)\ln y}{ax^{-p}+(1-a)y^{-p}}}{1}\\ &= -a\ln x-(1-a)\ln y\end{aligned}$$

Hence,

$$\lim_{p\to 0}\ln f(x,y)=\ln A + a\ln x + (1-a)\ln y$$

By continuity of the natural exponential function, it follows that

$$\exp\left[\lim_{p\to 0}\ln f(x,y)\right]=\exp\left[\ln A + a\ln x + (1-a)\ln y\right]$$

which is

$$\lim_{p\to 0} f(x,y) = Ax^{a}y^{1-a}$$

i.e., the Cobb-Douglas function. Hence, the CES function "converges" to Cobb-Douglas as $p \to 0$.

5. Logarithmic mean

An application to statistics and economics[10], we investigate in some detail the properties of the logarithmic mean, which stems on results from mean value theorem and L'Hospital's rule. By definition, the logarithmic mean of two distinct positive numbers a and b is given by the expression

$$\ell_{\mu}(a,b)=\frac{b-a}{\ln\left(\dfrac{b}{a}\right)},\quad b>a$$

With this definition, we raise the following questions: first, it is natural to ask, if this expression a *valid* definition of a mean[11] (i.e., a measure of central tendency), second, it is clear that this mapping is not continuous at a (is there a way out of this problem?), and third, is this related—in any way—to the natural logarithmic function?

[10] See Balk (2004:108fn) on some historical note, with applications in price indices by Dumagan (2013), and Dumagan and Balk (2016).

[11] Standard literature in theory of statistical distributions and expectations of random variables would require certain properties, as presented in various examples in Hogg McKean, and Craig (2013).

To answer the first question, recall the following (from advanced calculus and elementary statistics) that given two real numbers (without loss of generality, say, positive) a and b, we define the arithmetic mean μ and the geometric mean γ as

$$\mu(a,b) = \frac{a+b}{2}, \quad \gamma(a,b) = \sqrt{ab}$$

If $a = b$, then $\mu = \gamma = a$. Observe that as $b \to a$, the logarithmic mean approaches "0/0", and indeterminate form. Thus, we apply L'Hospital's rule and obtain

$$\lim_{b\to a} \ell_\mu(a,b) = \lim_{b\to a} \frac{b-a}{\ln\left(\frac{b}{a}\right)} = \lim_{b\to a} \frac{1}{\frac{1}{(b/a)} \cdot \frac{1}{a}} = \lim_{b\to a} b = a$$

applying chain rule on the denominator (of the middle expression above). Thus, the above shows that the logarithmic mean has a removable discontinuity at $b = a$: the logarithmic mean is not defined at a, but there is a limit as b approaches a. Hence,

$$\ell_\mu(a,b) = \begin{cases} \dfrac{b-a}{\ln(b/a)} & b \neq a \\ a & b = a \end{cases}$$

This is just consistent with the "finite version" of the laws of large numbers. It can be shown that the logarithmic mean is always positive (observe how the restrictions have changed), which is another property a valid mean. Thus, we have also answered the second question. Now, observe that the natural logarithmic function

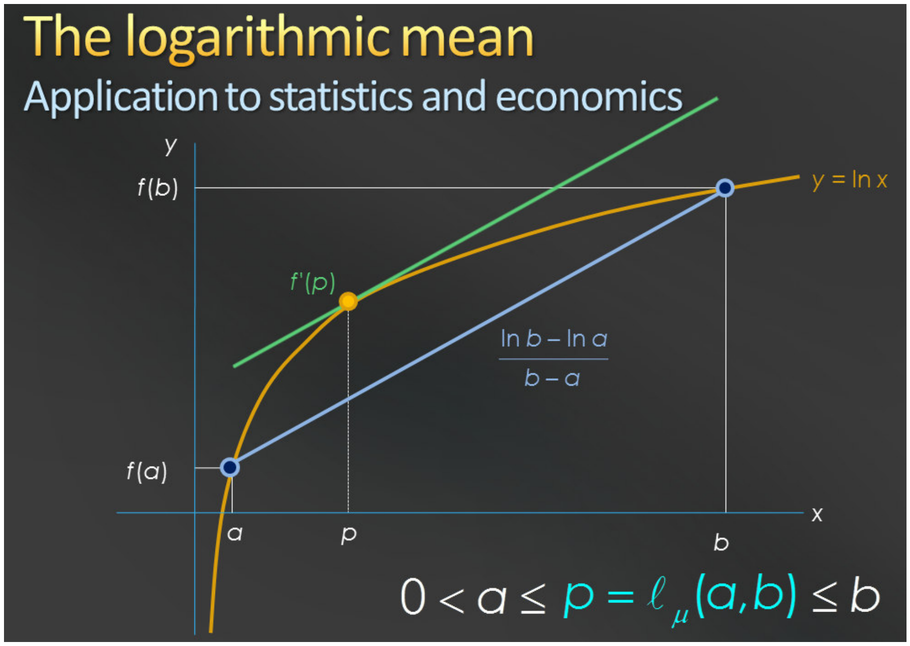
$$y = f(x) = \ln x, \quad x > 0$$

is continuously differentiable on its domain, Choose two points a and b, and apply the mean value theorem: there is a p between a and b such that

$$\frac{1}{p} = f'(p) = \frac{f(b) - f(a)}{b-a}$$

Solving for p, we obtain

$$p = \frac{b-a}{\ln(b/a)} = \ell_\mu(a,b)$$



In summary, we have shown that the logarithmic mean is a *valid* measure of central tendency for positive numbers: first, if $b = a$, then the mean is also a; second, the mean is *always* positive; and third, it is *always* between a and b. It is also continuously differentiable: inherent from the natural logarithmic function.

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