Corporate Voting

Miller, Alan

University of Haifa

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Alan D. Miller*†

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Abstract

I introduce a model of corporate voting. I characterize the shareholder majority rule as the unique corporate voting rule that satisfies four axioms: anonymity, neutrality, share monotonicity, and merger, a property that requires consistency in election outcomes following stock-for-stock mergers.

JEL classification: D71, D72, K22

1 Introduction

The shareholder franchise is understood to be an essential element of corporate governance. The most basic principle of shareholder voting is the "one share-one vote" principle, according to which each shareholder receives a number of votes proportional to the size of her holding.[1] Shareholder votes are commonly decided according to the majority of votes cast, although exceptions to this default rule exist.[2]

The positive effects of the one share-one vote rule were studied by **?** and **?**, who analyze conditions under a single class of equity stock and majority voting are optimal. However, despite the large literature in social choice theory devoted to voting and despite the economic importance of the rules of corporate governance, to

*Faculty of Law and Department of Economics, University of Haifa, Mount Carmel, Haifa, 31905, Israel. Email: admiller@econ.haifa.ac.il. Web: http://econ.haifa.ac.il/∼admiller/. Phone: +972.52.790.3793.

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[1] The one share-vote principle may be understood as requiring that all shares have an equal vote (that is, that there is only one class of shares), or alternatively, that the voting strength of a small shareholder must be linear in her holdings. The Delaware General Corporation Law provides, as a default, that: "Unless otherwise provided in the certificate of incorporation and subject to §213 of this title, each stockholder shall be entitled to 1 vote for each share of capital stock held by such stockholder." 8 Del. C. 1953, §212(a).

[2] "In all matters other than the election of directors, the affirmative vote of the majority of shares present in person or represented by proxy at the meeting and entitled to vote on the subject matter shall be the act of the stockholders". 8 Del. C. 1953, §216(2).

my knowledge there has never been a model to evaluate the normative properties of corporate voting rules.

In the model, there is a set of shareholders, each of whom has preferences on a shareholder resolution and each of whom owns a portion of the firms' common stock. Individuals may favor or oppose the resolution, or they many abstain from voting. A corporate voting rule takes into account the preferences of the individuals and their shareholdings, determines whether the resolution passes or fails, or whether the vote is indecisive.

There are four main axioms. The first two axioms, *anonymity* and *neutrality*, were introduced into the literature by **?**. Anonymity requires the corporate voting rule to treat each voter equally; it accomplishes this by requiring the result to be invariant to changes in the names of the individuals. Neutrality requires the corporate voting rule not to favor the passing of the resolution over its' failure. The third axiom, *share monotonicity*, is related to the positive responsiveness axiom of **?**; it requires that if the vote selects an outcome or is indecisive, and then a supporter of that outcome receives shares from a non-supporter, that outcome must now be chosen.

The fourth axiom, *merger*, requires a certain type of consistency in merged firms. Imagine that there are two firms (firm A and firm B), that wish to merge in a stock-for-stock transaction, and there is a common resolution in front of the shareholders of each firm. For example, that resolution may be to approve the merger, or it may related to post-merger plans. The merger axiom requires that, if the outcome of the vote held by the shareholders of firm A is the same as the outcome of the vote held the shareholders of firm B, then the outcome of the vote held by the shareholders of the combined firm (after the merger) combined firm must also be the same.

These four axioms are then used to characterize a voting rule, the "shareholder majority rule," under which each shareholder receives a vote whose strength is proportional to her shareholdings, and the winner is the option supported by the majority. The result is indecisive only in the event of a tie.

Three additional axioms are used to prove two additional results. The *reallocation invariance* axiom is motivated by the idea that individuals may be able to manipulate the identity of their shares owners' to the extent that ownership is relevant as far as voting rights are concerned. For example, if large blocks of shares were to receive disproportionately strong voting rights, likeminded shareholders may be able to combine their shares into a holding company, which becomes the sole owner of the shares.[3] The shareholders would then receive stock in the holding company. The transaction could be structured so that these shareholders could leave the holding company, and take their stock with them, in case that they wish to sell it or wish to vote differently from their fellow holding company participants. On the other hand, if small blocks of shares were to receive disproportionately strong voting rights, then a larger shareholder could partition her shares into several holding corporations. These are but a few of a wide variety of techniques that can be used to disguise the true

[3]Depending on its' size, such a transaction may trigger S.E.C. reporting requirements.

ownership of the shares; for more see **?**. The reallocation invariance axiom is stronger than anonymity; in conjunction with neutrality and share monotonicity it provides a separate characterization of the shareholder majority rule.

The *unanimity* axiom requires that the resolution be passed when all shareholders are in favor, and that the resolution fails when all shareholders are opposed. The *abstention* axiom requires the result to be invariant to abstentions, were abstaining votes to be not be counted in determining the total number of votes. It does not apply in the case that there are no non-abstaining votes. The unanimity and abstention axioms, along with anonymity and merger, characterize the *quotas rules*, a large family of rules that includes supermajority rules.

2 The Model

Let $\mathbb{N}$ be the set of all possible shareholders, and let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$. Let $\mathcal{R} \equiv \{-1, 0, 1\}$ be a set of preferences, with preferences R_i. For a set $N \in \mathcal{N}$, let $\mathbf{x} \in \Delta(N)$ be a distribution of shares. For $N \in \mathcal{N}$, let $\mathcal{Q}_N \equiv \mathcal{R}^N \times \Delta(N)$. The class of problems is the set $\mathcal{Q} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{Q}_N$.

For $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $N' \subseteq N$ for which $\sum_{i \in N'} \mathbf{x}_i = 1$, let $(R, \mathbf{x})|_{N'}$ denote the restriction of $(R, \mathbf{x})$ to $\mathcal{Q}_{N'}$. A function $f : \mathcal{Q} \to \mathcal{R}$ is **invariant to non-shareholders** if for $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $\sum_{i \in N'} \mathbf{x}_i = 1$ for $N' \subseteq N$ implies that $f(R, \mathbf{x}) = f\left((R, \mathbf{x})|_{N'}\right)$.

A **corporate voting rule** is a function $f : \mathcal{Q} \to \mathcal{R}$ that is invariant to non-shareholders.

The main result relies on four axioms. The first two, anonymity and neutrality, were introduced by **?**. For $N \in \mathcal{N}$, let Π_N refer to the set of permutations of N. For $\pi \in \Pi_N$, define $\pi R \equiv \left(R_{\pi(1)}, \ldots, R_{\pi(n)}\right)$ and $\pi \mathbf{x} \equiv \left(\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(n)}\right)$.

Anonymity For every $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$, $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$.

Next, for $R \in \mathcal{R}$, define $-R = (-R_1, \ldots, -R_n)$.

Neutrality For every $(R, \mathbf{x}) \in \mathcal{Q}$, $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

The third axiom, share monotonicity, requires that, if a particular resolution does not fail (that is, either it passes or there is an indecisive result), and then an individual who supports the resolution receives shares from an individual who does not support the resolution, result is that the resolution is now chosen. This axiom requires, essentially, that having more shares helps one's vote. It is related to the positive responsiveness axiom of **?**.

Share monotonicity: For every $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j \neq 0$, (b) $R_k \neq R_j$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, and (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \setminus \{j, k\}$, if $f(R, \mathbf{x}) \neq -R_j$ then $f(R, \mathbf{x}') = R_j$.

The fourth axiom, merger, requires a certain type of consistency in merged firms, as described in the introduction. The parameter λ represents the portion of the new firm that will be owned by the shareholders of first firm, while $1-\lambda$ represents the portion received by the shareholders of the second firm. Because the model allows for null shareholders, we can limit the axiom to the case where the sets of shareholders are the same.

Merger: For $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $\lambda \in (0,1)$, if $f(R, \mathbf{x}) = f(R, \mathbf{x}')$, then $f(R, \lambda \mathbf{x}_i + (1-\lambda)\mathbf{x}'_i) = f(R, \mathbf{x})$.

Having introduced these axioms, I now proceed to define the *shareholder majority rule*. For $x \in \mathbb{R}$, let

$$\tau(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0. \end{cases}$$

Shareholder majority rule: For all $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $f(R, \mathbf{x}) = \tau\left(\sum_{i \in N} R_i \mathbf{x}_i\right)$.

The shareholder majority rule is the unique corporate voting rule that satisfies these axioms. Furthermore, none of the four axioms is implied by the other three.

Theorem 1. *A corporate voting rule satisfies merger, anonymity, neutrality, and share monotonicity if and only if it is the shareholder majority rule. Furthermore, the four axioms are independent.*

The proof of this theorem is in the appendix.

2.1 Reallocation Invariance

The next axiom, *reallocation invariance*, is motivated by the idea that individuals may be able to manipulate the identity of their shares owners' to the extent that ownership is relevant as far as voting rights are concerned. For example, if large blocks of shares were to receive disproportionately strong voting rights, likeminded shareholders may be able to combine their shares into a holding company, which becomes the sole owner of the shares.[4] The shareholders would then receive stock in the holding company. The transaction could be structured so that these shareholders could leave the holding company, and take their stock with them, in case that they wish to sell it or wish to vote differently from their fellow holding company participants. On the other hand, if small blocks of shares were to receive disproportionately strong voting rights, then a larger shareholder could partition her shares into several holding corporations. These are but a few of a wide variety of techniques that can be used to disguise the true ownership of the shares; for more see **?**.

[4]Depending on its' size, such a transaction may trigger S.E.C. reporting requirements.

Reallocation invariance: For $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$, if for all $j, k \in S$, $R_j = R_k$, and for all $\ell \notin S$, $\mathbf{x}'_\ell = \mathbf{x}_\ell$, then $f(R, \mathbf{x}') = f(R, \mathbf{x})$.

The reallocation invariance axiom is stronger than anonymity axiom. It says not only that individuals should not be able to improve their position by hiding shares, but also that groups of individuals should not be able to improve their position by combining together, and that single individuals should not be able to improve their position by hiding their shares in several separate entities.

Lemma 1. *A corporate voting rule satisfies reallocation invariance only if it satisfies anonymity.*

The reallocation invariance axiom, in conjunction with neutrality and share monotonicity, characterizes with the shareholder majority rule.

Theorem 2. *A corporate voting rule satisfies reallocation invariance, neutrality, and share monotonicity if and only if it is the shareholder majority rule. Furthermore, the three axioms are independent.*

3 Quotas Rules

I next characterize a new family of rules, *quotas rules*, that includes the shareholder majority rule. Under a quotas rule, there are two thresholds that determine whether a resolution passes, fails, or ties. If the number of votes in favor, as a proportion of total votes cast in favor of or against the resolution, is above the higher threshold, then the resolution passes. If that number is below the lower threshold, it fails. If it is in between, then the vote leads to an indecisive result. The thresholds are set so that if there are some votes in favor, but none opposed, then the resolution must pass; similarly, if there are some votes opposed, but none in favor, the resolution must fail. The rule also specifies the result if the proportion of votes is exactly equal to one of the quotas, or if no votes are cast in favor of or against the resolution. The two quotas can be the same, and must be whenever the rule does not admit the possibility of an indecisive result.

Quotas rules: A corporate voting rule f is a *quotas rule* if there exist constants $p \in \mathcal{R}$, $q, r \in [-1, 1]$, and $s, t \in \{0, 1\}$ where $q \leq r$, $(q, s) \neq (-1, 0)$, $(r, t) \neq (1, 0)$, and $q = r$ implies $st = 0$, such that for all $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, if $\sum_i |R_i| \mathbf{x}_i = 0$ then $f(R, \mathbf{x}) = p$, and if $\sum_i |R_i| \mathbf{x}_i > 0$, then

$$f(R, \mathbf{x}) \geq 0 \text{ if and only if } \tau \left(\sum_i (R_i - q\,|R_i|)\, \mathbf{x}_i \right) \geq s, \text{ and}$$

$$f(R, \mathbf{x}) \leq 0 \text{ if and only if } \tau \left(\sum_i (R_i - r\,|R_i|)\, \mathbf{x}_i \right) \leq -t.$$

To demonstrate the wide flexibility of this class, consider the rule where $p = -1$, $q = 0$, $r = \frac{1}{5}$, and $s = t = 0$. In this rule, a resolution passes if it gets at least sixty percent of the non-abstaining vote, and it fails if it has no more than half of that vote. If fewer than sixty percent but greater than one-half of the non-abstaining vote is in favor, or if there are no non-abstaining votes, then this rule leads to an indecisive result.

The quotas rules also contains two important subclasses. First, the *supermajority* rules are those rules for which there is a single threshold. The resolution passes if the number of votes in favor exceeds the threshold; it fails if the number of votes in favor falls short. The class of "supermajority rules" includes the shareholder majority rule as a special case, as well as "submajority" rules that are biased in favor of the resolution.

Supermajority rules: A corporate voting rule f is a *supermajority rule* if it is a quotas rule for which $q = r$.

An example of a supermajority rule is the *two-thirds rule*, where $p = -1$, $q = r = \frac{1}{3}$, $s = 0$, and $t = 1$. Under the two-thirds rule, a resolution passes if it gets at least two-thirds of the non-abstaining vote, and fails otherwise.

Another example of a supermajority rule is *majority rule without indifference*, where $p = -1$, $q = r = s = 0$, and $t = 1$. This is a form of majority rule in which a resolution passes if it gets at greater than one half of the non-abstaining vote, and fails otherwise. Neither the two-thirds rule nor majority rule without indifference allow for the possibility of an indecisive result.

The shareholder majority rule is a quotas rule where $p = q = r = s = t = 0$. As described above, a resolution passes if it gets at greater than one half of the non-abstaining vote, fails if it gets less than one half, and leads to an indecisive result in the event of a tie.

The second important subclass of rules in this family is that of the balanced quotas rules, where the quotas are symmetric around zero.

Balanced quotas rules: A corporate voting rule f is a *balanced quotas rule* if it is a quotas rule for which $p = 0$, $q = -r$, and $s = t$.

For example, consider the rule where $p = 0$, $q = -\frac{1}{2}$, $r = \frac{1}{2}$, and $s = t = 0$. In this rule, a resolution passes if it gets at least three-fourths of the non-abstaining vote, and it fails if it has no more than one-fourth of that vote. If fewer than three-fourths but greater than one-fourth of the non-abstaining vote is in favor, or if there are no non-abstaining votes, then this rule leads to an indecisive result. The shareholder majority rule is also a balanced quota rule, as both of its' quotas are equal to zero.

As mentioned above, quotas rules have the feature that they ignore abstaining votes whenever some votes are non-abstaining. This property can be formalized in an axiom which requires that, for a given set of individuals and preferences, if the shares change, but the proportions of shares (out of non-abstaining votes) remains the same, and in one case, no votes are abstaining, then the outcome must not change.

Abstention: For every $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_i |R_i|\mathbf{x}_i > 0$, if for all $j \in N$, $\mathbf{x}'_j = \frac{|R_j|\mathbf{x}_j}{\sum_i |R_i|\mathbf{x}_i}$, then $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

Quota rules also have the property that if all shareholders support a resolution, that resolution passes. Similarly, if all shareholders are opposed to the resolution, then it fails.

Unanimity: For every $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, if there exists $k \in \{1, -1\}$ such that for all $i \in N$, $R_i = k$, then $f(R, \mathbf{x}) = k$.

Along with anonymity and merger, these properties characterize the quotas rules.

Theorem 3. *A corporate voting rule satisfies anonymity, abstention, unanimity, and merger if and only if it is a quotas rule. Furthermore, the four axioms are independent.*

The supermajority rules are quota rules that satisfy share monotonicity.

Proposition 1. *A corporate voting rule satisfies anonymity, abstention, unanimity, merger, and share monotonicity if and only if it is a supermajority rule. Furthermore, the five axioms are independent.*

The balanced quota rules are quota rules that satisfy neutrality.

Proposition 2. *A corporate voting rule satisfies anonymity, abstention, unanimity, merger, and neutrality if and only if it is a balanced quotas rule. Furthermore, the five axioms are independent.*

As described above, the shareholder majority rule is both a supermajority rule and a balanced quotas rule. It is the only such rule. This is straightforward; any rule that is both a supermajority rule and a balanced quotas rule must satisfy share monotonicity and neutrality; any rule that satisfies those axioms in combination with anonymity and merger must, by Theorem 1, be the shareholder majority rule.

Corollary 1. *A corporate voting rule is the shareholder majority rule if and only if it is a supermajority rule and a balanced quotas rule.*

4 Other rules

In this section I describe several important corporate voting rules that are worthy of further study. I do not provide full characterizations of these rules, but I do explain which of the axioms are satisfied by them. These claims are then used to prove the independence of the axioms used in the characterization results above.

Polynomial majority rules are those for which majority rule is applied to the shareholdings transformed by an exponent. Three important polynomial majority rules are (1) $\alpha = 0$, the *one person-one vote rule*, where each shareholder has an equal vote, (2) $\alpha = \frac{1}{2}$, *square-root voting* (**?**), where each shareholder receives a number of votes equal to the square-root of her holdings, and (3) $\alpha = 1$, the shareholder majority rule.

Polynomial majority rules: A corporate voting rule f is a *polynomial majority rule* if there is an $\alpha \in \mathbb{R}_+$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \tau\left(\sum_i R_i(\mathbf{x}_i)^\alpha\right)$.

The polynomial majority rules fail to satisfy reallocation invariance and merger, except in the case of the shareholder majority rule. They satisfy all of the other axioms, except for share monotonicity in the specific case of the one person-one vote rule.

Claim 1. *The polynomial majority rules satisfy anonymity, neutrality, abstention, and unanimity, but satisfy share monotonicity only for $\alpha > 0$ and reallocation invariance and merger only for $\alpha = 1$.*

I provide two examples of rules that fail anonymity, and therefore reallocation invariance. The first is the class of weighted majority rules, which assign a weight to each shareholder, by which the shareholdings are multiplied. The shareholder majority rule is a majority style rule where $\delta_i = \frac{1}{|N|}$ for all $i \in N$.

Weighted majority rules: A corporate voting rule f is a *weighted majority rule* if there is a strictly positive set of weights $\delta \in \operatorname{int}\{\Delta(N)\}$ for which $f(R, \mathbf{x}) = \tau\left(\sum_{i \in N} R_i \delta_i \mathbf{x}_i\right)$.

The weighted majority rules satisfy all axioms except for anonymity and reallocation invariance.

Claim 2. *The weighted majority rules satisfy neutrality, share monotonicity, merger, unanimity, and abstention, but do not necessarily satisfy reallocation invariance or anonymity.*

The second example is the lexicographic dictator rule. According to this rule, there is a list of individuals, and the rule proceeds by choosing the opinion of the first shareholder on the list with a strict preference and a positive holding. If no such individual exists, then the rule leads to an indecisive result.

For $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, let

$$d(R, \mathbf{x}) = \begin{cases} \min\{i : |R_i|\mathbf{x}_i > 0\}, & \text{if } \{i : |R_i|\mathbf{x}_i > 0\} \neq \varnothing \\ \min\{i : \mathbf{x}_i > 0\}, & \text{otherwise.} \end{cases}$$

Lexicographic dictator rule: $f(R, \mathbf{x}) = R_{d(R,\mathbf{x})}$.

The lexicographic dictator rule satisfies all axioms except for anonymity and reallocation invariance.

Claim 3. *The lexicographic dictator rule satisfies neutrality, share monotonicity, merger, unanimity, and abstention, but fails anonymity and reallocation invariance.*

The next rule is the constant rule, under which voting is irrelevant. The outcome under a constant rule is always the same, regardless of the preferences and the shareholdings.

Constant rules: There exists $k \in \mathcal{R}$ such that $f(R, \mathbf{x}) = k$.

All constant rules satisfy neutrality or share monotonicity, but not both. They fail unanimity, but satisfy the other axioms.

Claim 4. *The constant rules satisfy anonymity, reallocation invariance, merger, and abstention, satisfy neutrality if and only if $k = 0$, satisfy share monotonicity if and only if $k \neq 0$, and fail to satisfy unanimity for all $k \in \mathcal{R}$.*

A quorum rule is one in which the outcome is determined by the shareholder majority rule, under the condition that not too many shares abstain. If too many shares abstain, then there is an indecisive result.

Quorum rules: There exists an $r \in (0, 1)$ such that $f(R, \mathbf{x}) = \tau\left(\sum_i R_i \mathbf{x}_i\right)$ if $\sum_i |R_i| \mathbf{x}_i > r$; otherwise $f(R, \mathbf{x}) = 0$.

The quorum rules satisfy all axioms except for share monotonicity, merger, and abstention. In conjunction with Claim 2, it establishes that reallocation invariance and merger are logically independent.

Claim 5. *The quorum rules satisfy anonymity, neutrality, reallocation invariance, and unanimity, but do not satisfy share monotonicity, merger, or abstention.*

The absolute majority rule ignores abstaining votes. The resolution passes if greater than one-half of all votes, including those that abstain, are in favor. It fails if greater than one-half of all votes are opposed. If neither of these events occur, otherwise the result is indecisive.

Absolute majority rule: For $k \in \{-1, 1\}$, $f(R, \mathbf{x}) = k$ if $\sum_{i:R_i=k} \mathbf{x}_i > \frac{1}{2}$.

The absolute majority rule satisfies all axioms except for share monotonicity and abstention.

Claim 6. *The absolute majority rule satisfies anonymity, neutrality, reallocation invariance, merger, and unanimity, but does not satisfy share monotonicity or abstention.*

The last class of rules that I describe are *phantom voter rules.*[5] These rules are similar to majority rule, but with a handicap; they operate as if there are "phantom" voters who have already voted their shares. Here, t is the number of phantom votes, as a fraction of the total outstanding stock, that are in favor of the resolution. A handicap of $t < 0$ implies that these phantom votes are opposed. The shareholder majority rule is a phantom voter rule where $t = 0$.

Phantom voter rules: A corporate voting rule f is a *phantom voter rule* if there is a $t \in (-1, 1)$ such that $f(R, \mathbf{x}) = \tau\left(t + \sum_{i \in N} R_i \mathbf{x}_i\right)$.

The phantom voter rules satisfy all axioms except for neutrality and abstention.

Claim 7. *The phantom voter rules satisfy anonymity, share monotonicity, merger, reallocation invariance, and unanimity, but may fail to satisfy neutrality and abstention.*

Appendix

First, for a domain $\mathcal{Q}^* \subseteq \mathcal{Q}$, I define the following property:

Shareholder majority on $\mathcal{Q}^*$ property : For all $N \in \mathcal{N}$ and all $(R, \mathbf{x}) \in \mathcal{Q}_N \cap \mathcal{Q}^*$,

$$f(R, \mathbf{x}) = \tau\left(\sum_{i \in N} R_i \mathbf{x}_i\right)$$

It is straightforward to see that a corporate voting rule satisfies the shareholder majority on $\mathcal{Q}$ property if and only if it is identical to the shareholder majority rule.

Let $\mathcal{N}^3 \equiv \{N \in \mathcal{N} : |N| = 3\}$. Let $\mathcal{Q}^3 \subseteq \bigcup_{N \in \mathcal{N}^3} \mathcal{Q}_N$ be the set of problems for which, for all $N \in \mathcal{N}^3$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $R_i \neq R_j$ for all $\{i, j\} \subseteq N$.

Lemma 2. *A corporate voting rule satisfies anonymity, neutrality, and share monotonicity only if it satisfies the shareholder majority on $\mathcal{Q}^3$ property.*

Proof of Lemma 2. Let f satisfy anonymity, neutrality, and share monotonicity. Let $N \in \mathcal{N}^3$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N \cap \mathcal{Q}^3$. Let $j, k, \ell \in N$ such that $R_j = 1$, $R_k = -1$, and $R_\ell = 0$. Let $\pi \in \Pi_N$ such that $\pi(j) = k$ and $\pi(k) = j$. Note that $\sum_i R_i \mathbf{x}_i = \mathbf{x}_j - \mathbf{x}_k$.

Step one: I show that $f(R, \mathbf{x}) = -f(R, \pi\mathbf{x})$. By anonymity, $f(R, \mathbf{x}) = f(\pi R, \pi\mathbf{x})$. Because $\pi R = -R$, it follows that $f(R, \mathbf{x}) = f(-R, \pi\mathbf{x})$. By neutrality, $f(-R, \pi\mathbf{x}) = -f(R, \pi\mathbf{x})$, and therefore $f(R, \mathbf{x}) = -f(R, \pi\mathbf{x})$.

Step two: I show that if $\tau\left(\sum_i R_i \mathbf{x}_i\right) = 0$ then $f(R, \mathbf{x}) = 0$. Let $\tau\left(\sum_i R_i \mathbf{x}_i\right) = 0$. Then $\mathbf{x}_j = \mathbf{x}_k$, which implies that $\mathbf{x} = \pi\mathbf{x}$. From step one it follows that $f(R, \mathbf{x}) = -f(R, \mathbf{x}) = 0$.

[5] These rules are clearly different from the "phantom voter" result of **?**, but are similar in that they operate as if shares have been voted.

Step three: I show that if $\tau\left(\sum_i R_i\mathbf{x}_i\right) = 1$ then $f(R,\mathbf{x}) = 1$. Let $\tau\left(\sum_i R_i\mathbf{x}_i\right) = 1$ and assume contrariwise that that $f(R,\mathbf{x}) \neq 1$. Then by step one, $f(R,\pi\mathbf{x}) \in (0,1)$. Because $\tau\left(\sum_i R_i\mathbf{x}_i\right) = 1$, it follows that $\mathbf{x}_j > \mathbf{x}_k$. Because (a) $R_j = 1$, (b) $R_k = -1$, (c) $\pi\mathbf{x}_j < \mathbf{x}_j$, and (d) $\pi\mathbf{x}_\ell = \mathbf{x}_\ell$, it follows from share monotonicity that $f(R,\mathbf{x}) = 1$, a contradiction.

Step four: I show that if $\tau\left(\sum_i R_i\mathbf{x}_i\right) = -1$ then $f(R,\mathbf{x}) = -1$. Let $\tau\left(\sum_i R_i\mathbf{x}_i\right) = -1$. Then $\mathbf{x}_j < \mathbf{x}_k$. By step three, $f(R,\pi\mathbf{x}) = 1$. By step one, $f(R,\mathbf{x}) = -1$. □

Proof of Theorem 1. **Only if:** Let f satisfy anonymity, neutrality, share monotonicity, and merger. Let $N \in \mathcal{N}$ and let $(R,\mathbf{x}) \in \mathcal{Q}_N$. Without loss of generality, assume that $\mathbf{x}_i > 0$ for all $i \in N$. For $k \in \mathcal{R}$, define $S^k \equiv \{i \in N : R_i = k\}$. Let $\mathbf{y} \in \Delta(\mathcal{R})$ such that for $k \in \mathcal{R}$, $\mathbf{y}_k = \sum_{i\in S^k} \mathbf{x}_i$.

For $k \in \mathcal{R}$ and $i \in S^k$, let $\omega_i = \frac{\mathbf{x}_i}{\mathbf{y}_k}$ and let $\mathbf{z}^i \in [0,1]^N$ such that $\mathbf{z}^i_i = \mathbf{y}_k$ and $\mathbf{z}^i_j = 0$ for $j \neq i$. For $k \in \mathcal{R}$, if $|S^k| \neq \varnothing$, then let $\mathcal{S}^k = S^k$. If $|S^k| = \varnothing$, let $\mathcal{S}^k = \{i^k\}$, where $\mathcal{S}^k \cap N = \varnothing$, where $\omega_{i^k} = 1$, and where $\mathbf{z}^{i^k}_j = 0$ for all $j \in N$. For $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, let $\mathbf{x}^{jk\ell} \in \Delta(N)$ such that $\mathbf{x}^{jk\ell} = \mathbf{z}^j + \mathbf{z}^k + \mathbf{z}^\ell$.

$$\text{Note that } \mathbf{x} = \sum_{j\in\mathcal{S}^1}\sum_{k\in\mathcal{S}^{-1}}\sum_{\ell\in\mathcal{S}^0} \omega_j\omega_k\omega_\ell\mathbf{x}^{jk\ell}.$$

Because f satisfies anonymity, neutrality, and share monotonicity, it follows from Lemma 2 that, for $j,j' \in \mathcal{S}^1$, $k,k' \in \mathcal{S}^{-1}$, and $\ell,\ell' \in \mathcal{S}^0$, $f((R,\mathbf{x}^{jk\ell})|_{\{j,k,\ell\}\cap N}) = f((R,\mathbf{x}^{j'k'\ell'})|_{\{j',k',\ell'\}\cap N})$. Therefore by invariance to non-shareholders, $f(R,\mathbf{x}^{jk\ell}) = f(R,\mathbf{x}^{j'k'\ell'})$. By construction, the sets $\mathcal{S}^k$ are finite; thus, it follows from merger that for all $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, $f(R,\mathbf{x}) = f(R,\mathbf{x}^{jk\ell})$.

Furthermore, by Lemma 2, for all $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, $f(R,\mathbf{x}^{jk\ell}) = \tau\left(\sum_{i\in N} R_i\mathbf{x}^{jk\ell}_i\right)$. It follows that $f(R,\mathbf{x}) = \tau\left(\sum_{i\in N} R_i\mathbf{x}_i\right)$; *i.e.*, that f is the shareholder majority rule.

If: The shareholder majority rule is both a polynomial majority rule and a weighted majority rule. Therefore, it satisfies anonymity and neutrality (by Claim 1) and share monotonicity and merger (by Claim 2).

Independence of the Axioms: That the four axioms are independent follows from Claims 1, 2, and 4. □

Proof of Lemma 1. Let f be a corporate voting rule that satisfies reallocation invariance. Let $N \in \mathcal{N}$ and let $(R,\mathbf{x}) \in \mathcal{Q}_N$ and $\pi \in \Pi_N$. Without loss of generality, let $N = \{1,...,n\}$. Suppose, contrariwise, that $f(R,\mathbf{x}) \neq f(\pi R,\pi\mathbf{x})$.

Step one: I show that if there is a set $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$, then for $(R',\mathbf{x}') \in \mathcal{Q}_{N'}$, if there is a one-to-one mapping $\omega : N' \to N$ such that $(R,\mathbf{x}) = (\omega R',\omega\mathbf{x}')$, then $f(R,\mathbf{x}) = f(R',\mathbf{x}')$.

Let $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$. Without loss of generality, let $N' = \{n+1,...,2n\}$. Let $R' \in \mathcal{R}^{N'}$ such that $R'_i = R_{i-n}$ for $i \in N'$. Let $\mathbf{x}' \in \Delta(N')$ such that $\mathbf{x}'_i = \mathbf{x}_{i-n}$ for $i \in N'$. Note that, for $\omega(i) = n - i$, $(R,\mathbf{x}) = (\omega R',\omega\mathbf{x}')$.

Let $R^* \in \mathcal{R}^{N \cup N'}$ such that (i) $R^*_i = R_i$ for $i \in N$ and (ii) $R^*_i = R'_i$ for $i \in N'$. Let $\mathbf{x}^{\circ\circ} \in \Delta(N \cup N')$ such that (a) $\mathbf{x}^{\circ\circ}_i = \mathbf{x}_i$ for $i \in N$ and (b) $\mathbf{x}^{\circ\circ}_i = 0$ for $i \in N'$. Note that $(R^*, \mathbf{x}^{\circ\circ})|_N = (R, \mathbf{x})$.

Let $\mathbf{x}^{\circ\bullet} \in \Delta(N \cup N')$ such that (a) for $i \in N$, (i) $\mathbf{x}^{\circ\bullet}_i = 0$ if $R^*_i = 1$ and (ii) $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}_i$ if $R^*_i \neq 1$, and (b) for $i \in N'$, (i) $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}'_i$ if $R^*_i = 1$ and (ii) $\mathbf{x}^{\circ\bullet}_i = 0$ if $R^*_i \neq 1$. Let $S^1 \equiv \{i \in N \cup N' : R^*_i = 1\}$. For all $i \notin S^1$, $\mathbf{x}^{\circ\circ}_i = \mathbf{x}^{\circ\bullet}_i$. Thus, by reallocation invariance, $f(R^*, \mathbf{x}^{\circ\circ}) = f(R^*, \mathbf{x}^{\circ\bullet})$.

Let $\mathbf{x}^{\bullet\circ} \in \Delta(N \cup N')$ such that (a) for $i \in N$, (i) $\mathbf{x}^{\bullet\circ}_i = 0$ if $R^*_i \neq 0$ and (ii) $\mathbf{x}^{\bullet\circ}_i = \mathbf{x}_i$ if $R^*_i = 0$, and (b) for $i \in N'$, (i) $\mathbf{x}^{\bullet\circ}_i = \mathbf{x}'_i$ if $R^*_i \neq 0$ and (ii) $\mathbf{x}^{\bullet\circ}_i = 0$ if $R^*_i = 0$. Let $S^{-1} \equiv \{i \in N \cup N' : R^*_i = -1\}$. For all $i \notin S^{-1}$, $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}^{\bullet\circ}_i$. Thus, by reallocation invariance, $f(R^*, \mathbf{x}^{\circ\bullet}) = f(R^*, \mathbf{x}^{\bullet\circ})$.

Let $\mathbf{x}^{\bullet\bullet} \in \Delta(N \cup N')$ such that (a) $\mathbf{x}^{\bullet\bullet}_i = 0$ for $i \in N$ and (b) $\mathbf{x}^{\bullet\bullet}_i = \mathbf{x}'_i$ for $i \in N'$. Note that $(R^*, \mathbf{x}^{\bullet\bullet})|_{N'} = (R', \mathbf{x}')$. Let $S^0 \equiv \{i \in N \cup N' : R^*_i = 0\}$. For all $i \notin S^0$, $\mathbf{x}^{\bullet\circ}_i = \mathbf{x}^{\bullet\bullet}_i$. Thus, by reallocation invariance, $f(R^*, \mathbf{x}^{\bullet\circ}) = f(R^*, \mathbf{x}^{\bullet\bullet})$. As a consequence, it follows that $f(R, \mathbf{x}) = f\left(f(R^*, \mathbf{x}^{\bullet\bullet})|_{N'}\right) = f(R', \mathbf{x}')$.

Step two: Let $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$, and let ω be a one-to-one mapping from N' to N. Let $(R', \mathbf{x}') \in \mathcal{Q}_{N'}$ such that $(R, \mathbf{x}) = (\omega R', \omega \mathbf{x}')$. It follows from step one that $f(R, \mathbf{x}) = f(R', \mathbf{x}')$.

Let ω' be a one-to-one mapping from N' to N such that for all $i \in N'$, $\omega'(i) = \pi(\omega(i))$. Then $(\pi R, \pi \mathbf{x}) = (\omega' R', \omega' \mathbf{x}')$. It follows from step one that $f(\pi R, \pi \mathbf{x}) = f(R', \mathbf{x}')$. Therefore, $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$. $\square$

Proof of Theorem 2. **Only if:** Let f satisfy neutrality, share monotonicity, and reallocation invariance. By Lemma 1, because f satisfies reallocation invariance it satisfies anonymity.

Let $N \in \mathcal{N}$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N$. Without loss of generality, assume that $\{1,2,3\} \cap N = \varnothing$. Let $(R^*, \mathbf{x}^*) \in \mathcal{Q}_{\{1,2,3\}}$ such that $R^* = (1, -1, 0)$ and, for all $i \in \{1,2,3\}$, $\mathbf{x}^*_i = \sum_{\{j \in N : R_j = R^*_i\}} \mathbf{x}_j$. Define $N^+ \equiv \{1,2,3\} \cup N$.

Step one. I show that $f(R, \mathbf{x}) = f(R^*, \mathbf{x}^*)$. Let $(R', \mathbf{x}'), (R', \mathbf{x}'') \in \mathcal{Q}_{N^+}$ such that (a) for $i \in \{1,2,3\}$, $R'_i = R^*_i$, $\mathbf{x}'_i = \mathbf{x}^*_i$, and $\mathbf{x}''_i = 0$, and (b) for $j \in N$, $R'_j = R_j$, $\mathbf{x}'_j = 0$, and $\mathbf{x}''_j = \mathbf{x}_j$. Note that $(R', \mathbf{x}')|_{\{1,2,3\}} = (R^*, \mathbf{x}^*)$ and that $(R', \mathbf{x}'')|_N = (R, \mathbf{x})$.

For $k \in R$, define $S^k \equiv \{i \in N^+ : R'_i = k\}$. Let $\mathbf{x}^\circ \in \Delta(N^+)$ such that $\mathbf{x}^\circ_1 = \mathbf{x}'_1$, $\mathbf{x}^\circ_i = 0$ for $i \in S^1 \setminus \{1\}$, and $\mathbf{x}^\circ_j = \mathbf{x}''_j$ for $j \notin S^1$. Let $\mathbf{x}^\bullet \in \Delta(N^+)$ such that $\mathbf{x}^\bullet_2 = \mathbf{x}'_2$, $\mathbf{x}^\bullet_i = 0$ for $i \in S^{-1} \setminus \{2\}$, and $\mathbf{x}^\bullet_j = \mathbf{x}^\circ_j$ for $j \notin S^{-1}$.

Because $R'_i = R'_j$ for all $i, j \in S^1$ and because $\mathbf{x}^\circ_k = \mathbf{x}''_k$ for $k \notin S^1$, it follows from reallocation invariance that $f(R', \mathbf{x}^\circ) = f(R', \mathbf{x}'')$. Because $R'_i = R'_j$ for all $i, j \in S^{-1}$ and because $\mathbf{x}^\bullet_k = \mathbf{x}^\circ_k$ for $k \notin S^{-1}$, it follows from reallocation invariance that $f(R', \mathbf{x}^\bullet) = f(R', \mathbf{x}^\circ)$. Because $R'_i = R'_j$ for all $i, j \in S^0$ and because $\mathbf{x}'_k = \mathbf{x}^\bullet_k$ for $k \notin S^0$, it follows from reallocation invariance that $f(R', \mathbf{x}') = f(R', \mathbf{x}^\bullet)$. Hence, $f(R', \mathbf{x}') = f(R', \mathbf{x}'')$. Because f is invariant to non-shareholders, it follows that $f(R, \mathbf{x}) = f(R^*, \mathbf{x}^*)$.

Step two. By construction, $\sum_{i \in N} R_i \mathbf{x}_i = \sum_{i \in N} R^*_i \mathbf{x}^*_i$ and $(R^*, \mathbf{x}^*) \in \mathcal{Q}^3$. By anonymity, neutrality, and share monotonicity, it follows from Lemma 2 that $f(R^*, \mathbf{x}^*) = \tau\left(\sum_i R_i^* \mathbf{x}_i^*\right)$. By step one, $f(R, \mathbf{x}) = f(R^*, \mathbf{x}^*)$ and therefore $f(R, \mathbf{x}) = \tau\left(\sum_i R_i \mathbf{x}_i\right)$.

If: The shareholder majority rule is a polynomial majority rule and a phantom voter rule. Therefore, it satisfies neutrality (by Claim 1) and share monotonicity and reallocation invariance (by Claim 7).

Independence of the Axioms: That the three axioms are independent follows from Claims 1 and 4. □

Proof of Theorem 3. **Only if:**

Let f satisfy anonymity, abstention, unanimity, and merger.

Part one. Let $N \in \mathcal{N}^3$, let $j, k, \ell \in N$, and let $R \in \mathcal{R}^N$ such that $R_j = 1$, $R_k = -1$, $R_\ell = 0$. For $z \in [0,1]$ let $\mathbf{x}^z \in \Delta(N)$ such that $\mathbf{x}_j^z = z$, $\mathbf{x}_k^z = 1 - z$, and $\mathbf{x}_\ell^z = 0$. By unanimity and invariance to non-shareholders, $f(R, \mathbf{x}^1) = 1$ and $f(R, \mathbf{x}^0) = -1$.

Let $\alpha = \sup\{z \in [0,1] : f(R, \mathbf{x}^z) = -1\}$, and let $\beta = \inf\{z \in [0,1] : f(R, \mathbf{x}^z) = 1\}$. Let $q = 2\alpha - 1$. Let $r = 2\beta - 1$. Because $\alpha, \beta \in [0,1]$ it follows that $q, r \in [-1,1]$. Let $s = 1$ if $f\left(R, \mathbf{x}^\alpha\right) = -1$, otherwise let $s = 0$. Note that if $q = -1$ then $\alpha = 0$, which implies (by unanimity) that $s = 1$. Let $t = 1$ if $f\left(R, \mathbf{x}^\beta\right) = 1$, otherwise let $s = 0$. Note that if $r = 1$ then $\beta = 1$, which implies (by unanimity) that $t = 1$. Note that if $q = r$ and $st = 1$ then $\alpha = \beta$ but $-1 = f\left(R, \mathbf{x}^\alpha\right) = f\left(R, \mathbf{x}^\beta\right) = 1$, a contradiction. Let $\ddot{\mathbf{x}} \in \Delta(N)$ such that $\ddot{\mathbf{x}}_j = \ddot{\mathbf{x}}_k = 0$ and $\ddot{\mathbf{x}}_\ell = 1$. Note that for $\mathbf{x} \in \Delta(N)$, $\sum_i |R_i| \mathbf{x}_i = 0$ if and only if $\mathbf{x} = \ddot{\mathbf{x}}$. Let $p = f(R, \ddot{\mathbf{x}})$.

Let $\mathbf{x} \in \Delta(N) \setminus \{\ddot{\mathbf{x}}\}$ such that $\tau\left(\sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}_i\right) \geq s$. I show that $f(R, \mathbf{x}) \geq 0$. Define $\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x}'_j = \frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k}$, $\mathbf{x}'_k = \frac{\mathbf{x}_k}{\mathbf{x}_j + \mathbf{x}_k}$, and $\mathbf{x}'_\ell = 0$. By abstention, $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Note that $\tau\left(\sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}'_i\right) = \tau\left(2\mathbf{x}'_j - 1 - q\right) = \tau\left(2(\mathbf{x}'_j - \alpha)\right)$. If $s = 0$ then $\mathbf{x}'_j \geq \alpha$; if $s = 1$ then this tells us that $\mathbf{x}'_j > \alpha$. Clearly, if $s = 0$ and $\mathbf{x}'_j = \alpha$ then by construction, $f(R, \mathbf{x}^\alpha) \geq 0$. Otherwise, if $\mathbf{x}_j > \alpha$, by construction, $f(R, \mathbf{x}) \geq 0$.

Suppose, contrariwise, that $\mathbf{x} \in \Delta(N) \setminus \{\ddot{\mathbf{x}}\}$ such that $\tau\left(\sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}_i\right) < s$ but that that $f(R, \mathbf{x}) \geq 0$. Define $\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x}'_j = \frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k}$, $\mathbf{x}'_k = \frac{\mathbf{x}_k}{\mathbf{x}_j + \mathbf{x}_k}$, and $\mathbf{x}'_\ell = 0$. Note that $\sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}_i = \left(\mathbf{x}_i + \mathbf{x}_j\right) \sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}'_i$ and therefore $\tau\left(\sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}_i\right) = \tau\left(\sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}'_i\right)$. By abstention, $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $\tau\left(\sum_i \left(R_i - q\,|R_i|\right) \mathbf{x}'_i\right) < s$ it follows that $\tau\left(2(\mathbf{x}'_j - \alpha)\right) < s$, and therefore that either (a) $s = 1$ and $\mathbf{x}'_j = \alpha$, (b) $s = 1$ and $\mathbf{x}'_j < \alpha$ or (c) $s = 0$ and $\mathbf{x}'_j < \alpha$. If (a) then $\mathbf{x}' = \mathbf{x}^\alpha$ and therefore $f\left(R, \mathbf{x}'\right) = -1$, a contradiction. If (b) then there exists $z' \in (\mathbf{x}'_j, \alpha)$ such that $f(R, \mathbf{x}^{z'}) = -1$. If (c) then there exists $z' \in (\mathbf{x}'_j, \alpha]$ such that $f(R, \mathbf{x}^{z'}) = -1$. Because there exists $\lambda \in (0,1)$ such that $\mathbf{x}' = \lambda \mathbf{x}^{z'} + (1 - \lambda)\mathbf{x}^0$, it follows from merger that $f(R, \mathbf{x}') = -1$, a contradiction.

Let $\mathbf{x} \in \Delta(N) \setminus \{\ddot{\mathbf{x}}\}$ such that $\tau\left(\sum_i \left(R_i - r\,|R_i|\right) \mathbf{x}_i\right) \leq -t$. I show that $f(R, \mathbf{x}) \leq 0$. Define $\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x}'_j = \frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k}$, $\mathbf{x}'_k = \frac{\mathbf{x}_k}{\mathbf{x}_j + \mathbf{x}_k}$, and $\mathbf{x}'_\ell = 0$. By abstention, $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Note that $\tau\left(\sum_i \left(R_i - r\,|R_i|\right) \mathbf{x}'_i\right) = \tau\left(2\mathbf{x}'_j - 1 - r\right) = \tau\left(2(\mathbf{x}'_j - \beta)\right)$. If $t = 0$ then $\mathbf{x}'_j \leq \beta$; if $t = 1$ then this tells us that $\mathbf{x}'_j < \beta$. Clearly, if $t = 0$ and $\mathbf{x}'_j = \beta$ then by construction, $f(R, \mathbf{x}^\beta) \leq 0$. Otherwise, if $\mathbf{x}_j < \beta$, by construction, $f(R, \mathbf{x}) \leq 0$.

Suppose, contrariwise, that $\mathbf{x} \in \Delta(N) \setminus \{\ddot{\mathbf{x}}\}$ such that $\tau\left(\sum_i (R_i - r|R_i|)\,\mathbf{x}_i\right) > -t$ but that that $f(R, \mathbf{x}) \leq 0$. Define $\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x}'_j = \frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k}$, $\mathbf{x}'_k = \frac{\mathbf{x}_k}{\mathbf{x}_j + \mathbf{x}_k}$, and $\mathbf{x}'_\ell = 0$. Note that $\sum_i (R_i - r|R_i|)\,\mathbf{x}_i = (\mathbf{x}_i + \mathbf{x}_j)\sum_i (R_i - r|R_i|)\,\mathbf{x}'_i$ and therefore $\tau\left(\sum_i (R_i - r|R_i|)\,\mathbf{x}_i\right) = \tau\left(\sum_i (R_i - r|R_i|)\,\mathbf{x}'_i\right)$. By abstention, $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $\tau\left(\sum_i (R_i - r|R_i|)\,\mathbf{x}'_i\right) > -t$ it follows that $\tau\left(2(\mathbf{x}'_j - \beta)\right) > -t$, and therefore that either (a) $t = 1$ and $\mathbf{x}'_j = \beta$, (b) $t = 1$ and $\mathbf{x}'_j > \beta$ or (c) $t = 0$ and $\mathbf{x}'_j > \beta$. If (a) then $\mathbf{x}' = \mathbf{x}^\beta$ and therefore $f(R, \mathbf{x}') = 1$, a contradiction. If (b) then there exists $z' \in (\beta, \mathbf{x}'_j)$ such that $f(R, \mathbf{x}^{z'}) = 1$. If (c) then there exists $z' \in [\beta, \mathbf{x}'_j)$ such that $f(R, \mathbf{x}^{z'}) = 1$. Because there exists $\lambda \in (0,1)$ such that $\mathbf{x}' = \lambda \mathbf{x}^{z'} + (1-\lambda)\mathbf{x}^1$, it follows from merger that $f(R, \mathbf{x}') = 1$, a contradiction.

Lastly, I show that $q \leq r$. Suppose, contrariwise, that $q > r$. Let $\mathbf{x}^{\frac{q+r+2}{4}} \in \Delta(N)$. Then $\sum_i (R_i - q|R_i|)\,\mathbf{x}_i^{\frac{q+r+2}{4}} = (1-q)\frac{q+r+2}{4} - (1+q)\frac{2-q-r}{4} = \frac{r-q}{2}$. Because, by supposition, $q > r$, it follows that $\tau\left(\sum_i (R_i - q|R_i|)\,\mathbf{x}_i\right) < s$ and therefore $f(R, \mathbf{x}) < 0$. Also, $\sum_i (R_i - r|R_i|)\,\mathbf{x}_i^{\frac{q+r+2}{4}} = (1-r)\frac{q+r+2}{4} - (1+r)\frac{2-q-r}{4} = \frac{q-r}{2}$. Because, by supposition, $q > r$, it follows that $\tau\left(\sum_i (R_i - r|R_i|)\,\mathbf{x}_i\right) > -t$ and therefore $f(R, \mathbf{x}) > 0$, a contradiction.

Part two. Let $N' \in \mathcal{N}^3$, let $j', k', \ell' \in N$, and let $R' \in \mathcal{R}^N$ such that $R'_{j'} = 1$, $R'_{k'} = -1$, $R'_{\ell'} = 0$. Let $\mathbf{x} \in \Delta(N)$ and $\mathbf{x} \in \Delta(N')$ such that $\mathbf{x}_j = \mathbf{x}'_{j'}$, $\mathbf{x}_k = \mathbf{x}'_{k'}$, and $\mathbf{x}_\ell = \mathbf{x}'_{\ell'}$. I show that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

Let $N^* = \{j^*, k^*, \ell^*\} \in \mathcal{N}^3$ such that $N \cap N^* = N' \cap N^* = \varnothing$. Let $R^\circ \in \mathcal{R}^{N \cup N^*}$ such that $R^\circ_j = R^\circ_{j^*} = 1$, $R^\circ_k = R^\circ_{k^*} = -1$, and $R^\circ_\ell = R^\circ_{\ell^*} = 0$. Let $\mathbf{x}^\circ, \mathbf{x}^{\circ\circ} \in \Delta(N \cup N^*)$ such that (a) $\mathbf{x}^\circ_j = \mathbf{x}_j$, $\mathbf{x}^\circ_k = \mathbf{x}_k$, $\mathbf{x}^\circ_\ell = \mathbf{x}_\ell$, and $\mathbf{x}^\circ_{j^*} = \mathbf{x}^\circ_{k^*} = \mathbf{x}^\circ_{\ell^*} = 0$, and (b) $\mathbf{x}^{\circ\circ}_{j^*} = \mathbf{x}_j$, $\mathbf{x}^{\circ\circ}_{k^*} = \mathbf{x}_k$, $\mathbf{x}^{\circ\circ}_{\ell^*} = \mathbf{x}_\ell$, and $\mathbf{x}^{\circ\circ}_j = \mathbf{x}^{\circ\circ}_k = \mathbf{x}^{\circ\circ}_\ell = 0$. Let $\pi \in \Pi_{N \cup N^*}$ such that $\pi(j) = j^*$, $\pi(k) = k^*$, $\pi(\ell) = \ell^*$, $\pi(j^*) = j$, $\pi(k^*) = k$ and $\pi(\ell^*) = \ell$. Then $\pi R^\circ = R^\circ$, $\pi \mathbf{x}^\circ = \mathbf{x}^{\circ\circ}$. It follows from anonymity that $f(R^\circ, \mathbf{x}^\circ) = f(\pi R^\circ, \pi \mathbf{x}^\circ) = f(R^\circ, \mathbf{x}^{\circ\circ})$. Because $(R, \mathbf{x}) = (R^\circ, \mathbf{x}^\circ)|_N$, it follows from invariance to non-shareholders that $f(R, \mathbf{x}) = f(R^\circ, \mathbf{x}^{\circ\circ})$.

Let $R^\bullet \in \mathcal{R}^{N' \cup N^*}$ such that $R^\bullet_j = R^\bullet_{j^*} = 1$, $R^\bullet_k = R^\bullet_{k^*} = -1$, and $R^\bullet_\ell = R^\bullet_{\ell^*} = 0$. Let $\mathbf{x}^\bullet, \mathbf{x}^{\bullet\bullet} \in \Delta(N' \cup N^*)$ such that (a) $\mathbf{x}^\bullet_{j'} = \mathbf{x}_j$, $\mathbf{x}^\bullet_{k'} = \mathbf{x}_k$, $\mathbf{x}^\bullet_{\ell'} = \mathbf{x}_\ell$, and $\mathbf{x}^\bullet_{j^*} = \mathbf{x}^\bullet_{k^*} = \mathbf{x}^\bullet_{\ell^*} = 0$, and (b) $\mathbf{x}^{\bullet\bullet}_{j^*} = \mathbf{x}_j$, $\mathbf{x}^{\bullet\bullet}_{k^*} = \mathbf{x}_k$, $\mathbf{x}^{\bullet\bullet}_{\ell^*} = \mathbf{x}_\ell$, and $\mathbf{x}^{\bullet\bullet}_{j'} = \mathbf{x}^{\bullet\bullet}_{k'} = \mathbf{x}^{\bullet\bullet}_{\ell'} = 0$. Let $\pi' \in \Pi_{N' \cup N^*}$ such that $\pi'(j') = j^*$, $\pi'(k') = k^*$, $\pi'(\ell') = \ell^*$, $\pi'(j^*) = j'$, $\pi'(k^*) = k'$ and $\pi'(\ell^*) = \ell'$. Then $\pi R^\bullet = R^\bullet$, $\pi \mathbf{x}^\bullet = \mathbf{x}^{\bullet\bullet}$. It follows from anonymity that $f(R^\bullet, \mathbf{x}^\bullet) = f(\pi R^\bullet, \pi \mathbf{x}^\bullet) = f(R^\bullet, \mathbf{x}^{\bullet\bullet})$. Because $(R', \mathbf{x}') = (R^\bullet, \mathbf{x}^\bullet)|_N$, it follows from invariance to non-shareholders that $f(R', \mathbf{x}') = f(R^\bullet, \mathbf{x}^{\bullet\bullet})$. Because $(R^\circ, \mathbf{x}^{\circ\circ})|_{N^*} = (R^\bullet, \mathbf{x}^{\bullet\bullet})|_{N^*}$, it follows from invariance to non-shareholders that $f(R, \mathbf{x}) = f(R', \mathbf{x}')$.

Part three. Let $N \in \mathcal{N}$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N$. Without loss of generality, assume that $\mathbf{x}_i > 0$ for all $i \in N$. For $k \in \mathcal{R}$, define $S^k \equiv \{i \in N : R_i = k\}$. Let $\mathbf{y} \in \Delta(\mathcal{R})$ such that for $k \in \mathcal{R}$, $\mathbf{y}_k = \sum_{i \in S^k} \mathbf{x}_i$.

For $k \in \mathcal{R}$ and $i \in S^k$, let $\omega_i = \frac{\mathbf{x}_i}{\mathbf{y}_k}$ and let $\mathbf{z}^i \in [0,1]^N$ such that $\mathbf{z}^i_i = \mathbf{y}_k$ and

$\mathbf{z}^i_j = 0$ for $j \neq i$. For $k \in \mathcal{R}$, if $|S^k| \neq \varnothing$, then let $\mathcal{S}^k = S^k$. If $|S^k| = \varnothing$, let $\mathcal{S}^k = \{i^k\}$, where $\mathcal{S}^k \cap N = \varnothing$, where $\omega_{i^k} = 1$, and where $\mathbf{z}^{i^k}_j = 0$ for all $j \in N$. For $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, let $\mathbf{x}^{jk\ell} \in \Delta(N)$ such that $\mathbf{x}^{jk\ell} = \mathbf{z}^j + \mathbf{z}^k + \mathbf{z}^\ell$.

$$\text{Note that } \mathbf{x} = \sum_{j \in \mathcal{S}^1} \sum_{k \in \mathcal{S}^{-1}} \sum_{\ell \in \mathcal{S}^0} \omega_j \omega_k \omega_\ell \mathbf{x}^{jk\ell}.$$

By step two, for $j, j' \in \mathcal{S}^1$, $k, k' \in \mathcal{S}^{-1}$, and $\ell, \ell' \in \mathcal{S}^0$, $f((R, \mathbf{x}^{jk\ell})|_{\{j,k,\ell\} \cap N}) = f((R, \mathbf{x}^{j'k'\ell'})|_{\{j',k',\ell'\} \cap N})$. Therefore by invariance to non-shareholders, $f(R, \mathbf{x}^{jk\ell}) = f(R, \mathbf{x}^{j'k'\ell'})$. By construction, the sets $\mathcal{S}^k$ are finite; thus, it follows from merger that for all $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, $f(R, \mathbf{x}) = f(R, \mathbf{x}^{jk\ell})$. This implies that f is the quotas rule with the constants defined in part one.

If: Let f be a quotas rule with constants p, q, r, s, t. I show that it satisfies the four axioms.

Anonymity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and let $\pi \in \Pi_N$. If $\sum_i |R_i| \mathbf{x}_i = 0$, then $\sum_i |R_{\pi(i)}| \mathbf{x}_{\pi(i)} = 0$, thus $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$. Otherwise, note that for $q \in [-1, 1]$, $\sum_i (R_i - q |R_i|) \mathbf{x}_i = \sum_{\pi(i)} \left(R_{\pi(i)} - q \left|R_{\pi(i)}\right|\right) \mathbf{x}_{\pi(i)} = \sum_i \left(R_{\pi(i)} - q \left|R_{\pi(i)}\right|\right) \mathbf{x}_{\pi(i)}$ and therefore $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$.

Abstention. Let $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_i |R_i| \mathbf{x}_i > 0$ and such that, for all $j \in N$, $\mathbf{x}'_j = \frac{|R_j| \mathbf{x}_j}{\sum_i |R_i| \mathbf{x}_i}$. By construction, $R_i \mathbf{x}_i \neq 0$ and $R_i \mathbf{x}'_i \neq 0$ for some $i \in N$. Note that for for all $i \in N$ such that $\mathbf{x}'_i > 0$, $\mathbf{x}'_i = \frac{\mathbf{x}_i}{\sum_j R_j \mathbf{x}_j}$. Consequently, for $q \in [-1, 1]$, $\sum_i (R_i - q |R_i|) \mathbf{x}'_i = (\sum_j R_j \mathbf{x}_j)^{-1} \sum_i (R_i - q |R_i|) \mathbf{x}_i$, and therefore $f(R, \mathbf{x}) = f(R, \mathbf{x}')$..

Unanimity. Let $k \in \{1, -1\}$ such that for all $i \in N$, $R_i = k$. By construction, $\sum_i |R_i| \mathbf{x}_i > 0$. Consequently, for $q \in [-1, 1]$, $\sum_i (R_i - q |R_i|) \mathbf{x}_i = (R_i - q |R_i|) \sum_i \mathbf{x}_i = R_i - q |R_i|$. If $R_i = 1$ then $R_i - q |R_i| = 1 - q$. First, for $r < 1$, $\tau (1 - r) > 0$. If $r = 1$ then $t = 1$, and thus $\tau (0) > -1$. Consequently, $f(R, \mathbf{x}) = 1$

Merger. Let $N \in \mathcal{N}$, $\lambda \in (0, 1)$, and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Note that if $\sum_i |R_i| \mathbf{x}_i = \sum_i |R_i| \mathbf{x}'_i = 0$, then $R_i (\lambda \mathbf{x}_i + (1 - \lambda) \mathbf{x}'_i) = 0$ for all $i \in N$, so merger is satisfied. Next, if $\sum_i |R_i| \mathbf{x}_i = 0$ but $\sum_i |R_i| \mathbf{x}_i > 0$, then $f(R, \mathbf{x}) = f(R, \mathbf{x}') = p$. For $q \in [-1, 1]$, $\sum_i (R_i - q |R_i|) (\lambda \mathbf{x}_i + (1 - \lambda) \mathbf{x}'_i) = (1 - \lambda) \sum_i (R_i - q |R_i|) \mathbf{x}'_i$ and consequently, $f(R, \lambda \mathbf{x} + (1 - \lambda) \mathbf{x}') = f(R, \mathbf{x}')$. Lastly, if $\sum_i |R_i| \mathbf{x}_i > 0$ and $R_i \mathbf{x}'_i \neq 0$ for some $i \in N$, then $\sum_i (R_i - q |R_i|) (\lambda \mathbf{x}_i + (1 - \lambda) \mathbf{x}'_i) = \lambda \sum_i (R_i - q |R_i|) \mathbf{x}_i + (1 - \lambda) \sum_i (R_i - q |R_i|) \mathbf{x}'_i$. Hence, it follows that $\tau \left(\sum_i (R_i - q |R_i|) (\lambda \mathbf{x}_i + (1 - \lambda) \mathbf{x}'_i)\right) = \tau \left(\sum_i (R_i - q |R_i|) \mathbf{x}_i\right)$ and therefore $f(R, \lambda \mathbf{x} + (1 - \lambda) \mathbf{x}') = f(R, \mathbf{x}')$.

Independence of the Axioms: That the four axioms are independent follows from Claims 1, 2, 4, and 6. □

Proof of Proposition 1. **Only if:** Let f be a quotas rule with constants p, q, r, s, t that satisfies share monotonicity, and suppose by means of contradiction that $q < r$. Let $N = \{j, k\} \mathcal{N}$, let $R \in \mathcal{R}^N$ such that $R_j = 1$ and $R_k = -1$. Note that for all $\mathbf{x}$, $\sum_i (R_i - q |R_i|) \mathbf{x}_i = 2\mathbf{x}_j - p - 1$.

Let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x}_j = \frac{2q+r+3}{6}$ and $\mathbf{x}'_j = \frac{q+2r+3}{6}$. Then $\tau\left(\sum_i (R_i - q\,|R_i|)\,\mathbf{x}_i\right) = \tau\left(2\frac{2q+r+3}{6} - q - 1\right) = \tau\left(\frac{r-q}{3}\right)$. Because, by supposition, $q < r$, it follows that $\frac{r-q}{3} > 0$ and therefore $f(R, \mathbf{x}) \geq 0$. Furthermore $\tau\left(\sum_i (R_i - r\,|R_i|)\,\mathbf{x}_i\right) = \tau\left(2\frac{2q+r+3}{6} - r - 1\right) = \tau\left(\frac{2q-2r}{3}\right)$. Because $\frac{2q-2r}{3} < 0$ it follows that $f(R, \mathbf{x}) \leq 0$ and consequently that $f(R, \mathbf{x}) = 0$.

Next, $\tau\left(\sum_i (R_i - q\,|R_i|)\,\mathbf{x}'_i\right) = \tau\left(2\frac{q+2r+3}{6} - q - 1\right) = \tau\left(\frac{-2q+2r}{3}\right)$. Because $\frac{-2q+2r}{3} > 0$ it follows that $f(R, \mathbf{x}') \geq 0$. Furthermore $\tau\left(\sum_i (R_i - r\,|R_i|)\,\mathbf{x}'_i\right) = \tau\left(2\frac{q+2r+3}{6} - r - 1\right) = \tau\left(\frac{q-r}{3}\right)$. Because $\frac{q-r}{3} < 0$ it follows that $f(R, \mathbf{x}') \leq 0$ and consequently that $f(R, \mathbf{x}') = 0$. By share monotonicity, $R_j = 1$, $R_k = -1$, $\mathbf{x}_j < \mathbf{x}'_j$, and $f(R, \mathbf{x}) = 0$; it follows that $f(R, \mathbf{x}) = 1$, a contradiction.

If: Let f be a quota rule with constants p, q, r, s, t such that $q = r$. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}) \in \mathcal{Q}_N$, and $j, k \in N$ such that $R_j = 1$, $R_k \neq 1$, $\mathbf{x}_j < \mathbf{x}'_j$, $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \setminus \{j, k\}$, and such that $f(R, \mathbf{x}) \in \{0, 1\}$. I will show that $f(R, \mathbf{x}') = 1$.

Because $f(R, \mathbf{x}) \in \{0, 1\}$, it follows that $\tau\left(\sum_i (R_i - q\,|R_i|)\,\mathbf{x}_i\right) \geq s$, which implies that $\sum_i (R_i - q\,|R_i|)\,\mathbf{x}_i \geq 0$. This last expression is equal to $(R_j - q\,|R_j|)\,\mathbf{x}_j + (R_k - q\,|R_k|)\,\mathbf{x}_k \geq 0 - \sum_{i \neq j,k} (R_\ell - q\,|R_\ell|)\,\mathbf{x}_\ell$. Because $\mathbf{x}_j < \mathbf{x}'_j$ it follows that $(R_j - q\,|R_j|)\,\mathbf{x}'_j + (R_k - q\,|R_k|)\,\mathbf{x}'_k > (R_j - q\,|R_j|)\,\mathbf{x}_j + (R_k - q\,|R_k|)\,\mathbf{x}_k$. Becaues $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for $\ell \neq j, k$, it follows that $\sum_{\ell \neq j,k} (R_\ell - q\,|R_\ell|)\,\mathbf{x}_\ell = \sum_{\ell \neq j,k} (R_\ell - q\,|R_\ell|)\,\mathbf{x}'_\ell$. Thus $(R_j - q\,|R_j|)\,\mathbf{x}'_j + (R_k - q\,|R_k|)\,\mathbf{x}'_k > 0 - \sum_{\ell \neq j,k} (R_\ell - q\,|R_\ell|)\,\mathbf{x}'_\ell$ and therefore that $\sum_i (R_i - q\,|R_i|)\,\mathbf{x}'_i > 0$. Because $q = r$ it follows that $\sum_i (R_i - r\,|R_i|)\,\mathbf{x}'_i > 0$, and that $\tau\left(\sum_i (R_i - r\,|R_i|)\,\mathbf{x}'_i\right) = 1 > -t$. Therefore $f(R, \mathbf{x}') = 1$.

Independence of the Axioms: A quota rule for which $q \neq r$ satisfies anonymity, merger, abstention, or unanimity by Theorem 3, but fails share monotonicity because it is not a supermajority rule. This fact, in conjunction with Claims 1, 2, 4, and 7, is sufficient to establish the independence of the axioms. □

Proof of Proposition 2. **Only if:** Let f be a quotas rule with constants p, q, r, s, t that satisfies neutrality. Let $N = \{j, k\}\mathcal{N}$, let $R \in \mathcal{R}^N$ such that $R_j = 1$ and $R_k = -1$. Note that for all $\mathbf{x}$, $\sum_i (R_i - q\,|R_i|)\,\mathbf{x}_i = 2\mathbf{x}_j - q - 1$ and $\sum_i (-R_i - q\,|-R_i|)\,\mathbf{x}_i = 1 - q - 2\mathbf{x}_j$.

Let $\mathbf{x} \in \Delta(N)$ such that $\mathbf{x}_j = \frac{q-r+2}{4}$ Then $\sum_i (R_i - q\,|R_i|)\,\mathbf{x}_i = 2\frac{q-r+2}{4} - q - 1 = \frac{-q-r}{2}$. Consequently $f(R, \mathbf{x}) \geq 0$ if and only if $\tau(-q-r) \geq s$.

By neutrality, $f(R, \mathbf{x}) = -f(-R, \mathbf{x})$. Then $\sum_i (-R_i - r\,|-R_i|)\,\mathbf{x}_i = 1 - r - 2\frac{q-r+2}{4} = \frac{-r-q}{2}$. Consequently $f(-R, \mathbf{x}) \leq 0$ if and only if $\tau(-q-r) \leq -t$.

It follows from neutrality that (a) $\tau(-q-r) \geq s$ if and only if (b) $\tau(-q-r) \leq -t$. If $-q-r > 0$ then (a) holds (for all s) but not (b) (for any t). Thus $-q-r \leq 0$. If $-q-r < 0$ then (b) holds (for all t) but not (a) (for any (s). Thus $-q-r = 0$. If $s = 0$ then (a) holds, which implies that (b) holds, which implies that $t = 0$. If $s = 1$ then (a) does not hold, which implies that (b) does not hold, which implies that $t = 1$.

Let $N = \{'ell\}\mathcal{N}$, let $R \in \mathcal{R}^N$ such that $R_\ell = 0$, and let $\mathbf{x} \in \Delta(N)$ such that $\mathbf{x}_\ell = 1$. By the definition of the quotas rule, $f(R, \mathbf{x}) = p$. By neutrality, $f(-R, \mathbf{x}) = -p$.

Because $R = -R$, this implies that $p = -p = 0$.

If: Let f be a quota rule with constants p, q, r, s, t such that $p = 0$, $q = -r$, and $s = t$. I will show that for all $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, that $f(R, \mathbf{x}) \geq 0$ implies that $f(-R, \mathbf{x}) \leq 0$ and that $f(R, \mathbf{x}) \leq 0$ implies that $f(-R, \mathbf{x}) \geq 0$.

Let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \mathbf{x}_i \neq 0$ for some $i \in N$.. If $f(R, \mathbf{x}) \geq 0$ then $\tau\left(\sum_i (R_i - q\,|R_i|)\,\mathbf{x}_i\right) \geq s$ This implies that $\tau\left(-\sum_i (R_i - q\,|R_i|)\,\mathbf{x}_i\right) \leq -s$, which in turn implies that $\tau\left(\sum_i (-R_i + q\,|-R_i|)\,\mathbf{x}_i\right) \leq -s$. Substituting $r = -q$ and $s = t$ we have that $\tau\left(\sum_i (-R_i - r\,|-R_i|)\,\mathbf{x}_i\right) \leq -t$. Consequently, we know that $f(R, \mathbf{x}) \geq 0$ implies that $f(-R, \mathbf{x}) \leq 0$.

If $f(R, \mathbf{x}) \leq 0$ then $\tau\left(\sum_i (R_i - r\,|R_i|)\,\mathbf{x}_i\right) \leq -t$. This implies that $\tau\left(-\sum_i (R_i - r\,|R_i|)\,\mathbf{x}_i\right) \geq t$, which in turn implies that $\tau\left(\sum_i (-R_i + r\,|-R_i|)\,\mathbf{x}_i\right) \geq t$. Substituting $r = -q$ and $s = t$ we have that $\tau\left(\sum_i (-R_i - q\,|-R_i|)\,\mathbf{x}_i\right) \geq s$. Consequently, $f(R, \mathbf{x}) \leq 0$ implies that $f(-R, \mathbf{x}) \geq 0$.

Next, let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \mathbf{x}_i = 0$ for every $i \in N$. Then $f(R, \mathbf{x}) = p = 0$. Because $R_i \mathbf{x}_i = 0$ for every $i \in N$ it follows that $-R_i \mathbf{x}_i = 0$ for every $i \in N$. Consequently, it follows that $f(-R, \mathbf{x}) = 0$, and therefore $-f(-R, \mathbf{x}) = 0 = -p$.

Independence of the Axioms: A quota rule for which $q \neq -r$ satisfies anonymity, merger, abstention, or unanimity by Theorem 3, but fails share monotonicity because it is not a supermajority rule. This fact, in conjunction with Claims 1, 2, 4, and 6, is sufficient to establish the independence of the axioms. $\square$

Proof of Claim 1. I show that all polynomial majority rules satisfy anonymity, neutrality, abstention, and unanimity, that share monotonicity is satisfied if and only if $\alpha > 0$, and that reallocation invariance and merger are satisfied if and only if $\alpha = 1$.

Anonymity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, $\pi \in \Pi_N$, and $\alpha \in \mathbb{R}_+$. Note that $\sum_{i \in N} R_i(\mathbf{x}_i)^\alpha = \sum_{\pi(i) \in N} R_{\pi(i)}(\mathbf{x}_{\pi(i)})^\alpha = \sum_{i \in N} R_{\pi(i)}(\mathbf{x}_{\pi(i)})^\alpha$; thus $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$.

Neutrality. Let $(R, \mathbf{x}) \in \mathcal{Q}$. Because $\sum_i -R_i(\mathbf{x}_i)^\alpha = -\sum_i R_i(\mathbf{x}_i)^\alpha$, it follows that $\tau\left(\sum_i -R_i(\mathbf{x}_i)^\alpha\right) = -\tau\left(\sum_i R_i(\mathbf{x}_i)^\alpha\right)$, and therefore that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Abstention. Let $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_i |R_i| \mathbf{x}_i > 0$ and, for all $j \in N$, $\mathbf{x}'_j = \frac{|R_j| \mathbf{x}_j}{\sum_i |R_i| \mathbf{x}_i}$. Then $\sum_i R_i(\mathbf{x}'_i)^\alpha = \sum_i R_i\left(\frac{|R_i i \mathbf{x}_i}{\sum_j |R_j| \mathbf{x}_j}\right)^\alpha = \frac{1}{\sum_j |R_j| \mathbf{x}_j} \sum_i R_i(|R_i| \mathbf{x}_i)^\alpha$, which implies that $\tau\left(\sum_i R_i(\mathbf{x}'_i)^\alpha\right) = \tau\left(\sum_i R_i(\mathbf{x}_i)^\alpha\right)$. Therefore $f(R, \mathbf{x}) = f(R, \mathbf{x}')$

Unanimity. Let $k \in \{-1, 1\}$ and let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i = k$ for all $i \in N$. Then $\sum_i R_i(\mathbf{x}_i)^\alpha = k \sum_i (\mathbf{x}_i)^\alpha$, and therefore $\tau\left(\sum_i R_i(\mathbf{x}_i)^\alpha\right) = \tau(k) = k$.

Share monotonicity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j \neq 0$, (b) $R_k \neq R_j$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \notin N \setminus \{j, k\}$, and (e) $f(R, \mathbf{x}) \neq -R_j$. If $\alpha = 0$, then $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$, a contradiction. Let $\alpha > 0$. From (e) it follows that $R_j(\sum_i R_i(\mathbf{x}_i)^\alpha) \geq 0$. Therefore, $R_j\left(R_j(\mathbf{x}_j)^\alpha + R_k(\mathbf{x}_k)^\alpha\right) \geq R_j\left(0 - \sum_{\ell \in N \setminus \{j,k\}} R_\ell(\mathbf{x}_\ell)^\alpha\right)$. From (c) it follows that $(\mathbf{x}_j)^\alpha < (\mathbf{x}'_j)^\alpha$. From (d) it follows that $\mathbf{x}_j + \mathbf{x}_k = \mathbf{x}'_j + \mathbf{x}'_k$ and thus from (c) that $(\mathbf{x}'_k)^\alpha < (\mathbf{x}_k)^\alpha$. Together this implies that $(\mathbf{x}'_j)^\alpha - (\mathbf{x}'_k)^\alpha > (\mathbf{x}_j)^\alpha - (\mathbf{x}_k)^\alpha$. If $R_k = -R_j$, then $(\mathbf{x}'_j)^\alpha - (\mathbf{x}'_k)^\alpha > (\mathbf{x}_j)^\alpha - (\mathbf{x}_k)^\alpha$ implies that $R_j \left(R_j(\mathbf{x}'_j)^\alpha + R_k(\mathbf{x}'_k)^\alpha\right) > R_j \left(R_j(\mathbf{x}_j)^\alpha + R_k(\mathbf{x}_k)^\alpha\right)$. If $R_k = 0$, then this is implied by the fact that $(\mathbf{x}_j)^\alpha < (\mathbf{x}'_j)^\alpha$. By (d) it follows that $\sum_{\ell\in N\setminus\{j,k\}} R_\ell(\mathbf{x}_\ell)^\alpha = \sum_{\ell\in N\setminus\{j,k\}} R_\ell(\mathbf{x}'_\ell)^\alpha$. Putting this together, we have that $R_j(R_j(\mathbf{x}'_j)^\alpha + R_k(\mathbf{x}'_k)^\alpha) > R_j(0 - \sum_{\ell\in N\setminus\{j,k\}} R_\ell(\mathbf{x}'_\ell)^\alpha)$, and therefore that $R_j\left(\sum_i R_i(\mathbf{x}'_i)^\alpha\right) > 0$. Therefore $f(R,\mathbf{x}') = R_j$.

Merger. If $\alpha = 1$ then f is the shareholder majority rule, which is a weighted majority rule, and therefore it satisfies merger by Claim 2. Let $\alpha \neq 1$. Let $N = \{1,2,3\}$, and let $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1,-1,-1)$, $\mathbf{x} = (\frac{1}{2},\frac{1}{2},0)$, and $\mathbf{x}' = (\frac{1}{2},0,\frac{1}{2})$. In this case, $f(R,\mathbf{x}) = f(R,\mathbf{x}') = 0$. However, $f(R,\frac{1}{2}\mathbf{x}+\frac{1}{2}\mathbf{x}') = 0$ if and only if $(\frac{1}{2})^\alpha = 2(\frac{1}{4})^\alpha$ which is false for $\alpha \neq 1$.

Reallocation invariance. If $\alpha = 1$ then f is the shareholder majority rule, which is a phantom voter rule, and therefore it satisfies reallocation invariance by Claim 7. Let $\alpha \neq 1$. Let $N = \{1,2,3\}$, and let $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1,-1,-1)$, $\mathbf{x} = (\frac{1}{2},\frac{1}{2},0)$, and $\mathbf{x}' = (\frac{1}{2},\frac{1}{4},\frac{1}{4})$. Note that for $S = \{2,3\}$ the predicate of the axiom is satisfied. However, for $\sum_i R_i(\mathbf{x}_i)^\alpha = 0$ while $\sum_i R_i(\mathbf{x}'_i)^\alpha = \left(\frac{1}{2}\right)^\alpha - 2\left(\frac{1}{4}\right)^\alpha$. By reallocation invariance, $\left(\frac{1}{2}\right)^\alpha - 2\left(\frac{1}{4}\right)^\alpha = 0$, which is false for $\alpha \neq 1$. $\square$

Proof of Claim 2. I show that weighted majority rules satisfy neutrality, share monotonicity, merger, abstention, and unanimity, but may fail to satisfy anonymity and reallocation invariance.

Neutrality. Let $(R,\mathbf{x}) \in \mathcal{Q}$. Because $\sum_i -R_i\delta_i\mathbf{x}_i = -\sum_i R_i\delta_i\mathbf{x}_i$, it follows that $\tau\left(\sum_i -R_i\delta_i\mathbf{x}_i\right) = -\tau\left(\sum_i R_i\delta_i\mathbf{x}_i\right)$, and therefore that $f(-R,\mathbf{x}) = -f(R,\mathbf{x})$.

Share monotonicity. Let $N \in \mathcal{N}$, $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$, and $j,k \in N$ such that (a) $R_j \neq 0$, (b) $R_k \neq R_j$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \notin N\setminus\{j,k\}$, and (e) $f(R,\mathbf{x}) \neq -R_j$. From (e) it follows that $R_j(\sum_i R_i\delta_i\mathbf{x}_i) \geq 0$. Therefore, $R_j\left(R_j\delta_j\mathbf{x}_j + R_k\delta_k\mathbf{x}_k\right) \geq R_j\left(0 - \sum_{\ell\in N\setminus\{j,k\}} R_\ell\delta_\ell\mathbf{x}_\ell\right)$. From (d) it follows that $\mathbf{x}_j + \mathbf{x}_k = \mathbf{x}'_j + \mathbf{x}'_k$ and thus from (c) it follows that $\mathbf{x}'_j - \mathbf{x}'_k > \mathbf{x}_j - \mathbf{x}_k$. If $R_k = -R_j$, then $\mathbf{x}'_j - \mathbf{x}'_k > \mathbf{x}_j - \mathbf{x}_k$ implies that $R_j\left(R_j\delta_j\mathbf{x}'_j + R_k\delta_k\mathbf{x}'_k\right) > R_j\left(R_j\delta_j\mathbf{x}_j + R_k\delta_k\mathbf{x}_k\right)$. If $R_k = 0$, then this fact is implied by (c). By (d) it follows that $\sum_{\ell\in N\setminus\{j,k\}} R_\ell\delta_\ell\mathbf{x}_\ell = \sum_{\ell\in N\setminus\{j,k\}} R_\ell\delta_\ell\mathbf{x}'_\ell$. Putting this together, we have that $R_j(R_j\delta_j\mathbf{x}'_j + R_k\delta_k\mathbf{x}'_k) > R_j(0 - \sum_{\ell\in N\setminus\{j,k\}} R_\ell\delta_\ell\mathbf{x}'_\ell)$, and therefore that $R_j\left(\sum_i R_i\delta_i\mathbf{x}'_i\right) > 0$. Therefore $f(R,\mathbf{x}') = R_j$.

Merger. Let $N \in \mathcal{N}$, $\lambda \in (0,1)$, and $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$ such that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$. Because $\sum_i R_i\delta_i\left(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i\right) = \lambda\left(\sum_i R_i\delta_i\mathbf{x}_i\right) + (1-\lambda)\left(\sum_i R_i\delta_i\mathbf{x}'_i\right)$, it follows that

$$\max\{\sum_i R_i\delta_i\mathbf{x}_i, \sum_i R_i\delta_i\mathbf{x}'_i\} \geq \sum_i R_i\delta_i\left(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i\right) \geq \min\{\sum_i R_i\delta_i\mathbf{x}_i, \sum_i R_i\delta_i\mathbf{x}'_i\},$$

and consequently that $\tau(\sum_i R_i\delta_i\mathbf{x}_i) = \tau(\sum_i R_i\delta_i\left(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i\right))$. Therefore $f(R,\mathbf{x}) = f(R,\mathbf{x}') = f(R,\lambda\mathbf{x} + (1-\lambda)\mathbf{x}')$.

Abstention. Let $N \in \mathcal{N}$ and $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_i |R_i|\mathbf{x}_i > 0$ and, for all $j \in N$, $\mathbf{x}'_j = \frac{|R_j|\mathbf{x}_j}{\sum_i |R_i|\mathbf{x}_i}$. Then $\sum_i R_i\delta_i\mathbf{x}'_i = \sum_i R_i\delta_i \frac{|R_i i\mathbf{x}_i}{\sum_j |R_j|\mathbf{x}_j} = \frac{1}{\sum_j |R_j|\mathbf{x}_j}\sum_i R_i\delta_i\mathbf{x}_i$, which implies that $\tau\left(\sum_i R_i\delta_i\mathbf{x}'_i\right) = \tau\left(\sum_i R_i\delta_i\mathbf{x}_i\right)$. Therefore $f(R,\mathbf{x}) = f(R,\mathbf{x}')$

Unanimity. Let $k \in \{-1,1\}$ and let $N \in \mathcal{N}$ and $(R,\mathbf{x}) \in \mathcal{Q}_N$ such that $R_i = k$ for all $i \in N$. Then $\sum_i R_i\delta_i\mathbf{x}_i = k\sum_i \delta_i\mathbf{x}_i$, and therefore $\tau\left(\sum_i R_i\delta_i\mathbf{x}_i\right) = \tau(k) = k$.

Anonymity. Let $(R,\mathbf{x}) \in \mathcal{Q}_{\{1,2\}}$ such that $R = (1,-1)$ and $\mathbf{x} = (\frac{1}{2},\frac{1}{2})$, let $\delta = (\frac{2}{3},\frac{1}{3}) \in \operatorname{int}\{\Delta(\{1,2\})\}$, and let $\pi \in \Pi_{\{1,2\}}$ such that $\pi(1) = 2$ and $\pi(2) = 1$. Then $f(R,\mathbf{x}) = \tau(\frac{1}{6}) = 1$ but $f(\pi R,\pi\mathbf{x}) = \tau(-\frac{1}{6}) = -1$.

Reallocation invariance. Because weighted majority rules may fail to satisfy anonymity, they may fail to satisfy reallocation invariance, by Lemma 1. $\square$

Proof of Claim 3. I show that the lexicographic dictator rule satisfies neutrality, share monotonicity, merger, abstention, and unanimity, but fails anonymity and reallocation invariance.

Neutrality. Let $N \in \mathcal{N}$ and let $(R,\mathbf{x}) \in \mathcal{Q}_N$. By the lexicographic dictator rule, $f(-R,\mathbf{x}) = -R_{d(-R,\mathbf{x})}$ and $-f(R,\mathbf{x}) = -R_{d(R,\mathbf{x})}$. By the definition of $d(R,\mathbf{x})$, $d(R,\mathbf{x}) = d(-R,\mathbf{x})$, and consequently, $f(-R,\mathbf{x}) = -f(R,\mathbf{x})$.

Share monotonicity. Let $N \in \mathcal{N}$, $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$, and $i,j \in N$ such that (a) $R_i \neq 0$, (b) $R_j \neq R_i$, (c) $\mathbf{x}_i < \mathbf{x}'_i$, (d) $\mathbf{x}_k = \mathbf{x}'_k$ for all $k \notin N \setminus \{i,j\}$, and (e) $f(R,\mathbf{x}) \neq -R_i$. From (e) there are two cases, $f(R,\mathbf{x}) = R_i$ and $f(R,\mathbf{x}) = 0$.

Case 1: $f(R,\mathbf{x}) = R_i$. If $i \geq d(R,\mathbf{x})$, then $i \geq d(R,\mathbf{x}')$. So then $d(R,\mathbf{x}) = d(R,\mathbf{x}')$ and therefore $f(R,\mathbf{x}') = R_{d(R,\mathbf{x}')} = R_{d(R,\mathbf{x})} = R_i$. If $i < d(R,\mathbf{x})$, then $d(R,\mathbf{x}') = i$, and therefore, by (a), $f(R,\mathbf{x}') = R_i$.

Case 2: $f(R,\mathbf{x}) = 0$. In this case, by the definition of $d(R,\mathbf{x})$, there does not exist an individual $\ell \in N$ for which $x_\ell > 0$ and $R_\ell \neq 0$. Thus it must be the case that $R_j = \mathbf{x}_i = 0$. By (c) and (d) it follows that $d(R,\mathbf{x}') = i$, and therefore that $f(R,\mathbf{x}') = R_i$.

Merger. Let $N \in \mathcal{N}$, $\lambda \in (0,1)$, and $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$ such that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$. Because $f(R,\mathbf{x}) = f(R,\mathbf{x}')$ it follows that $R_{d(R,\mathbf{x})} = R_{d(R,\mathbf{x}')}$. Because $\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i > 0$ if and only if $\max\{\mathbf{x}_i,\mathbf{x}'_i\} > 0$, it follows that $d(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}') \in \{d(R,\mathbf{x}),d(R,\mathbf{x}')\}$ and hence $f(R,\mathbf{x}) = f(R,\mathbf{x}') = f(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}')$.

Abstention. Let $N \in \mathcal{N}$ and $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_i |R_i|\mathbf{x}_i > 0$ and, for all $j \in N$, $\mathbf{x}'_j = \frac{|R_j|\mathbf{x}_j}{\sum_i |R_i|\mathbf{x}_i}$. Because, for all $j \in N$ such that $R_j \neq 0$, $\mathbf{x}_j > 0$ if and only if $\mathbf{x}'_j > 0$, it follows that $d(R,\mathbf{x}) = d(R,\mathbf{x}')$. Consequently, $f(R,\mathbf{x}) = f(R,\mathbf{x}')$.

Unanimity. Let $k \in \{-1,1\}$ and let $N \in \mathcal{N}$ and $(R,\mathbf{x}) \in \mathcal{Q}_N$ such that $R_i = k$ for all $i \in N$. Then $R_{d(R,\mathbf{x})} = k$ and therefore $f(R,\mathbf{x}) = k$.

Anonymity. Let $(R,\mathbf{x}) \in \mathcal{Q}_{\{1,2\}}$ such that $R = (1,-1)$ and $\mathbf{x} = (\frac{1}{2},\frac{1}{2})$, and let $\pi \in \Pi$ such that $\pi(1) = 2$ and $\pi(2) = 1$. Note that $d(R,\mathbf{x}) = 1$, while $d(\pi R,\pi\mathbf{x}) = 2$. Then $f(R,\mathbf{x}) = R_1 = 1 \neq f(\pi R,\pi\mathbf{x}) = R_2 = -1$.

Reallocation invariance. Because the lexicographic dictator rule fails to satisfy anonymity, it fails to satisfy reallocation invariance, by Lemma 1. $\square$

Proof of Claim 4. The constant rules satisfy anonymity, reallocation invariance, merger, and abstention because these axioms require invariance. I show that the constant rules satisfy neutrality if and only if $k = 0$, share monotonicity if and only if $k \neq 0$, and fail to satisfy unanimity for all $k \in \mathcal{R}$.

NEUTRALITY. Let $k = 0$. Then clearly $f(R, \mathbf{x}) = 0$ and $f(-R, \mathbf{x}) = -f(R, \mathbf{x}) = 0$. Let $k \neq 0$. Then $f(R, \mathbf{x}) = k$ and $f(-R, \mathbf{x}) = -f(R, \mathbf{x}) = -k \neq k$, a contradiction.

SHARE MONOTONICITY. Let $k \neq 0$. Then for all $\mathbf{x}, \mathbf{x}'$, $f(R, \mathbf{x}) \in \{0, 1\}$ implies that $f(R, \mathbf{x}') = 1$. Let $k = 0$. Let $N = \{1, 2\}$, let $R_1 = 1$, let $R_2 \neq 1$, and let $\mathbf{x}_1 < \mathbf{x}'_1$. Then $f(R, \mathbf{x}) = 0$ but $f(R, \mathbf{x}') = 0$, a contradiction.

UNANIMITY. If $k = 0$, let $R_i = 1$ for all $i \in N$. In this case, $f(R, \mathbf{x}) = 0 \neq 1$, a contradiction. If $k \neq 0$ let $R_i = -k$ for all $i \in N$. In this case, $f(R, \mathbf{x}) = k \neq -k$, a contradiction. $\square$

Proof of Claim 5. I show that quorum rules satisfies anonymity, neutrality, reallocation invariance, and unanimity, but fails to satisfy share monotonicity, merger, and abstention. ANONYMITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$. Anonymity follows from the facts that $\sum_i R_i \mathbf{x}_i = \sum_{\pi(i)} R_{\pi(i)} \mathbf{x}_{\pi(i)} = \sum_i R_{\pi(i)} \mathbf{x}_{\pi(i)}$ and that $\sum_i |R_i| \mathbf{x}_i = \sum_{\pi(i)} |R_{\pi(i)}| \mathbf{x}_{\pi(i)} = \sum_i |R_{\pi(i)}| \mathbf{x}_{\pi(i)}$.

NEUTRALITY. Let $(R, \mathbf{x}) \in \mathcal{Q}$. That $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$ follows from the facts that $\sum_i -R_i \mathbf{x}_i = -\sum_i R_i \mathbf{x}_i$ and that $\sum_i |-R_i| \mathbf{x}_i = \sum_i |-R_i| \mathbf{x}_i$.

REALLOCATION INVARIANCE. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$ such that, for all $i, j \in S$, $R_i = R_j$ and for all $k \notin S$, $\mathbf{x}_k = \mathbf{x}'_k$. From the fact that $\mathbf{x}_k = \mathbf{x}'_k$ for $k \notin S$, it follows that (a) $\sum_{k \notin S} R_k \mathbf{x}_k = \sum_{k \notin S} R_k \mathbf{x}'_k$, (b) $\sum_{k \notin S} |R_k| \mathbf{x}_k = \sum_{k \notin S} |R_k| \mathbf{x}'_k$, and (c) $\sum_{i \in S} \mathbf{x}_i = \sum_{i \in S} \mathbf{x}'_i$. From the fact that $R_i = R_j$ for all $i, j \in S$, it follows that (d) $\sum_{i \in S} R_i \mathbf{x}_i = R_i \sum_{i \in S} \mathbf{x}_i$ and $\sum_{i \in S} R_i \mathbf{x}'_i = R_i \sum_{i \in S} \mathbf{x}'_i$ and (e) $\sum_{i \in S} |R_i| \mathbf{x}_i = |R_i| \sum_{i \in S} \mathbf{x}_i$ and $\sum_{i \in S} |R_i| \mathbf{x}'_i = |R_i| \sum_{i \in S} \mathbf{x}'_i$. Combining (a), (c), and (d), it follows that $\sum_i R_i \mathbf{x}_i = \sum_i R_i \mathbf{x}'_i$. Combining (b), (c), and (e), it follows that $\sum_i |R_i| \mathbf{x}_i = \sum_i |R_i| \mathbf{x}'_i$. Thus it follows that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

UNANIMITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $sum_i |R_i| \mathbf{x}_i = 1$ and $\sum_i R_i \mathbf{x}_i = k$. Thus $f(R, \mathbf{x}) = \tau(k) = k$.

SHARE MONOTONICITY. Let $N = \{1, 2\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$ and $R_2 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0, 1)$ and $\mathbf{x}' = (r, 1 - r)$. Then $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$. By share monotonicity, because $R_1 = 1$, $R_2 \neq 1$, $\mathbf{x}_1 < \mathbf{x}'_1$, and $f(R, \mathbf{x}) = 0$, it follows that $f(R, \mathbf{x}') = 1$, a contradiction.

MERGER. Let $N = \{1, 2, 3\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$, $R_2 = -1$, and $R_3 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.5, 0.5, 0)$ and $\mathbf{x}' = (r, 0, 1 - r)$. In this case, $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$. Then $\sum_i |R_i|(\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i) = \lambda + (1 - \lambda)r > r$, thus $f(R, \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}') = \tau\left(\sum_i R_i \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}'\right)$. However, $\sum_i R_i \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}' = \lambda \frac{1}{2} + (1 - \lambda)r - \lambda \frac{1}{2} = (1 - \lambda)r > 0$, which implies that $f(R, \mathbf{x}) = 1$, a contradiction.

ABSTENTION. Let $N = \{1, 2, 3\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$, $R_2 = -1$, and $R_3 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (\frac{2r}{6}, \frac{r}{6}, 1 - \frac{r}{2})$ and $\mathbf{x}' = (\frac{2}{3}, \frac{1}{3}, 0)$. Because $\mathbf{x}'_1 = \frac{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{x}_2}$, $\mathbf{x}'_2 = \frac{\mathbf{x}_2}{\mathbf{x}_1 + \mathbf{x}_2}$, and $\mathbf{x}'_3 = 0$, it follows that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because

$\sum_i |R_i|\mathbf{x}'_i = 1$ and $\sum_i R_i\mathbf{x}'_i = \frac{1}{3}$, $f(R,\mathbf{x}') = 1$. However, because $\sum_i |R_i|\mathbf{x}_i = \frac{r}{2} < r$, $f(R,\mathbf{x}) = 0$, a contradiction. □

Proof of Claim 6. I prove that the absolute majority rule satisfies anonymity, neutrality, merger, reallocation invariance, and unanimity, but does not satisfy share monotonicity or abstention.

ANONYMITY. Let $N \in \mathcal{N}$, $(R,\mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$. Anonymity follows from the fact that $\sum_{i:R_i=k} \mathbf{x}_i = \sum_{\{i\in N:R_i=k\}} \mathbf{x}_i = \sum_{\{\pi(i)\in N:R_{\pi(i)}=k\}} \mathbf{x}_{\pi(i)} = \sum_{\{i\in N:R_{\pi(i)}=k\}} \mathbf{x}_{\pi(i)} = \sum_{i:R_{\pi(i)}=k} \mathbf{x}_{\pi(i)}$.

NEUTRALITY. Let $(R,\mathbf{x}) \in \mathcal{Q}$. Note that $\sum_{i:-R_i=k} \mathbf{x}_i = \sum_{\{i\in N:-R_i=k\}} \mathbf{x}_i = \sum_{\{i\in N:R_i=-k\}} \mathbf{x}_i = \sum_{i:R_i=-k} \mathbf{x}_i$. For $k \in \{-1,1\}$, if $f(R,\mathbf{x}) = k$ then $\sum_{i:R_i=k} \mathbf{x}_i > \frac{1}{2}$, which implies that $\sum_{i:-R_i=-k} \mathbf{x}_i > \frac{1}{2}$ and therefore $f(-R,\mathbf{x}) = -k$. If $f(R,\mathbf{x}) = 0$ then
$sum_{i:R_i=k}\mathbf{x}_i \leq \frac{1}{2}$ for all $k \in \{-1,1\}$. This implies that $\sum_{i:R_i=-k} \mathbf{x}_i = \sum_{i:-R_i=k} \mathbf{x}_i \leq \frac{1}{2}$ for all $k \in \{-1,1\}$ and therefore that $f(-R,\mathbf{x}) = 0$.

MERGER. Let $N \in \mathcal{N}$, $\lambda \in (0,1)$, and $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$ such that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$. Because $f(R,\mathbf{x}) = f(R,\mathbf{x}')$ it follows that either (a) there is a $k \in \{-1,1\}$ such that $f(R,\mathbf{x}) = k$ and therefore $\sum_{i:R_i=k} \mathbf{x}_i, \sum_{i:R_i=k} \mathbf{x}'_i > \frac{1}{2}$ or (b) for all $k \in \{-1,1\}$ $\sum_{i:R_i=k} \mathbf{x}_i, \sum_{i:R_i=k} \mathbf{x}'_i \leq \frac{1}{2}$. Note that $\sum_{i:R_i=k} \lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i = \lambda\sum_{i:R_i=k} \mathbf{x}_i + (1-\lambda)\sum_{i:R_i=k} \mathbf{x}'_i$. If (a) is true, and there is $k \in \{-1,1\}$ such that $\sum_{i:R_i=k} \mathbf{x}_i, \sum_{i:R_i=k} \mathbf{x}'_i > \frac{1}{2}$, then $\sum_{i:R_i=k} \lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i > \frac{1}{2}$ and therefore $f(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}) = f(R,\mathbf{x})$. If (b) is true then for all $k \in \{-1,1\}$, $\sum_{i:R_i=k} \lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i \leq \frac{1}{2}$ and therefore $f(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}) = 0 = f(R,\mathbf{x})$.

REALLOCATION INVARIANCE. Let $N \in \mathcal{N}$, $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$ such that, for all $i,j \in S$, $R_i = R_j$ and for all $k \notin S$, $\mathbf{x}_k = \mathbf{x}'_k$. Let $\ell \in \mathcal{R}$ such that $S \subseteq \{i \in N : R_i = \ell\}$. For all $k \notin S$, $\mathbf{x}_k = \mathbf{x}'_k$, which implies that for $\ell' \neq \ell$, $\sum_{i:R_i=\ell'} \mathbf{x}_i = \sum_{i:R_i=\ell'} \mathbf{x}'_i$. Because, for $\ell \in \mathcal{R}$, $\sum_{i:R_i=\ell} \mathbf{x}_i = 1 - \sum_{i:R_i\neq\ell} \mathbf{x}_i$, it follows that $\sum_{i:R_i=\ell} \mathbf{x}_i = \sum_{i:R_i=\ell} \mathbf{x}'_i$. Consequently, $f(R,\mathbf{x}) = f(R,\mathbf{x}')$.

UNANIMITY. Let $N \in \mathcal{N}$, $(R,\mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1,1\}$ such that $R_i = k$ for all $i \in N$. Then $\sum_{i:R_i=k} \mathbf{x}_i = 1$. Thus $f(R,\mathbf{x}) = k$.

SHARE MONOTONICITY. Let $N = \{1,2,3\}$, let $R = (1,-1,0) \in \mathcal{R}^N$, and let $\mathbf{x},\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.35, 0.45, 0.2)$ and $\mathbf{x}' = (0.45, 0.35, 0.2)$. In this case, $\sum_{i:R_i=1} \mathbf{x}_i = 0.35 \leq \frac{1}{2}$ and $\sum_{i:R_i=-1} \mathbf{x}_i = 0.45 \leq \frac{1}{2}$ so $f(R,\mathbf{x}) = 0$. By share monotonicity, because $R_1 = 1$, $R_2 \neq 1$, $\mathbf{x}_1 < \mathbf{x}'_1$, $\mathbf{x}_3 = \mathbf{x}'_3$, and $f(R,\mathbf{x}) = 0$, it follows that $f(R,\mathbf{x}') = 1$. But $\sum_{i:R_i=1} \mathbf{x}_i = 0.45 \leq \frac{1}{2}$, a contradiction.

ABSTENTION. Let $N = \{1,2\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$ and $R_2 = 0$, and let $\mathbf{x},\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.4, 0.6)$ and $\mathbf{x}' = (1,0)$. Because $\mathbf{x}'_1 = \frac{\mathbf{x}_1}{\mathbf{x}_1}$ it follows from abstention that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$. Because $\sum_{i:R_i=1} \mathbf{x}_i = 0.4 \leq \frac{1}{2}$ it must be that $f(R,\mathbf{x}) = 0$. However, because $\sum_{i:R_i=1} \mathbf{x}_i = 1 > \frac{1}{2}$, $f(R,\mathbf{x}') = 1$, a contradiction. □

Proof of Claim 7. I show that phantom voter rules satisfy anonymity, share monotonicity, merger, reallocation invariance, and unanimity, but may fail to satisfy neutrality and abstention.

ANONYMITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$. Note that $\sum_{i \in N} R_i \mathbf{x}_i = \sum_{\pi(i) \in N} R_{\pi(i)} \mathbf{x}_{\pi(i)} = \sum_{i \in N} R_{\pi(i)} \mathbf{x}_{\pi(i)}$. It follows that $\tau\left(t + \sum_{i \in N} R_i \mathbf{x}_i\right) = \tau\left(t + \sum_{i \in N} R_{\pi(i)} \mathbf{x}_{\pi(i)}\right)$, and therefore that $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$.

SHARE MONOTONICITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j \neq 0$, (b) $R_k \neq R_j$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \notin N \setminus \{j, k\}$, and (e) $f(R, \mathbf{x}) \neq -R_j$. From (e) it follows that $R_j(\sum_i R_i \mathbf{x}_i + t) \geq 0$. Therefore, $R_j\left(R_j \mathbf{x}_j + R_k \mathbf{x}_k\right) \geq R_j\left(-t - \sum_{\ell \in N \setminus \{j,k\}} R_\ell \mathbf{x}_\ell\right)$. From (d) it follows that $\mathbf{x}_j + \mathbf{x}_k = \mathbf{x}'_j + \mathbf{x}'_k$ and thus from (c) it follows that $\mathbf{x}'_j - \mathbf{x}'_k > \mathbf{x}_j - \mathbf{x}_k$. If $R_k = -R_j$, then $\mathbf{x}'_j - \mathbf{x}'_k > \mathbf{x}_j - \mathbf{x}_k$ implies that $R_j\left(R_j \mathbf{x}'_j + R_k \mathbf{x}'_k\right) > R_j\left(R_j \mathbf{x}_j + R_k \mathbf{x}_k\right)$. If $R_k = 0$, then this fact is implied by (c). By (d) it follows that $\sum_{\ell \in N \setminus \{j,k\}} R_\ell \mathbf{x}_\ell = \sum_{\ell \in N \setminus \{j,k\}} R_\ell \mathbf{x}'_\ell$. Putting this together, we have that $R_j(R_j \mathbf{x}'_j + R_k \mathbf{x}'_k) > R_j(-t - \sum_{\ell \in N \setminus \{j,k\}} R_\ell \mathbf{x}'_\ell)$, and therefore that $R_j\left(\sum_i R_i \mathbf{x}'_i + t\right) > 0$. Therefore $f(R, \mathbf{x}') = R_j$.

MERGER. Let $N \in \mathcal{N}$, $\lambda \in (0, 1)$, and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $\sum_i R_i\left(\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i\right) = \lambda\left(\sum_i R_i \mathbf{x}_i\right) + (1 - \lambda)\left(\sum_i R_i \mathbf{x}'_i\right)$, it follows that

$$\max\{\sum_i R_i \mathbf{x}_i, \sum_i R_i \mathbf{x}'_i\} \geq \sum_i R_i\left(\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i\right) \geq \min\{\sum_i R_i \mathbf{x}_i, \sum_i R_i \mathbf{x}'_i\},$$

and consequently that $\tau(t + \sum_i R_i \mathbf{x}_i) = \tau(t + \sum_i R_i\left(\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i\right))$. Therefore $f(R, \mathbf{x}) = f(R, \mathbf{x}') = f(R, \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}')$.

REALLOCATION INVARIANCE. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$ such that (a) for all $i, j \in S$, $R_i = R_j = R_S$, and (b) for all $k \notin S$, $\mathbf{x}'_k = \mathbf{x}_k$. From (b) it follows that $\sum_{i \in S} \mathbf{x}'_i = \sum_{i \in S} \mathbf{x}_i$, and from (a) that $R_S \sum_{i \in S} \mathbf{x}'_i = \sum_{i \in S} R_i \mathbf{x}'_i$ and that $R_S \sum_{i \in S} \mathbf{x}_i = \sum_{i \in S} R_i \mathbf{x}_i$. From (b) it also follows that $\sum_{k \notin S} R_k \mathbf{x}'_k = \sum_{k \notin S} R_k \mathbf{x}_k$. Together, this implies that $\sum_{i \in N} R_i \mathbf{x}'_i = \sum_{i \in N} R_i \mathbf{x}_i$ and therefore that $f(R, \mathbf{x}') = f(R, \mathbf{x})$.

UNANIMITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $f(R, \mathbf{x}) = \tau(t + k)$. Because $t \in (-1, 1)$, $\tau(t + k) = \tau(k) = k$.

NEUTRALITY. Let $t = \frac{1}{2}$, and let $(R, \mathbf{x}) \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, 0)$ and $\mathbf{x} = (\frac{2}{3}, \frac{1}{3}, 0)$. Then $f(-R, x) = \tau\left(\frac{1}{6}\right) \neq -\tau\left(\frac{5}{6}\right) = -f(R, x)$.

ABSTENTION. Let $-t = \frac{1}{2}$, and let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_{\{1,2\}}$ such that $R = (1, 0)$, $\mathbf{x} = (\frac{1}{3}, \frac{2}{3})$, and $\mathbf{x}' = (1, 0)$. Because $\mathbf{x}'_1 = \frac{\mathbf{x}_1}{\mathbf{x}_1}$ it follows from abstention that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. However, $f(R, \mathbf{x}) = \tau\left(t - \frac{1}{3}\right) = -1$, but $f(R, \mathbf{x}') = \tau\left(t + 1\right) = 1$, a contradiction. $\square$