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1 February 2017

On periodic ergodicity of a general periodic mixed Poisson autoregression

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Abstract

We propose a general class of non-linear mixed Poisson autoregressions whose form and parameters are periodic over time. Under a *periodic contraction* condition on the forms of the conditional mean, we show the existence of a unique nonanticipative solution to the model, which is strictly periodically stationary, periodically ergodic and periodically weakly dependent having in the pure Poisson case finite higher-order moments. Applications to some well-known integer-valued time series models are considered.

Keywords: Periodic mixed Poisson autoregression, periodic $INGARCH$ models, non-linear $INGARCH$ models, weak dependence, strict periodic stationarity, periodic ergodicity, periodic contraction condition.

1. Introduction

Poisson autoregressions proposed by Grunwald et *al* (2000) and Rydberg and Shephard (2000) have gained increasing interest over the past two decades (see e.g. Davis et al, 2016 and the references therein). Aside from their ability to model various integer-valued time series characteristics, the study of their probability structure has been very challenging (e.g. Grunwald et *al*, 2000; Ferland et *al*, 2006; Fokianos et *al*, 2009; Franke, 2010; Neuman, 2011; Fokianos and Tjostheim, 2011; Doukhan et *al*, 2012; Douc et *al*, 2013; Davis and Liu, 2016). Numerous extensions of the original Poisson autoregression have been introduced. Among them, Bentarzi and Bentarzi (2017) proposed a periodic Poisson autoregressive model in order to account for seasonality which is often observed in integer-valued time series applications. In this model, the conditional mean is a linear function of its lagged values and of the observations with periodically time-varying coefficients. Bentarzi and Bentarzi (2017) focused on the properties "in mean" of their model like higher order periodic stationarity and moment structure. They also showed its usefulness on some real data. However, neither the properties "in probability" (strict periodic stationarity, periodic ergodicity, periodic weak dependence...) nor the statistical properties of the maximum likelihood estimate they used have been studied. Moreover, their conditionally Poisson model excludes modeling overdispersed phenomena that are very observed in integer-valued time series applications (e.g. Davis et *al*, 2016).

In this paper we propose a general periodic mixed Poisson autoregression whose conditional distribution is a mixture of Poisson laws and whose conditional mean is a general non-linear periodic function of its lagged values and of the observations. Depending on the mixing variable, the conditional distribution of the proposed model encompasses a broad range of distributions including the Poisson distribution, the negative binomial distribution, the double Poisson distribution, the Poisson stopped-sum distribution and the Tweedie-Poisson model. Moreover, except for the pure Poisson case, the proposed model is conditionally overdispersed and reduces in the aperiodic case to the mixed Poisson autoregression introduced by Christou and Fokianos (2014, 2015). We study some probability properties of the model, namely strict periodic stationarity, periodic ergodicity, periodic weak dependence and existence of higher order moments. These properties serve, among other things, to establish the asymptotic properties of the quasi-maximum likelihood estimate of the underlying model. For this, a key assumption is a periodic contraction condition that we define below and which is jointly satisfied by the systems of conditional mean functions corresponding to the different seasons.

The rest of this work is organized as follows. Section 2 defines the model and Section 3 examines its probability structure. Section 4 concludes while the proofs of the main results are left to Section 5.

2. Periodic mixed Poisson autoregression

Let $\{N_t(.), t \in \mathbb{Z}\}$ be an independent sequence of homogeneous Poisson processes with unit intensity. Consider a positive independent and S-periodically distributed (ipd_S) sequence $\{Z_t, t \in \mathbb{Z}\}$ with mean 1 and variance σ_t^2. The S-periodicity of $\{Z_t, t \in \mathbb{Z}\}$ which is also assumed to be independent of $\{N_t(.), t \in \mathbb{Z}\}$ is understood in the sense that $Z_t \overset{d}{=} Z_{kS+t}$ for all $k, t \in \mathbb{Z}$, where $\overset{d}{=}$ denotes equality in distribution. In fact, the period S is the smallest positive integer satisfying the latter equality which necessarily implies that $\sigma_t^2 = \sigma_{t+kS}^2$ for all $k, t \in \mathbb{Z}$. An integer-valued stochastic process $\{Y_t, t \in \mathbb{Z}\}$ is said to be a *periodic mixed Poisson autoregression* if it is a solution to the following equation

$$\begin{cases} Y_t = N_t(Z_t \lambda_t) \\ \lambda_t = f_t(Y_{t-1}, \lambda_{t-1}; \theta_t) \end{cases}, \quad t \in \mathbb{Z}, \tag{2.1}$$

where $\{\theta_t, t \in \mathbb{Z}\}$ is a S-periodic sequence of real parameter vectors, that is $\theta_t = \theta_{t+kS}$ for all $k, t \in \mathbb{Z}$ with $\theta_t \in \Theta_t \subset \mathbb{R}^{m_t}$ and $m_t \in \mathbb{N}^*$. The sequence of positive real functions $\{f_t, t \in \mathbb{Z}\}$

defined by $f_t : \mathbb{N} \times \mathbb{R}_+^* \times \Theta_t \to \mathbb{R}_+^*$ is also S-periodic in the sense that $f_t = f_{t+kS}$ for all $k, t \in \mathbb{Z}$. Under the properties of $\{N_t(.), t \in \mathbb{Z}\}$ and $\{Z_t, t \in \mathbb{Z}\}$ given above, it is clear that $E(Y_t/\mathcal{F}_{t-1}) = \lambda_t$ and $Var(Y_t/\mathcal{F}_{t-1}) = \lambda_t(1 + \sigma_t^2 \lambda_t) \geq E(Y_t/\mathcal{F}_{t-1})$ where $\mathcal{F}_t$ is the σ-algebra generated by $Y_t, Y_{t-1}, ...$ Thus, apart the pure Poisson case corresponding to $\sigma_t^2 \equiv 0$, model (2.1) is periodically overdispersed. Moreover, the conditional distribution of Y_t can be given explicitly for some specific distributions of Z_t. Indeed, if Z_t is degenerate at 1 for all $t \in \mathbb{Z}$ then $Y_t/\mathcal{F}_{t-1} \sim \mathcal{P}(\lambda_t)$ is Poisson distributed with parameter λ_t. Similarly, if $Z_t \sim G\left(\sigma_t^{-2}, \sigma_t^{-2}\right)$ for all $t \in \mathbb{Z}$ then $Y_t/\mathcal{F}_{t-1} \sim \mathcal{BN}\left(\sigma_t^{-2}, \frac{\lambda_t}{\sigma_t^{-2}+\lambda_t}\right)$, where $G(a, b)$ stands for the Gamma distribution with shape parameter $a > 0$ and rate parameter $b > 0$ and $\mathcal{BN}(k, p)$ denotes the negative binomial distribution with parameters $k > 0$ and $p \in (0, 1)$.

To highlight the periodicity of model (2.1) it is possible to write it in the following representation

$$\begin{cases} Y_{nS+v} = N_{nS+v}(Z_{nS+v}\lambda_{nS+v}) \\ \lambda_{nS+v} = f_v(Y_{nS+v-1}, \lambda_{nS+v-1}; \theta_v) \end{cases}, \ n \in \mathbb{Z}, \ 1 \leq v \leq S, \tag{2.2}$$

which retains S functions f_v and S parameters $\theta_v \in \Theta_v \subset \mathbb{R}^{m_v}$ $(1 \leq v \leq S)$ corresponding to the different seasons. By season $v \in \{1, ..., S\}$ we mean the set $\{..., v - S, v, v + S, ...\}$. Thus, model (2.1) is fairly general and covers a wide range of well-known integer-valued time series models. For example, when $S = 1$ we find the non-linear mixed Poisson autoregressive model proposed by Christou and Fokianos (2014, 2015). Other particularly important cases of (2.1) are given by the following examples.

Example 2.1 (Linear conditional mean) Let

$$f_v(y, \lambda; \theta_v) = \omega_v + \alpha_v y + \beta_v \lambda, \tag{2.3}$$

where $\theta_v = (\omega_v, \alpha_v, \beta_v)' \in \Theta_v \subset \mathbb{R}_+^{*3}$, $1 \leq v \leq S$.

$i)$ When Z_v is degenerate at 1 for all $1 \leq v \leq S$, model (2.1) reduces to the Poisson periodic $INGARCH$ (INteger-valued Generalized AutoRegressive Conditionally Heteroskedastic) model proposed by Bentarzi and Bentarzi (2017).

$ii)$ When $Z_v \sim G(\sigma_v^{-2}, \sigma_v^{-2})$ $(1 \leq v \leq S)$ we call the resulting model negative binomial periodic $INGARCH$. The latter is a periodic generalization of the negative binomial $INGARCH$ model proposed by Zhu (2011) and Christou and Fokianos (2014). $\square$

Example 2.2 (Exponential conditional mean) Consider model (2.1) with

$$f_v(y, \lambda; \theta_v) = \omega_v + \alpha_v y + \left(\beta_v + \delta_v \exp\left(-\gamma_v \lambda^2\right)\right)\lambda, \tag{2.4}$$

where $\theta_v = (\omega_v, \alpha_v, \beta_v, \delta_v, \gamma_v)' \in \Theta_v \subset \mathbb{R}_+^{*5}$ $(1 \leq v \leq S)$.

$i)$ When Z_v is degenerate for all $1 \leq v \leq S$, representation (2.4) reduces to a periodic version of the specification proposed by Fokianos et *al* (2009) (see also Doukhan et *al*, 2012).

$ii)$ When $Z_v \sim G\left(\sigma_{0v}^{-2}, \sigma_{0v}^{-2}\right)$, model (2.4) is an extension of the exponential negative binomial autoregression (Christou and Fokianos, 2014-2015). $\square$

Example 2.3 (Perturbed linear conditional mean) Let

$$f_v(y,\lambda;\theta_v) = \omega_v (1+\lambda)^{-\gamma_v} + \alpha_v y + \beta_v \lambda. \tag{2.5}$$

As γ_v approaches zero for all $1 \leq v \leq S$, the resulting model approaches the linear conditional mean model (2.3) of which it is a perturbation (see also Christou and Fokianos, 2014-2015). For the latter model the parameter of the model is $\theta_v = (\omega_v, \alpha_v, \beta_v, \gamma_v)' \in \Theta_v \subset \mathbb{R}_+^{*4}$ $(1 \leq v \leq S)$.

Example 2.4 (Mixed-season conditional mean specifications) Model (2.1) also allows different specifications along seasons (see also Aknouche et *al*, 2017 in the context of real-valued $GARCH$ models). As an illustration consider $S = 2$, $f_1(y,\lambda;\theta_1) = \omega_1 + \alpha_1 y + \beta_1 \lambda$ and $f_2(y,\lambda;\theta_2) = \omega_2 + \alpha_2 y + \left(\beta_2 + \delta_2 \exp\left(-\gamma_2 \lambda^2\right)\right)\lambda$. The parameters of this model are $\boldsymbol{\sigma}^2 = (\sigma_1^2, \sigma_2^2)'$ and $\theta = (\theta_1', \theta_2')'$ with $\theta_1 = (\omega_1, \alpha_1, \beta_1)'$ and $\theta_2 = (\omega_2, \alpha_2, \beta_2, \delta_2, \gamma_2)'$. Obviously, when Z_v is degenerate for all $1 \leq v \leq S$ then only θ is retained. $\square$

3. Some probabilistic properties of the model

We now give a sufficient condition on the functions $f_1, ..., f_S$ such that (2.1) admits a strictly periodically stationary, periodically ergodic and periodically weakly dependent solution having finite means. Under additional conditions, this solution also has finite higher (integer) order moments. Recall that a stochastic process $\{Y_t, t \in \mathbb{Z}\}$ is said to be strictly periodically stationary (resp. periodically ergodic) if and only if all its subprocesses $\{Y_{nS+v}, n \in \mathbb{Z}\}$ $(1 \leq v \leq S)$ are strictly stationary (resp. ergodic) in the usual sense. For a more explicit definitions of these properties see e.g. Aknouche et *al* (2017). Similarly, $\{Y_t, t \in \mathbb{Z}\}$ is said to be *periodically weakly dependent* if and only if for all $1 \leq v \leq S$, $\{Y_{nS+v}, n \in \mathbb{Z}\}$ is weakly dependent in the sense of Dedecker and Prieur (2004).

Consider on $f_1, ..., f_S$ the following assumption which we call *periodic contraction condition.*

A1 *For all* $v \in \{1, ..., S\}$, $y, y' \in \mathbb{N}$ *and* $\lambda, \lambda' > 0$,

$$|f_v(y,\lambda) - f_v(y',\lambda')| < \kappa_{v1}|y - y'| + \kappa_{v2}|\lambda - \lambda'|, \tag{3.1a}$$

where κ_{v1} *and* κ_{v2} *are non-negative constants satisfying*

$$\prod_{v=1}^{S} (\kappa_{v1} + \kappa_{v2}) < 1. \tag{3.1b}$$

Through (3.1a), assumption **A1** simply expresses that the system $f_1, ..., f_S$ are Lipschitz functions with the additional constraint that (3.1b) holds. Notice that if $f_1, ..., f_S$ are contracting in the standard sense, i.e. $f_1, ..., f_S$ are Lipschitz functions with

$$(\kappa_{v1} + \kappa_{v2}) < 1 \text{ for all } v \in \{1, ..., S\}, \tag{3.1c}$$

then they are periodically contracting in the sense of (3.1b). Obviously the converse is not true, so periodic contraction is weaker than contraction along seasons. For Example 2.1 the functions $f_1, ..., f_S$ being linear, condition (3.1) reduces to

$$\prod_{v=1}^{S} (\alpha_v + \beta_v) < 1, \tag{3.2}$$

which is the same as the one given by Bentarzi and Bentarzi (2017). On the other hand, for Example 2.2, since $\frac{\partial f_v(y,\lambda)}{\partial y} = \alpha_v$ and $\left|\frac{\partial f_v(y,\lambda)}{\partial \lambda}\right| < \beta_v + \delta_v$, condition (3.1) becomes

$$\prod_{v=1}^{S} (\alpha_v + \beta_v + \delta_v) < 1. \tag{3.3}$$

For Example 2.3, as $\frac{\partial f_v(y,\lambda)}{\partial y} = \alpha_v$ and $\left|\frac{\partial f_v(y,\lambda)}{\partial \lambda}\right| < \beta_v + \omega_v \gamma_v$ condition (3.1) simplifies to

$$\prod_{v=1}^{S} (\omega_v \gamma_v + \alpha_v + \beta_v) < 1. \tag{3.4}$$

Finally, for Example 2.4 the periodic contraction condition results in

$$(\alpha_1 + \beta_1)(\alpha_2 + \beta_2 + \delta_2) < 1.$$

Let $X_t = (Y_t, \lambda_t)$, $\zeta_t = (N_t, Z_t)$ and

$$F_t(X_{t-1}, \zeta_t) = (N_t(Z_t f_t(Y_{t-1}, \lambda_{t-1}; \theta_t)), f_t(Y_{t-1}, \lambda_{t-1}; \theta_t)).$$

Then the sequence of functions $\{F_t, t \in \mathbb{Z}\}$ is S-periodic and model (2.2) may be written in the following non-homogeneous Markov form

$$X_{nS+v} = F_v\left(X_{nS+v-1}, \zeta_{nS+v}\right), \quad n \in \mathbb{Z},\ 1 \le v \le S, \tag{3.5}$$

where $\{\zeta_t, t \in \mathbb{Z}\}$ is ipd_S. Our main result is the following.

Theorem 3.1 *i) Under* (3.1), *equation* (3.5) *admits a strictly periodically stationary, periodically ergodic and periodically weakly dependent solution* $\{(Y_t, \lambda_t), t \in \mathbb{Z}\}$ *having finite mean. Moreover, this solution is unique and is given by the following S nonanticipative schemes*

$$X_{nS+v} = H_v\left(\xi_{nS+v}, \xi_{(n-1)S+v}, ...\right), \quad n \in \mathbb{Z},\ 1 \le v \le S, \tag{3.6}$$

for some measurable functions $H_1, ..., H_S : \left(\mathbb{N} \times \mathbb{R}_+^*\right)^{\mathbb{N}} \to \mathbb{N} \times \mathbb{R}_+^*$.

ii) If, in addition, Z_v is degenerate for all $1 \le v \le S$ then the solution (3.6) *is such that* $E(Y_v^r + \lambda_v^r) < \infty$ *for all $r \in \mathbb{N}$ and $1 \le v \le S$.*

The above result shows that in the pure Poisson case, solution (3.6) has finite moments of any orders under the same condition (3.1). For the Poisson cases of Examples 2.1-2.3, Theorem 3.1 simplifies as follows.

Corollary 3.1 *Under* (3.2) *the Poisson periodic $INGARCH$ equation (cf. Example 2.1, i)) with linear conditional mean (cf.* (2.3)*) admits a unique nonanticipative solution* $\{(Y_t, \lambda_t), t \in \mathbb{Z}\}$, *which is periodically ergodic, periodically weakly dependent and satisfies* $E(Y_v^r + \lambda_v^r) < \infty$ *for all* $r \in \mathbb{N}$ *and* $1 \leq v \leq S$.

Corollary 3.2 *Under* (3.3) *the Poisson periodic $INGARCH$ equation with exponential conditional mean (cf.* (2.4)*) admits a unique nonanticipative solution* $\{(Y_t, \lambda_t), t \in \mathbb{Z}\}$ *which is periodically ergodic and periodically weakly dependent such that* $E(Y_v^r + \lambda_v^r) < \infty$ *for all* $r \in \mathbb{N}$ *and* $1 \leq v \leq S$.

Corollary 3.3 *Under* (3.4) and $\sigma_t^2 \equiv 0$ *the Poisson periodic $INGARCH$ equation with perturbed conditional mean (cf. Example 2.3) admits a unique nonanticipative solution* $\{(Y_t, \lambda_t), t \in \mathbb{Z}\}$, *which is periodically ergodic and periodically weakly dependent with* $E(Y_v^r + \lambda_v^r) < \infty$ *for all* $r \in \mathbb{N}$ *and* $1 \leq v \leq S$.

In the non-Poisson case, the conditions of existence of moments of order larger than one may depend on the mixture variances $\boldsymbol{\sigma}^2 = (\sigma_1^2, ..., \sigma_S^2)'$. In particular, for the negative binomial periodic $INGARCH$ model with linear conditional mean (cf. Example 2.1, *ii*)), the following result shows that these conditions vary according to the order of the underlying moment.

Proposition 3.1 *The negative binomial periodic $INGARCH$ model with linear conditional mean (cf. Example 2.1, ii)) admits a unique nonanticipative periodically ergodic solution* $\{Y_t, t \in \mathbb{Z}\}$ *such that:*

i) $E(Y_v) < \infty$ $(1 \leq v \leq S)$ *if and only if* (3.2) *hold;*

ii) $E(Y_v^2) < \infty$ $(1 \leq v \leq S)$ *if and only if*

$$\prod_{v=1}^{S} \left(\sigma_v^2 \alpha_v^2 + (\alpha_v + \beta_v)^2\right) < 1; \tag{3.7}$$

iii) $E(Y_v^4) < \infty$ $(1 \leq v \leq S)$ *if and only if*

$$\prod_{v=1}^{S} \left((\alpha_v + \beta_v)^4 + 6\sigma_v^2 \alpha_v^2 (\alpha_v + \beta_v)^2 + \sigma_v^4 \alpha_v^3 (11\alpha_v + 8\beta_v) + 6\sigma_v^6 \alpha_v^4\right) < 1. \tag{3.8}$$

From the previous result it follows that for this model the periodic contraction condition (3.2) is only necessary for the existence of moments of order larger than one.

4. Conclusion

In this paper we proposed to enlarge the class of mixed Poisson autoregressions so as to include periodicity in their conditional distribution. The proposed model encompasses a large class of conditional distributions as well as conditional mean forms. Periodic ergodicity and other related properties of the proposed model have been established under a simple periodic contraction condition (3.1) which is weaker than the standard contraction conditions on the S forms of the conditional mean of the model. A particular subclass of model (2.1) which has not been mentioned here is the periodic threshold conditional form (cf. Wang et *al*, 2014 in the non-periodic case) for which the periodic contraction condition remains true.

5. Proofs

5.1. Proof of Theorem 3.1

i) Iterating equation (3.5) S times we obtain the S homogeneous Markov equations

$$X_{nS+v} = \mathbb{F}_v \left(X_{(n-1)S+v}, \xi_{nS+v} \right), \qquad n \in \mathbb{Z},\ 1 \leq v \leq S, \tag{5.1}$$

where $\mathbb{F}_v = F_v \circ F_{v-1} \circ .... \circ F_{v-S+1}$ and $\left\{ \xi_{nS+v}, n \in \mathbb{Z} \right\}$ is an independent and identically distributed (iid) sequence for all $v \in \{1, ..., S\}$ with $\xi_{nS+v} = \left(\zeta_{nS+v}, \zeta_{nS+v-1}, ..., \zeta_{nS+v-S+1} \right)'$. The proof is then based on checking condition (3.1) of Doukhan and Wintenberger (2008) as conditions (3.2) and (3.3) in the same paper seem trivial. For all $x = (y, \lambda) \in \mathbb{R}^2$ and $\epsilon > 0$ let $\|.\|_\epsilon$ be a norm on $\mathbb{R}^2$ defined by $\|x\|_\epsilon = |y| + \epsilon |\lambda|$. In view of (5.1), (3.1$a$), the Poisson property of the process $N_t(.)$ and the independence of this latter with the independent sequence $\{Z_t, t \in \mathbb{Z}\}$ which satisfies $E(Z_v) = 1$ for all $v \in \{1, ..., S\}$, it follows that

$$\begin{aligned} & E\left(\left\| \mathbb{F}_v \left(x, \xi_{nS+v} \right) - \mathbb{F}_v \left(x', \xi_{nS+v} \right) \right\|_\epsilon \right) \leq \\ & (1+\epsilon) \prod_{k=1}^{v-S+2} \left(\kappa_{k1} + \kappa_{k2} \right) \left[\kappa_{v-S+1,1} \left| y - y' \right| + \kappa_{v-S+1,2} \left| \lambda - \lambda' \right| \right] \\ & \leq (1+\epsilon) \prod_{k=1}^{v-S+2} \left(\kappa_{k1} + \kappa_{k2} \right) \max\left(\tfrac{\kappa_{v-S+1,1}}{\epsilon}, \kappa_{v-S+1,2} \right) \left\| x - x' \right\|_\epsilon . \end{aligned} \tag{5.2}$$

Taking $\epsilon = \frac{\kappa_{v-S+1,1}}{\kappa_{v-S+1,2}}$, inequality (5.2) becomes

$$E\left(\left\| F_{v,S} \left(x, \xi_{nS+v} \right) - F_{v,S} \left(x', \xi_{nS+v} \right) \right\|_\epsilon \right) \leq \prod_{k=1}^{v-S+1} \left(\kappa_{k1} + \kappa_{k2} \right) \left\| x - x' \right\|_\epsilon ,$$

where by $(3.1b)$,

$$\prod_{k=1}^{v-S+1}(\kappa_{k1}+\kappa_{k2})=\prod_{k=1}^{S}(\kappa_{k1}+\kappa_{k2})<1.$$

Thus, we have shown that under **A1**, condition (3.1) of Doukhan et Wintenberger (2008) is satisfied for the norm $\|.\|_{\epsilon}$ (with $\epsilon=\frac{\kappa_{v-S+1,1}}{\kappa_{v-S+1,2}}$) and the identity *Orlicz* function. Therefore, by Theorem 3.1 of Doukhan et Wintenberger (2008) there exists for all $v\in\{1,...,S\}$ a unique nonanticipative solution $\{(Y_{nS+v},\lambda_{nS+v}),n\in\mathbb{Z}\}$ of (5.1), which is strictly stationary, ergodic, weakly dependent, having finite mean and whose expression is given by (3.6). This is equivalent to say that $\{(Y_t,\lambda_t),t\in\mathbb{Z}\}$ is a unique nonanticipative strictly periodically stationary, periodically ergodic and periodically weakly dependent solution of (3.5) having S finite means.

ii) We will show that the condition $\kappa^r<1$, which is implied by $(3.1b)$ entails $E(y_v^r)<\infty$ and $E(\lambda_v^r)<\infty$ for all $r\in\mathbb{N}$ and $v\in\{1,...,S\}$, where $\kappa=\prod_{v=1}^{S}\kappa_v$ et $\kappa_v=\kappa_{v1}+\kappa_{v2}$. For $r\in\mathbb{N}$ and $x\in\mathbb{N}\times\mathbb{R}_+^*$ consider the norm $\|x\|_{\epsilon,r}=(y^r+\epsilon\lambda^r)^{1/r}$. We have

$$E\left(\|X_t\|_{\epsilon,r}^r\right)=E\left(Y_t^r+\epsilon\lambda_t^r\right)=E\left(E\left(Y_t^r/\mathcal{F}_{t-1}\right)\right)+\epsilon E\left(\lambda_t^r\right).$$

Since $Y_t/\mathcal{F}_{t-1}\sim\mathcal{P}(\lambda_t)$ the latter equality becomes

$$E\left(\|X_t\|_{\epsilon,r}^r\right)=(1+\epsilon)E\left(\lambda_t^r\right)+\sum_{i=0}^{r-1}\left\{{r\atop i}\right\}E\left(\lambda_t^i\right), \tag{5.3}$$

where $\left\{{r\atop i}\right\}$ is the Stirling number of second kind (e.g. Ferland et *al*, 2006; Doukhan et *al*, 2012). Now we show by induction on $r\in\mathbb{N}$ that there exists $\epsilon>0$ such that $E\left(\|X_v\|_{\epsilon,r}^r\right)<\infty$ for all $v\in\{1,...,S\}$. From (2.1) and $(3.1a)$ we have

$$\begin{aligned}E\left(\lambda_t^r\right)&=E\left(f_t\left(Y_{t-1},\lambda_{t-1};\theta_t\right)^r\right)=E\left(\left(f_t\left(Y_{t-1},\lambda_{t-1};\theta_t\right)-f_t\left(0,0;\theta_t\right)+f_t\left(0,0;\theta_t\right)\right)^r\right)\\&\leq E\left(\left(\kappa_{t1}\left|Y_{t-1}\right|+\kappa_{t1}\left|\lambda_{t-1}\right|+f_t\left(0,0;\theta_t\right)\right)^r\right):=E\left(\left(g_t\left(Y_{t-1},\lambda_{t-1}\right)+b_t\right)^r\right)\\&=E\left(g_t\left(Y_{t-1},\lambda_{t-1}\right)^r\right)+R_{t,r-1},\end{aligned} \tag{5.4}$$

where $g_t(Y_{t-1},\lambda_{t-1})=\kappa_{t1}Y_{t-1}+\kappa_{t1}\lambda_{t-1}$, $b_t=f_t(0,0;\theta_t)$ and

$$R_{t,r-1}=\sum_{i=0}^{r-1}\binom{r}{i}E\left(g_t^i\left(Y_{t-1},\lambda_{t-1}\right)\right)b_t^{r-i}<\infty,$$

is a polynomial of degree $r-1$ and is thus finite by the induction hypothesis, $\binom{r}{i}$ being the binomial coefficient. On the other hand, Jensen's inequality yields

$$\begin{aligned}E\left(g_t\left(Y_{t-1},\lambda_{t-1}\right)^r\right)&=\kappa_t^rE\left(\tfrac{\kappa_{t1}}{\kappa_t}Y_{t-1}+\tfrac{\kappa_{t1}}{\kappa_t}\lambda_{t-1}\right)^r\\&\leq\kappa_t^{r-1}\left(\kappa_{t1}E\left(Y_{t-1}^r\right)+\kappa_{t2}E\left(\lambda_{t-1}^r\right)\right)\\&\leq\kappa_t^rE\left(\|X_{t-1}\|_{\epsilon,r}^r\right),\end{aligned}$$

so that (5.4) takes the form

$$E\left(\lambda_t^r\right) \leq \kappa_t^r \left\|X_{t-1}\right\|_{\epsilon,r}^r + R_{t,r-1}. \tag{5.5}$$

Combining (5.3) and (5.5), one obtains the following linear periodic difference inequation

$$E\left(\left\|X_t\right\|_{\epsilon,r}^r\right) \leq (1+\epsilon)\,\kappa_t^r E\left(\left\|X_{t-1}\right\|_{\epsilon,r}^r\right) + C_t, \tag{5.6}$$

where $C_t = (1+\epsilon)\,R_{t,r-1} + \sum\limits_{i=0}^{r-1}\left\{{r \atop i}\right\}E\left(\lambda_t^i\right)$ is finite by the induction hypothesis. By successive replacements S times in (5.6) and by virtue of the periodic stationarity of $\{X_t, t\in\mathbb{Z}\}$ which implies that $\left\|X_t\right\|_{\epsilon,r}^r = \left\|X_{t-S}\right\|_{\epsilon,r}^r$, it follows that

$$E\left(\left\|X_t\right\|_{\epsilon,r}^r\right) \leq (1+\epsilon)^S \kappa^r E\left(\left\|X_t\right\|_{\epsilon,r}^r\right) + K_t,$$

where $K_t = \sum_{j=0}^{S-1}\prod\limits_{i=0}^{j-1}\kappa_{t-i}^r C_{t-j}\,(1+\epsilon)^j < \infty$. It suffices to take $\epsilon < \frac{1}{\kappa^{r/S}}$, it follows that

$$E\left(\left\|X_t\right\|_{\epsilon,r}^r\right) \leq \frac{K_t}{1-(1+\epsilon)^S\kappa^r} < \infty,$$

for all $r\in\mathbb{N}$ and $t\in\{1,...,S\}$.

5.2. Proof of Proposition 3.1

i) We show the necessity of (3.2) as its sufficiency for $E\left(Y_v\right) < \infty$ stems from Theorem 3.1. Taking expectation of Y_t in (2.1) and using the linear form of f_t (see Example 2.1, ii)), we find the linear periodic difference equation $E\left(\lambda_t\right) = \left(\alpha_t + \beta_t\right)E\left(\lambda_{t-1}\right) + \omega_t$ whose solution exists if and only if (3.2) is satisfied.

ii) For the existence of the second moments $E\left(Y_v^2\right)$, it is enough to find a necessary and sufficient condition for $E\left(\lambda_t^2\right) < \infty$ $(1\leq t\leq S)$ (see e.g. Ahmad and Francq, 2016 in the aperiodic case $S=1$). By a direct calculation we find the following periodic linear difference equation

$$E\left(\lambda_t^2\right) = \left(\alpha_t^2\left(\sigma_t^2+1\right) + 2\alpha_t\beta_t + \beta_t^2\right)E\left(\lambda_{t-1}^2\right) + \left(2\alpha_t\omega_t + 2\beta_t\omega_t + \alpha_t^2\right)E\left(\lambda_{t-1}\right) + \omega_t^2,$$

whose solution exists if and only if (3.7) is satisfied.

iii) The proof is similar to ii). We shall find a necessary and sufficient condition for $E\left(\lambda_t^4\right) < \infty$ $(1\leq t\leq S)$. From the expression of the third and fourth moments of the negative binomial distribution, we find the following linear periodic difference equation

$$E\left(\lambda_t^4\right) = \left(\left(\alpha_t+\beta_t\right)^4 + 6\sigma_t^2\alpha_t^2\left(\alpha_t+\beta_t\right)^2 + \sigma_t^4\alpha_t^3\left(11\alpha_t + 8\beta_t\right) + 6\sigma_t^6\alpha_t^4\right)E\left(\lambda_{t-1}^4\right) + R_t,$$

whose solution exists if and only if (3.8) is satisfied, R_t being a finite generic constant whose value is unimportant.

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