Gravitation of market prices towards normal prices: some new results

Bellino, Enrico and Serrano, Franklin

Università Cattolica del Sacro Cuore, Piacenza (Italy), Instituto de Economia, Universidade Federal do Rio de Janeiro (Brazil)

22 May 2017

Gravitation of market prices towards normal prices: some new results

by Enrico Bellino[*] and Franklin Serrano[**]

May 2017

Abstract. The gravitation process of market prices towards production prices is here presented by means of an analytical framework where the classical capital mobility principle is coupled with a determination of the deviation of market from normal (natural) prices which closely follows the description provided by Adam Smith: each period the *level* of the market price of a commodity will be higher (lower) than its production price if the quantity brought to the market falls short (exceeds) the level of effectual demand. This approach also simplifies the results with respect to those obtained in cross-dual literature. At the same time, anchoring market prices to effectual demands and quantities brought to the markets requires a careful study of the dynamics of the 'dimensions' along with that of the 'proportions' of the system. Three different versions of the model are thus proposed, to study the gravitation process: i) assuming a given level of aggregate employment; ii) assuming a sort of Say's law; iii) and on the basis of an explicit adjustment of actual outputs to effectual demands. All these cases describe dynamics in which market prices can converge asymptotically towards production prices.

Key words: Market prices, normal prices, Classical competition, gravitation, effectual demand

J.E.L. classification: B12, D20, E11, E30

[*] Università Cattolica del Sacro Cuore, Piacenza (Italy); enrico.bellino@unicatt.it

[**] Instituto de Economia, Universidade Federal do Rio de Janeiro (Brazil); franklinserrano@gmail.com

1. Introduction

The Classical analysis of gravitation consists of two basic steps. The first step, as argued by Adam Smith, Ricardo and Marx, is that whenever the quantity brought to market is larger than the effectual demand for a commodity the *level* of market prices of that commodity will fall below the normal price[1]. Conversely when the quantity brought to market is lower than effectual demand the *level* of market prices will be higher than the normal price.

In the second step the Classics argued that the quantities brought to market would decrease in the sectors where profits were below average and increase in those sectors where profits were above average. This process would ensure the tendency of market prices to gravitate towards (or oscillate around) normal prices.

The Classical process of gravitation of market prices around normal prices has generally been formalized during the 1980s and 1990s by models where relative prices interact with sectoral output proportions through a 'cross-dual' dynamics. In these models the *rates of change* of sectoral output proportions react to deviations between sectoral and average rates of profits; symmetrically, the *rates of change* of market prices react to deviations between demand and the quantity brought to the market of the respective commodity. Yet, a quite puzzling outcome emerged immediately in the main contributions: the basic dynamics arising from this interaction was intrinsically *unstable*. Convergence results were then obtained only after introducing a variety of suitable (even if reasonable) modifications of the basic cross-dual model.

In the present paper, following the lead of Garegnani (1997), we propose a reformulation of the formal analysis of the Classical gravitation process, closer to what is written in Smith's and Ricardo's own texts, in which the *levels* of market (relative to natural) prices that react to the difference between effectual demands and quantities brought to the market. A convergence result is then proved, confirming thus the classical conjecture of gravitation of market prices towards natural prices.

Section 2 introduces the notation adopted and outlines the system of normal prices. In Section 3 we will recall intrinsic limits of cross-dual models of gravitation. An alternative approach will be thus presented in the rest of the paper. A convergence result is firstly proved in Section 4 in a very simple model, where the gravitation process is formulated in *relative* terms for both prices and quantities. This structure may

[1] Normal prices here of course mean what Smith and Ricardo called natural price and Marx called price of production. And effectual demand was called "social need" by Marx.

be represented through one first degree difference equation. The simplicity of this model allows us to catch immediately the fundamental forces operating in a capitalistic system in engendering the convergence towards the normal position. Yet, this simplicity leads to an asymmetry between industries in the adjustment rule of market to normal price, which is not justified from the economic point of view. We thus reformulate in Section 5 the entire process in terms of the absolute output levels of the two industries. We obtain thus a system of *two* first degree difference equations. Unfortunately, the generalization of this simple model entails another analytical problem. It displays a *continuum* of steady states. All the situations where output levels reflect the *proportions* between normal output could be resting point for the dynamic system. In other words, for production prices prevail it is sufficient that relative sectoral outputs have the same proportion between normal outputs. Their absolute values do not matter. In this way production prices would prevail in situations of a general glut, as well as, in situations of a general shortage. As the formalization adopted to represent the principle of capital mobility reveals to be able to affect only the proportions of the system, we need to introduce a further principle capable to control the dimension of the system. This will be done in three alternative steps. In Section 6 we will adopt the simplest solution: that of keeping the dimension of the economic system measured in terms of employment fixed at a given level. In Section 7 we will determine the dimension of the system endogenously, by a sort of Say's law, in coherence with the approach accepted by classical economists. In Section 8 we will determine the dimension of the system, through a set of relations similar to those constituting the open Leontief model. Section 9 presents brief final remarks.

2. Notation

Consider an economic system with two commodities, $c = 1, 2$, and two industries, $i = 1, 2$. The technology of this system is represented by a (2×2) socio-technical matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} m_{11} + \ell_1 b_1 & m_{12} + \ell_1 b_2 \\ m_{21} + \ell_2 b_1 & m_{22} + \ell_2 b_2 \end{bmatrix} = \mathbf{M} + \boldsymbol{\ell}\mathbf{b}^{\mathrm{T}}$$

where m_{ic}, is the quantity of commodity c employed to produce 1 unit of commodity i, $\mathbf{b} = [b_c]$ is the real wage bundle assumed to be paid in advance, $\boldsymbol{\ell} = [\ell_c]$ is the vector of direct labour quantities, and $^{\mathrm{T}}$ is the transposition symbol (vectors are thought as column vectors; row vectors are denoted by the transposition symbol). Methods of production are represented on the rows of the matrix. Technical coefficients are supposed to be constant with respect to changes in output levels. The normal price equations are

$$p_1 = (1 + r)(a_{11}p_1 + a_{21}p_2) \tag{1a}$$

$$p_2 = (1 + r)(a_{21}p_1 + a_{22}p_2), \tag{1b}$$

where p_c is the price of commodity c and r is the uniform rate of profit. As known, system (1) determines a *unique* economically meaningful relative price and a unique positive rate of profit given by

$$p^* = \left(\frac{p_1}{p_2}\right)^* = \frac{a_{11} - a_{22} + \sqrt{\Delta}}{2a_{21}} \quad \text{and} \quad r^* = \frac{2}{a_{11} + a_{22} + \sqrt{\Delta}}, \tag{2}$$

where $\Delta = (a_{11} - a_{22})^2 + 4a_{12}a_{21} > 0$, if the dominant eigenvalue of $\mathbf{A}$, denoted by λ^*, is smaller than 1. The normal relative price of commodity 1 in terms of commodity 2, p^*, and the normal rate of profit, r^*, satisfy equations (1), that is,

$$p^*/(1 + r^*) = (a_{11}p^* + a_{21}) \tag{3a}$$

$$1/(1 + r^*) = (a_{21}p^* + a_{22}). \tag{3b}$$

3. The pure cross-dual model

The dynamics of proportions in a pure cross dual model can be summarized as follows. Suppose that all profits are saved and invested ('accumulated'). Let $q_t = q_{1t}/q_{2t}$ the ratio between actual output levels at time t. In discrete time the competitive process is described by the following difference system:

$$q_{t+1} = q_t \, \frac{1 + r_t + \gamma(r_{1t} - r_t)}{1 + r_t + \gamma(r_{2t} - r_t)} \tag{4a}$$

$$p_{t+1} = p_t \, \frac{1 + \beta\left(\dfrac{d_{1t} - q_t}{q_t}\right)}{1 + \beta(d_{2t} - 1)}, \tag{4b}$$

where

$$r_t(q_t, p_t) = \frac{q_t p_t + 1}{q_t a_{11} p_t + q_t a_{12} + a_{21} p_t + a_{22}} - 1, \tag{5a}$$

$$r_{1t}(p_t) = \frac{p_t}{a_{11}p_t + a_{12}} - 1 \quad \text{and} \quad r_{2t}(p_t) = \frac{1}{a_{21}p_t + a_{22}} - 1. \tag{5b}$$

$$d_{ct} = \frac{\delta_{ct}}{q_{2t}} = q_t[1 + r_t + \gamma(r_{1t} - r_t)]a_{1c} + [1 + r_t + \gamma(r_{2t} - r_t)]a_{2c}, \quad c = 1, 2, \tag{5c}$$

r denotes the general (average) rate of profit, r_1 and r_2 denote the sectoral rates of profit, $\delta_{ct} = q_{1t+1}a_{1c} + q_{2t+1}a_{2c}$ denotes the demand of commodity c in period t, d_{ct} denotes the ratio between the demand of commodity c and the output of commodity 2, and β and γ are two reaction parameters (for further detail see Boggio, 1992).

The dynamic system represented by difference equations (4), with symbols defined in (5) admits a unique economically meaningful (steady state) equilibrium, (q^*, p^*) where

$$q^* = \frac{a_{11} - a_{22} + \sqrt{\Delta}}{2a_{12}}$$

and p^* is defined in (2).[2] It can be verified that the Jacobian matrix of the difference system evaluated at the steady state is

$$\mathbf{J}^* = \begin{bmatrix} 1 & \gamma U \\ -\beta V & 1 + \beta\gamma Z \end{bmatrix}, \tag{6}$$

where $U = \rho(q^*/p^*)M > 0$, $V = (p^*/q^*)(1 - \rho^2|\mathbf{A}|) > 0$, $Z = \rho^3 M|\mathbf{A}|$, $\rho = (1 + r^*)$ and $M = (a_{12}/p^* + a_{21}p^*)$. From (6) it is quite easy to verify that the steady state (q^*, p^*) is asymptotically unstable for any positive level of the reaction coefficients β and γ (for details see Boggio, 1984, 1985 and 1992, or Duménil and Lévy, 1993, appendix to chap. 6).

The causes of such a negative conclusion have been explained in detail by Lippi (1990) and by Garegnani (1990, Section 27 and the Appendix available only in the revised version of the paper, that is, Garegnani, 1997); see also the discussion in Serrano, 2011, Section V). They can be summarized by observing that the dynamics

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[2] Observe that q^* is the proportion between sectoral outputs characterizing the 'balanced growth path' obtained when all profits calculated at the rate r^* are entirely re-invested. Such a path is defined by the conditions

$$q_{ct+1} = (1 + g)q_{ct}, \quad c = 1, 2, \tag{*}$$

where

$$q_{ct} = q_{1t+1}a_{1c} + q_{2t+1}a_{2c}, \quad c = 1, 2, \tag{**}$$

and g is the uniform growth rate. Substitute (*) into (**); we yield

$$q_c = (1 + g)(q_1a_{1c} + q_2a_{2c}), \quad c = 1, 2, \tag{***}$$

where the time index has been omitted because all variables are here contemporaneous. By expressing equations (***) in relative terms we obtain

$$q = (1 + g)(qa_{11} + a_{21})$$
$$1 = (1 + g)(qa_{12} + a_{22}),$$

whose (positive) solution is in fact $q = q^*$ and $g = r^*$.

engendered by equations (4) give rise to a dynamic behaviour which is not sensible from the economic point of view, and do not correspond to the first step as argued by the classics mentioned above. Cross-dual models make the rate of change (instead of the level) of market prices react to deviation between effectual demands and quantities brought to the market. Therefore, as long as the quantity brought to the market (output) happens to be lower than the *level* of effectual demand by a particular amount, output will be increasing but the market prices will also be increasing (their rate of change will be positive), leading to overshooting. Then when output reaches effectual demand the market price will stop changing but will be at a level above the normal price and thus, in spite of the effectual demand being equal to the quantity brought to the market, the rate of profits of the sector will be above normal and thus the quantities brought to the market will continue to increase, and market prices will also start falling, leading to overshooting. Conversely, the same process happens symmetrically in reverse, if we start from a situation in which the quantity brought to market is greater than the effectual demand and a market price lower than normal.

Following Lippi (1990), this argument can be represented graphically. Let us approximate the demand of commodity c at time t, given by $\delta_{ct} = q_{1t+1}a_{1c} + q_{2t+1}a_{2c}$, by $\widetilde{\delta}_{ct} = q_{1t}a_{1c} + q_{2t}a_{2c}$, so that the dynamics of the relative price is given by

$$p_{t+1} = p_t \frac{1+\beta\left(\dfrac{q_t a_{11} + a_{21}}{q_t} - 1\right)}{1+\beta(q_t a_{12} + a_{22} - 1)} = \frac{1+\beta\left(\dfrac{\widetilde{d}_1(q_t)}{q_t} - 1\right)}{1+\beta[\widetilde{d}_2(q_t) - 1]}. \tag{4b′}$$

It is quite easy to prove that $\widetilde{d}_1(q)/q \overset{>}{\underset{<}{-}} \widetilde{d}_2(q)$ if and only if $q \overset{<}{\underset{>}{-}} q^*$. Hence, by (4b′), we can state that

$$p_t \begin{Bmatrix} \text{increases} \\ \text{remains constant} \\ \text{decreases} \end{Bmatrix} \text{ as long as } q_t \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} q^*. \tag{7}$$

Suppose the system initially is in its normal position, $q = q^*$ and $p = p^*$, represented by point E in Figure 1, which was originally presented by Lippi (1990, p. 63). Then a shock displaces q to $q_0 < q^*$, i.e. to point R_0. Then p increases so that $r_1 > r_2$ attracts capital from industry ‘2’ to industry ‘1’: the system moves thus from R_0 to R_1, to R_2 etc.. But when q has reached q^*, that is, point R', the relative market price p, which was initially equal to p^*, happens to be *greater* than p^*, so that $r_1 > r_2$ and q *continues to increase* beyond q^*, i.e. system *overshoots* moving toward R''.

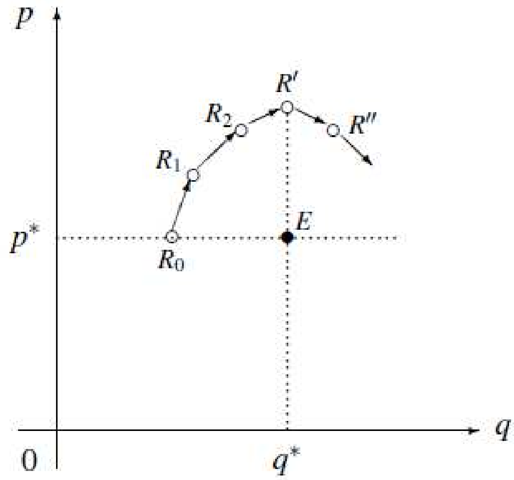

Figure 1 – Movements of q and p around q^* and p^* in the pure cross-dual model

Relative market price overshoots because its dynamics described by equation (4b′), is totally unconnected with normal output q^*. As recalled by Lippi, these elements explain the reason why the dynamics ensuing from cross dual models *oscillates* around the steady state equilibrium. An analytical study of the model verifies that these oscillations are *divergent*, that is, their amplitudes increases as time goes by, so that when the economy happens to be out of its long-run equilibrium it moves away from it.

Several devices have been proposed to ‘adjust’ this destabilizing dynamics.[3] Yet, this destabilizing dynamics seems more inherent to this type of model rather than to the classical competitive process as described verbally by the classical economists.

We can move towards a more satisfactory formal representation of the basic Classical gravitation process if we produce a simple model that adheres to the first step of the Classical argument.[4] As it is well known, the Classics did not conceive market prices theoretical magnitudes and allowed for the fact that many transactions could occur at different market prices for the same commodity. Even so they thought that

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[3] For example, Dumenil-Levy(1993, chap. 6 and its appendix) proposed to consider the ‘realized’ rates of profit $r_{it} = (d_{it}\, p_{it} - q_{it}\, \mathbf{a}_i^{\mathrm{T}}\mathbf{p}_t)/(q_{it}\, \mathbf{a}_i^{\mathrm{T}}\mathbf{p}_t)$, instead of the ‘appropriated’ rates of profit, defined in (5): in this way, the calculation of revenues of industry i on the basis of the amount of output actually demanded of commodity i is sufficient to counter-balances the destabilizing forces contained in the pure cross-dual model. A similar amendment of the pure cross-dual model, still proposed by (Dumenil-Levy, 1993, chap. 6), is obtained by introducing a sort of ‘direct adjustment of quantities’, which contrasts the destabilizing forces of pure cross-dual dynamics.

[4] An unpublished pioneering example of this is the model put forward by Silveira (2002) using Lyapunov distance functions.

some average level of market prices for a commodity would be higher or lower relative to the (single) normal price when effectual demand was higher or lower than the quantity brought to market. Because of the variability of the causes that led to these deviations they did not conceive this relationship between market prices and normal price of a commodity as a definite formal let alone linear function; we shall do so just to illustrate the idea of gravitation. We shall also assume, for simplicity, that all producers have access only to the same single dominant technique to produce each commodity.[5]

Note that this Classical relationship between market prices and normal price does not involve market clearing, since it determines a particular level of (average) prices in the market taking it into account the reactions of both producers and user or consumers of the commodity.[6] This relationship certainly does not represent a "demand function', even when we formally assume this relationship to be a given and linear function because the extent by which market prices rise above or fall below normal prices will reflect both by the behaviour of those demanding the commodity (other people that are not effectual demanders but can afford buying the commodity when price are sufficient below normal, for instance) and those supplying it (reservation prices, firm's decisions concerning holding inventories, etc.).

Moreover, it is important to point out that this relationship is not assumed to be known by agents in the economy since it is a description of the results of the 'haggling of the market' as a whole under given circumstances and not a description of the behaviour of a particular agent. This was the perspective adopted by Smith, who aimed to provide a description of the general *outcome* of the competitive process, rather than to give a detailed description of the actual moves of *each actor* of the process by which market prices were determined (about which there could be no fully general theory). Gravitation analysis concerns only the operation of the capital mobility principle and the firms are only assumed to increase the quantity brought to the market if the actual rate of profit is higher than that of the other sector(s) and reduce it if it is lower.

Steedman (1984) criticized the second step of the Classical gravitation analysis showing that with three or more commodities a sector in which the market price is higher (lower) than normal could possibly have a rate of profit *below* (above) average if the market prices of its inputs where proportionally much higher (lower) than their normal price. To this Ciampalini and Vianello (2000, p.365 footnote 9) countered that

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[5] On the relationship between normal and market prices in the works of the Classics see Garegnani (1976), Ciccone (1999), Vianello (1989) and Aspromourgous (2009)
[6] Garegnani (1997, Appendix).

Smith was thinking about the price and rate of profits of the vertically integrated industry (or subsystem) producing a commodity. Garegnani (1990, 1997) on the other hand, has shown that this possibility exists for a particular sector but could only really endanger the gravitation process if the rates of profits of all sectors could all be above (or below) the normal rate at the same time, something that is logically impossible in a Classical framework for a given technique and level of the real wage. Therefore, the second step in the Classical process of gravitation is not strictly necessary to guarantee that market prices gravitate towards normal prices. We only need the first and the third steps. In any case the Steedman critique does not affect the analysis presented in the following Section—as well as the main part of the cross-dual gravitation models—as the changes in relative outputs are presented as functions of the observed profit rate differentials (and not directly of the divergence between market and normal prices).

4. A gravitation model with a Smithian behaviour of market prices: the simplest formulation

We will consider for simplicity a system with no capital accumulation, where all profits and wages are consumed. Moreover, normal output of the two commodities will be taken as given and will be denoted by q_1^* and q_2^*. Let $q^* = q_1^* / q_2^*$ be the normal output proportion. Actual (relative) output dynamics is described by the following difference equation:

$$q_{t+1} = q_t \frac{1+\gamma(r_{1t} - r_t)}{1+\gamma(r_{2t} - r_t)}, \tag{8}$$

where

$$r_{1t} = \frac{p_t}{a_{11}p_t + a_{12}} - 1, \quad r_{2t} = \frac{1}{a_{21}p_t + a_{22}} - 1, \quad r_t = \frac{q_t p_t + 1}{q_t a_{11} p_t + q_t a_{12} + a_{21} p_t + a_{22}} - 1 \tag{9a}$$

and

$$p_t = p^* + \beta(q^* - q_t). \tag{9b}$$

The analytical structure of this system is very simple: by (8) the future level of relative output, q_{t+1}, depends on the present level, q_t, and on r_{1t}, r_{2t} and r_t, which depend on p_t and q_t. But as also p_t depends on q_t; then we have that q_{t+1} depends ultimately on q_t only. The steady state(s) of difference equation (8) can be found by setting $q_{t+1} = q_t = q$ which, once substituted in (8) yields $r_1 = r_2 = r$. As known, there is a unique positive relative price ensuring a uniform rate of profit: $p = p^*$, which guarantees $r = r^*$, where

p^* and r^* are defined in (2). From (9b) we get $q_t = q^*$, which is the unique meaningful (i.e. positive) steady-state of equation (8). Thus the following Proposition holds.

Proposition 1. *Difference equation (8) with r_1, r_2, r and p defined by (9) admits a unique meaningful equilibrium, $q_t = q^*$, where $r_1 = r_2 = r = r^*$ and $p = p^*$.*

Quite simple calculations obtain (see Appendix A.1)

$$\left.\frac{\mathrm{d}\,q_{t+1}}{\mathrm{d}\,q_t}\right| = 1 - \beta\gamma\omega Z\,, \tag{10}$$

where $\omega = q^*/p^*$ and

$$M = \rho^2\left(\frac{a_{12}}{p^*} + a_{21}p^*\right) > 0. \tag{11}$$

From (10) and (11) the following preposition holds.

Proposition 2. *The steady state equilibrium of difference equation (8) with r_1, r_2, r and p defined by (9) is locally asymptotically stable if the reaction coefficients are such that their product $\beta\gamma$ is sufficiently small.*

This simple formulation has the merit to let emerge immediately the stabilizing force of capital mobility when it is coupled with a principle that regulates the deviation of market prices from their normal (natural) values similar to that described by Adam Smith.

On a simple graph it is possible to represent the dynamics of this simple model. Differently to what happens in cross-dual models, where when one state variable moves towards the equilibrium level the other one move away from it (see Figure 1), the dynamics of the pairs (q_t, p_t) is bounded to take place only in the one-dimensional space, represented by the straight line (9b) (see Figure 2): hence, as q_t moves towards q^*, p_t is forced to move towards p^*.

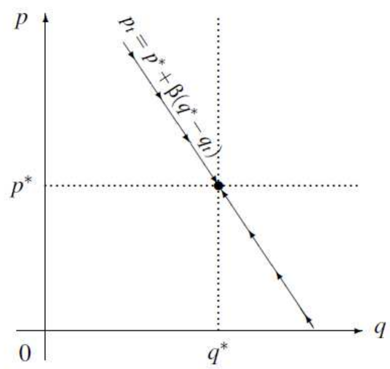

Figure 2 – Movements of q and p around q^* and p^* in the Smithian simple model

The limit of this simple formulation is the *asymmetry* in the formulation of this principle: in fact, if we deduce from equation (9b), that is, $p_1/p_2 = (p_1/p_2)^* + \beta(q_1^*/q_2^* - q_1/q_2)$, the price of commodity 2 expressed in terms of commodity 1 we get

$$\frac{p_2}{p_1} = \frac{1}{\left(\frac{p_1}{p_2}\right)^* + \beta\left(\frac{q_1^*}{q_2^*} - \frac{q_1}{q_2}\right)}.$$

While a *linear* relation describes the formation of market price of commodity 1 in terms of commodity 2, a *non-linear* relation describes the formation of the market price of commodity 2 in terms of commodity 1. If, on the one hand, a linear function may be considered the simplest approximation of any linear continuous function, there are no reasons to imagine such a different behaviour of the deviation of market prices from production prices. In following sections the gravitation process will be re-formulated in order to avoid this asymmetry.

5. Towards a more general formulation and the problem of proportions and scale

One way to re-express the mechanism regulating the deviations of market prices from production prices in a form where each commodity is treated symmetrically is to adopt the following equation

$$p_t = p^* \cdot \frac{1 + \beta_1\left(1 - \frac{q_{1t}}{q_1^*}\right)}{1 + \beta_2\left(1 - \frac{q_{2t}}{q_2^*}\right)}, \tag{12}$$

where q_{1t} and q_{2t} denote the actual output of each commodity and β_1 and β_2 are two reaction coefficients, which in general need not to be equal. It is straightforward to observe that no asymmetry between the market prices of the commodities is entailed in this formulation. At the same time, possible different degree of price flexibility between industries may find space by a suitable choice of coefficients β_1 and β_2 (this possibility was never explicitly considered in cross-dual models, even though it should not alter the main results). Moreover, generalizations to any number of commodities are quite straightforward with the formulation of the market price equation given by (12).

This reformulation of the market price equation requires that output dynamics originating from the capital mobility principle is expressed in terms of *absolute* rather than in relative terms. We have thus *two* difference equations:

$$q_{1t+1} = q_{1t}\,[1 + \gamma(r_{1t} - r_t)] \tag{13a}$$

$$q_{2t+1} = q_{2t}\,[1 + \gamma(r_{2t} - r_t)] \tag{13b}$$

where

$$r_{1t} = \frac{p_t}{a_{11}p_t + a_{12}} - 1, \quad r_{2t} = \frac{1}{a_{21}p_t + a_{22}} - 1, \tag{14a}$$

$$r_t = \frac{q_{1t}p_t + q_{2t}}{q_{1t}a_{11}p_t + q_{1t}a_{12} + q_{2t}a_{21}p_t + q_{2t}a_{22}} - 1 \tag{14b}$$

are the industrial and the average rates of profit (sectoral rates of profit coincide with those defined in equation (9a); the average rate of profit now contains absolute instead of relative output levels). The relative market price is determined by equation (12).

This formulation displays an analytical problem: the *scale* of activity of the industries is undetermined in steady state. In fact, if we impose

$$q_{1t+1} = q_{1t} = q_1 \quad \text{and} \quad q_{2t+1} = q_{2t} = q_2.$$

in equations (13) we obtain $r_1 = r_2 = r$. This uniformity holds if $p = p^*$—ensuring $r = r^*$, where p^* and r^* are defined in (2)—which entails by (12):

$$\beta_1\left(1 - \frac{q_1}{q_1^*}\right) = \beta_2\left(1 - \frac{q_2}{q_2^*}\right). \tag{15}$$

In this way, *any pair* (q_1, q_2) satisfying condition (15) is a steady state of the model: we have a *continuum* of steady states. Just to understand, suppose that $\beta_1 = \beta_2$: condition (15) reduces $q_1 / q_2 = q_1^* / q_2^*$. Any situation where actual output levels happen to be in the *proportion* characterizing normal output levels is a steady state. In such steady states nothing guarantees that in steady state $q_1 = q_1^*$ and $q_2 = q_2^*$. In other words, in this model the normal relative price (p^*) and the uniform rate of profit (r^*) would be compatible with an imbalance *of the same sign* and of the same percent entity between actual and normal output—a general glut as well as a general shortage. A simple numerical simulation of this model reveals that when the reaction coefficients, β_1, β_2 and γ are sufficiently small, actual output levels, q_{1t} and q_{2t}, tend to two (finite) levels which depend on their initial levels—so starting from two different initial conditions,

(q'_{10}, q'_{20}) and (q''_{10}, q''_{20}) market prices converge towards their normal level, p^*, the rates of profit converge towards the (uniform) normal level, r^*, while output levels converge to two different resting points, $(\overline{q}_{10}, \overline{q}_{20})$ and $(\overline{\overline{q}}_{10}, \overline{\overline{q}}_{20})$, generally different from the effectual demand, (q_1^*, q_2^*). *But* the *proportions* between these resting output levels always coincide with the normal proportions, that is, $\overline{q}_{10} / \overline{q}_{20} = \overline{\overline{q}}_{10} / \overline{\overline{q}}_{20} = q_1^* / q_2^*$. This is quite obvious. Keep, for simplicity, the assumption that $\beta_1 = \beta_2$: once the system has reached one of the infinitely many steady states the same pressure is exerted on the market prices of each commodity so that their relative value would remain constant at p^*: no further deviations of the relative market price from the relative normal price comes up to correct the general disequilibrium.[7] This is clearly a misspecification or, better, an insufficient specification of the model. The result just emerged shows how the principle of capital mobility as described in (13) and the market price determination contained in (12) affect the *proportions* but not the *dimension* of the system. We need a further force to control the scale of activity of the industries. This will be done in the following sections in three alternative ways.

6. A model with a given level of employment

The simplest way to manage the problem of indeterminacy of the scale of activity emerged in the previous Section is to study the gravitation process in a situation where the scale of the economy is (artificially) kept fixed in terms of its aggregate level of labour employment. In principle, the level of aggregate employment is affected by a set of elements that are not directly connectable with the gravitation process. Hence, we can take these elements as given when studying the gravitation process as done by Garegnani (1990, 1997). To this purpose, the aggregate level of labour employment will be artificially forced in each period at a given level, L^*, not necessarily the full employment level. As our reference outputs are q_1^* and q_2^*, it is reasonable to consider

$$L^* = q_1^* \ell_1 + q_2^* \ell_2 \tag{16}$$

i.e., the amount of labour necessary to produce the normal output. We assume thus that in each period the actual aggregate level of labour employment is equal to L^*.[8] Then, re-scale the outputs determined by equations (13) by a factor, σ_t,

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[7] For the case $\beta_1 \neq \beta_2$ a similar situation conditions the interpretation of the steady state of the model.
[8] This assumption has been justified on the same lines by Garegnani:

$$q_{1t+1} = \sigma_t \, q_{1t} \, [1 + \gamma(r_{1t} - r_t)] \tag{17a}$$

$$q_{2t+1} = \sigma_t \, q_{2t} \, [1 + \gamma(r_{2t} - r_t)] \tag{17b}$$

in such a way that the labour employed in each period is L^*:

$$q_{1t+1}\ell_1 + q_{2t+1}\ell_2 = L^*, \qquad t = 0, 1, 2, \ldots \tag{19-t}$$

that is,

$$\sigma_t \, q_{1t} \, [1 + \gamma(r_{1t} - r_t)]\ell_1 + \sigma_t \, q_{2t} \, [1 + \gamma(r_{2t} - r_t)]\ell_2 = L^*, \qquad t = 0, 1, 2, \ldots .$$

the re-scaling factor is thus given by[9]

$$\sigma_t = \frac{L^*}{q_{1t}[1+\gamma(r_{1t}-r_t)]\ell_1 + q_{2t}[1+\gamma(r_{2t}-r_t)]\ell_2}, \qquad t = 0, 1, 2, \ldots . \tag{18}$$

Moreover, assume that initial actual output levels, q_{10} and q_{20}, which are not obtained by equations (17), satisfy

$$q_{10}\ell_1 + q_{20}\ell_2 = L^* \tag{19-0}$$

too.

Though the subject is beyond the aim of the present chapter some observations may here be necessary with respect to the assumption, implied in the above postulate, [of given effectual demand, e.b.] that the *aggregate* economic activity (on which the effectual demands of the individual commodities evidently depend) can be taken as given in analysing market prices. A first view which may be in that respect is that the deviations of the actual outputs from the respective effectual demands (and therefore their changes during the process of adjustment) will in general broadly compensate each other with respect to their effect on aggregate demand and its determinants. However, the classical postulate of given effectual demands does not appear to ultimately rest on any such eventual compensation of deviations. Here also what needs in effect be assumed is only the possibility of *separating* the two analysis. Thus, if we had reason to think that the effects on aggregate demand of the circumstances causing (or arising out of) certain kinds of deviation of actual from normal relative outputs were sufficiently important – then, it would seem, those effects could be considered in the separate analysis of the determinants of aggregate economic activity and hence of the individual effectual demands. In this chapter, the level of aggregate demand is assumed constant in terms of the level of aggregate labour employment (Garegnani, 1997, pp. 140–1).

[9] As, by construction, the output of each period satisfies equation (19-t), when the dynamics of one of the two outputs is determined (by (17a) or by (17b)), the dynamics of the other one can be determined residually by (19-t). In fact, thanks to (18) we can re-write (19-t) as

$$q_{1t+1}\ell_1 + q_{2t+1}\ell_2 = \sigma_t \, q_{1t} \, [1 + \gamma(r_{1t} - r_t)]\ell_1 + \sigma_t \, q_{2t} \, [1 + \gamma(r_{2t} - r_t)]\ell_2 ; \tag{19-t'}$$

If, for example, q_{2t+1} is determined by (17b), then (19-t') reduces to

$$q_{1t+1}\ell_1 = \sigma_t \, q_{1t} \, [1 + \gamma(r_{1t} - r_t)]\ell_1,$$

that is, to equation (17a) (in a similar way, just one initial condition can be chosen at will, the other being determined residually by (19-0)). We will return later on this point.

We now study the difference system (17), with r_{1t}, r_{2t} and r_t defined by (14), p_t defined by (12) and σ_t defined by (18).

Steady state. In steady state

$$q_{ct+1} = q_{ct} = q_c,\ c = 1, 2. \tag{20}$$

Substitute (20) into (17) and obtain (after simplification)

$$1 = \sigma\,[1 + \gamma(r_1 - r)] \tag{17a′}$$

$$1 = \sigma\,[1 + \gamma(r_2 - r)] \tag{17b′}$$

from which one gets $r_1 - r = r_2 - r$, i.e., $r_1 = r_2$, which entails, at the same time,

$$r_1 = r_2 = r = r^* \tag{21a}$$

$$p = p^*. \tag{21b}$$

Substitute (21a) into (17a′) (or in (17b′)) and obtain

$$\sigma = 1. \tag{22}$$

Substitute (21a) and (22) into (18) and obtains $q_1\ell_1 + q_2\ell_2 = L^*$ which, thanks to (16) yields

$$q_1\ell_1 + q_2\ell_2 = q_1^*\ell_1 + q_2^*\ell_2\,. \tag{23}$$

Substitute (21b) into (12) and obtain

$$\beta_1\left(1-\frac{q_1}{q_1^*}\right)=\beta_2\left(1-\frac{q_2}{q_2^*}\right). \tag{24}$$

Equation (23) and (24) define two straight lines in space (q_1, q_2). Both equations (23) and (24) pass through point (q_1^*, q_2^*). As (23) is a *decreasing* and (24) is *increasing* (q_1^*, q_2^*) is their *unique* intersection. This proves the following:

Proposition 3. (q_1^*, q_1^*) *is the* unique *economically meaningful steady state of difference system (17) with r_{1t}, r_{2t} and r_t defined by (14), p_t defined by (12) and σ_t defined by (18).*

Local asymptotic stability of the steady state. On the basis of the preliminary derivatives calculated in the Appendix (Section 2), the Jacobian matrix of the difference system evaluated at the steady state is:

$$\mathbf{J}^* = \begin{bmatrix} \omega_2(1-\beta_1\gamma M) & -\omega_2 \dfrac{q_1^*}{q_2^*}(1-\beta_2\gamma M) \\ -\omega_1 \dfrac{q_2^*}{q_1^*}(1-\beta_1\gamma M) & \omega_1(1-\beta_2\gamma M) \end{bmatrix},$$

where

$$\omega_1 = \ell_1 q_1^* / L^*, \quad \omega_2 = \ell_2 q_2^* / L^* \quad \text{and } \omega_1 + \omega_2 = 1.$$

and M is defined in (11).

$$M = \rho^2\left(\frac{a_{12}}{p^*} + a_{21}p^*\right) > 0. \tag{25}$$

It is easy to verify that the characteristic equation of $\mathbf{J}^*$, that is, det $(\mathbf{J}^* - \lambda\mathbf{I}) = 0$, is

$$\lambda\{\lambda - [1 - (\omega_2\beta_1 + \omega_1\beta_2)\gamma M]\} = 0.$$

The constant terms is disappeared: therefore it has a null solution, $\lambda_1 = 0$, and a second solution given by

$$\lambda_2 = 1 - (\omega_2\beta_1 + \omega_1\beta_2)\gamma M.$$

In order to prove the asymptotic stability of the steady state it is sufficient to verify that $|\lambda_2| < 1$:

a) $\lambda_2 > -1$, that is, $1 - (\omega_2\beta_1 + \omega_1\beta_2)\gamma M > -1$, which entails

$$(\omega_2\beta_1 + \omega_1\beta_2)\gamma < 2/M; \tag{26}$$

condition (26) is verified if the reaction coefficients β_1, β_2 and γ are sufficiently small.

b) $\lambda_2 < 1$, that is, $1 - (\omega_2\beta_1 + \omega_1\beta_2)\gamma M < 1$, which is ever verified, as it reduces to $(\omega_2\beta_1 + \omega_1\beta_2)\gamma > 0$.

We have thus proved the following:

Proposition 4. The steady (q_1^*, q_1^*) *of difference system (17) with* r_{1t}, r_{2t} *and* r_t *defined by (14),* p_t *defined by (12) and* σ_t *defined by* (18) *is locally asymptotically stable if the reaction coefficients* β_1, β_2 *and* γ *are sufficiently small.*

Remark. The result that one of the eigenvalues of **J*** is null, confirms what said in footnote 9 about the residual character of one output level, once determined the other one, in order to satisfy the constraint to keep the employment level constant in each period. Eigenvalue λ_2 is the eigenvalue which determines the dynamics of the *proportions*; the dynamics of *dimension* is here completely determined by the necessity to keep the employment level constant. It does not add any further tendency to output levels. For this reason the corresponding eigenvalue is zero.

7. A model with Say's law

An alternative way to control the dimension of the system is to suppose that a sort of Say's law holds, according to which the value of output level determined in each period equal the (normal) value of (effectual) demand: $\mathbf{q}_{t+1}^T\mathbf{p}^* = \mathbf{q}^{*T}\mathbf{p}^*$, that is,

$$q_{1t+1}p_1^* + q_{2t+1}p_2^* = q_1^* p_2^* + q_2^* p_2^* . \tag{27}$$

In principle, it should be better to use contemporary market prices to evaluate actual output of period $t + 1$, imposing thus $\mathbf{q}_{t+1}^T\mathbf{p}_{t+1} = \mathbf{q}^{*T}\mathbf{p}^*$. But, at time t, when output of period t +1 are determined on the basis of capital mobility principle, the price vector of period t +1 is not determined yet. For this reason we adopt the normal price vector to evaluate the future output vector. Similarly to what did in Section 6 the output levels of each period, still determined by the capital mobility equations (17), will be rescaled by factor σ_t which this time is determined in such a way to satisfy equation (27) that is equivalent to $\sigma_t q_{1t}[1 + \gamma(r_{1t} - r_t)]p^* + \sigma_t q_{2t}[1 + \gamma(r_{2t} - r_t)] = q_1^* p^* + q_2^*$. The re-scaling factor is thus

$$\sigma_t = \frac{q_1^* p^* + q_2^*}{q_{1t}[1+\gamma(r_{1t} - r_t)]p^* + [1+\gamma(r_{2t} - r_t)]} . \tag{28}$$

We now study the difference system (17), with r_{1t}, r_{2t} and r_t defined by (14), p_t defined by (12) and σ_t defined by (28).

Steady state. In steady state

$$q_{ct+1} = q_{ct} = q_c, \quad c = 1, 2. \tag{29}$$

As in Section 6 we obtain equations (21), (24) and

$$\sigma = 1. \tag{30}$$

Substitute (21a) and (30) into (28) and obtains

$$q_1^* p^* + q_2^* = q_1 p^* + q_2 . \tag{31}$$

Equations (31) and (24) define two straight lines in space (q_1, q_2). As before, both equations (31) and (24) pass through point (q_1^*, q_2^*). As (31) is a *decreasing* and (24) is *increasing* (q_1^*, q_2^*) is their *unique* intersection. This proves the following:

Proposition 5. (q_1^*, q_1^*) *is the* unique *economically meaningful steady state of difference system (17) with* r_{1t}, r_{2t} *and* r_t *defined by (14),* p_t *defined by (12) and* σ_t *defined by (28).*

Local asymptotic stability of the steady state. On the basis of the preliminary derivatives calculated in the Appendix (Section 3), the Jacobian matrix of difference system evaluated at the steady state is:

$$\mathbf{J}^{**} = \begin{bmatrix} \psi_2(1-\beta_1\gamma M) & -\frac{\psi_1}{p^*}(1-\beta_2\gamma M) \\ -p^*\psi_2(1-\beta_1\gamma M) & \psi_1(1-\beta_2\gamma M) \end{bmatrix},$$

where M is defined in (11) and

$$\psi_1 = \frac{q_1^* p^*}{q_1^* p^* + q_2^*} \quad \text{and} \quad \psi_2 = \frac{q_2^*}{q_1^* p^* + q_2^*}, \quad \text{where} \quad \psi_1 + \psi_2 = 1;$$

It is easy to verify that det $\mathbf{J}^{**} = 0$; hence, the characteristic equation of $\mathbf{J}^{**}$ reduces to

$$\lambda[\lambda - \psi_2(1 - \beta_1\gamma M) - \psi_1(1 - \beta_2\gamma M)] = 0,$$

whose solutions are $\lambda_1 = 0$, and a $\lambda_2 = 1 - (\psi_2\beta_1 + \psi_1\beta_2)\gamma M$. From the formal point of view these eigenvalues coincides with those obtained in the model with a given level of employment. Hence the following proposition holds

Proposition 6. The steady (q_1^*, q_1^*) *of difference system (17) with* r_{1t}, r_{2t} *and* r_t *defined by (14),* p_t *defined by (12) and* σ_t *defined by* (28) *is locally asymptotically stable if the reaction coefficients* β_1, β_2 *and* γ *are sufficiently small.*

8. Market effectual demands

An alternative solution to the problem of determining the scale of the system can be solved by supposing that the output of each period is determined on the basis of the "market effectual demands", that is, taking into account the endogeneity of the induced components of the effectual demands exerted in each period of the adjustment process.[10] We can outline the quantities relation like in an open Leontief model, by defining the market effectual demand of the two commodities by equations

[10] This notion of 'market effectual demands' is based on Ciccone(1999).

$$d_{1t} = q_{1t}a_{11} + q_{2t}a_{21} + c_1, \tag{32a}$$

$$d_{2t} = q_{1t}a_{12} + q_{2t}a_{22} + c_2, \tag{32b}$$

where c_1 and c_2 represent the final demand of commodities 1 and 2.[11] The output dynamics is thus reformulated as follows:

$$q_{1t+1} = d_{1t}\,[1 + \gamma(r_{1t} - r_t)] \tag{33a}$$

$$q_{2t+1} = d_{2t}\,[1 + \gamma(r_{2t} - r_t)], \tag{33b}$$

or, after substitution of (32) into (33),

$$q_{1t+1} = (q_{1t}a_{11} + q_{2t}a_{21} + c_1)\,[1 + \gamma(r_{1t} - r_t)] \tag{33a$'$}$$

$$q_{2t+1} = (q_{1t}a_{12} + q_{2t}a_{22} + c_2)\,[1 + \gamma(r_{2t} - r_t)], \tag{33b$'$}$$

where r_{1t}, r_{2t} and r_t are still defined by (14) and p_t is defined by

$$p_t = p^* \frac{1 + \beta_1\left(1 - \dfrac{q_{1t}}{d_{1t}}\right)}{1 + \beta_2\left(1 - \dfrac{q_{2t}}{d_{2t}}\right)}. \tag{34}$$

Steady state of system (33). The search of the steady state of this model and, in particular, the prove of its uniqueness will be ascertained in two steps: i) we will prove the pair of output levels (q_1^*, q_2^*), corresponding to the solution of the open Leontief system,

$$q_1 = q_1a_{11} + q_2a_{21} + c_1 \tag{35a}$$

$$q_2 = q_1a_{12} + q_2a_{22} + c_2, \tag{35b}$$

is a steady state of difference equation system (33); ii) later we will prove the (local) uniqueness of this solution together with its local asymptotic stability.

Proposition 7. *The pair* (q_1^*, q_2^*) *is a steady state of difference system (33) in correspondence of which* $r_1 = r_2 = r = r^*$ *and* $p = p^*$.

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[11] More rigorously, the demand of each commodity c exerted in period t should be

$$d_{ct} = q_{1t+1}a_{1c} + q_{2t+1}a_{2c} + c_c, \qquad c = 1, 2, \tag{32$'$}$$

that is, it should depend on the output levels planned for the *subsequent* period. But the adoption of such definition of demand would give rise to a loop in difference system (33): the output of period $t + 1$ would depend on the demand of period t which, on its turn, would depend on the output of period $t + 1$.

Proof. Firstly, observe that a steady state of system (33) is an output configuration such that $q_{ct+1} = q_{ct} = q_c$, $c = 1, 2$. In steady state difference equations (33) take thus the form

$$q_1 = (q_1 a_{11} + q_{2t} a_{21} + c_1)\,[1 + \gamma(r_1 - r)] \tag{36a}$$

$$q_2 = (q_1 a_{12} + q_2 a_{22} + c_2)\,[1 + \gamma(r_2 - r)]. \tag{36b}$$

As by construction (q_1^*, q_2^*) satisfy equations (35), equations (36) reduce to $[1 + \gamma(r_1 - r)] = 1 = [1 + \gamma(r_2 - r)]$, that is, to $r_1 = r_2 = r$, hence $r = r^*$ and, consequently, $p = p^*$. □

Local uniqueness and local asymptotic stability of the steady state. The Jacobian matrix of difference system (33′) wit, with r_{1t}, r_{2t} and r_t defined by (14) and p_t defined by (34) evaluated at the steady state is

$$\mathbf{J}^{**} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \gamma \begin{bmatrix} q_1^* & 0 \\ 0 & q_2^* \end{bmatrix} \begin{bmatrix} -\rho^2 \dfrac{a_{12}}{p^{*2}} + r_p & \rho^2 \dfrac{a_{12}}{p^{*2}} - r_p \\ \rho^2 a_{21} + r_p & -(\rho^2 a_{21} + r_p) \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\mu} & 0 \\ 0 & \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\nu} \end{bmatrix},$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} p^* \dfrac{q_2^* a_{21} + c_1}{q_1^{*2}} \\ p^* \dfrac{a_{12}}{q_2^*} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\nu} = \begin{bmatrix} p^* \dfrac{a_{21}}{q_1^*} \\ p^* \dfrac{q_1^* a_{12} + c_2}{q_2^{*2}} \end{bmatrix}.$$

By observing $\mathbf{J}^{**}$ we observe that, if $\gamma = 0$, then $\mathbf{J}^{**} = \mathbf{A}^{\mathrm{T}}$. $\mathbf{A} > \mathbf{O}$ is the input-output matrix; Perron-Frobenius theorems hold. In particular, $\lambda_M(\mathbf{A}) > 0$ and $|\,\lambda_m(\mathbf{A})\,| \leq \lambda_M(\mathbf{A})$, where $\lambda_M(\mathbf{A})$ denotes the dominant eigenvalue of $\mathbf{A}$ and $\lambda_m(\mathbf{A})$ denotes the other eigenvalue. As technology is viable, then $\lambda_M(\mathbf{A}) < 1$. Hence both eigenvalue of $\mathbf{A}$ are smaller than 1 in modulus:

$$|\,\lambda_m(\mathbf{A})\,| \leq \lambda_M(\mathbf{A}) < 1. \tag{37}$$

By (37) and by continuity arguments, if β_1, β_2 and γ are sufficiently small then:

1) matrix $\mathbf{J}^{**}$ has no eigenvalues equal to 1; by Lemma 1 (see the Appendix) (q_1^*, q_2^*) is a locally unique steady state of difference system (33);

2) (q_1^*, q_2^*) is a locally asymptotically stable steady state.

The following Proposition then holds.

Proposition 8. *The steady* (q_1^*, q_1^*) *of difference system* (33) *with d_{ct} defined by (32), r_{1t}, r_{2t}, r_t defined by* (14) *and p_t defined by (34) is: i) locally unique and ii) locally asymptotically stable, if the reaction coefficients β_1, β_2 and γ are sufficiently small.*

9. Comparison and contrast with the literature

In this Section we shall briefly comment the few formal works on the Classical gravitation process in which the focus has been also posed on the *level* of market prices (instead of its rates of change) and the market price level is determined by a comparison between actual output and (effectual) demand.

In Benetti (1979, 1981) the *level* of market prices is determined directly by the proportion between the 'value of effectual demand' (normal price times effectual demand) and the quantity brought to the market.[12] In formulas:

$$p_{ct}q_{ct} = p_c^* q_c^*, \qquad c = 1, 2, ..., C. \qquad (38)$$

This 'value of effectual demand'[13] formulation implies that, for example, a 10% short fall of the quantity brought to market relative to effectual demand (d) would entail a market price 10% above normal price and a 30% excess of effectual demand relative to quantity brought to market would lead to a market price exactly 30% below the normal price.

Moreover, this approach seems to introduce a form of market clearing in value terms in each market and in each period of time with respect to the normal configuration, a condition which restricts unsuitably the description of the Classical competitive process: (38) entails that market prices always will rise or fall to the extent that is necessary to sell to consumers (or users) the whole of the quantities brought to the market. In addition, Benetti's formulations are also subject to the Steedman (1984) critique, as sectoral outputs are supposed to respond directly to deviations between market and normal prices.

A similar approach as regards the determination of the market price level has been developed and extended by Kubin (1989 and 1991). She distinguishes the agents who exert the demand of the various commodities in two classes: consumers and producers. Moreover, she supposes that the demand of producers is ever satisfied in terms of quantities, while consumers' demand is brought into equality with the residual supply of commodities by the price *level*. Prices, again, clear markets (there is no holding of inventories by firms).

The weak point of these formulation is that the *equality* between the actual value of the quantity brought to market of each commodity and its effectual demand evaluated

[12] See also Benetti (1981), Kubin (1989, 1991).
[13] See Kubin (1998).

at the normal price is a very restrictive assumption. Moreover, it is not really consistent with the views of the Classics, who argued that the extent by which market prices would fall or rise relative to normal would be quite variable and irregular and certainly not proportional.[14] Ricardo, for instance, argued that: “the effects of plenty or scarcity, in the price of corn, are incalculably greater than in proportion to the increase or deficiency of quantity […] the exchangeable value of corn does not rise in proportion only to the deficiency of supply, but two, three, four, times as much, according to the amount of the deficiency” (apud Signorino & Salvadori, 2013 p.13 n.11)[15]

Observe that the case considered in Section 7 where Say’s law is supposed to hold in each period has nothing to do with these approaches, because the equality between the value of supply and the value of (effectual) demand was there established *in the aggregate*, not at the level of each industry.

Interpreting the principle that market prices are ‘*regulated* by the proportion’ between the effectual demand and the quantity brought to market in the sense that market price is directly and univocally determined exactly by the proportion $p_c^* q_c^* / q_c$ involves interpreting ‘effectual demand’ not as a physical quantity of the commodity (which being homogeneous with the quantity brought to market may be directly compared to it) but as a value magnitude.

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14 Smith (1776) in chapter 7, book I on the *Wealth of Nations* says “the market price will rise more or less above the natural price, according as either the greatness of the deficiency, or the wealth and wanton luxury of the competitors, happen to animate more or less the eagerness of the competition. Among competitors of equal wealth and luxury the same deficiency will generally occasion a more or less eager competition, according as the acquisition of the commodity happens to be of more or less importance to them. Hence the exorbitant price of the necessaries of life during the blockade of a town or in a famine.” and in the opposite case “The market price will sink more or less below the natural price, according as the greatness of the excess increases more or less the competition of the sellers, or according as it happens to be more or less important to them to get immediately rid of the commodity. The same excess in the importation of perishable, will occasion a much greater competition than in that of durable commodities; in the importation of oranges, for example, than in that of old iron.” These passages are clearly inconsistent with the ‘value of effectual demand’ formulation.

15 Ricardo also writes that ’When the quantity of corn at market, from a succession of good crops, is abundant, it falls in price, not in the same proportion as the quantity exceeds the ordinary demand, but very considerably more […] No principle can be better established, than that a small excess of quantity operates very powerfully on price. This is true of all commodities; but of none can it be so certainly asserted as of corn, which forms the principal article of the food of the people’ apud Salvadori & Signorino (2013, p.13, fn 11)

In the *Palgrave Dictionary*, Boggio (1987) proposed a model that, differently from all his previous and subsequent contributions, contains drastic simplifications, the main purpose of this essay being that of sketching the essentials of the Classical gravitation process. However, in Boggio's model the output dynamics of each commodity is regulated by the difference between actual and normal prices, and not, as in our case, by profit rates differentials. Although this approach follows literally Smith's assertions, it makes the whole argument vulnerable to Steedman's critique, as Boggio himself alerts.[16]

Finally, another formal model where the *level* of market prices is determined on the basis of a comparison between actual output and effectual demand is proposed by Nell (1998, Ch. 8). The essential difference with respect to the models here presented is that Nell adopts a formulation of the principle of capital mobility where the deviations of industrial rates of profits are calculated with respect to the *normal* rate of profit, r^*, which is a magnitude *not* known by capitalists when the system is out of its normal position.

10. Final remarks

The literature on formal models of the Classical gravitation process has tended to give the impression that the Classical principle of capital mobility in general is not able, by itself, to insure a tendency of market prices to converge or oscillate around normal prices without resorting to very specific and arbitrary assumptions about technology (restricting analysis to two goods and excluding self-intensive goods for instance) and/or the help of other principles extraneous to the Classical process of competition (as consumer substitution effects). Even in Caminati (1990), Petri (2010) and Aspromourgous (2009) we can find here and there an echo of this generally negative tone. On the contrary, the formal analysis presented in this paper confirms Garegnani's (1990, 1997) and Serrano (2011) more positive views that the Classical principle of competition through capital mobility is enough to ensure gravitation under quite general conditions concerning technology and effectual demands. Of course there is still a lot of interesting things to be done regarding the analysis of stylized patterns of

[16] The dynamic process described by Boggio is constituted by two sets of equations:

$$p_{it} - p_i^* = g_i(d_{it} - q_{it}), \qquad i = 1, 2, ..., n,$$

$$\mathrm{d}q_i/\mathrm{d}t = s_i(p_{it} - p_i^*), \qquad i = 1, 2, ..., n,$$

where g_i and s_i are continuous sign-preserving functions.

disequilibrium reactions and their implications for the possible dynamics of average market prices (especially regarding expectations and speculation). But we are convinced that the simple model here presented should be considered the starting point for further formal studies on gravitation as, differently from other approaches, it permits to fully appreciate the stabilizing properties of the Classical principle of capital mobility in conveying the system towards its normal position.[17]

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[17] Fratini and Naccarato (2016) followed a completely different approach and proposed a re-formulation of the gravitation process in a probabilistic form. They consider the deviations from the normal configuration as the outcome of a stochastic process whose formal properties are such to guarantee that the probability of the means of market prices are very close to natural prices. This result depends on the following assumptions: "(i) market prices depend on natural prices and on random deviations, (ii) entrepreneurs as a whole do not make systematic errors about the quantities produced and (iii) the structure of market-price determination (whatever it may be) is persistent over time"(fratini & naccarato (2016, p.17), These assumptions imply that normal prices are a kind of "statistical equilibrium" , which we do not think represents the views of the classical economists on gravitation which seem to have been way more "Newtonian". In particular the classics did not think that deviations of market prices from normal prices were really random , as they argued that there were systematic reasons for the *sign* of such deviations. Note also that random shocks to the adjustment parameters of our model can be easily added in simulations , making the pattern of market prices more irregular but without implying that normal prices represent a 'statistical equilibrium'.

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Appendix

A.1 Difference equation (8), Section 4

From the definition of r given in (9a) we get:

$$\frac{\partial r}{\partial q} = \frac{p[q(a_{11}p + a_{12}) + (a_{21}p + a_{22})] - (a_{11}p + a_{12})(qp+1)}{[q(a_{11}p + a_{12}) + (a_{21}p + a_{22})]^2}.$$

At the steady state defined in Proposition 1, $p = p^*$ and $r = r^*$, where p^* and r^* satisfy equations (3); let $\lambda^* = 1/(1 + r^*)$. Consequently

$$\left.\frac{\partial r}{\partial q}\right|_* = \frac{p^*(q^* p^* \lambda^* + \lambda^*) - (p^* \lambda^*)(q^* p^* + 1)}{(q^* p^* \lambda^* + \lambda^*)^2} = 0.$$

Moreover,

$$\frac{\partial r}{\partial p} = \frac{q[q(a_{11}p + a_{12}) + (a_{21}p + a_{22})] - (qa_{11} + a_{21})(qp+1)}{[q(a_{11}p + a_{12}) + (a_{21}p + a_{22})]^2}.$$

In steady state one yields

$$r_p \equiv \left.\frac{\partial r}{\partial p}\right|_* = \frac{(q^* p^* + 1)[q^* \lambda^* - (q^* a_{11} + a_{21})]}{(q^* p^* \lambda^* + \lambda^*)^2} \neq 0.$$

The sign of this derivative is undefined: it can be either positive or negative). Moreover, from the definitions of r_1 and r_2 given in (9a)

$$\frac{\partial r_1}{\partial p} = \frac{a_{12}}{(a_{11}p + a_{12})^2} \quad \text{and} \quad \frac{\partial r_2}{\partial p} = \frac{-a_{21}}{(a_{21}p + a_{22})^2}\,. \tag{39}$$

In steady state:

$$\left.\frac{\partial r_1}{\partial p}\right|_* = \rho^2 \frac{a_{12}}{p^{*2}} \quad \text{and} \quad \left.\frac{\partial r_2}{\partial p}\right|_* = -\rho^2 a_{21}\,,$$

where $\rho = 1 + r^*$. Finally,

$$\frac{\mathrm{d}\,p_t}{\mathrm{d}\,q_t} = -\beta\,, \quad \text{so that,} \quad \left.\frac{\mathrm{d}\,p_t}{\mathrm{d}\,q_t}\right|_* = -\beta\,.$$

By deriving difference equation (8) one obtains

$$\begin{aligned}\frac{\mathrm{d}\,q_{t+1}}{\mathrm{d}\,q_t} \quad = \quad & \frac{1+\gamma(\cdot)}{1+\gamma(\cdot)} + q_t \frac{1}{[1+\gamma(\cdot)]^2}\left\{\gamma\left(\frac{\partial r_{1t}}{\partial p_t}\frac{\mathrm{d}\,p_t}{\mathrm{d}\,q_t} - \frac{\partial r_t}{\partial q} - \frac{\partial r_t}{\partial p_t}\frac{\mathrm{d}\,p_t}{\mathrm{d}\,q_t}\right)[1+\gamma(\cdot)] + \right.\\ - \quad & \left.\gamma\left(\frac{\partial r_{2t}}{\partial p_t}\frac{\mathrm{d}\,p_t}{\mathrm{d}\,q_t} - \frac{\partial r_t}{\partial q} - \frac{\partial r_t}{\partial p_t}\frac{\mathrm{d}\,p_t}{\mathrm{d}\,q_t}\right)[1+\gamma(\cdot)]\right\}\end{aligned}$$

Evaluating at the steady state one yields:

$$\left.\frac{\mathrm{d}\,q_{t+1}}{\mathrm{d}\,q_t}\right| = 1 - \beta\gamma q^*\left(\rho^2\frac{a_{12}}{p^{*2}} - r_p + \rho^2 a_{21} + r_p\right) = 1 - \beta\gamma\rho^2\frac{q^*}{p^*}\left(\frac{a_{12}}{p^*} + a_{21}p^*\right) = 1 - \beta\gamma\omega M\,,$$

where

$$\omega = q^*/p^*$$

and

$$M = \rho^2\left(\frac{a_{12}}{p^*} + a_{21}p^*\right) > 0;$$

(observe that the terms with r_p cancel out).

A.2 Model with a given level of employment (Section 6)

The derivatives of the sectoral rates of profit with respect to the relative price have been already calculated in (39). Sectoral output levels now appear in the expression of the average rate of profit.

$$r = \frac{q_1 p + q_2}{q_1 a_{11} p + q_1 a_{12} + q_2 a_{21} p + q_2 a_{22}} - 1 .$$

Thus r can be derived with respect to q_1, q_2 and p.

$$\frac{\partial r}{\partial q_1} = \frac{p[q_1(a_{11}p + a_{12}) + q_2(a_{21}p + a_{22})] - (a_{11}p + a_{12})(q_1 p + q_2)}{[q_1(a_{11}p + a_{12}) + q_2(a_{21}p + a_{22})]^2}$$

and

$$\frac{\partial r}{\partial q_2} = \frac{[q_1(a_{11}p + a_{12}) + q_2(a_{21}p + a_{22})] - (a_{21}p + a_{22})(q_1 p + q_2)}{[q_1(a_{11}p + a_{12}) + q_2(a_{21}p + a_{22})]^2} .$$

In steady state $p = p^*$; thus we have

$$\boxed{\left.\frac{\partial r}{\partial q_1}\right|_* = \frac{p^*(q_1 p^* \lambda^* + q_2 \lambda^*) - p^* \lambda^*(q_1 p^* + q_2)}{(q_1 p^* \lambda^* + q_2 \lambda^*)^2} = 0} \tag{40a}$$

and

$$\boxed{\left.\frac{\partial r}{\partial q_2}\right|_* = \frac{(q_1 p^* \lambda^* + q_2 \lambda^*) - \lambda^*(q_1 p^* + q_2)}{(q_1 p^* \lambda^* + q_2 \lambda^*)^2} = 0} . \tag{40b}$$

Moreover,

$$\frac{\partial r}{\partial p} = \frac{q_1[q_1(a_{11}p + a_{12}) + q_2(a_{21}p + a_{22})] - (q_1 a_{11} + q_2 a_{21})(q_1 p + q_2)}{[q_1(a_{11}p + a_{12}) + q_2(a_{21}p + a_{22})]^2} .$$

In steady state equations (3) holds, $q_1 = q_1^*$, $q_2 = q_2^*$ and we have

$$r_p \equiv \left.\frac{\partial r}{\partial p}\right|_* = \frac{q_1^*(q_1^* a_{12} + q_2^* a_{22}) - q_2^*(q_1^* a_{11} + q_2^* a_{21})}{(\lambda^* p^* q_1^* + \lambda^* q_2^*)^2} \neq 0 . \tag{41}$$

Again, the sign of r_p is undefined; but in this case too we will see that it cancels out from all the terms of the Jacobian matrix of the system.

The relative market price p_t defined in (12) depends on q_{1t} and on q_{2t}; hence we can calculate

$$\frac{\partial p}{\partial q_1} = p^* \frac{\beta_1\left(-\dfrac{1}{q_1^*}\right)}{1 + \beta_2(\cdot)}, \quad \frac{\partial p}{\partial q_2} = p^* \frac{-\beta_2\left(-\dfrac{1}{q_2^*}\right)[1 + \beta_1(\cdot)]}{[1 + \beta_2(\cdot)]^2} .$$

In steady state, $\beta_1(\cdot) = \beta_2(\cdot) = 0$:

$$\boxed{\left.\frac{\partial p}{\partial q_1}\right|_* = -\beta_1 \frac{p^*}{q_1^*}}, \quad \boxed{\left.\frac{\partial p}{\partial q_2}\right|_* = \beta_2 \frac{p^*}{q_2^*}}. \tag{42}$$

The rescaling factor (18) depends on q_{1t} and on q_{2t}; thus we can calculate:

$$\frac{\partial \sigma}{\partial q_1} = \frac{-L^*}{\{q_1[1+\gamma(\cdot)]\ell_1 + q_2[1+\gamma(\cdot)]\ell_2\}^2}\left\{[1+\gamma(\cdot)]\ell_1 + q_1\gamma\frac{\partial r_1}{\partial p}\frac{\partial p}{\partial q_1}\ell_1 - q_1\gamma\frac{\partial r}{\partial p}\frac{\partial p}{\partial q_1}\ell_1 + \right.$$
$$\left. + q_2\gamma\frac{\partial r_2}{\partial p}\frac{\partial p}{\partial q_1}\ell_2 - q_2\gamma\frac{\partial r}{\partial p}\frac{\partial p}{\partial q_1}\ell_2\right\}$$

$$\frac{\partial \sigma}{\partial q_2} = \frac{-L^*}{\{q_1[1+\gamma(\cdot)]\ell_1 + q_2[1+\gamma(\cdot)]\ell_2\}^2}\left\{q_1\gamma\frac{\partial r_1}{\partial p}\frac{\partial p}{\partial q_2}\ell_1 - q_1\gamma\frac{\partial r}{\partial p}\frac{\partial p}{\partial q_2}\ell_1 + [1+\gamma(\cdot)]\ell_2 + \right.$$
$$\left. + q_2\gamma\frac{\partial r_2}{\partial p}\frac{\partial p}{\partial q_2}\ell_2 - q_2\gamma\frac{\partial r}{\partial p}\frac{\partial p}{\partial q_2}\ell_2\right\}$$

Here the terms containing $\partial r/\partial q_c$, $c = 1, 2$, have been omitted, as they are zero in steady state (see equations (40)). Moreover, in steady state, $\gamma(\cdot) = 0$, $q_1^*\ell_1 + q_2^*\ell_2 = L^*$; therefore:

$$\boxed{\left.\frac{\partial \sigma}{\partial q_1}\right|_* = -\frac{\ell_1}{L^*} + \beta_1\gamma\left(\omega_1\rho^2\frac{a_{12}}{p^{*2}} - \omega_2\rho^2 a_{21} - r_p\right)\frac{p^*}{q_1^*}},$$

$$\boxed{\left.\frac{\partial \sigma}{\partial q_2}\right|_* = -\frac{\ell_2}{L^*} - \beta_2\gamma\left(\omega_1\rho^2\frac{a_{12}}{p^{*2}} - \omega_2\rho^2 a_{21} - r_p\right)\frac{p^*}{q_2^*}},$$

where

$$\omega_1 = \ell_1 q_1^*/L^*, \quad \omega_2 = \ell_2 q_2^*/L^* \quad \text{and } \omega_1 + \omega_2 = 1.$$

We can now calculate the Jacobian matrix of difference system (17)

$$\frac{\partial q_{1t+1}}{\partial q_{1t}} = \frac{\partial \sigma_t}{\partial q_{1t}} q_{1t}[1+\gamma(\cdot)] + \sigma_t[1+\gamma(\cdot)] + \sigma_t q_{1t}\gamma\frac{\partial r_{1t}}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}} - \sigma_t q_{1t}\gamma\frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}} \tag{43a}$$

$$\frac{\partial q_{1t+1}}{\partial q_{2t}} = \frac{\partial \sigma_t}{\partial q_{2t}} q_{1t}[1+\gamma(\cdot)] + \sigma_t q_{1t}\gamma\frac{\partial r_{1t}}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}} - \sigma_t q_{1t}\gamma\frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}} \tag{43b}$$

$$\frac{\partial q_{2t+1}}{\partial q_{1t}} = \frac{\partial \sigma_t}{\partial q_{1t}} q_{2t}[1+\gamma(\cdot)] + \sigma_t q_{2t}\gamma\frac{\partial r_{2t}}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}} - \sigma_t q_{2t}\gamma\frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}} \tag{43c}$$

$$\frac{\partial q_{2t+1}}{\partial q_{2t}} = \frac{\partial \sigma_t}{\partial q_{2t}} q_{2t}[1+\gamma(\cdot)] + \sigma_t[1+\gamma(\cdot)] + \sigma_t q_{2t}\gamma \frac{\partial r_{2t}}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}} - \sigma_t q_{2t}\gamma \frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}} \tag{43d}$$

In steady state

$$\boxed{\left.\frac{\partial q_{1t+1}}{\partial q_{1t}}\right|_* = \omega_2(1-\beta_1\gamma M)},$$

$$\boxed{\left.\frac{\partial q_{1t+1}}{\partial q_{2t}}\right|_* = -\omega_2\frac{q_1^*}{q_2^*}(1-\beta_2\gamma M)}$$

$$\boxed{\left.\frac{\partial q_{2t+1}}{\partial q_{1t}}\right|_* = -\omega_1\frac{q_2^*}{q_1^*}(1-\beta_1\gamma M)}$$

$$\boxed{\left.\frac{\partial q_{2t+1}}{\partial q_{2t}}\right|_* = \omega_1(1-\beta_2\gamma M)},$$

where

$$M = \rho^2\left(\frac{a_{12}}{p^*} + a_{21}p^*\right) > 0$$

(observe that all the terms containing r_p cancel out).

A.3 Model with Say's law (Section 7)

The difference system presented in Section 7 differs from that of Section 6 for the rescaling factor only, which is now given in (28). Hence, the derivatives of the sectoral rates of profit with respect to the relative price coincide with those calculated in (39). The derivatives of the average rate of profit coincide with those calculated in (40) and (41). The derivatives of market price have been calculated in (42). The derivatives of the rescaling factor, defined in (28), are given by:

$$\frac{\partial \sigma}{\partial q_1} = \frac{-(q_1^* p^* + q_2^*)}{\left\{q_{1t}[1+\gamma(\cdot)]p^* + q_{2t}[1+\gamma(\cdot)]\right\}^2}\cdot\left\{[1+\gamma(\cdot)]p^* + q_1\gamma\left(\frac{\partial r_1}{\partial p}\frac{\partial p}{\partial q_1} - \frac{\partial r}{\partial p}\frac{\partial p}{\partial q_1}\right)p^* + q_2\gamma\left(\frac{\partial r_2}{\partial p}\frac{\partial p}{\partial q_1} - \frac{\partial r}{\partial p}\frac{\partial p}{\partial q_1}\right)\right\}$$

$$\frac{\partial \sigma}{\partial q_2} = \frac{-(q_1^* p^* + q_2^*)}{\left\{q_{1t}[1+\gamma(\cdot)]p^* + q_{2t}[1+\gamma(\cdot)]\right\}^2}\cdot\left\{q_1\gamma\left(\frac{\partial r_1}{\partial p}\frac{\partial p}{\partial q_2} - \frac{\partial r}{\partial p}\frac{\partial p}{\partial q_2}\right)p^* + [1+\gamma(\cdot)] + q_2\gamma\left(\frac{\partial r_2}{\partial p}\frac{\partial p}{\partial q_2} - \frac{\partial r}{\partial p}\frac{\partial p}{\partial q_2}\right)\right\}$$

Again, here the terms containing $\partial r/\partial q_c$, c = 1, 2, have been omitted, as they are zero in steady state (see equations (40)). Evaluating these derivatives in the steady state:

$$\left.\frac{\partial\sigma}{\partial q_1}\right|_* = \frac{-p^*}{q_1^* p^* + q_2^*} + \beta_1\gamma\rho^2\left(a_{12} - \frac{q_2^*}{q_1^*}a_{21}p^*\right)\frac{1}{q_1^* p^* + q_2^*} - \beta_1\gamma r_p \frac{p^*}{q_1^*}$$

$$\left.\frac{\partial\sigma}{\partial q_2}\right|_* = \frac{-1}{q_1^* p^* + q_2^*} - \beta_2\gamma\rho^2\left(\frac{q_1^*}{q_2^*}a_{12} - a_{21}p^*\right)\frac{1}{q_1^* p^* + q_2^*} + \beta_2\gamma r_p \frac{p^*}{q_2^*}$$

The elements of the Jacobian matrix coincide with those derived in (43). In steady state we have:

$$\boxed{\left.\frac{\partial q_{1t+1}}{\partial q_{1t}}\right|_* = \psi_2(1-\beta_1\gamma M)},$$

$$\boxed{\left.\frac{\partial q_{1t+1}}{\partial q_{2t}}\right|_* = -\frac{\psi_1}{p^*}(1-\beta_2\gamma M)}$$

$$\boxed{\left.\frac{\partial q_{2t+1}}{\partial q_{1t}}\right|_* = -p^*\psi_2(1-\beta_1\gamma M)}$$

$$\boxed{\left.\frac{\partial q_{2t+1}}{\partial q_{2t}}\right|_* = \psi_1(1-\beta_2\gamma M)},$$

where M is defined in (25) and

$$\psi_1 = \frac{q_1^* p^*}{q_1^* p^* + q_2^*} \quad \text{and} \quad \psi_2 = \frac{q_2^*}{q_1^* p^* + q_2^*}, \quad \text{where} \quad \psi_1 + \psi_2 = 1;$$

observe that, again, all the terms containing r_p cancel out.

A.4 Market prices of with market effectual demand

Substitute (32) into (12) and obtain

$$p_t = p^* \frac{1+\beta_1\left(1-\dfrac{q_{1t}}{q_1 a_{11} + q_2 a_{21} + c_1}\right)}{1+\beta_2\left(1-\dfrac{q_{2t}}{q_1 a_{12} + q_2 a_{22} + c_2}\right)}.$$

$$\frac{\partial p}{\partial q_1} = \frac{p^*}{[1+\beta_2(\cdot)]^2}\left\{\beta_1\left(-\frac{q_1 a_{11} + q_2 a_{21} + c_1 - a_{11}q_1}{(q_1 a_{11} + q_2 a_{21} + c_1)^2}\right)[1+\beta_2(\cdot)] - \beta_2\left(-\frac{-a_{12}q_2}{(q_1 a_{12} + q_2 a_{22} + c_2)^2}\right)[1+\beta_1(\cdot)]\right\}$$

$$\frac{\partial p}{\partial q_2}=\frac{p^*}{[1+\beta_2(\cdot)]^2}\left\{\beta_1\left(-\frac{-a_{21}q_1}{(q_1a_{11}+q_2a_{21}+c_1)^2}\right)[1+\beta_2(\cdot)]-\beta_2\left(-\frac{q_1a_{12}+q_2a_{22}+c_2-a_{22}q_2}{(q_1a_{12}+q_2a_{22}+c_2)^2}\right)[1+\beta_1(\cdot)]\right\}$$

In steady state (q_1^*, q_2^*), equations (35) holds; hence,

$$\left.\frac{\partial p}{\partial q_1}\right|_* = -\beta_1 p^* \frac{q_2^* a_{21}+c_1}{q_1^{*2}} - \beta_2 p^* \frac{a_{12}}{q_2^*} = -\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\mu},$$

and

$$\left.\frac{\partial p}{\partial q_2}\right|_* = \beta_1 p^* \frac{a_{21}}{q_1^*} + \beta_2 p^* \frac{q_1^* a_{12}+c_2}{q_2^{*2}} = \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\nu},$$

where

$$\boldsymbol{\beta}=\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix},\ \boldsymbol{\mu}=\begin{bmatrix}p^*\dfrac{q_2^*a_{21}+c_1}{q_1^{*2}}\\ p^*\dfrac{a_{12}}{q_2^*}\end{bmatrix}\ \text{ and }\ \boldsymbol{\nu}=\begin{bmatrix}p^*\dfrac{a_{21}}{q_1^*}\\ p^*\dfrac{q_1^*a_{12}+c_2}{q_2^{*2}}\end{bmatrix}.$$

5. Elements of the Jacobian matrix of difference system (33).

$$\frac{\partial q_{1t+1}}{\partial q_{1t}}=\frac{\partial d_{1t}}{\partial q_{1t}}[1+\gamma(\cdot)]+d_{1t}\left(\gamma\frac{\partial r_{1t}}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}}-\gamma\frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}}\right)$$

$$\frac{\partial q_{1t+1}}{\partial q_{2t}}=\frac{\partial d_{1t}}{\partial q_{2t}}[1+\gamma(\cdot)]+d_{1t}\left(\gamma\frac{\partial r_{1t}}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}}-\gamma\frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}}\right)$$

$$\frac{\partial q_{2t+1}}{\partial q_{1t}}=\frac{\partial d_{2t}}{\partial q_{1t}}[1+\gamma(\cdot)]+d_{2t}\left(\gamma\frac{\partial r_{2t}}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}}-\gamma\frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{1t}}\right)$$

$$\frac{\partial q_{2t+1}}{\partial q_{2t}}=\frac{\partial d_{2t}}{\partial q_{2t}}[1+\gamma(\cdot)]+d_{2t}\left(\gamma\frac{\partial r_{2t}}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}}-\gamma\frac{\partial r_t}{\partial p_t}\frac{\partial p_t}{\partial q_{2t}}\right)$$

In steady state

$$\boxed{\left.\frac{\partial q_{1t+1}}{\partial q_{1t}}\right|_* = a_{11}-\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\mu}\gamma\rho^2 q_1^*\frac{a_{12}}{p^{*2}}+\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\mu}\gamma q_1^* r_p}$$

$$\boxed{\left.\frac{\partial q_{1t+1}}{\partial q_{2t}}\right|_* = a_{21}+\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\nu}\gamma\rho^2 q_1^*\frac{a_{12}}{p^{*2}}-\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\nu}\gamma q_1^* r_p}$$

$$\boxed{\left.\frac{\partial q_{2t+1}}{\partial q_{1t}}\right|_* = a_{12} + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\mu}\gamma\rho^2 q_2^* a_{21} + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\mu}\gamma q_2^* r_p}$$

$$\boxed{\left.\frac{\partial q_{2t+1}}{\partial q_{2t}}\right|_* = a_{22} - \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\nu}\gamma\rho^2 q_2^* a_{21} - \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\nu}\gamma q_2^* r_p}.$$

Lemma A1. *Let* **x*** *be a steady state for the difference system*

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t), \tag{44}$$

and let **J*** *be the Jacobian matrix of* **f** *evaluated at* **x***. *If matrix* **J*** *has no eigenvalues equal to 1, then* **x*** *is a* locally unique *steady state for difference system (44).*

Proof. A steady state of system (44) is a solution of

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}. \tag{45}$$

Equation (45) defines an implicit function: $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{x} = \mathbf{o}$. As $\mathbf{x}^*$ is a steady state of (44), $\mathbf{g}(\mathbf{x}^*) \equiv \mathbf{o}$. The Jacobian matrix of $\mathbf{g}(\mathbf{x})$ evaluated at $\mathbf{x}^*$ is $\mathbf{J}^* - \mathbf{I}$. As matrix $\mathbf{J}^*$ has no eigenvalues equal to 1, matrix $\mathbf{J}^* - \mathbf{I}$ is non-singular. Then, by the implicit function theorem, $\mathbf{x} = \mathbf{x}^*$ is a locally unique solution of $\mathbf{g}(\mathbf{x}) \equiv \mathbf{o}$, that is, $\mathbf{x}^*$ is a locally unique steady state of difference system (44). □