Hopf Bifurcation from new-Keynesian Taylor rule to Ramsey Optimal Policy

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Abstract

This paper shows that a shift from Ramsey optimal policy under short term commitment (based on a negative-feedback mechanism) to a Taylor rule (based on positive-feedback mechanism) in the new-Keynesian model is in fact a Hopf bifurcation, with opposite policy advice. The number of stable eigenvalues corresponds to the number of predetermined variables including the interest rate and its lag as policy instruments for Ramsey optimal policy. With a new-Keynesian Taylor rule, however, these policy instruments are arbitrarily assumed to be forward-looking variables when policy targets (inflation and output gap) are forward-looking variables. For new-Keynesian Taylor rule, this Hopf bifurcation implies a lack of robustness and multiple equilibria if public debt is not set to zero for all observation.

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Keywords: Bifurcations, Taylor rule, Taylor principle, new-Keynesian model, Ramsey optimal policy, Finite horizon commitment

1 Introduction

Negative feedback is the core mechanism for stabilizing dynamic systems with optimal control (Aström and Kumar (2014)) since a negative-feedback mechanism prevents bifurcations of dynamic systems. Barnett and Chen (2015) and Barnett and Duzhak (2010, 2008) emphasize the importance of bifurcations in the reference new-Keynesian macroeconomic model (Gali (2015)). The Taylor rule parameters (the response of the interest rate to inflation or to the output gap) are bifurcation parameters. A small change of their values may lead to big changes of the dynamic path, from stability to instability and conversely.

This paper highlights additional results:

(1) Barnett and Duzhak's (2008) Hopf bifurcation, Barnett and Duzhak's (2010) period-doubling (flip) bifurcations, and a saddle-node (fold) bifurcation for the closed economy new-Keynesian model necessarily occur when the response of the interest rate to inflation in the Taylor rule is *strictly positive* and the response of the interest rate to the output gap is *strictly negative*.

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(2) The seemingly surprising negative sign of the output-gap Taylor-rule parameter is nonetheless consistent with a *negative-feedback mechanism.*

(3) Bifurcations necessarily occur when the response of the interest rate to inflation in the Taylor rule is *strictly positive* and the response of the interest rate to the output gap is *strictly positive*, if ever there is a *negative* intertemporal elasticity of substitution (related to limited asset market participation) and an accelerationist Phillips curve, such that a positive output gap is correlated with an increase of *future* inflation.

(4) For Ramsey optimal policy with finite horizon commitment (Debortoli and Nunes (2014))[1], the Ljungqvist and Sargent (2012, chapter 19) algorithm can be used to find the Taylor rule parameters: the response of the interest rate to inflation in the feedback Taylor rule is strictly above one and that the response of the interest rate to the output gap is *strictly negative.*

(5) Shifting from Ramsey optimal policy to a new-Keynesian Taylor rule corresponds to a Hopf bifurcation, because the policy instrument (the interest rate) and its lag are predetermined variables in Ramsey optimal policy, whereas they are assumed to be forward-looking for the *ad hoc* new-Keynesian Taylor rule. Ramsey optimal policy has two additional stable eigenvalues with respect to the new-Keynesian Taylor rule equilibrium. Ramsey optimal policy is based on a negative-feedback mechanism. The new-Keynesian Taylor rule is based on a positive-feedback mechanism.

The welfare loss of time-inconsistency, used as an argument against implausible *infinite-horizon* commitment, is in fact *negligible* for Ramsey optimal policy with *short-horizon* commitment. It is plausible to assume at least a few periods (two quarters or two weeks) before policy maker's re-optimization (Schaumburg and Tamballoti (2006), Debortoli and Nunes (2014)), because the deviation of the optimal path following re-optimization is small. A negative-feedback mechanism is also saved for a non-zero, such as an infinitesimal or very small probability (say 1%) of not reneging commitment at each period (Schaumburg and Tamballoti (2006), Debortoli and Nunes (2014)).

By contrast, the bifurcation leading to the decrease of the number of stable eigenvalues based on positive-feedback mechanism when shifting to a new-Keynesian Taylor rule implies at least two drawbacks:

(1) In case of imperfect knowledge of the monetary policy transmission mechanism, the positive-feedback mechanism of the new-Keynesian Taylor rule is only robust to a much smaller set of misspecifications by the evil-agent than robust Ramsey optimal policy (Giordani and Söderlind (2003), Zhou, Doyle and Glover (1995), Hansen and Sargent (2008, 2011)). This lack of robustness is why control theory is based on the principle of negative-feedback mechanism instead of *positive-feedback* mechanism.

(2) There is uniqueness of Ramsey optimal policy, but we get multiple equilibria of the new-Keynesian Taylor rule, unless all dynamic equations of controllable predetermined state variables of the private sector (wealth, debt, capital stock) are arbitrarily replaced by non-controllable auto-regressive equations of exogenous forcing-variables at the final step of computations. Leeper (1993) is a classic example of multiple equilibria with one controllable forward-looking variable and one controllable backward-looking variable, including the "fiscal theory of the price level" as one equilibrium. Each of these equilibria will not be robust to a wide set of misspecification.

Section two shows that Barnett and Duzhak's (2008, 2010) bifurcations occur for a negative output-gap rule-parameter. Section three shows that Ramsey optimal policy

[1] These models are related to models with quasi-commitment (Roberds (1987), Schaumburg and Tamballoti (2007), Kara (2007), Fujiwara, Kam and Sunakawa (2016)).

has a representation of its policy rule with a negative output-gap rule-parameter. Section four shows that Barnett and Duzhak's (2008) Hopf bifurcation corresponds to a shift from Ramsey optimal policy to a new-Keynesian Taylor rule. Section five states that this bifurcation implies major changes with respect to robustness and multiple equilibria. Section five concludes.

2 Bifurcations in the New-Keynesian Model

The new-Keynesian private sector's four-equations model is written with all variables as log-deviations of an equilibrium (Gali (2015)). In the representative household's intertemporal substitution (IS) consumption Euler equation, current output gap x_t is positively correlated with *expected* output gap and negatively correlated x_t with real rate of interest, equal to the nominal rate i_t minus *expected* inflation $E_t\pi_{t+1}$. The intertemporal elasticity of substitution (IES) $\gamma = 1/\sigma$ is a measure of the responsiveness of the growth rate of consumption to the interest rate, usually considered to be smaller than one. It is the inverse or the relative degree of resistance to intertemporal substitution of consumption (RISC) of σ (the relative fluctuation aversion), which measure the strength of the preference for smoothing consumption over time, usually considered to be larger than one.

$$x_t = E_t x_{t+1} - \gamma\left(i_t - E_t\pi_{t+1}\right) + z_{x,t} \text{ where } \gamma > 0 \tag{1}$$

A non-controllable exogenous stationary and predetermined variable $z_{x,t}$ is auto-regressive of order one $(0 < |\rho_{z,x}| < 1)$ where $\varepsilon_{g,t}$ are zero-mean, normally, independently and identically distributed additive disturbances. Initial values of predetermined forcing variable are given.

$$z_{x,t} = \rho_{z,x} z_{x,t-1} + \varepsilon_{x,t} \text{ where } \varepsilon_{x,t} \text{ is i.i.d. } N\left(0, s_x^2\right),\ z_{x,0} \text{ given,} \tag{2}$$

In the new-Keynesian Phillips curve, β discounted expected inflation is equal to current inflation plus a negative linear function of current output gap with a sensitivity $-\kappa$. *Sign restrictions* are such that parameters γ, β, κ are all strictly positive.

$$\pi_t = \beta E_t\pi_{t+1} + \kappa x_t + u_{\pi,t} \text{ where } \beta > 0 \text{ and } \kappa > 0$$

A non-controllable exogenous stationary and predetermined variable "cost-push shock" $u_{\pi,t}$ is auto-regressive of order one ($0 < |\rho_u| < 1$) where $\varepsilon_{u,t}$ are zero-mean, normally, independently and identically distributed additive disturbances. Initial values of predetermined forcing variable $u_{\pi,0}$ is given.

$$u_{\pi,t} = \rho_u u_{\pi,t-1} + \varepsilon_{u,t} \text{ where } \varepsilon_{u,t} \text{ is i.i.d. } N\left(0, s_\pi^2\right),\ u_{\pi,0} \text{ given.} \tag{3}$$

For initial values of the two forcing variables set to zero, the new-Keynesian model with its feedback Taylor rule can be written as follows. The matrices notations in bold below corresponds to the general notations of the augmented linear quadratic regulator (LQR) in Anderson, Hansen, McGrattan and Sargent (1996, p.203). Firstly, the controllable variables dynamics are:

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix}}_{=\mathbf{A}_{yy}} \underbrace{\begin{pmatrix} x_t \\ \pi_t \end{pmatrix}}_{=\mathbf{y}_t} + \underbrace{\begin{pmatrix} \gamma \\ 0 \end{pmatrix}}_{=\mathbf{B}_y} i_t + \underbrace{\begin{pmatrix} -1 & \frac{\gamma}{\beta} \\ 0 & -\frac{1}{\beta} \end{pmatrix}}_{=\mathbf{A}_{yz}} \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{=\mathbf{z}_t}$$

Secondly, the non-controllable forcing variables dynamics are:

$$\begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \rho_z & 0 \\ 0 & \rho_u \end{pmatrix}}_{=\mathbf{A}_{zz}} \begin{pmatrix} z_t \\ u_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{z,t} \\ \varepsilon_{u,t} \end{pmatrix}$$

Thirdly, the feedback linear policy rule is:

$$i_t = \underbrace{\begin{pmatrix} F_x & F_\pi \end{pmatrix}}_{=\mathbf{F}_y} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} + \underbrace{\begin{pmatrix} F_z & F_u \end{pmatrix}}_{=\mathbf{F}_z} \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

The two policy targets are two-time-step controllable by a single policy instrument (the effect of the interest rate on the expected output gap on the first period is followed on the next period by an effect of the output gap on expected inflation):

$$\operatorname{rank}\left(\mathbf{B}_y, \mathbf{A}_{yy}\mathbf{B}_y\right) = \operatorname{rank}\left(\begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \begin{pmatrix} 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \gamma \\ 0 \end{pmatrix}\right) = 2$$

Kalman (1960) controllability (here, one policy instrument can control two policy targets in *two* periods) is the generalization to linear dynamic systems of Tinbergen (1952) principle for a static linear system of equations (n policy instruments can control n policy targets in a *single* period).

The closed loop matrix $\mathbf{A}_{yy}+\mathbf{B}_y\mathbf{F}_y$ of the controllable part of the new-Keynesian model is:

$$\begin{pmatrix} 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} + \begin{pmatrix} \gamma \\ 0 \end{pmatrix} \begin{pmatrix} F_x & F_\pi \end{pmatrix} = \begin{pmatrix} 1+\frac{\gamma\kappa}{\beta}+\gamma F_x & -\frac{\gamma}{\beta}+\gamma F_\pi \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix}$$

Wonham's (1967) pole placement theorem for a controllable dynamic linear system and for a *single* instrument (single input) implies that for any set of symmetric complex (or real) eigenvalues (λ_1, λ_2) of the closed loop matrix of the new-Keynesian model, there exist a *unique* set of linear feedback policy rule parameters (F_x, F_π). *Wonham's (1967) pole placement theorem states that linear feedback rule parameters are always bifurcation parameters of controllable linear systems.* Close to bifurcations limit values, a small change of the policy rule parameters leads to big qualitative change of the dynamics of the system to be controlled, through a change of stability of eigenvalues.

The characteristic polynomial of the closed loop matrix of the new-Keynesian model is function of the trace T and determinant D:

$$0 = P(X) = X^2 - T.X + D,$$

Its eigenvalues (the roots of the characteristic polynomial) are:

$$\lambda_1 = \frac{T+\sqrt{T^2-4D}}{2} \text{ and } \lambda_2 = \frac{T-\sqrt{T^2-4D}}{2}$$

Trace T and determinant D are affine function of the Taylor rule parameters F_x and F_π:

$$T = 1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta} + \gamma F_x \text{ and } D = \frac{1}{\beta} + \frac{\gamma}{\beta} F_x + \frac{\gamma\kappa}{\beta} F_\pi \tag{4}$$

Conversely, the Taylor rule parameters F_x and F_π are affine function of the trace T and determinant D:

$$F_\pi = \frac{1}{\beta} + \frac{1}{\gamma\kappa\beta} - \frac{T}{\kappa\gamma} + \frac{\beta}{\gamma\kappa} D \text{ and } F_x = \frac{-1}{\gamma}\left(1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta}\right) + \frac{1}{\gamma} T \tag{5}$$

In Azariadis (1993, chapter 8), a stability triangle with Hopf, flip and saddle-node bifurcations borders and a quadratic function delimiting complex conjugate versus non-complex solutions (discriminant $\Delta \leq 0$) are described in the plane of the trace and determinant (T, D). This defines areas with with zero, one or two stable eigenvalues (λ_1, λ_2) in the plane (T, D).

Using the *affine* relationship between (T, D) and Taylor rule parameters (F_π, F_x) and based on table 1 computations, figure 1 draws a *stability triangle bordered by Hopf, period-doubling (flip) and saddle-node bifurcations* in the plane of Taylor rule parameters (F_π, F_x), which are then two bifurcation parameters. This defines areas with with zero, one or two stable eigenvalues (λ_1, λ_2) in the plane (F_π, F_x), with a unique relation between Taylor rule parameters (F_π, F_x) and (λ_1, λ_2) as a particular single instrument case of Wonham (1967) pole placement theorem for controllable system.

Table 1 summarizes apexes of the stability triangle. It also includes its center with both eigenvalues equal to zero ($\lambda_1 = \lambda_2 = 0$). This provides the Taylor rule parameters with the fastest stabilization. It includes the laissez-faire case, where both Taylor rule parameters are zero. The laissez-faire ("open loop") private sector's model is described by the Fed following a fixed interest rate target or peg: $i_t - i^* = 0$. In this case, the transition matrix of the new-Keynesian model has one stable eigenvalue less than one in absolute value and the other eigenvalue is unstable (larger than one).

Table 1: Apexes and center of the stability triangle and laissez-faire of the new-Keynesian model ($\gamma = 0.5$ and $\kappa = 0.1$).

(λ_1, λ_2)	T	D	F_π	F_x	Point
$(1, 1)$	2	1	$\frac{1}{\beta} + \frac{1}{\gamma\kappa\beta} - \frac{2}{\kappa\gamma} + \frac{\beta}{\gamma\kappa} = 1.01$	$\frac{-1}{\gamma}\left(1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta}\right) + \frac{2}{\gamma} = -0.12$	A
$(-1, -1)$	-2	1	$\frac{1}{\beta} + \frac{1}{\gamma\kappa\beta} + \frac{2}{\kappa\gamma} + \frac{\beta}{\gamma\kappa} = 81.0$	$\frac{-1}{\gamma}\left(1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta}\right) - \frac{2}{\gamma} = -8.12$	B
$(-1, 1)$	0	-1	$\frac{1}{\beta} + \frac{1}{\gamma\kappa\beta} - \frac{\beta}{\gamma\kappa} = 1.41$	$\frac{-1}{\gamma}\left(1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta}\right) = -4.12$	C
$(0, 0)$	0	0	$\frac{1}{\beta} + \frac{1}{\gamma\kappa\beta} = 21$	$\frac{-1}{\gamma}\left(1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta}\right) = -4.12$	D
$\left(\lambda_1, \frac{1}{\beta\lambda_1}\right)$	$1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta}$	$\frac{1}{\beta}$	0	0	Origin

Point A is the Hopf and Saddle-node bifurcations intersection. Point B is the Hopf and Flip bifurcation intersection. Point C is the Saddle-node and Flip bifurcation intersection.

Figures 1 to 3 represents bifurcations lines delimiting the number of stable eigenvalues for the new-Keynesian model for positive sign restrictions on monetary policy transmission mechanism (figures 1 and 3) and for negative sign restrictions on monetary policy transmission mechanism (figure 2).

Figure 1: $\gamma = 0.5$, $\kappa = 0.1$, $\beta = 0.99$: stability triangle in the plane (F_π, F_x)

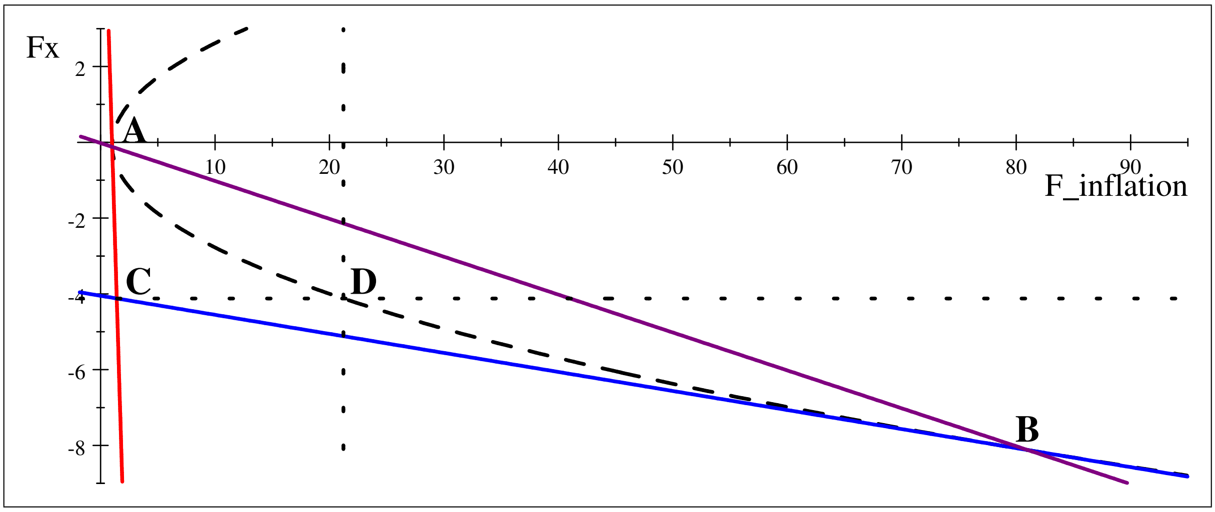

Figure 2: $\gamma = -0.5$**,** $\kappa = -0.1$**,** $\beta = 0.99$**: stability triangle in the plane** (F_π, F_x)

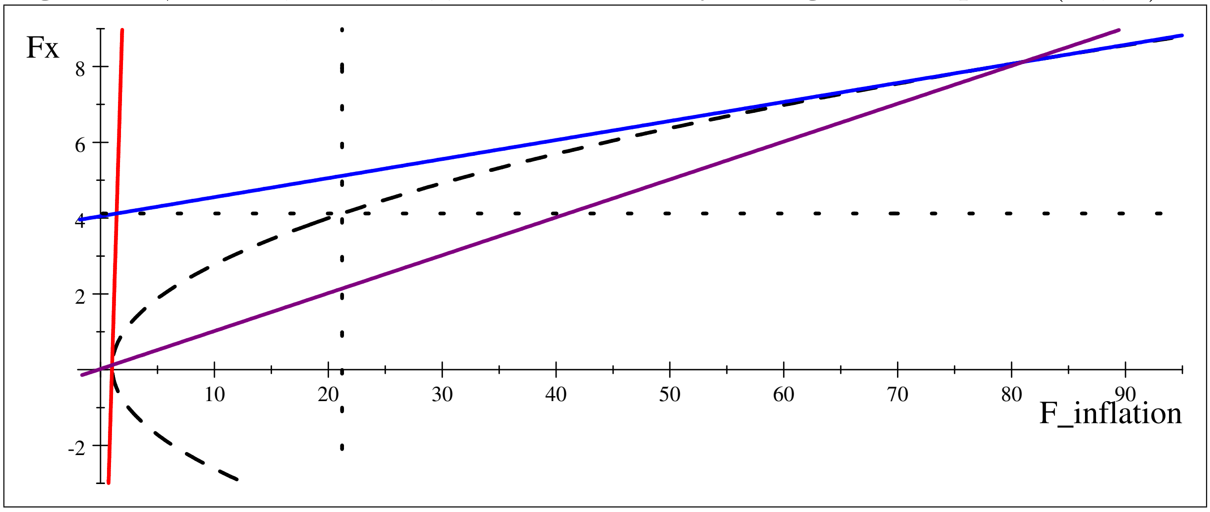

Figure 3: to be inserted here, see last page.

(1) The **Hopf bifurcation** inequality condition is $D < 1$ computed by Barnett and Duzhak (2008) with a limit pair of complex conjugate solution of absolute value equal to one.

$$D < 1 \Rightarrow F_x < \frac{\beta - 1}{\gamma} - \kappa F_\pi$$

$$\text{If } F_\pi > 0 \Rightarrow F_x < \frac{\beta - 1}{\gamma} < 0.$$

The Hopf bifurcation corresponds to the segment between point A and point B. With conjugate complex roots on both sides of the segment, the economy shifts from damped oscillations dynamics to exploding oscillations. It is an upper limit of the triangle in figure 1 and an lower limit of the triangle in figure 2.

(2) The **saddle-node bifurcation** inequality condition is $P(1) > 0$ with one limit eigenvalue equal to 1. It corresponds to equation 4.4 in Barnett and He (2015) although they do not explicitly mention it may correspond to a saddle-node bifurcation:

$$P(1) = 1 - T + D > 0 \Rightarrow F_x > \frac{\kappa}{1-\beta}(1 - F_\pi) \text{ or } F_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right) F_x \tag{6}$$

$$\text{if } F_x < 0 \Rightarrow F_\pi > 1 \tag{7}$$

The saddle-node bifurcation corresponds to the near-vertical segment between point A and point C. It is the Taylor principle condition: the inflation Taylor rule inflation parameter should be above one: $F_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right) F_x > 1$. This bifurcation boundary corresponds the case where one root of the characteristic polynomial of the transition matrix of the new-Keynesian model is equal to one: $P(1) = 0$. It corresponds to a shift from two real stable eigenvalues (inside the triangle) to one unstable and one stable on the left-hand side of the triangle. The near-vertical line $F_\pi = 1 - \left(\frac{1-\beta}{\kappa}\right) F_x$ above the point A of the triangle corresponds to a shift from one unstable and one stable eigenvalue one the left-hand side to two unstable eigenvalue on the right hand side. The near-vertical line $F_\pi = 1 - \left(\frac{1-\beta}{\kappa}\right) F_x$ below the point C of the triangle corresponds to a shift from one unstable and one stable eigenvalue one the left-hand side to two unstable eigenvalue on the right hand side.

(3) The **period-doubling (flip) bifurcation** inequality $P(-1) > 0$ with one limit eigenvalue equal to -1 is computed by Barnett and Duzhak (2010).

$$P(-1) = 1 + T + D > 0 \Rightarrow F_x > -\frac{2}{\gamma} - \frac{\kappa}{1+\beta} - \frac{\kappa}{1+\beta} F_\pi \tag{8}$$

The period-doubling (Flip) bifurcation corresponds to the segment between point C to point B, with is the lower side of the triangle in figure 1 and the upper side of the triangle in figure 2. The economy shifts from convergent and sign change every period dynamics (inside the triangle) to divergent and sign change every period dynamics (below the triangle in figure 1 and above the triangle in figure 2).

(4) A quadratic function delimits complex conjugate solutions. It crosses the case when both eigenvalues are zero.

$$\Delta \leq 0 \Leftrightarrow T^2 - 4D \leq 0 \Leftrightarrow D \geq \frac{T^2}{4}$$

$$\frac{1}{\beta} + \frac{\gamma}{\beta} F_x + \frac{\gamma\kappa}{\beta} F_\pi > \frac{1}{4}\left(1 + \frac{1}{\beta} + \frac{\gamma\kappa}{\beta} + \gamma F_x\right)^2$$

(5) The case of fastest stabilization with two identical eigenvalues equal to zero corresponds to point D with Taylor rule parameters $F_\pi = 5.7$ and $F_\pi = -3.6$ at the intersection of a vertical and dot lines and the parabola $\Delta = 0$ delimiting complex conjugate (damped oscillations dynamics) versus real solutions (damped dynamics) on figure 1. It corresponds to $F_\pi = 5.7$ and $F_\pi = 3.6$ in figure 2 for opposite signs of the monetary policy transmission mechanism.

(6) When the Taylor rule parameters are both equal to zero (interest rate peg or laissez-faire) with correspond to the origin of the plane $F_\pi = F_x = 0$ in figures 1 and 2, the economy is outside the stability triangle. The Taylor principle where the Taylor rule inflation parameter should be larger than one is not fulfilled. There is one stable and one unstable eigenvalue.

Proposition 1. *For strictly positive intertemporal elasticity of substitution* $\gamma > 0$

and a strictly positive slope of the new-Keynesian Phillips curve $\kappa > 0$ and β positive, close and below to one (for example $\beta \approx 0.99$), a necessary condition for having two stable eigenvalues ($|\lambda_1| < 1$ and $|\lambda_2| < 1$) in the closed-loop transition matrix of the new-Keynesian model is that if rule parameters (F_x, F_π) are inside in a triangle such that the output gap parameter is strictly negative $(F_x < 0)$ and the inflation rule parameters is larger than one $\left(F_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right) F_x > 1\right)$, according to the Taylor principle. The stability triangle with two stable eigenvalues lies into the second quadrant of the plane (F_π, F_x).

Proof. Eigenvalues are both stable ($|\lambda_1| < 1$ and $|\lambda_2| < 1$) implies:

$$F_x > -\frac{2}{\gamma} - \frac{\kappa}{1+\beta} - \frac{\kappa}{1+\beta} F_\pi \text{ for } P(-1) > 0 \tag{9}$$

As $0 < \beta \approx 0.99 < 1$, (e.g. $\beta \approx 0.99$), the line $(D = 1)$ is over the line $(P(-1) = 0)$ for $F_\pi > 1$ and $F_x < 0$:

$$\frac{\beta - 1}{\gamma} > -\frac{2}{\gamma} - \frac{\kappa}{1+\beta} \text{ and } -\kappa > -\frac{\kappa}{1+\beta} \tag{10}$$

With opposite sign restrictions $\gamma < 0$ and $\kappa < 0$, triangles of figure 1 are found by symmetry with respect to the horizontal axis in the positive quadrant of ($F_x > 0$,$F_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right) F_x$).

Q.E.D.

Proposition 2. *For a strictly positive intertemporal elasticity of substitution $\gamma > 0$ and a strictly positive slope of the new-Keynesian Phillips curve, $\kappa > 0$, a negative Taylor rule parameter on the output gap is a necessary condition for a negative-feedback mechanism.*

The new-Keynesian transition matrix is:

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\gamma\kappa}{\beta} + \gamma F_x & -\frac{\gamma}{\beta} + \gamma F_\pi \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}$$

$E_t x_{t+1}$ converges faster to equilibrium with respect to laissez-faire (negative-feedback mechanism) if:

$$\begin{aligned} &0 < 1 + \frac{\gamma\kappa}{\beta} + \gamma F_x < 1 + \frac{\gamma\kappa}{\beta} \text{ if } \gamma F_x < 0; \text{ if } F_x < 0 \text{ when } \gamma > 0 \\ &0 > -\frac{\gamma}{\beta} + \gamma F_\pi > -\frac{\gamma}{\beta} \text{if } \gamma F_\pi > 0; \text{ if } F_\pi > 0 \text{ when } \gamma > 0 \end{aligned}$$

As Taylor rule parameters have only a correlation on inflation after two periods, we iterate for inflation:

$$E_t(\pi_{t+2}) = \left(\frac{1+\kappa\gamma}{\beta^2} - \frac{\kappa}{\beta}\gamma F_\pi\right)\pi_t - \frac{\kappa}{\beta}\left(\frac{1+\kappa\gamma}{\beta} + 1 + \gamma F_x\right) x_t$$

$E_t x_{t+1}$ converges faster to equilibrium with respect to laissez-faire (negative-feedback mechanism) if:

$$\begin{aligned} &0 < \frac{1+\kappa\gamma}{\beta^2} - \frac{\kappa}{\beta}\gamma F_\pi < \frac{1+\kappa\gamma}{\beta^2} \text{ if } -\frac{\kappa}{\beta}\gamma F_\pi < 0; \text{ if } F_\pi > 0 \text{ when } -\frac{\kappa}{\beta}\gamma < 0 \\ &0 > -\frac{\kappa}{\beta}\left(\frac{1+\kappa\gamma}{\beta} + 1 + \gamma F_x\right) > -\frac{\kappa}{\beta}\left(\frac{1+\kappa\gamma}{\beta} + 1\right) \text{ if } -\frac{\kappa}{\beta}\gamma F_x > 0; \text{ if } F_x < 0 \text{ when } -\frac{\kappa}{\beta}\gamma < 0 \end{aligned}$$

The *negative response of the interest rate to the output gap is decreasing the absolute value of eigenvalues with respect to laissez-faire* (negative feedback mechanism) is caused by the intertemporal substitution effects in the Euler consumption equation and in the new-Keynesian Phillips curve. A rise of current interest rate leads to an increase of future consumption next period (and a decrease of current consumption). An increase of consumption leads to a decrease of future inflation (and a decrease of current inflation). Negative-feedback targets to decrease the auto-correlation of future values of policy targets, in order to fasten convergence back to the set-point long run equilibrium.

$$i_t \uparrow \Longleftrightarrow E_t\left(x_{t+1}\right) \uparrow \Longleftrightarrow E_t\left(\pi_{t+2}\right) \downarrow$$

Q.E.D.

Proposition 3. *For strictly negative intertemporal elasticity of substitution $\gamma > 0$ and a strictly negative slope of the new-Keynesian Phillips curve $\kappa > 0$ and β positive, close and below to one (for example $\beta \approx 0.99$), a necessary condition for having two stable eigenvalues ($|\lambda_1| < 1$ and $|\lambda_2| < 1$) in the closed-loop transition matrix of the new-Keynesian model is that if rule parameters (F_x, F_π) are inside in a triangle such that the output gap parameter is strictly positive ($F_x > 0$) and the inflation rule parameters is at least slightly below one $\left(F_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right) F_x\right)$, according to the Taylor principle.*

Proof. Same computations than for proposition 2 with opposite signs for γ and κ.

Q.E.D.

Alternative monetary policy transmission mechanisms

In figure 2, an increase of the interest rate leads to a decrease of future production ($\gamma < 0$) (due to limited asset market participation (Bilbiie (2008), Bilbiie and Straub (2013)) which leads to a decrease of inflation next period, through the accelerationist Phillips curve ($\kappa < 0$) as in Romer (2012, chapter 11). Ball (1994) started a theoretical debate on the sign of κ. Mavroeidis *et al.* (2014) found hundreds of negative estimates of κ using U.S. data.

$$i_t \uparrow \Longleftrightarrow E_t\left(x_{t+1}\right) \downarrow \Longleftrightarrow E_t\left(\pi_{t+2}\right) \downarrow$$

Taylor (1999) assume no intertemporal substitution effect ($\gamma = 0$), with a *user cost of capital channel* effect only on current output. Gali (2015) assumes the production function does not depend on capital. Hence, Gali (2015) assumes that the user cost of capital channel does not exist. Taylor (1999) also assumes an accelerationist Phillips curve effect, with a positive correlation between output gap and future inflation, instead of the negative correlation:

$$i_t \uparrow \Longleftrightarrow x_t \downarrow \Longleftrightarrow E_t\left(\pi_{t+1}\right) \downarrow$$

Assuming inflation is forward-looking with the new-Keynesian Phillips curve with a marginal cost of working capital including the interest rate into the marginal cost (Christiano *et al.* (2010), Bratsiotis and Robinson (2016))), the monetary policy transmission channel is a based on the correlation between expected inflation and the cost of capital:

$$i_t \uparrow \Longleftrightarrow E_t\left(\pi_{t+1}\right) \downarrow$$

Changing the sign of the transmission mechanism may change the sign of negative-feedback rule parameters.

3 Ramsey optimal policy with finite horizon commitment

The policy maker optimizes again on its previously given Ramsey optimal policy plans at least three periods after the previous optimization (a commitment of not optimizing again until the duration $T \geq 3$), as in Debortoli and Nunes (2014, appendix 4). A short finite horizon commitment is close to a dynamic version of static "rational inattention" (Sims (2010)), assuming discrete re-optimization over time, instead of continuous-time optimization. For example, the European Central Bank has meetings every two Thursdays: the duration of commitment T could be viewed as a duration of at least six weeks. One may also re-scale each discrete time period as three shorter sub-periods.

The policy maker could also re-optimize on each future period with exogenous probability ("stochastic replanning" (Roberds, 1987), "quasi commitment" (Schaumburg and Tambalotti, 2007; Kara 2007) or "loose commitment" (Debortoli and Nunes, 2014)). This assumption is observationally equivalent to Chari and Kehoe (1990) optimal policy under sustainable plans facing a punishment threat at a given horizon in case of deviation of an optimal plan (Fujiwara, Kam, Sunakawa (2016)).

The policy maker minimizes the expectation of the present value W of a discounted quadratic loss function L_t over a finite horizon of duration $T \geq 3$. He chooses respect to the policy targets (inflation and the output gap) with a positive weight on the second policy target (output gap) $\lambda_x \geq 0$ and a weight normalized to $\lambda_\pi = 1$ for inflation (the limit case $\lambda_\pi = 0$ can also be taken into account) and with a *strictly* positive adjustment cost parameter $\lambda_i > 0$ on the volatility of her policy instrument (the interest rate) and a discount factor β:

$$E\left[W\right] = -E_t \sum_{t=0}^{T} \beta^t \left\{ \lambda_\pi \frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} + \lambda_i \frac{i_t^2}{2} \right\}, T \geq 3 \tag{11}$$

subject to the private sector's new-Keynesian four equations model (equations (1) to (4)), with initial conditions for predetermined state variables and natural boundary conditions for private sector's forward variables. Denoting Lagrangian multipliers $\phi_{x,t}$ for the consumption Euler equation and $\phi_{\pi,t}$ for the new-Keynesian Phillips curve, the Lagrangian $\mathcal{L}$ is:

$$\mathcal{L} = -E_0 \sum_{t=0}^{T} \beta^t \left\{ \lambda_\pi \frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} + \lambda_i \frac{i_t^2}{2} + \phi_{x,t}\left[x_t - x_{t+1} + \sigma\left(i_t - \pi_{t+1}\right)\right] + \phi_{\pi,t}\left[\pi_t - \beta_t \pi_{t+1} - \kappa x_t\right] \right\} \tag{12}$$

The law of iterated expectations has been used to eliminate the condition expectations that appeared in each constraint. Because of the certainty equivalence principle for determining optimal policy in the linear quadratic regulator including additive normal random shocks (Simon (1956)), the expectations of random variables u_t are set to zero and do not appear in the Lagrangian.

Because inflation and the output gap are assumed to be forward-looking, they are optimally chosen at the initial date $t = 0$ and at the final date $t = T$ according to transversality conditions, also called natural boundary conditions (Debortoli and Nunes (2014), appendix 4):

$$\phi_{\pi,t=0} = \phi_{x,t=0} = \phi_{\pi,t=T} = \phi_{x,t=T} = 0 \text{ (Transversality conditions)} \tag{13}$$

The Hamiltonian system with Lagrange multipliers of the linear quadratic regulator includes two stable roots and two unstable roots:

$$x_t = x_{t+1} + \sigma\left(i_t - \pi_{t+1}\right) \tag{14}$$

$$\pi_t = \beta_t \pi_{t+1} + \kappa x_t \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \pi_t + \phi_{\pi,t} - \phi_{\pi,t-1} - \frac{\sigma}{\beta}\phi_{x,t-1} = 0 \text{ for } 0 \leq t \leq T, \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial x_t} = 0 \Rightarrow \lambda_x x_t - \kappa \phi_{\pi,t} + \phi_{x,t} - \frac{1}{\beta}\phi_{x,t-1} = 0 \text{ for } 0 \leq t \leq T, \tag{17}$$

A bounded optimal plan is a set of bounded processes $\{\pi_t, x_{t,}i_t, \phi_{x,t}, \phi_{\pi,t}\}$ for date $0 \leq t \leq T$ of the Hamiltonian system that satisfy the monetary transmission mechanism four equations, the first order equations and the optimal initial conditions.

The link between the Lagrange multipliers and the interest rate is given by:

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \lambda_i i_t + \sigma \phi_{x,t} = 0 \Rightarrow \ \phi_{x,t} = -\frac{\lambda_i}{\sigma} i_t \text{ for } 0 \leq t \leq T,$$

The boundary conditions and the marginal conditions leads to these constraints on the interest rate:

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \lambda_i i_t + \sigma \phi_{x,t} = 0 \Rightarrow \ \phi_{x,t} = -\frac{\lambda_i}{\sigma} i_t \text{ for } 0 \leq t \leq T, \tag{18}$$

$$\phi_{\pi,t=0} = \phi_{x,t=0} = \phi_{\pi,t=T} = \phi_{x,t=T} = 0 \text{ hence} \tag{19}$$

$$i_{-1} = i_{-2} = 0 = i_{T-1} = i_{T-2} \ (i_0 \neq 0 \text{ for the shortest horizon } T = 3) \tag{20}$$

Using the method of undetermined coefficients, Ljungqvist and Sargent (2012, chapter 19.3.1, *step 1*, p.769) solve a Riccati equation which allow to compute endogenous optimal negative-feedback rule parameters $(F_{x,R}, F_{\pi,R})$. Anderson, Hansen, McGrattan and Sargent (1996, p.203) first consider that auto-regressive shocks are initially set to zero ($\mathbf{z}_0 = \mathbf{0}$) to solve this Riccati equation. They do not depend on auto-regressive parameters of exogenous variables. Then, Anderson, Hansen, McGrattan and Sargent (1996, p.203) take into account the exogenous auto-regressive process of forcing variables. They solve the rule parameter responding to shocks $F_{z,R}, F_{u,R}$. This step one representation of the optimal Ramsey policy rule depends on all four variables of the private sector:

$$i_t = F_x x_t + F_\pi \pi_t + F_z z_t + F_u u_t \tag{21}$$

Besides the two stable eigenvalues of the block of endogenous variables, there are the two stable eigenvalues which are the auto-regressive parameters of the block of the two predetermined and exogenous forcing variables (productivity and cost-push shock).

Ljunqgvist and Sargent (2012, chapter 19, step 2) computes the linear stable subspace constraint between Lagrange multiplier (in this paper: $\phi_{x,t}, \phi_{\pi,t}$) and the variables of the private sector $\phi_t = \mathbf{P}_y \mathbf{y}_t + \mathbf{P}_z \mathbf{z}_t$. The matrix $\mathbf{P}_y$ is the unique solution of a discrete time Riccati equation (Sargent and Ljungqvist (2012), chapter 19) and $\mathbf{P}_z$ is the unique solution of a Sylvester equation in the augmented discounted linear quadratic regulator

(Anderson, Hansen, McGrattan and Sargent (1996), p.202-204).

Ljunqgvist and Sargent (2012, chapter 19, step 3) substitute the forward-looking variables (in this paper: x_t, π_t) in the rule by their Lagrange multiplier (in this paper: $\phi_{x,t}, \phi_{\pi,t}$) using the linear constraint $\phi_t = \mathbf{P}_y \mathbf{y}_t + \mathbf{P}_z \mathbf{z}_t$. This representation of the Ramsey optimal policy rule is a function of four predetermined variables $(\phi_t, \mathbf{z}_t)$ which are not observable.

Ljunqgvist and Sargent (2012, chapter 19, step 4) solve for the optimal initial anchor of forward-looking variables on predetermined variables:

$$\phi_t = \mathbf{P}_y \mathbf{y}_t + \mathbf{P}_z \mathbf{z}_t,\ \phi_0 = \mathbf{0},\ \mathbf{P_y} \text{ invertible} \Longleftrightarrow \mathbf{y}_0 = -\mathbf{P}_y^{-1} \mathbf{P}_z \mathbf{z}_0. \tag{22}$$

Ljunqgvist and Sargent (2012, chapter 19.3.7) mention another representation of Ramsey optimal policy rule which depends lags of the policy instruments and of predetermined variables.

Giannoni and Woodford (2003) is a different representation of Ramsey optimal policy rule which depends on lags of policy instruments and of forward-looking variables (inflation and the output gap). All variables are observable. They substituted the predetermined auto-regressive forcing variables (z_t, u_t) by lag one and two of the interest rate (i_{t-1}, i_{t-2}). Other observationally equivalent representations of Ramsey optimal policy rule can use leads or lags of other variables than the policy instrument.

All the various representations of the rule of Ramsey optimal policy corresponds to linear substitutions change of variables which are taken as vector basis. They are all *observationally equivalent* taking into account the other equations of the stable solution of the Hamiltonian system. The eigenvalues of the Hamiltonian system are *invariant* to these changes of vector basis related to a change of representation of the rule of Ramsey optimal policy.

Step 1 representation of the rule of Ramsey optimal policy is such that the rule parameters with respect to inflation and the output gap are represented in figure 1 (for $\mathbf{z}_0 = \mathbf{0}$), with bifurcations and areas for stable and unstable eigenvalues. For each of the infinite number of other observationally equivalent representations of the rule of Ramsey optimal policy, including leads or lags and/or linear change of variables (vector basis) within the stable subspace of Ramsey optimal policy, another specific figure could be done to highlight bifurcations.

Proposition 4. *The Taylor rule parameters on inflation and on the output gap in Ljunqgvist and Sargent (2012, chapter 19) step 1 representation are located in a subset of the stability triangle of section 2.*

Proof. Ramsey optimal policy assumes that inflation and the output gap are forward-looking *and that the policy instrument (the interest rate) and its lag are predetermined.* Two stable eigenvalues in the block of two endogenous variables of the new-Keynesian model are required to satisfy Blanchard and Kahn's (1980) determinacy condition. The reduced form Taylor rule parameter have to lie in the areas where both eigenvalues are stable, that is, in the stability triangle.

Q.E.D.

The key principle is that negative-feedback mechanism is stabilizing forward-looking variables, such as output and inflation, during a finite short horizon (the duration of a monetary policy regime), leaning against inflation and output gap spirals.

For a numerical example, let us consider that the monetary policy regime last at least ten years ($T = 80$ quarters), which may correspond to Benati and Goodhart (2010) duration of recent monetary policy regimes for the Fed since the 1960s. It can be roughly approximated by the infinite horizon LQR solution of the matrix Riccati equation which can only be solved numerically with a system of two policy targets. Shorter horizon T solutions can be computed using Schaumburg and Tambalotti (2006) algorithm. Table 2 and figure 3 provides the boundaries of the LQR reduced form Taylor rule parameters triangle, obtained by a simulation grid varying *the weights in the loss function in three dimensions*. The apexes of the LQR triangles corresponds to the cases where the central bank minimizes only inflation, or only the output gap or only the interest rate.

Table 2: Step 1: Linear quadratic regulator rule parameters triangle, $\kappa = 0.1, \gamma = 0.5, \beta = 0.99, \rho_z = \rho_u = 0.9$.

Minimize only:	λ_π	λ_x	λ_i	$\lambda_1, \|\lambda_1\|$	$\lambda_2, \|\lambda_2\|$	F_π	F_x	F_z	F_u
Inflation	1	0	10^{-7}	$\mathbf{7.10^{-5}}$	**0.006**	**21.21**	$\mathbf{-3.92}$	$\mathbf{-2.01}$	**39.5**
Inflation output gap	4	1	10^{-7}	4.10^{-7}	0.819	4.76	-2.27	-2.01	17.6
Inflation output gap	1	1	10^{-7}	4.10^{-7}	0.905	3.03	-2.10	-2.01	12.8
Inflation output gap	1/4	1	10^{-7}	4.10^{-7}	0.951	2.10	-2.01	-2.01	8.90
Output gap	0	1	10^{-7}	$\mathbf{4.10^{-7}}$	**0.995**	**1.21**	$\mathbf{-1.92}$	$\mathbf{-2.01}$	**2.95**
Output gap interest	0	4	1	0.348	0.953	1.70	-1.31	-2.21	6.78
Output gap interest	0	1	1	0.541	0.918	1.83	-0.98	-2.43	7.38
Output gap interest	0	1/4	1	0.663	0.878	1.87	-0.82	-2.42	7.55
Interest rate	0	0	$1(+\infty)$	**0.748**	**0.832**	**1.89**	$\mathbf{-0.74}$	$\mathbf{-2.46}$	**7.60**
Inflation interest	1/4	0	1	0.784	0.784	1.99	-0.77	-2.43	7.85
Inflation interest	1	0	1	0.772	0.772	2.22	-0.83	-2.37	8.45
Inflation interest	4	0	1	0.742	0.742	2.82	-0.98	-2.26	9.95

Table 3: Step 4: optimal initial anchor, $\kappa = 0.1, \gamma = 0.5, \beta = 0.99, \rho_{z,x} = \rho_{z,\pi} = 0.9$.

Minimize only:	λ_π	λ_x	λ_i		z_0	u_0		z_0	u_0
Inflation	1	0	10^{-7}	x_0	10^{-4}	$\mathbf{-10.1}$	π_0	10^{-5}	10^{-6}
Inflation output gap	4	1	10^{-7}	x_0	10^{-6}	-1.25	π_0	10^{-8}	3.14
Inflation output gap	1	1	10^{-7}	x_0	10^{-6}	-0.49	π_0	10^{-8}	4.91
Inflation output gap	1/4	1	10^{-7}	x_0	10^{-6}	-0.16	π_0	10^{-6}	6.66
Output gap	0	1	10^{-7}	x_0	10^{-6}	10^{-6}	π_0	-10^{-6}	**9.61**
Output gap interest	0	4	1	x_0	0.35	1.53	π_0	-0.56	7.18
Output gap interest	0	1	1	x_0	0.58	2.52	π_0	-0.79	6.20
Output gap interest	0	1/4	1	x_0	0.72	3.13	π_0	-0.92	5.63
Interest rate	0	0	$1(+\infty)$	x_0	**0.73**	**3.14**	π_0	$\mathbf{-0.92}$	**5.63**
Inflation interest	1/4	0	1	x_0	5.00	-10.3	π_0	0.71	-0.04
Inflation interest	1	0	1	x_0	4.45	-10.3	π_0	0.61	-0.03
Inflation interest	4	0	1	x_0	3.45	-10.2	π_0	0.42	-0.02

In figure 3, the LQR triangle is within the stability triangle. In table 3, vertexes of the LQR triangle corresponds to the central bank minimizing only the variance of inflation (inflation nutter) without notice of the zero lower bound constraint on the policy interest rate ($\lambda_i = 10^{-7} > 0$), or minimizing only the variance of output gap without notice of the zero lower bound ($\lambda_i = 10^{-7} > 0$) or seeking only maximal inertia of the policy rate ($\lambda_i \rightarrow +\infty$).

- When the central bank is an inflation nutters, both eigenvalues are close to zero (it corresponds to point D). To stabilize inflation in the second step of the transmission mechanism, one needs first to stabilize the output gap on the first step. This is the reason why both eigenvalues are equal to zero. The optimal anchor leads to an initial jump of inflation to zero, whereas the output gap jump is not zero.

- The lower side of the LQR triangle correponds to no lower bound constraint (no cost of changing the policy rate) and to changes of the relative weight on the output gap variance with respect to inflation variance. The lower the weight λ_π for a given weight λ_x, the lower F_π the response of interest rate to inflation in the Taylor rule, the larger the initial jump of inflation, the lower the initial jump of the output gap.

- When the central bank is an output gap nutter, one eigenvalue is zero (related to output gap stabilization in next period based on the Euler consumption equation). However, the other eigenvalue tends to one when the weight of inflation is zero. There is no margin of errors on the inflation Taylor rule parameters with respect to the saddle-node bifurcation This second eigenvalue is related to the new-Keynesian Phillips curve and inflation auto-regressive parameter, which only occurs on the second period of the monetary policy transmission mechanism.

- A second side of the LQR triangle is the closest of the saddle-node Taylor principle. It corresponds zero inflation weight $\lambda_\pi = 0$ and a rising relative cost of changing the policy instruments with respect to output gap weight. The lower the weight λ_x for a given weight on the policy instrument λ_i, the higher F_π the response of interest rate to inflation in the Taylor rule, the lower the initial jump of inflation, the larger the initial jump of the output gap. "Mostly inflation eigenvalue" decreases from 0.995 to 0.83. "Mostly output gap eigenvalue" increases from zero to 0.75.

- When the central bank uses minimum energy control (it minimizes only the volatility of its policy instrument (or "input"): $\lambda_x = \lambda_\pi = 0$, $\lambda_i > 0$), the eigenvalue 0.75 corresponds to the stable eigenvalue of laissez-faire matrix $\sqrt{\beta}\mathbf{A}_{yy}$ and the second stable eigenvalue $0.83 = 1/1.20$ is the inverse of the unstable eigenvalue 1.20 of the laissez-faire matrix $\sqrt{\beta}\mathbf{A}_{yy}$.

- The second upper side of the LQR triangle corresponds to zero weight on the output gap, and a relative increase of the weight of inflation with respect to the weight of the policy instrument. The higher the weight λ_π for a given weight on the policy instrument λ_i, the inflation Taylor rule parameter rises from 1.89 to 21.2 (inflation nutter case). The inflation initial jump decreases towards zero.

4 Hopf bifurcation from Ramsey optimal policy to New-Keynesian Taylor rule

4.1 Hopf bifurcation

The very first paper (Currie and Levine (1985)) inventing the equilibrium of simple rule with forward-looking variables considered an ad hoc general linear stochastic rational expectations model which includes $n - m$ predetermined policy targets and m forward-looking policy targets.

For Ramsey optimal policy, n Euler equations on Lagrange multipliers (missing in the *ad hoc* simple rule system) belongs to the Hamiltonian system besides n equations of the transmission mechanism. The Hamiltonian system of the linear quadratic regulator includes n stable eigenvalues (related to $n - m$ predetermined state variables and m predetermined costate variables of forward-looking state variables). They mention that for Ramsey optimal policy "*we have n stable and n unstable roots*" (Currie and Levine (1985) p.238). The determinacy condition is "*just the set of necessary conditions for maximization of a quadratic function subject to linear constraints*" (Blanchard and Kahn

(1980), p.1309).

For simple rule with forward-looking variables, Currie and Levine (1985) are *"concerned only"* with solutions that the number of eigenvalues of the closed loop transition matrix which are stable equals $n-m$ (eigenvalues having *negative* real parts in continuous time or modulus strictly below one in discrete time), which is smaller than n stable roots for Ramsey optimal policy. They *"confine attention"* to representations of linear time invariant feedback rule on the $n-m$ predetermined variables (Currie and Levine (1985) p.234-235). In this case, policy instruments and their lags only respond to predetermined variables: they are necessarily forward-looking variables.

The underlying assumption consistent with control theory is that the *policy maker only considers as policy targets the subset of predetermined variables and exclude forward-looking variables from policy targets.* Such an underlying assumption contradicts the narrative of the new-Keynesian model, because the policy targets (inflation and output gap) are assumed to forward-looking variables. All the predetermined variables are exogenous (hence, non-controllable) auto-regressive forcing variables. They are never described as policy targets in new-Keynesian theory. The trick to conceal this contradiction is to use the property that the two forward-looking policy targets (inflation and the output gap) are linear functions of the two predetermined forcing variables of the new-Keynesian model. New-Keynesian theory confine attention to representation of the linear time invariant feedback rule responding on the $m=2$ forward-looking variables instead of the representation of the linear time invariant feedback rule responding to $n-m=2$ predetermined variables, by contrast to Currie and Levine (1985).

When all the predetermined variables are non-controllable and when their number $n-m$ is assumed to be equal to the number m of forward-looking variables, the linear substitution of predetermined variables by forward-looking variables in the representation of the policy rule demonstrates that for *ad hoc* simple rule, forward-looking policy targets are seemingly related to forward-looking policy instruments. This is the opposite of the Ramsey optimal policy with finite or infinite horizon: forward-looking policy targets are related to predetermined policy instrument (and some of their lags, if the number of policy instruments is smaller than the number of policy targets).

Proposition 5. *Shifting from optimal policy under finite horizon commitment to new-Keynesian Taylor rule with positive Taylor rule parameters corresponds to a Hopf bifurcation in the new-Keynesian model.*

Proof. The area for positive Taylor rule parameters correspond to two unstable eigenvalues of the new-Keynesian controllable and endogenous block transition matrix, whereas the area for Ramsey optimal policy lies within the stability triangle, with two stable eigenvalues. The bifurcation border to cross corresponds when shifting from Ramsey optimal policy to positive new-Keynesian Taylor rule parameter to Hopf bifurcation, where at the Hopf bifurcation border, the two eigenvalues are complex conjugate.

Q.E.D.

This bifurcation has consequences for robust control and multiple equilibria.

4.2 Bifurcation and robust control

For robust optimal control, the central bank is uncertain about the model. The policy maker thus design a robust rule minimizing the maximum of losses as if he was facing an evil agent.

For robust simple rule, Giordani and Söderlind (2004, p.2386-8) "*propose to be more* *specific about the set of model the evil agent can choose (that is, the type of misspecification feared), by imposing that he sets his instruments as a linear function of predetermined variables... Else, by strategically exploiting expectations, an agent free to commit can drive the loss function to infinity for any degree of robustness... The misspecification feared is then a trend increase (or decrease) of inflation.*" Giordani and Söderlind (2004) propose to restrict evil agent misspecifications. He turns to be an "angel agent" forbidden to select bifurcations of expectations outside the stable subspace of a given simple rule. Without this restriction, if ever the Central Bank makes the slightest mistake on evaluating forward-looking expectations, the new-Keynesian Taylor rule leads to trend increase or decrease of inflation and/or output gap. Expectations are supposed to be perfectly known by contrast to backward-looking variables, whereas practitioners may perceive the opposite.

For robust Ramsey optimal policy, the evil agent can set his instruments as a linear function of predetermined variables and forward-looking variables. Hence, robust Ramsey optimal policy (Hansen and Sargent (2008, 2011)) is *robust to a much broader set of evil-agent misspecifications* than new-Keynesian Taylor rule, because its stable subspace is of dimension n (due to negative-feedback mechanism) instead of dimension $n-m$ (due to positive-feedback mechanism).

4.3 Bifurcation and multiple equilibria

Kalman (1960) demonstrated the uniqueness of the Riccati solution and the optimal policy rule in the case of the linear quadratic regulator with boundary conditions leading to n stable eigenvalues of the closed loop matrix $\mathbf{A}_{yy}+\mathbf{B}_y\mathbf{F}_y$ for n controllable policy targets for $2n$ equation of the Hamiltonian systems. This corresponds to Ramsey optimal policy. The determinacy condition is "*just the set of necessary conditions for maximization of a quadratic function subject to linear constraints*" (Blanchard and Kahn (1980), p.1309).

When *varying* the policy rule $\mathbf{F}_y$ (instead of assuming it is given), if one seeks a number of stable eigenvalues $0 < n-m < n$ lower than the number n of controllable policy targets and *strictly* above zero, the Riccati equation may have multiple solutions (Freiling (2003)). There are as many equilibria as the number of combinations picking $n-m$ stable eigenvalues in a total set of n controllable eigenvalue (that could be stabilized by the policy rule according to Wonham's theorem, with relationships $\lambda_i(\mathbf{F}_y)$). There is only one combination to pick n stable eigenvalues in a set of n eigenvalues (Kalman's (1960) LQR solution). There is only one combination to pick 0 stable eigenvalues in a set of n controllable eigenvalues (Gali (2015) new-Keynesian model). For a DSGE macro-prudential model including two agents (households and financially constrained banks) with two controllable forward-looking flows of saving variables and two controllable predetermined stocks of debt, there are already *six* combinations (equilibria) to pick 2 stable eigenvalues in a set of 4 controllable eigenvalues.

A famous example of multiple solutions of matrix Riccati equation is Leeper's (1993) fiscal theory of the price level (FTPL) equilibrium. The FTPL is an equilibrium such as "inflation eigenvalue" is stable and "public debt eigenvalue" is unstable, by contrast to another equilibrium where "inflation eigenvalue" is unstable and "public debt eigenvalue" is stable. These multiple equilibria "à la Leeper" happen if the private sector variables includes at least one controllable predetermined state variable (the stock of wealth, the stock of private and/or public debt, the stock of capital) and at least one controllable forward-looking costate variable such as consumption. If all the policy instruments of ad hoc policy rules are arbitrarily assumed to be forward-looking, there should be at least one stable eigenvalue and one unstable eigenvalue. When varying the policy rule $\mathbf{F}_y$, any values of the eigenvalues of the controllable system can be reached. Two types of equilibria can be reached: one with $|\lambda_1(\mathbf{F}_y)| > 1$ and $|\lambda_2(\mathbf{F}_y)| < 1$ and the second one with $|\lambda_1(\mathbf{F}_y)| < 1$ and $|\lambda_2(\mathbf{F}_y)| > 1$.

On figure 1, if ever we assumed arbitrarily that inflation is predetermined and output gap is forward-looking (or the reverse), the first type of equilibria à la Leeper with only one stable eigenvalue is reached by policy rule parameters in *upper-left* area of the plane of rule parameters. The second type of equilibria à la Leeper with only one stable eigenvalue is reached by policy rule parameters in *lower-right* area of the plane of rule parameters.

Gali (2015) rules out multiple equilibria à la Leeper in footnote (3): "*The assumptions of a representative household and a zero net supply of one-period debt, guarantee that [transversality condition] (6) is always satisfied in equilibrium, since $B_t = 0$ for all observation allows us to ignore that condition in much of what follows.*"

Once eliminating all the endogenous controllable predetermined state variable, there remains only forward-looking controllable inflation and output. A degenerate rational expectations equilibrium without predetermined variables is such that determinacy imposes that all forward-looking always instantly jump on their long-run equilibrium values.

In order to avoid (1) a degenerate rational expectations equilibrium and (2) multiple equilibria "à la Leeper", Gali (2015) replaces household's controllable state variable by at least one non-controllable auto-regressive forcing variable.

From the point of view of control, if the private sector state variable (the stock of wealth, which is fully invested in one-period public debt) is always zero for all observation, it is not controllable and it is always stable at its equilibrium value. Then, household's Euler consumption first order equation of the private sector is an irrelevant condition.

5 Conclusion

Barnett and Duzhak's (2008) Hopf bifurcation corresponds to a shift from Ramsey optimal policy to a new-Keynesian Taylor rule for a given new-Keynesian monetary policy transmission mechanism including the new-Keynesian Phillips curve and the intertemporal substitution Euler equation of consumption.

To base stabilization on a positive-feedback rule-mechanism contradicts the key principle of a negative-feedback mechanism of control (Aström and Kumar (2014)). A positive-feedback rule for stabilization would require more robust theoretical foundations, the practical evidence of its use, and the empirical evidence of its success for stabilizing the economy.

Distinguishing positive-feedback mechanism from negative-feedback mechanism requires a simultaneous estimation of the policy transmission mechanism and of the policy rule. Testing the Taylor rule principle or the sign of the response of the interest rate to the output gap in an estimation of a single equation of the Taylor rule cannot inform about new-Keynesian positive-feedback versus negative-feedback macroeconomic stabilization without estimating the signs and the magnitude of the effects involved in the policy transmission mechanism.

Further research may test Ramsey optimal policy versus a new-Keynesian Taylor rule for the joint model of Euler consumption equation and the new-Keynesian Phillips curve, following Chatelain and Ralf (2017) tests and estimations in the case where the transmission mechanism is only the new-Keynesian Phillips curve (Gali (2015), chapter 5).

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6 Appendix 1: Scilab code

Download the open source software Scilab and copy and paste the following code in the command window, for given preferences of the central bank (Qpi, Qx, R) and given monetary policy transmission mechanism parameters (beta1, gamma1, kappa, rho1, rho2). Transition matrix are multiplied by $\sqrt{\beta}$ in order to take into account discounting as proposed by Anderson, Hansen, McGrattan and Sargent (1996). Formulas for Riccati, Sylvester and rule parameters are taken in Anderson, Hansen, McGrattan and Sargent (1996).

```
Qpi=4;Qx=0;R=1;Qxpi=0;
beta1=0.99; gamma1=0.5; kappa=0.1;
rho1=0.9; rho2=0.9; rho12=0;
Qxrho1=0; Qpirho1=0;
Qxrho2=0; Qpirho2 =0;
A1=[1-(kappa*gamma1/beta1) -gamma1/beta1 ; -kappa/beta1 1/beta1] ;
A=sqrt(beta1)*A1;
B1=[gamma1 ; 0];
B=sqrt(beta1)*B1;
Q=[Qx Qxpi ;Qxpi Qpi ];
Big=sysdiag(Q,R);
[w,wp]=fullrf(Big);
C1=wp(:,1:2);
D12=wp(:,3:$);
M=syslin('d',A,B,C1,D12);
[Fy,Py]=lqr(M);
A+B*Fy;
Py
Fy
spec(A+B*Fy)
abs(spec(A+B*Fy))
A+B*Fy
A
B
Ayz=[-1 gamma1/beta1 ; 0 -1/beta1 ];
Azz=[rho1 rho12; rho12 rho2 ];
Qyz=[Qxrho1 Qpirho1 ; Qxrho2 Qpirho2 ];
BS=-Azz;
AS=(A+B*Fy)';
CS=Qyz+AS*Py*Ayz;
Pz=sylv(AS, BS, CS, 'd');
AS*Pz*BS+Pz-CS;
norm (AS*Pz*BS+Pz-CS);
N=-inv(Py)*Pz;
Fz=inv(R+B'*Py*B)*B'*(Py*Ayz + Pz*Azz);
sp1=spec(A+B*Fy)
sp1t=sp1'
Py
Pz
Spectrum=[sp1t rho1 rho2 ]
abs(spec(A+B*Fy))
F=[Fy Fz ]
N
```

Figure 3: Ramsey optimal policy LQR dark triangle within the stability triangle (determinacy with two stable eigenvalues). New-Keynesian Taylor rule area in light grey (determinacy with zero stable eigenvalues). Inflation Taylor rule parameter on the horizontal axis, output gap Taylor rule parameter on the vertical axis.

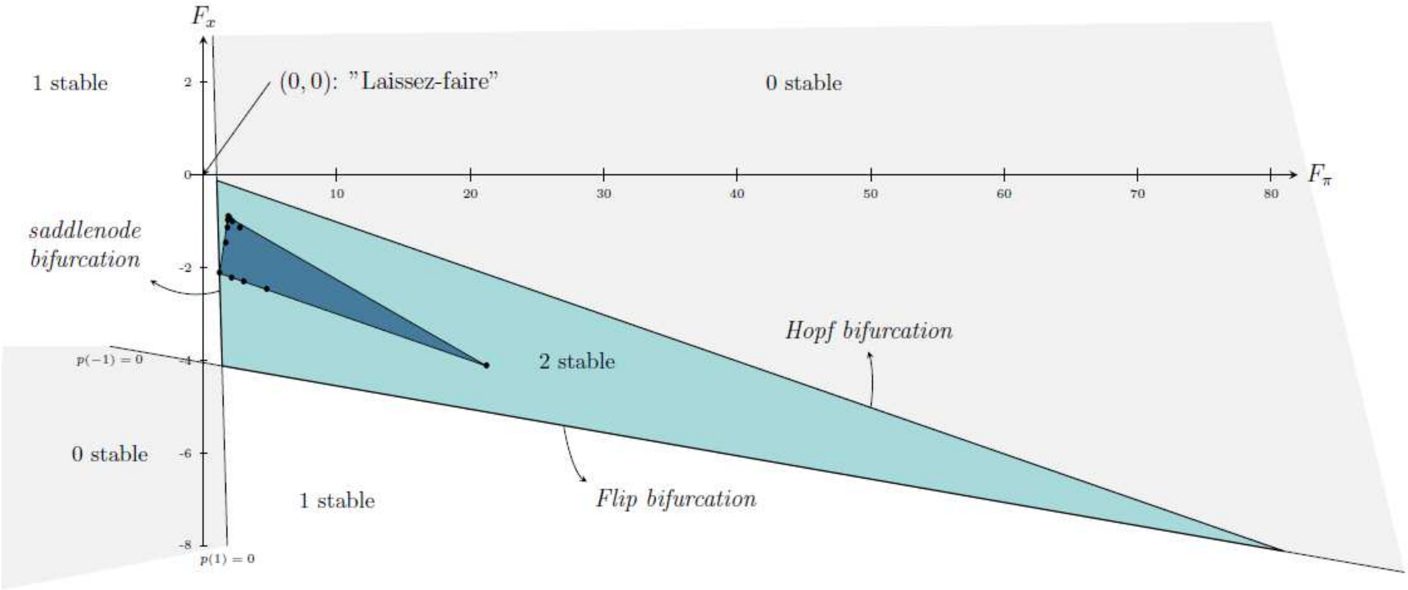

FIGURE: LQR TRIANGLE WITHIN THE STABILITY TRIANGLE

Figure 3bis: Stability triangle in the trace and determinant plane (Azariadis (1993), figure 8.4). (not for publication).