A Reconsideration of the Equity Premium Puzzle

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Abstract

This paper develops an equilibrium asset pricing framework that allows for investor aggregation, and assumes a log-normally distributed aggregate endowment growth. This framework allows me to derive the equilibrium risk free rate, the expected market return, and expected returns for individual securities. To test how reasonable the results are, I use data of several developed economies from Campbell (2003, 2017) to find a median value of relative risk aversion of 1.57, and a time preference rate of 4.58%. The framework allows me to estimate a version of the CAPM and a multi-period pricing model. JEL Codes D53, E10, E21, G12, G13, G30, G32.

1 Introduction

Consumption based asset pricing (CBAP), pioneered by Rubinstein (1976) and Breeden (1979), is one of the most widely used frameworks in finance and macroeconomics. Consumption based asset pricing lends itself to study issues of great interest, such as the estimation of relative risk aversion and time preference rates for an economy at large, and the calculation of prices and expected returns for individual securities.

Regarding the estimation of utility parameters, Mehra and Prescott (1985) found that in a CBAP framework, the relative risk aversion parameters needed to justify the observed equity risk premium were unreasonably large. Weil (1989) found a similar result about the time preference rates consistent with observed risk free rates. A huge literature emerged to verify and explain the equity premium and risk free rate puzzles. Any interested reader is referred to Campbell (2003, 2017) for an excellent explanation of the different models involved in that research program.

*I would like to thank Prof. John Campbell for sharing data on aggregate dividends. All errors are of course my responsibility

In terms of individual security pricing, the initial models by Sharpe (1964), Black (1972), and Ross (1976) were followed by a large empirical literature in finance. A myriad of papers searched for factors that affected expected returns. This search took force in large part because, as Fama and French (1993) argued, the CAPM had empirically failed to perform adequately. Harvey et al. (2016) give a very thorough recount of the hundreds of papers written on this topic.

In this note, I propose to go back to a complete markets framework with investor aggregation and log-normally distributed aggregate endowment growth. Using slightly relaxed conditions from a typical representative agent model, I reproduce some results found in the literature, namely a formula for the risk free rate. This model also yields a simple formula for the expected market return and equity premium. I use data from Campbell (2003, 2017) to recalculate the relative risk aversion and time preference rate for several economies using the theoretical framework developed here. I find a median relative risk aversion parameter of 1.57 and a median time preference rate of 4.58%.

The model can also be used to price individual securities. I find that for a class of cash flows, one can reproduce a non-linear version of the CAPM. Finally, I extend the model in a multi-period setting and find formulas for a very simple yield curve.

This article is structured as follows: section 2 develops the complete markets model. Section 3 derives the formulas for the risk free rate and the expected market return. These formulas are used to estimate the relative risk aversion and time preference rate parameters. The fourth section develops a pricing framework for individual securities, and looks at a special class of cash flows. In that framework, a modified version of the CAPM emerges. The fifth section extends the model in a multi period framework, and the sixth section concludes.

2 The Model

This model uses a complete markets framework developed by Debreu (1959), and found in standard finance texts such as Huang and Litzenberger (1988). The specific assumptions here are:

1. An exchange economy with exogenous individual endowments.

2. The economy has two periods, $t = 0, 1$, with uncertainty in period $t = 1$. This is extended in the last section of this article.

3. In period $t = 1$ there are a number states of nature, that can be either finite or continuous. The set of states of nature is S. Any given state of nature is called $s \epsilon S$. Agents have homogenous beliefs about the probability of a state s, defined as π_s. For discrete states, π_s is a simple probability, while if there are continuous states, π_s is a density function. The price of an Arrow-Debreu security is p_s for $s \epsilon S$. I will define everything in terms of the goods at time $t = 0$, that is $p_0 \equiv 1$.

4. There are I investors with separable cardinal utilities, so for $i = 1, 2..., I$:

$$U_i = u_i(c_{i0}) + e^{-\delta_i} u_i(c_{it})$$

$$u_i(c_{it}) = \frac{c_{it}^{1-\gamma_i} - 1}{1-\gamma_i} \rightarrow u_i'(c_{it}) = c_{it}^{-\gamma_i}$$

The time preference rate for each investor is δ_i , and their relative risk aversion is γ_i. Each investor i has endowments e_{i0} and e_{is} $\forall s \epsilon S$. Aggregate endowments are $E_0 \equiv \sum_{i=1}^{I} e_{i0}$ and $E_s \equiv \sum_{i=1}^{I} e_{is}$.

5. There are another K autarkic agents who receive wages w_{it} in both periods for $i = I+1, ...I+K$. The fixed aggregate value of these wages is $W \equiv \sum_{i=I+1}^{I+K} w_{i0} = \sum_{i=I+1}^{I+K} w_{i1}$. Aggregate consumption in this economy is $C_0 = E_0 + W$ and $C_s = E_s + W$.

6. There are J complex securities with a value V_j for $j = 1, ..., J$. A complex security with cash flows cf_{js} is valued by a no arbitrage condition, yielding $V_j = \sum_{s \epsilon S} p_s cf_{js}$ for finite states, and $V_j = \int_{s \epsilon S} p_s cf_{js} ds$ for continuous states.

The decentralized equilibrium is found by solving an individual maximization, that is:

$$L_i = u_i(c_{i0}) + e^{-\delta_i} \sum_{s \epsilon S} \pi_s u_i(c_{is}) ds + \lambda_i \left[(e_{i0} - c_{i0}) + \sum_{s \epsilon S} p_s \left(e_{is} - c_{is} \right) \right]$$

In the case of continuous states, one would use $\int_{s \epsilon S} ds$ instead of $\sum_{s \epsilon S}$. The first order conditions to this problem are:

$$\frac{\partial L_i}{\partial c_{i0}} = u_i'(c_{i0}) - \lambda_i = 0 \rightarrow \lambda_i = u_i'(c_{i0})$$

$$\frac{\partial L_i}{\partial c_{is}} = e^{-\delta_i} \pi_s u_i'(c_{is}) - \lambda_i p_s = 0 \rightarrow \lambda_i = \frac{e^{-\delta_i} \pi_s}{p_s} u_i'(c_{is}) = u_i'(c_{i0})$$

Using the power utilities, we get the following well known result:

$$p_s = e^{-\delta_i} \pi_s \left(\frac{c_{i0}}{c_{is}} \right)^{\gamma_i}$$

These tell us how individual consumptions should adjust to state prices and probabilities. To estimate equilibrium prices and returns based on aggregate investor endowments, I assume that $\delta_i = \delta$[1] and $\gamma_i = \gamma$, for $i = 1, ..., I$.[2] With this simplification, I get partial aggregation and a representative agent-like solution as follows:

$$c_{i0} = \left(\frac{p_s}{e^{-\delta}\pi_s}\right)^{1/\gamma} c_{is}$$

$$E_0 = \sum_{i=1}^{I} c_{i0} = \left(\frac{p_s}{e^{-\delta}\pi_s}\right)^{1/\gamma} \sum_{i=1}^{I} c_{is} = \left(\frac{p_s}{e^{-\delta}\pi_s}\right)^{1/\gamma} E_s$$

The equilibrium state prices can be re-written as:

$$p_s = e^{-\delta}\pi_s \left(\frac{E_s}{E_0}\right)^{-\gamma} \qquad (1)$$

State prices rise with a fall in δ and E_s. p_s go up with increases in π_s and E_0. For states of nature, that is $\frac{E_s}{E_0} < 1$, p_s increases as γ rises, while for abundant resources the opposite is true. If we assume that $\frac{E_s}{E_0}$ has a lognormal distribution, with $ln\left(\frac{E_s}{E_0}\right) \sim N(\mu, \sigma^2)$, the state prices become:

$$p_s = \pi_s exp\left(-\delta - \gamma\mu - \gamma\sigma s\right) = \frac{1}{\sqrt{2\pi}} exp\left(-\delta - \gamma\mu - \gamma\sigma s - \frac{s^2}{2}\right)$$

Where $s \sim N(0,1)$ and hence $\pi_s = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{s^2}{2}\right)$. I will use equation (1) throughout, since it is the one that yields more economic intuition.

3 Asset prices and expected returns of two fundamental assets

3.1 The risk free asset

Consider an economy with continuous states. A risk free security is the one that offers one unit for every state of nature at $t = 1$, and has a value V_{r_f} given by:

$$V_{r_f} \equiv \int_{s\epsilon S} p_s ds = e^{-\delta} \int_{s\epsilon S} \pi_s \left(\frac{E_s}{E_0}\right)^{-\gamma} = e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right]$$

Consider a continuously compounded rate r_f, where $V_{r_f} \equiv e^{-r_f}$. If $ln\left(\frac{E_s}{E_0}\right) \sim N(\mu, \sigma^2)$, where μ growth rate of the logs of aggregate endowments, and σ^2 its

[1] In theory, we could have aggregation even for heterogenous beliefs $\pi_{is}$and time preference rates δ_i, as long as $\pi_{is}e^{-\delta_i}$ were constant for all $i = 1, ..I$.However, this case does not make much sense to me

[2] No such assumption is needed for agents $i = I + 1, ..I + K$.

variance, then $\left(\frac{E_s}{E_0}\right)^{-\gamma}$ is also lognormal, since $ln\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right] = -\gamma ln\left[\left(\frac{E_s}{E_0}\right)\right] \sim N(-\gamma\mu, \gamma^2\sigma^2)$. The expected value of the lognormal variable is $E\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right] = e^{-\gamma\mu+\frac{1}{2}\gamma^2\sigma^2}$. Putting all these results together we obtain:

$$V_{r_f} = e^{-r_f} = e^{-\delta-\gamma\mu+\frac{1}{2}\gamma^2\sigma^2}$$

Taking the natural logarithm and re-arranging terms, we obtain a formula for the continuously compounded risk free rate:

$$r_f = \delta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 \tag{2}$$

This is identical to the formula developed in standard consumption based asset pricing textbooks, such as Campbell (2017). The continuously compounded risk free rate increases as the time preference rate δ rises, and as expected growth μ increases. The effect of expected growth on r_f is greater in economies with high risk aversion. Expected volatility lowers the risk free rate, as it depresses p_s generally. This effect is again stronger for economies with highly risk averse agents. Finally, risk aversion has an ambiguous effect on r_f, as $\frac{\partial r_f}{\partial\gamma} = \mu - \gamma\sigma^2$. It is positive for $\gamma < \frac{\mu}{\sigma^2}$ and negative for $\gamma > \frac{\mu}{\sigma^2}$. We will find with the data in section 3 that all of the economies studied there fall under the second case.

3.2 The market portfolio

The market consists of all the traded assets in this economy, with an aggregate output of E_s. Its value is given by:

$$V_m = \int_{s\epsilon S} E_s p_s = E_0 e^{-\delta}\int_{s\epsilon S}\pi_s\left(\frac{E_s}{E_0}\right)^{1-\gamma} = E_0 e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{1-\gamma}\right]$$

$$V_m = E_0 e^{(1-\gamma)\mu+\frac{1}{2}(1-\gamma)^2\sigma^2-\delta} = E_0 exp\left(\mu - \gamma\sigma^2 + \frac{\sigma^2}{2} - r_f\right)$$

The last part of the equality exploits the definition of r_f found in equation (2). Denote r_{ms} as the continuously compounded return:

$$r_{ms} \equiv ln(1+\widetilde{r}_{ms}) = ln\left(\frac{E_s}{V_m}\right) = ln\left(\frac{E_s}{E_0}\right) - ln\left(\frac{V_m}{E_0}\right)$$

Using the above result about the value of the market portfolio V_m and using the log-normality of $\frac{E_s}{E_0}$ we have:

$$r_{ms} = \mu + \sigma s + \left(r_f - \mu + \gamma\sigma^2 - \frac{\sigma^2}{2}\right)$$

$$r_{ms} = r_f + \gamma\sigma^2 - \frac{\sigma^2}{2} + \sigma s = \delta + \gamma\mu - \frac{\sigma^2}{2}(\gamma-1)^2 + \sigma s$$

$$Var(r_{ms}) \equiv \sigma_m^2 = \sigma^2$$

This last equation is key in understanding the difference between a consumption based asset pricing framework and this model, since the variance of the endowments is equal to the variance of the market returns. On the other hand, aggregate consumption growth in this model is:

$$\frac{C_s}{C_0} = \frac{E_s + W}{E_0 + W} = \alpha \left(\frac{E_s}{E_0} \right) + (1 - \alpha)W$$

Where $\alpha \equiv \frac{E_0}{E_0+W}$ is the share of total period 0 endowments invested in Arrow Debreu securities. It follows that $var\left(\frac{C_s}{C_0}\right) = \alpha^2 var\left(\frac{E_s}{E_0}\right) \ll var\left(\frac{E_s}{E_0}\right)$. If one wishes to use a proxy for the volatility of endowments, the closest in this model is the volatility of the log change of aggregate dividends[3]. The expected return of the market is:

$$E(r_m) = r_f + \gamma\sigma^2 - \frac{\sigma^2}{2} \tag{3}$$

The simple returns are $\widetilde{r}_{ms} \equiv e^{r_{ms}} - 1$, and their expected value is $E(\widetilde{r}_{ms}) = exp(r_f + \gamma\sigma^2) - 1$. As can be seen, the same factors that affect the risk free rate will also change the expected market return, with some subtle differences. The expected market return increases with a rise in the time preference rate δ of investors, with the expected growth rate of endowments μ. An increase in relative risk aversion and volatility raises the expected simple market return $E(\widetilde{r}_{ms})$. Equation (3) can be re-written to estimate the relative risk aversion:

$$\gamma = \frac{E(r_m) - r_f}{\sigma^2} + \frac{1}{2} \tag{4}$$

Compare equation (4) with the standard CBAP implementation:

$$\gamma_{CBAP} = \frac{E(r_m) - r_f + \frac{\sigma_m^2}{2}}{\sigma_{cm}}$$

The complete market framework takes the CBAP to this implacable conclusion: general equilibrium implies that for those agents who determine the price of securities, their consumption has in fact the property that $\sigma_{cm} = \sigma_m^2$.

3.3 Numerical estimation of the equity risk premium

I estimate various utility parameters using two proxies for aggregate endowment volatility: stock market volatility, denoted σ_m, and the variance of log dividend growth, called here σ_d.

[3] After the early 1980s, this became a worse proxy, as share buybacks became an important source of income for investors

Table 1: Source data of returns, volatilities, and growth rates

Country	Range	r_m	r_f	σ_m	σ_d	σ_c	μ_d
AUL	1970.1-2011.2	3.840	2.000	20.750	11.890	1.770	1.654
CAN	1970.1-2011.2	5.470	2.070	17.850	8.848	1.930	1.548
FR	1973.2-2011.2	7.060	2.080	23.100	12.561	1.800	2.081
GER	1978.4-2011.2	7.540	2.380	23.850	13.971	4.190	2.174
ITA	1971.2-2011.2	1.510	1.860	25.740	23.667	2.230	-1.184
JAP	1970.2-2011.2	2.700	1.030	21.410	11.053	2.920	-0.249
NTH	1977.2-2011.2	8.570	2.290	19.760	12.687	2.210	1.112
SWT	1982.2-2011.2	8.140	0.870	20.050	12.312	1.300	3.652
SWD	1920-1998	7.084	2.209	18.641	12.894	2.816	1.551
UK	1919-1998	7.713	1.255	22.170	7.824	2.886	1.990
USA	1891-1998	7.169	2.020	18.599	14.019	3.218	1.516
Median		7.084	2.020	20.750	12.561	2.230	1.551

Numbers stated in percentage points. Here $d \equiv ln\left(\frac{DIV_t}{DIV_{t-1}}\right)$ and $c \equiv ln\left(\frac{C_t}{C_{t-1}}\right)$. Source: Campbell (2003), Tables 1 and 2. r_m: average of the log return for the market. r_f log return for a short term risk free asset, σ_m standard deviation of the log market return. σ_c standard deviation for the log growth in aggregate consumption.. σ_d standard deviation for the log growth in annual aggregate dividends, μ_d average of the log growth in annual dividends. Source: author calculation from Campbell data for all countries except Sweden, UK, and USA, which come from Campbell (2003).

Table 1 shows data from Campbell (2003, 2017), in addition to the average log dividend average growth and volatility[4]. I have also transcribed the volatility of aggregate log consumption growth σ_c.

Table 2 presents the different estimates for γ and δ.

The first four columns of Tables 2 calculate the relative risk aversion parameter γ. γ_m uses σ_m, while γ_d uses σ_d as a proxy for log aggregate endowment growth volatility σ. The third and fourth columns show the estimates of γ using the consumption based asset pricing (CBAP) framework as reported by Campbell (2003, 2017) for two cases: γ_{CBAP} where σ_{cm} is directly used, and γ_{2CBAP} where the correlation between aggregate consumption and market returns is set to one, so $\sigma_{cm} = \sigma_m \sigma_c$. When I use market volatility as the proxy for log endowment volatility, I get positive, stable, and reasonable relative risk aversion parameters, ranging from 0.437 in Italy to 2.308 for Switzerland, with a median value of 1.567. When I use the volatility of the log change in aggregate dividends, the estimate for γ increases in all but one country (Italy), and the median value is 3.432. The third and fourth columns show the results reported

[4]Which I have updated with data kindly obtained from John Campbell. I calculated annual real dividends from this data, and taken log growth rates, calculating μ_d and σ_d from 1970 to 2011. The annual data is chosen because of strong seasonality in almost all the economies shown here. The data for μ_d and σ_d for the last three rows, comes from Campbell (2003).

Table 2: Parameter Estimates of Complete Markets Model vs. CBAP

	Variable	γ_m	γ_d	γ_{CPAB}	γ_{2CPAB}	δ_m	δ_d	δ_{2CBAP}
						%	%	%
Country	Period							
AUL	1970.1-2011.2	0.927	1.802	<0	10.890	2.32	1.31	-15.93
CAN	1970.1-2011.2	1.567	4.843	166.97	14.510	3.56	3.76	-18.55
FR	1973.2-2011.2	1.433	3.656	<0	18.340	4.58	5.02	17.80
GER	1978.4-2011.2	1.407	3.144	<0	8.010	4.95	5.19	-5.97
ITA	1971.2-2011.2	0.447	0.438	66.96	5.150	3.05	2.91	-8.73
JAP	1970.2-2011.2	0.864	1.867	118.09	6.320	2.96	3.62	-8.15
NTH	1977.2-2011.2	2.108	4.402	141.29	18.900	8.62	12.99	-8.86
SWT	1982.2-2011.2	2.308	5.296	483.74	35.600	3.15	2.79	-15.24
SWD	1920-1998	1.903	3.432	74.062	12.400	5.55	6.68	-13.17
UK	1919-1998	1.814	11.050	41.233	14.483	5.73	16.64	-11.75
USA	1891-1998	1.988	3.120	22.827	11.293	5.84	6.86	-11.25
	Median value	1.567	3.432	96.08	12.400	4.58	5.02	-11.25

$\gamma_m = \frac{E(r_m)-r_f}{\sigma_m^2} + \frac{1}{2}$, $\gamma_d = \frac{E(r_m)-r_f}{\sigma_d^2} + \frac{1}{2}$, and $\gamma_{CBAP} = \frac{E(r_m)-r_f+\frac{\sigma_m^2}{2}}{\sigma_c\sigma_m}$ is RRA 1 $\gamma_{2CBAP} = \frac{E(r_m)-r_f+\frac{\sigma_m^2}{2}}{\sigma_c\sigma_m}$ is RRA2 in Campbell (2003, Table 4) and Campbell (2017, Table 6.2). $TPR_c \equiv \delta_c = r_f - \gamma_c\mu_d + \frac{1}{2}\gamma_c^2\sigma_m^2$,$TPR_d \equiv \delta_d = r_f - \gamma_d\mu_d + \frac{1}{2}\gamma_d^2\sigma_m^2$, δ_{CBAP} is TPR1 TPR_{2CBAP} is TPR2 in Table 5 from Campbell (2003) and Campbell (2017), Table 6.3.

by Campbell in 2003 and 2017. Table 2 shows that this model does deliver a framework that estimates reasonable relative risk aversion parameters. The second parameter of interest is the time preference rate δ, shown in the last three columns of table 2. δ would be the risk free rate under risk neutrality. δ_m which uses σ_m has a median value of 4.58%, while δ_d has a median value of 5.02% when I use σ_d as a proxy for log aggregate endowment growth. The last column reports the time preference rate under the CBAP, as reported by Campbell (2003, 2017), for the case when the correlation between market returns and consumption growth is set to 1. Again, the model developed here delivers reasonable results. I also calculate the effect of increasing risk aversion on the risk free rate, with the general result that an increase in γ lowers the risk free rate.

4 Pricing and returns of individual securities

With the estimates of the preference parameters, we could recalculate the Arrow-Debreu state prices. For example, using the USA data, we would have:

$$p_s = \pi_s exp\left(-r_f - 1.976\sigma^2 - 1.988\sigma s\right)$$

One could use readily observable forward looking market information such as r_f and expected market volatilities using, for example, the VIX. Any arbitrary asset with cash flows cf_{js} is worth $V_j = \int p_s cf_{js} ds$. This method, while mathematically straightforward, lacks a good deal of economic intuition.

To study one specific case of interest, consider a corporation j that produces the following cash flows:

$$cf_{js} = a_j + b_j E_s^{\beta_j} + \varepsilon_{js}$$

Where ε_{js} is an idiosyncratic risk that is uncorrelated to the pricing kernel, i.e., has $E(\varepsilon_{js}) = E(\varepsilon_{js} E_s^{-\gamma}) = 0$. I will show that while it is very easy to price this asset, it is more complicated to state its expected returns. Let us begin by looking at the value of the complex security:

$$V_j = a_j e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right] + b_j E_0^{\beta_i} e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{\beta_i - \gamma}\right] + \frac{e^{-\delta}}{E_0^{-\gamma}} E\left[\varepsilon_{js} E_s^{-\gamma}\right]$$

$$V_j = a_j e^{-\delta - \gamma\mu + \frac{\gamma^2 \sigma^2}{2}} + b_j E_0^{\beta_j} e^{-\delta} e^{(\beta_j - \gamma)\mu + \frac{1}{2}(\beta_j - \gamma)^2 \sigma^2}$$

$$V_j = a_j e^{-r_f} + b_j E_0^{\beta_j} e^{-\delta} e^{(\beta_j - \gamma)\mu + \frac{1}{2}(\beta_j - \gamma)^2 \sigma^2} \equiv V_{dj} + V_{ej}$$

This exploits the fact that the a_j (which can be negative) cash flows look like a riskless zero coupon bond, that $ln\left(\frac{E_s}{E_0}\right)^{\beta_i - \gamma} \sim N((\beta_i - \gamma)\mu, (\beta_i - \gamma)^2 \sigma^2)$, and that $\int_{s \epsilon S} p_s \varepsilon_{js} = \frac{e^{-\delta}}{E_0^{-\gamma}} E(\varepsilon_{js} E_s^{-\gamma}) = 0$. This last result means that the economy does not subtract value from idiosyncratic risks, just as in the standard CAPM model by Sharpe (1964). The value V_j can be partitioned into a riskless debt component worth V_{dj} and an equity component worth V_{ej}. The expected simple rate of return of security j is $E(\widetilde{r}_j)$ looks like a weighted average cost of capital, i.e.:

$$E(\widetilde{r}_j) = \frac{V_{dj}}{V_j} \widetilde{r}_f + \frac{V_{ej}}{V_j} E[\widetilde{r}_{ej}]$$

Where $\widetilde{r}_f = e^{r_f} - 1$. It is a well known result by Stiglitz (1969) that in a complete market with no bankruptcy costs, the Modigliani Miller proposition I holds, i.e. the enterprise value does not vary with changes in capital structure, even with risky debt. This implies that the Modigliani Miller proposition II also holds, i.e. that $E(\widetilde{r}_j)$ is invariant to changes in capital structure. Hence, it is valid to look at this specific capital structure to obtain the expected return on security j. I will now focus on the equity-like part of the cash flows, and look at the simple systematic returns:

$$\widetilde{r}_{ejs} = \frac{cf_{ejs}}{V_{ej}} - 1 = \frac{b_j E_0^{\beta_j} \left(\frac{E_s}{E_0}\right)^{\beta_j}}{b_j E_0^{\beta_j} e^{-\delta} e^{(\beta_j - \gamma)\mu + \frac{1}{2}(\beta_j - \gamma)^2 \sigma^2}} - 1$$

For the following calculations it is easier to work with continuously compounded rates of return:

$$r_{ejs} \equiv ln(1+\widetilde{r}_{ejs}) = \delta - (\beta_j - \gamma)\mu - \frac{1}{2}(\beta_j - \gamma)^2\sigma^2 + \beta_j\mu + \beta_j\sigma s$$

This equation can be simplified using equations (2) and (3) to obtain the following result:

$$r_{ejs} = r_f + \beta_j(r_{ms} - r_f) + \frac{\sigma^2}{2}\beta_j(1-\beta_j)$$

In expected returns, the above equation would yield:

$$E(r_{ej}) = r_f + \frac{\sigma^2}{2}\beta_j(1-\beta_j) + \beta_j\left[E(r_m) - r_f\right] \quad (5)$$

If we define excess expected returns as $z_s \equiv r_s - r_f$, the above equation becomes:

$$E(z_{ej}) = \frac{\sigma^2}{2}\beta_j(1-\beta_j) + \beta_j E(z_m) \equiv \alpha_j + \beta_j E(z_m)$$

Where $\alpha_j \equiv \frac{\sigma^2}{2}\beta_j(1-\beta_j)$ and σ^2 is the market volatility. This equation implies that for $\beta_j < 1$ then $\alpha_j > 0$, and for $\beta_j > 1$, then $\alpha_j < 0$. This theoretical prediction is similar to the CAPM in Black (1972) and found in the earliest empirical studies, such as Black, Jensen, and Scholes (1972). However, in our case, this is due to the log-normality of returns. To obtain the expected simple rate of return, re-arrange equation (5) as follows:

$$E(r_{ej}) + \frac{\beta_j^2\sigma^2}{2} = r_f + \beta_j\left[E(r_m) + \frac{\sigma^2}{2} - r_f\right]$$

remember that for a continuously compounded rate $r_{xs} \sim N(\mu_x, \sigma_x^2)$, then its simple counterpart has $E(1+\widetilde{r}_x) = 1 + E[\widetilde{r}_x] = E(e^{r_{xs}}) = e^{\mu_x + \frac{\sigma_x^2}{2}}$, and $ln\left(1 + E[\widetilde{r}_x]\right) = \mu_x + \frac{\sigma_x^2}{2}$, hence

$$ln(1+E[\widetilde{r}_{ej}]) = ln(1+\widetilde{r}_f) + \beta_j\left[ln\left(1+E[\widetilde{r}_m]\right) - ln(1+\widetilde{r}_f)\right] \quad (6)$$

Where $\beta = \frac{cov(r_{ej}, r_m)}{var(r_m)}$ with the continuously compounded rates. This is a kind of logarithmic CAPM. There are several ways to look at equation (6):

1. If we elevate it to an exponential, we get:

$$1 + E[\widetilde{r}_{ej}] = (1+\widetilde{r}_f)\left(\frac{1+E[\widetilde{r}_m]}{1+\widetilde{r}_f}\right)^{\beta_j}$$

2. If we define a discount factor rate $\hat{r}_x = ln(1+E[\widetilde{r}_x])$ as one that establishes the present value of an expected cash flow, i.e. $V_x = e^{-\hat{r}_x} E(cf_x)$, then equation 6) becomes:

$$\hat{r}_{ej} = r_f + \beta_j \left(\hat{r}_m - r_f\right)$$

   Where $\hat{r}_m = r_f + \gamma\sigma^2$.

3. If we replace the right hand side of the above equation with its fundamental formulas, found in section 3, we obtain:

$$\hat{r}_{ej} = \delta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 + \beta_j\gamma\sigma^2$$

4. If we linearize equation (6) with a Taylor expansion such that $ln(a+x) \approx ln(a) + \frac{x}{a} \mid_{a=1} = x$, then Sharpe's CAPM can be seen as an approximation of equation (6):

$$E[\widetilde{r}_{ej}] \approx \widetilde{r}_f + \beta_j \left(E[\widetilde{r}_m] - r_f\right)$$

I would like to end this section with a word of caution. For an empirical examination of these results, we need to distinguish between fundamental assumptions (used in sections 2 and 3) and auxiliary premises (used in this section to specify the company cash flows cf_{js}). The failure of any of these types of assumptions would lead to an empirical falsification of equation (6). Such a rejection would not be so damaging if I have simply mis-specified the cash flows, but it would be more consequential if it is due to the failure of the pricing from equation (1). The proper formulation of corporate cash flows is an important area for future research. Indeed, papers by Mclean and Pontiff (2016), Harvey et al. (2016), and Hou et al. (2017) have shown that most of the 'factors' discovered in empirical asset pricing are due to either data mining, or to market inefficiencies that were relatively quickly corrected. There are, however, some factors that seem to endure. It would be interesting to then look at the behavior of firms affected with such relevant factors, to explore if their cash flows depart in a significant way from the specification set forth here. If such a departure in cash flow specification is true, then the framework developed in sections 2 and 3 could still account for their pricing and expected returns.

5 A simple multi-period extension

A multi-period extension of the model in sections 2 and 3 can be done in continuous or discrete time with periods up to T. Consider the following: each investor $i = 1, ..., I$ is given endowments e_{ist}. Each period $t > 0$, there is an aggregate endowment E_{st}. Define the set of states of nature in period t as S_t, and any particular element as $s_t \epsilon S_t$, with a probability π_{st} with the condition that either $\sum_{s_t \epsilon S_t} \pi_{st} = 1$ for discrete states and $\int\limits_{s_t \epsilon S_t} \pi_{st} ds_t = 1$ for continuous states for every t. This is similar to an Arrow (1964) setup where probabilities are viewed on a period by period case. The generalized problem that each investor faces is:

$$L_i = u_i(c_{i0}) + \sum_{t=1}^{T} \left[e^{-t\delta_{it}} \int\limits_{s_t \epsilon S_t} \pi_{st} u_i(c_{ist}) ds_t \right] + \lambda_i \left[(e_{i0} - c_{i0}) + \sum_{t=1}^{T} \int\limits_{s_t \epsilon S_t} p_{st} \left(e_{ist} - c_{ist} \right) ds_t \right]$$

$$L_i = u_i(c_{i0}) + \int_0^T \left[e^{-t\delta_{it}} \int\limits_{s_t \epsilon S_t} \pi_{st} u_i(c_{ist}) ds_t \right] dt + \lambda_i \left[(e_{i0} - c_{i0}) + \int_0^T \int\limits_{s_t \epsilon S_t} p_{st} \left(e_{ist} - c_{ist} \right) ds_t dt \right]$$

The problems are stated in discrete and continuous time setups, with continuous states of nature. For each period we have power utility functions as in the two period case. I have allowed for the time preference rate δ_{it} to vary with time. The pricing solution is straightforward by exploiting the aggregation that comes if $\delta_{it} = \delta_t$ and $\gamma_i = \gamma$, and the general equilibrium properties of the problem. State prices are:

$$p_{st} = e^{-t\delta_t} \pi_{st} \left(\frac{E_{st}}{E_0} \right)^{-\gamma}$$

Any complex security j with cash flows cf_{jst} is worth $V_j = \sum_{t=1}^{T} \left(\int_{s_t \epsilon S_s} p_{st} cf_{jst} \right)$ for a discrete period framework, and $V_j = \int_0^T \left(\int_{s_t \epsilon S_s} p_{st} cf_{jst} \right) dt$ for its continuous counterpart.

To establish the properties of $\left(\frac{E_{st}}{E_0} \right)$, consider discrete time case. Suppose that change in aggregate endowments, conditional on information at time zero, has:

$$ln \left(\frac{E_{st}}{E_{st-1}} \mid \mathcal{F}_0 \right) = \mu_t + \sigma_t s_t$$

Where $s_t \sim N(0,1)$ has a unit normal distribution. This implies that:

$$ln \left(\frac{E_{st}}{E_0} \right) = \sum_{\tau=1}^{t} ln \left(\frac{E_{s\tau}}{E_{s\tau-1}} \right) = \sum_{\tau=1}^{t} (\mu_\tau + \sigma_\tau s_\tau)$$

It is clear that $\frac{E_{st}}{E_0}$ is log-normally distributed, given $ln \left(\frac{E_{st}}{E_0} \right)$, the sum of normal shocks, is normally distributed. To present the simplest result, consider the situation where $\mu_t = \mu$, $\sigma_t = \sigma$ and s_t is independently distributed. In that case $E \left[ln \left(\frac{E_{st}}{E_0} \right) \right] = t\mu$ and $var \left[ln \left(\frac{E_{st}}{E_0} \right) \right] = t\sigma^2$. In summary, we have that $ln \left(\frac{E_{st}}{E_0} \right) \sim N(t\mu, t\sigma^2)$. The continuously compounded risk free yield for a zero coupon bond with maturity t is r_{ft} . The price of such bond is $V_{r_{ft}} = e^{-tr_{ft}}$, and using the results in equation (2) yields:

$$r_{ft} = \rho_t + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 \qquad (7)$$

The yield curve in this simple case would only depend on the shape of δ_t, but this could of course be enriched by modeling more complex behaviors for μ_t, σ_t and autocorrelations for s_t. For the expected market discount rate (expressed as per one unit of time), we would also obtain:

$$E(r_{mst}) = \rho_t + \gamma\mu - \frac{\sigma^2}{2}(\gamma - 1)^2 = r_{ft} + \gamma\sigma^2 - \frac{\sigma^2}{2} \qquad (8)$$

The market risk premium is constant in this very simple specification of the multi-period model.

6 Conclusion

I have used a complete markets framework with additive power utilities that allow for aggregation, and lognormal aggregate endowments. These assumptions have been used before, but the approach here is to calculate asset prices and returns directly, rather than first solving the optimal consumption and investment decisions. Such an approach yields the following simple yet powerful results:

- The most important result is that the Arrow-Debreu securities have a price given by $p_s = e^{-\delta}\pi_s\left(\frac{E_s}{E_0}\right)^{-\gamma}$, where $\frac{E_s}{E_0}$ are the aggregate endowments for investors, which are then assumed to be log-normally distributed. Using this pricing kernel, I find familiar and surprising results. The first familiar result is the formula for the continuously compounded risk free rate, which is the same as in consumption based models, i.e. $r_f = \delta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2$.

- The continuously compounded expected market return, on the other hand, is $E(r_m) = r_f + \gamma\sigma^2 - \frac{\sigma^2}{2}$, and so the estimate for the coefficient of relative risk aversion becomes $\gamma = \frac{E(r_{ms})-r_f}{\sigma^2} + \frac{1}{2}$ .The difference between the model here and the consumption based asset pricing comes from general equilibrium consideration of asset pricing. This implies that the volatility of the market portfolio is identical to the volatility of the log change of the aggregate endowments of those agents who determine security prices. I argue that using aggregate consumption is misleading if there is a significant part of the wealth that is not traded.

- Using data from Campbell (2003, 2017) I obtain a median relative risk aversion value of 1.567 and 3.432 depending on the proxy used for σ. Using the above calculations also yields a median time preference rate 4.58% and 5.02%.

- In the case where the cash flows of an individual security can be written as $cf_{js} = b_j E_s^{\beta_j} + \epsilon_{js}$ with $E(\epsilon_{js}) = E(\epsilon_{js} E_s^{-\gamma}) = 0$, the expected return of that security is $ln(1+E[\widetilde{r}_{ej}]) = ln(1+\widetilde{r}_f)+\beta_j\left[ln(1+E[\widetilde{r}_m]) - ln(1+\widetilde{r}_f)\right]$, i.e. a nonlinear version of the CAPM.

- The model is extended to a multi-period setting, yielding the same insights as with the two period setup. This extension allows for a simple yield curve that depends on the time preference rates, and on the behavior of short versus long term expected endowment growth and volatility.

In addition to the above results, this model is easy to extend according to the needs of the researcher, and so clarify even more the inner workings of financial economics.

7 References

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