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Abstract:

In this paper, we propose a quick and efficient method to examine whether a time series X_t possesses any nonlinear feature by testing a kind of dependence remained in the residuals after fitting X_t with a linear model. The advantage of our proposed nonlinearity test is that it is not required to know the exact nonlinear features and the detailed nonlinear forms of the variable being examined. Another advantage of our proposed test is that there is no over-rejection problem which exists in some famous nonlinearity tests. Our proposed test can also be used to test whether the hypothesized model, including linear and nonlinear, to the variable being examined is appropriate as long as the residuals of the model being used can be estimated. Our simulation study shows that our proposed test is stable and powerful. We apply our proposed statistic to test whether there is any nonlinear feature in the sunspot data. The conclusion drawn from our proposed test is consistent with those from other well-established tests.

Keywords: Nonlinearity, Dependence, Nonlinear test, Dependent test, Volterra expansion, Sunspots

JEL Classification: C01, C12

1 Introduction

It is well-known that nonlinearity always appear in many time series like natural data and economic and financial time series, including some well-known datasets like the sunspots (Moran, 1954), Canadian lynx (Tong, 1990), and inflation rate (Engle, 1982). In practice, nonlinearity is common in both stationary or non-stationary time series. Nevertheless, detecting nonlinearity in time series is very important because very often academics and practitioners have to know this feature in the data before conducting their analysis. For example, Fourier analysis assumes the time series to be linear and stationary while, on the other hand, the wavelet analysis (Cheng, et al., 1996) is raised for linear but nonstationary. Thus, before academics and practitioners apply Fourier analysis and/or wavelet analysis in their work, they have to examine whether there is any nonlinearity in the time series.

There is a growing interest in the testing, estimation, specification, and developing properties for nonlinearity for decades. There are many nonlinear features including asymmetric cycles, nonlinear relationship among the variables being studied and their lags, time irreversibility, sensitivity to initial conditions, and others. The early development of nonlinear models include bilinear models (Granger and Andersen, 1978), threshold autoregressive models (Tong, 1978), state-dependent model (Priestley, 1980), exponential autoregressive model (Haggan and Ozaki, 1981), ARCH model (Engle, 1982), Markov switching model (Hamilton, 1989), and nonlinear state-space model (Carlin, et al., 1992). In addition, Chen and Tsay (1993a) use an arranged local regression procedure to construct functional-coefficient autoregressive models while Chen and Tsay (1993b) develop some new techniques for a class of nonlinear additive autoregressive models with exogenous variables. On the other hand, Tjϕstheim (1994) uses nonparametric regression techniques as an alternative nonlinear time series model. Tiao and Tsay (1994) discuss the advances in non-linear modelling and in Bayesian inference via the Gibbs sampler. On the other hand, Tjϕstheim (1994) uses nonparametric regression techniques as an alternative nonlinear time series model. Zhao (2011) shows that many popular nonlinear time series models can be viewed as Hidden Markov models HMMs. There are also many breakthroughs in limiting theory of nonlinear time series, such as Hsing and Wu (2004), Wu and Min (2005), Shao and Wu (2007).

Nonetheless, the most general form of a nonlinear stationary process is the Volterra expansion. Using the idea of Volterra expansions, Keenan (1985) applies the one-degree-of freedom test (Tukey, 1949) for nonadditivity to derive a time-domain statistic for discriminating nonlinear from linear models. Tsay (1986) extends the work of Keenan to establish a more powerful test. Other nonlinear tests include a simple portmanteau test (Petruccelli and Davies, 1986), the quasi-likelihood ratio test (Chan and Tong, 1990), and the Wald test (Hansen, 1996). In addition, Li and Li (2011) develop a quasi-likelihood ratio test statistic for an autoregressive moving average model against its threshold extension. Zhou (2012) proposes a distance correlation approach to measure nonlinear dependence in time series.

Since the number of parameters of the nonlinearity part could be very large, this could affect the performance of the existing nonlinear tests. In addition, nonlinearity may occur in many ways. Brock, et al. (BDS, 1996) present a nonparametric method for testing a kind of serial dependence and nonlinear structure in a time series. The advantage of this test is that it is not required to know the exact nonlinear features and the detailed nonlinear forms of a time series. But the level of BDS test is right bias; in other words, this test has a over-rejection problem even when sample are very large in practice.

The objective in this paper is to circumvent the limitation of Volterra expansion or other similar approaches by developing a new method to test the nonlinearity for a time series that does not involve many parameters. Most importantly, our proposed test does not have any over-rejection problem.

The remainder of the paper is organized as follows. In Section 2, we first discuss the Volterra expansion and state the nonlinearity tests developed by Brock, et al. (1996) in Subsection 2.1. Thereafter, we develop our proposed new nonlinearity test in Subsection 2.2. Section 3 displays the superiority of the nonlinearity test we developed in Subsection 2.2 by conducting a simulation to examine its performance over the tests developed by Brock, et al. (1996). In Section 4, we illustrate the applicability of our proposed nonlinearity test by applying it to examine whether there is any nonlinear feature in the sunspot data. Section 5 wraps up the paper by providing several well-grounded observations. Readers may refer to Hui, et al. (2017) for the proof.

2 Theory

We suppose that X_t is stationary and follows a time series model of the current and past independent and identically distributed (iid) shocks such that $X_t = f(\varepsilon_t, \varepsilon_{t-1}, \cdots)$. If $f(\cdot)$ is a linear function of the shocks, the model is linear; otherwise, it is nonlinear. One of the most commonly used linear models is an ARMA process that could be presented as an AR and/or MA representation (Box, et al., 1994). If the null hypothesis of linearity is true, residuals of the hypothesized linear model are independent. This is the basic idea used in the development of various nonlinearity tests. There are many approaches, for example, parametric, semi-parametric, and nonparametric approaches, to identify the nonlinear forms of the models. There are also several nonlinearity tests available. For example, Fan and Yao (2003) establish a likelihood ratio test to test for a linear model versus a TAR model with two regimes.

One of the most commonly used approaches is to apply the Volterra expansion (Wiener, 1958) to expand a nonlinear and stationary time series, say, X_t, to be in terms of the linear, quadratic, cubic, etc. such that

$$X_t = \mu + \sum_{-\infty}^{\infty} a_u \varepsilon_{t-u} + \sum_{u,v=-\infty}^{\infty} a_{uv} \varepsilon_{t-u} \varepsilon_{t-v} + \sum_{u,v,w=-\infty}^{\infty} a_{uvw} \varepsilon_{t-u} \varepsilon_{t-v} \varepsilon_{t-w} + \cdots \ , \quad (1)$$

where ε_t $(-\infty < t < \infty)$ is an iid innovation with zero mean.

There are several methods test for nonlinearity based on this expansion, such as Tsay (1986). Cox (1981) suggests using quadratic or cubic regression to test for nonlinearity.

The major drawback of applying the Volterra expansion is that there are too many parameters in the model. To circumvent the limitation, one could assume a_u, a_{uv}, and a_{uvw} in equation (1) to be functions of small numbers of parameters. However, the problem of this approach is that we do not know the forms of "functions" and, in fact, such "functions" may not exist.

2.1 BDS Test

Brock, et al. (1996) introduce an approach to circumvent the limitation of the Volterra expansion of getting too many parameters. Follow the idea of Tsay (1986), Brock, et al. (1996) use the following AR model to remove autocorrelation in the data:

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + e_t \ , \tag{2}$$

where $e_t \overset{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$ and WN stands for 'white noise.'

After removing the linear components in $\{X_t\}$ by introducing the linear model in (2), the residual $\hat{e}_t$ is denoted to be Y_t. Brock, et al. (1996) examine the iid assumption on $\{Y_t\}$. In other words, the null hypothesis of BDS test is:

$$H_0 \quad : \quad \{Y_t\} \text{ is iid.} \tag{3}$$

The basic idea of the BDS test is to use the concept of a "correlation integral". Let $Y_t^k \equiv \left(Y_t, Y_{t+1}, \cdots, Y_{t+k-1}\right)$, define

$$C_l(\delta, T) \equiv \frac{2}{T_k(T_k - 1)} \sum_{i<j} I_\delta \left(Y_i^*, Y_j^*\right) \ , \ l = 1, k, \tag{4}$$

where T is the length of $\{Y_t\}$, $T_k = T - k + 1$, $Y_i^* = Y_i$ if $l = 1$, $Y_i^* = Y_i^k$ if $l = k$. Under the null hypothesis that Y_t are iid with a nondegenerated distribution function $F(\cdot)$, for any fixed k and δ, the statistic $\sqrt{T}\{C_k(\delta, T) - [C_1(\delta, T)]^k\}$ is asymptotically distributed as normal with mean zero and variance

$$\sigma_k^2(\delta) = 4(N^k + 2\sum_{j=1}^{k-1} N^{k-j}C^{2j} + (k-1)^2 C^{2k} - k^2 N C^{2k-2}) \ ,$$

where $C = \int [F(z+\delta) - F(z-\delta)]dF(z)$ and $N = \int [F(z+\delta) - F(z-\delta)]^2 dF(z)$. Since $C_1(\delta, T)$ is a consistent estimate of C and $N(\delta, T) = \frac{6}{T_k(T_k-1)(T_k-2)} \sum_{t<s<u} I_\delta(Y_t, Y_s) I_\delta(Y_s, Y_u)$ is a consistent estimate of N, replacing C and N by $C_1(\delta, T)$ and $N(\delta, T)$, respectively, in $\sigma_k^2(\delta)$ will yield a consistent estimate $\sigma_k^2(\delta, T)$.

The BDS test statistic is $D_k(\delta, T) = \sqrt{T}\{C_k(\delta, T) - [C_1(\delta, T)]^k\}/\sigma_k(\delta, T)$, the hypothesis H_0 defined in (3) is rejected at level α if $\left|D_k(\delta, T)\right| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the upper

$\alpha/2$ quantile of the standard normal distribution $\mathcal{N}\Big(0,1\Big)$, and the original sequence X_t is concluded to possess nonlinearity.

Readers may refer to Brock, et al. (1996) for more details of the BDS test.

2.2 New Non-Linearity Test

From our simulation, we find that the BDS test has a very serious over-rejection problem which discounts its applicability in practice. To circumvent the limitation, in this paper we develop a new independence test to test for independence for any residual series $\{Y_t\}$. We first state the following definition:

Definition 2.1 *Let $\{Y_t\}$ be the residuals series as discussed in the above, series $\{X_t\}$* ***does not possess any nonlinearity*** *if and only if for any t the law of corresponding residuals $\{Y_t\}$ satisfies*

$$\mathcal{L}\left(Y_t \mid Y_{t-1}\right) = \mathcal{L}\left(Y_t\right) \,. \tag{5}$$

In addition, we define

$$\begin{aligned} C_1\Big(\eta\Big) &\equiv Pr\left(Y_{t-1} < \eta, Y_t < \eta\right) \ , \\ C_2\Big(\eta\Big) &\equiv Pr\left(Y_{t-1} < \eta\right) \ , \\ C_3\Big(\eta\Big) &\equiv Pr\left(Y_t < \eta\right) \ . \end{aligned} \tag{6}$$

Since

$$Pr\left(Y_t < \eta \mid Y_{t-1} < \eta\right) = \frac{C_1\Big(\eta\Big)}{C_2\Big(\eta\Big)},$$

when one tests the existence of the nonlinearity of a sequence $\{X_t\}$, we suggest to test the following hypothesis:

$$H_0 : \frac{C_1\Big(\eta\Big)}{C_2\Big(\eta\Big)} - C_3\Big(\eta\Big) = 0 \,. \tag{7}$$

The series $\{X_t\}$ is said to possess nonlinearity if the hypothesis H_0 in (7) is rejected.[1]

[1] We note that we should pay attention to the estimation effect by using two-step method in our proposed test. This approach is commonly used in econometrics and is used in some other nonlinearity tests like the BDS test. In practice, one needs to ensure zero autocorrelation in residuals $\{Y_t\}$.

For a residual sequence $\{y_t\}$, the dependence test statistic is given by

$$T_n = \sqrt{n}\left(\frac{C_1(\eta, n)}{C_2(\eta, n)} - C_3(\eta, n)\right) , \qquad (8)$$

where

$$C_1(\eta, n) \equiv \frac{1}{n}\sum_{t=2}^{T} I_{(y_{t-1}<\eta)} \cdot I_{(y_t<\eta)} ,$$
$$C_2(\eta, n) \equiv \frac{1}{n}\sum_{t=2}^{T} I_{(y_{t-1}<\eta)} ,$$
$$C_3(\eta, n) \equiv \frac{1}{n}\sum_{t=2}^{T} I_{(y_t<\eta)} ,$$

$n = T - 1$, and T is the length of residual $\{y_t\}$.

We establish the following property for our proposed test statistic T_n defined in (8).

Theorem 2.1 *If the residual $\{Y_t\}$ is iid, then the test statistic defined in (8) is distributed as $\mathcal{N}(0, \sigma^2(\eta))$ asymptotically.*

The asymptotic variance $\sigma^2(\eta)$ with its consistent estimator $\hat{\sigma}^2(\eta)$ and the proof of theorem 2.1 are given in Hui, et al. (2017). The hypothesis H_0 defined in (7) is rejected at level α if

$$|T_n|/\hat{\sigma}^2(\eta) > z_{\alpha/2} ,$$

where T_n is defined in (8) and $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution $\mathcal{N}(0, 1)$. In this situation, $\{X_t\}$ is concluded to possess nonlinearity.

3 Simulation

In this section, we illustrate the applicability and superiority of our proposed nonlinearity test we developed in Subsection 2.2. For simplicity, we call the test developed by Brock, et al. (1996) "BDS test" and our new test "HWBZ test". Let R be the times of rejecting the null hypothesis that $\{Y_t\}$ is iid in the 10,000 replications at α level, and thus, the rejection frequency is $R/10,000$. The length of testing sequences are chosen to be $T = 200$ and 400. In BDS test, the parameters δ and k are the same as Brock, et al. (1996): $\delta = 0.5\sqrt{\text{Var}(Y_t)}$, and $k = 2, 3$. In HWBZ test, we first standardize the residual sequence $\{Y_t\}$ and choose the parameter $\eta = 1$ and 1.5.

Since the type 1 error should be controlled first in practice, we begin our simulation study by presenting empirical sizes of both BDS and HWBZ tests displayed in Table 1. From Table 1, readers can find that the BDS test has a very serious over-rejection problem. For example, in the first panel of Table 1, when test level $\alpha = 0.05$, the frequency of rejection is 0.1654 (0.1049) for sample size of 200 (400), this is unacceptably high. We note that sample size of 400 is very high in many practical issues. We also note that even 200 observations could also be too high in many practical studies. For example, financial economists usually deal with weekly reruns of an asset or an index and 200 observations means to get return observations in four years in the analysis which could be too long for many studies because the economy condition could be completely different after four years and thus using four years to analyze the property of the returns in the same economic situation could be too long. Moreover, most economists could only obtain annual data to analyze. In this situation, getting 200 observations means to get 200 years data. The world has changed too much, not to mention in 200 years but in 20 years. So, getting 200 observations to analyze is nearly impossible to many economists to analyze economic problems.

Table 1: Empirical sizes of BDS test and HWBZ test.

$\alpha = 0.05$	BDS		HWBZ	
$Y_t \overset{\text{iid}}{\sim} N(0,1)$	$k=2$	$k=3$	$\eta=1$	$\eta=1.5$
$T = 200$	0.1654	0.1683	0.0517	0.05
$T = 400$	0.1049	0.1062	0.0513	0.0459
$\alpha = 0.1$	BDS		HWBZ	
$Y_t \overset{\text{iid}}{\sim} N(0,1)$	$k=2$	$k=3$	$\eta=1$	$\eta=1.5$
$T = 200$	0.2440	0.2452	0.1058	0.1013
$T = 400$	0.1803	0.1789	0.1023	0.0953

Note: Simulation times is 10000. T is the length of testing sequences. In BDS test, the parameters δ and k are the same as Brock, et al. (1996): $\delta = 0.5\sqrt{\text{Var}(Y_t)}$, and $k = 2, 3$. In HWBZ test, we first standardize the residual sequence $\{Y_t\}$ and choose the parameter $\eta = 1$ and 1.5.

Now we check the empirical power of the HWBZ test when $\{X_t\}$ are generated from the following three most representative models:

$$\begin{aligned} \text{Model A :} \quad & X_t = \varepsilon_t + 0.5\varepsilon_{t-1}\varepsilon_{t-1} + 0.5\varepsilon_{t-1}\varepsilon_{t-2} \ , \\ \text{Model B :} \quad & X_t = \begin{cases} 0.5X_{t-1} + \varepsilon_t & X_{t-1} \geq 0 \\ -0.5X_{t-1} + \varepsilon_t & X_{t-1} < 0 \end{cases} , \\ \text{Model C :} \quad & X_t = h_t\varepsilon_t \ , \quad \text{where} \quad h_t = \sqrt{1 + 0.8X_{t-1}^2 + 0.1h_{t-1}^2} \ . \end{aligned} \tag{9}$$

Model A is a typical example of Volterra expansion, Model B is a threshold autoregressive model which is another popular method in nonlinear analysis, and Model C is a GARCH model which plays an especially important role in modeling financial data. The error term ε_t in all the models are all assumed to be iid $\mathcal{N}(0, 1)$.

Brock, et al. (1996) point out that the above three models exhibit zero autocorrelation. For simplicity, we take $\{X_t\}$ as $\{Y_t\}$ in (5) directly in our simulation. Table 2 displays the empirical power of the HWBZ test as well as the rejecting frequency of the BDS test. It may not be proper to call the rejecting frequency of the BDS test to be power when $T = 200$ and 400 because the BDS test possesses very serious over-rejection problem as shown in Table 1. From Table 2, we can see that our test possesses decent power especially when $\{X_t\}$ are generated from Model B.

Table 2: Empirical power of the BDS and HWBZ tests.

$\alpha = 0.05$ Model A	BDS $m = 2$	BDS $m = 3$	HWBZ $\eta = 1$	HWBZ $\eta = 1.5$
$T = 200$	0.9400	0.9348	0.6195	0.6344
$T = 400$	0.9994	0.9994	0.8726	0.8367
$\alpha = 0.05$ Model B	$m = 2$	$m = 3$	$\eta = 1$	$\eta = 1.5$
$T = 200$	0.5791	0.5758	0.7044	0.6650
$T = 400$	0.8101	0.8084	0.9310	0.9026
$\alpha = 0.05$ Model C	$m = 2$	$m = 3$	$\eta = 1$	$\eta = 1.5$
$T = 200$	0.9983	0.9982	0.5625	0.6153
$T = 400$	1	1	0.8153	0.8400

Note: Simulation times is 10000. T is the length of testing sequences. In BDS test, the parameters δ and k are the same as Brock, et al. (1996): $\delta = 0.5\sqrt{\text{Var}(Y_t)}$, and $k = 2, 3$. In HWBZ test, we first standardize the residual sequence $\{Y_t\}$ and choose the parameter $\eta = 1$ and 1.5.

Our proposed HWBZ test can detect not only nonlinearity but also can be used as a test for model specification as Brock, et al. (1996) claim for their BDS test. For instance, after regressing one time series variable on another time series variable with a selected model, one important step is to check whether the residuals are dependent, including to test whether there is any nonlinear and linear dependence in the residuals. In our previous simulation, we have addressed the nonlinear issue. To test for any linear dependence, in this example, we examine the performance of our proposed test on the following MA(1) model:

$$\text{Model D}: \qquad Y_t = 0.3\varepsilon_{t-1} + \varepsilon_t \ . \tag{10}$$

We exhibit our simulation results in Table 3. The table shows that our proposed HWBZ test can detect linear dependence very well and is more powerful than the BDS test.

From our discussion in the above, we can claim that our proposed HWBZ test possesses good size and decent power. And we suggest academics and practitioners not to underestimate the importance of the over-rejection problem for the BDS test.

Table 3: Empirical power of BDS and HWBZ tests.

$\alpha = 0.05$	BDS		HWBZ	
Model D	$m = 2$	$m = 3$	$\eta = 1$	$\eta = 1.5$
$T = 200$	0.3958	0.3962	0.4653	0.4384
$T = 400$	0.6083	0.6049	0.7418	0.6117

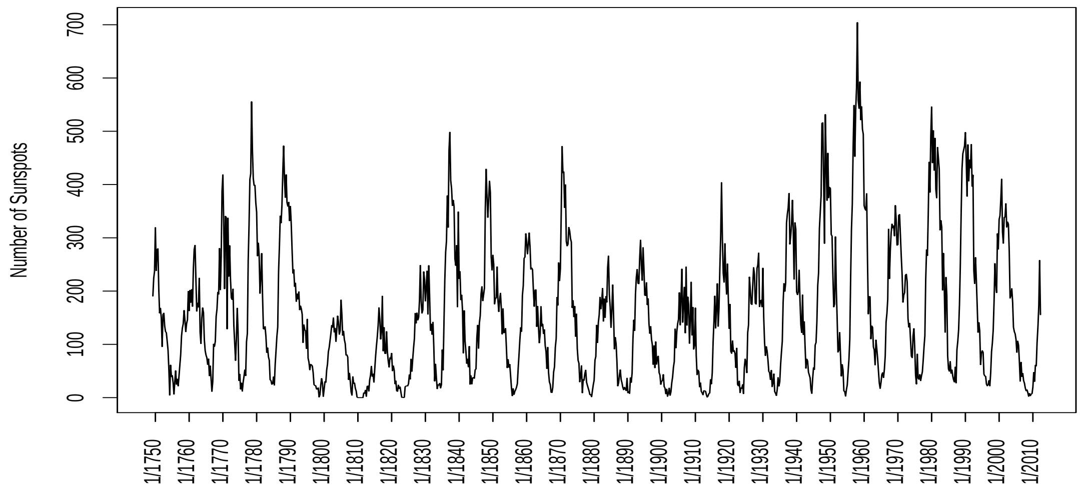
Note: Simulation times is 10000. T is the length of testing sequences. In BDS test, the parameters δ and k are the same as Brock, et al. (1996): $\delta = 0.5\sqrt{\mathrm{Var}(Y_t)}$, and $k = 2, 3$. In HWBZ test, we first standardize the residual sequence $\{Y_t\}$ and choose the parameter $\eta = 1$ and 1.5.

4 Illustration

In this section, we illustrate the applicability of the nonlinearity test we have developed in Subsection 2.2 by applying our proposed nonlinearity test and the BDS test.

Sunspots refer to dark spots on the surface of the sun related to the motion of the solar dynamo. Johann Rudolf Wolf introduces a formula for calculating the sunspot numbers: $R = k(10g + f)$, where g is the number of groups of sunspots, f is the total number of individual spots, and k is a constant for the observations. To honor the contribution by Johann Rudolf Wolf, it is common to call sunspot number "Wolf's sunspot number" (Izenman, 1983)

Figure 1: Wolf's Sunspots Numbers



Note: Quarterly Wolf's sunspot numbers from first quarter of 1749 to first quarter of 2012.

The earliest linear model built for the sunspot data is probably done by Yule (1927) who introduces the class of linear autoregressive models to analyze the data. Since then, the literature, see, for example, Moran (1954), of linear time series analysis of the sunspot data has been growing exponentially. However, some works, see, for example, Tong and Lim (1980) point out that linear model is not adequate for fitting the data and forecasting.

In this paper we illustrate the applicability of our proposed HWBZ test and the DBS test to examine the nonlinearity in the quarterly Wolf's sunspot numbers from the first quarter of 1749 to the first quarter of 2012. Let X_t be Wolf's quarterly sunspot numbers from the first quarter of 1749 to the first quarter of 2012. We exhibit the time series plot of the sunspot data in Figure 1. We first discuss how to use our test statistic to examine whether there is any nonlinearity in $\{X_t\}$. To do so, as discussed in Section 2, we first fit the data by using the following AR(p) model:

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + e_t, \ e_t \overset{\text{iid}}{\sim} \text{WN}(0, \sigma^2) \tag{11}$$

to the sunspot data. We find that the "best" linear model for the sunspot data is

$$\begin{aligned} X_t &= 19.8849 - 0.7051X_{t-1} - 0.1549X_{t-2} - 0.1873X_{t-3} - 0.0834X_{t-4} \ . \\ &\quad +0.1055X_{t-6} + 0.0712X_{t-7} + 0.0810X_{t-9} + \hat{e}_t \ . \end{aligned} \tag{12}$$

We exhibit the results in Table 4. Thereafter, we apply the Ljung-Box test to test the hypothesis of no autocorrelations up to lag k for the residuals and display the results in Table 5. In addition, we display the autocorrelations of the residuals in Figure 2. The results from Table 5 and Figure 2 show that the autocorrelations of the residuals are not significantly different from zero for any lag up to 30,[2] and thus, one may conclude that the AR model in (12) is adequate and there is no other linear relationship remained in the residuals.

Table 4: The Results of the Linear AR Model

Parameter	Estimate	Standard Error	t Value
intercept	19.8849	2.2872	8.694***
X_{t-1}	0.7029	0.0305	23.004***
X_{t-2}	0.1545	0.0375	4.114***
X_{t-3}	0.1872	0.0378	4.948***
X_{t-4}	0.0883	0.0353	2.497**
X_{t-6}	-0.1049	0.0353	-2.965***
X_{t-7}	-0.0722	0.0346	-2.083**
X_{t-9}	-0.0830	0.0247	-3.355***

Note: This table exhibits the results of the linear AR model as shown in (12).
*, **, and *** mean significant at levels 10%, 5%, and 1%, respectively.

One may believe that the linear model in (12) fits the sunspot data well. To check whether this is true, we apply the test we developed in Subsection 2.2 and the DBS test to examine whether there is sequential dependence within the standardized residuals, $\left(\hat{e}_t - \text{mean}(\hat{e}_t)\right) / \sqrt{\text{Var}(\hat{e}_t)}$, obtained from fitting the linear model in (11). The p values

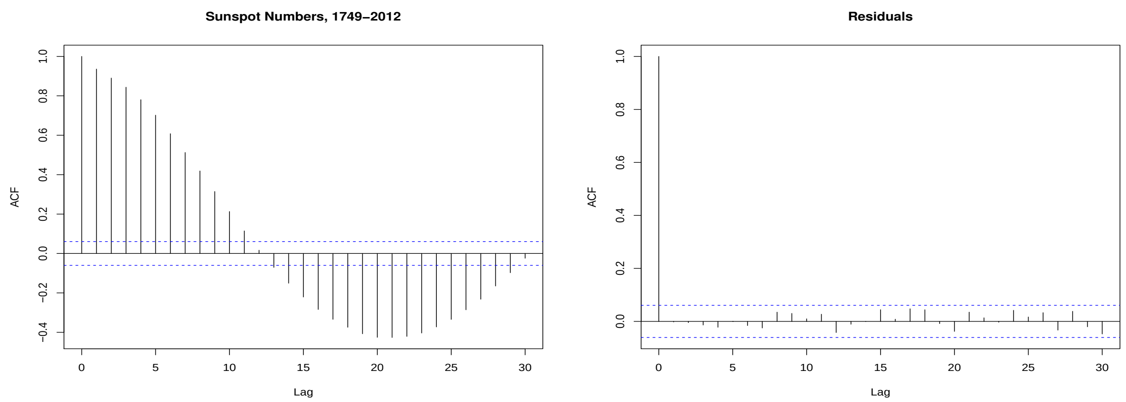
[2]Readers may consider to apply the approach developed by Li (1992) to correct the residual autocorrelations for nonlinear time series models.

Table 5: Autocorrelation Check: The Result of Ljung-Box Test

Check for Sunspots Numbers			Check for Residuals		
Lag (k)	df	$\chi^2(k)$	Lag (k)	df	$\chi^2(k)$
4	4	3160.750***	12	5	6.632
6	6	4075.119***	18	11	13.377
7	7	4353.965***	24	17	18.366
9	9	4645.693***	30	23	25.434

Note: The null hypothesis of Ljung-Box test is that the autocorrelations up to lag k in the population from which the sample is taken are 0. $\chi^2(k)$ is the test statistic with k degrees of freedom. Readers may refer to Ljung and Box (1978) for more details of the test. The left panel displays the values of $\chi^2(k)$ for the Sunspots numbers while the right panel shows the values for the residuals after fitting the linear AR model as shown in (12).
*, **, and *** mean significant at levels 10%, 5%, and 1%, respectively.

Figure 2: Plots of the Autocorrelation Functions



Note: The left panel exhibits the ACF for Sunspots numbers whereas the right panel displays the ACF for the residuals after fitting the linear AR model as shown in (12).

of the HWBZ tests which are present in Table 6 strongly reveals dependence within the residuals. Thus, applying our test, one could realize that there still exists nonlinearity component in the sunspot data. This result is consistent with the findings by Tong and Lim (1980), Tong (1983), and many others. In addition, results of the BDS test also show the nonlinearity in the Wolf's Sunspots numbers.

Readers should be aware of the limitation when using two-step method in both BDS test and our proposed test that the estimation effect could exist in practice. If one wants to test whether the series are independent, or test whether there is any linear and/or nonlinear dependence in the series, then one could apply our test direct to the series. In this situation, there is no estimation effect in the test. However, if one wants to separate the nonlinear dependence from the linear dependence, then one has to fit the linear model first, then apply our proposed test to the corresponding residuals as we do in the above sunspot data analysis. In this situation, the estimation effect cannot be avoided. We suggest practitioners to find linear models to ensure zero autocorrelation in the residuals before applying our proposed test or BDS test.

Table 6: p value of BDS test and HWBZ test.

$Y_t = [\hat{e}_t - \text{mean}(\hat{e}_t)]/\sqrt{\text{Var}(\hat{e}_t)}$	p value of BDS		p value of HWBZ	
	$m = 2$	$m = 3$	$\eta = 1$	$\eta = 1.5$
$T = 3160$	$5.24 * 10^{-20}$	$1.82 * 10^{-19}$	$2.91 * 10^{-3}$	$6.78 * 10^{-5}$

Note: T is the length of Sunspots numbers sequence. In BDS test, the parameters δ and k are the same as Brock, et al. (1996): $\delta = 0.5\sqrt{\text{Var}(Y_t)}$, and $k = 2, 3$. In HWBZ test the parameters $\eta = 1$ and 1.5.

5 Conclusion

There are many works in the development of nonlinearity tests. A nonlinearity test could be parametric, semi-parametric, or nonparametric. However in general nonlinearity may occur in many and could be infinite ways. Thus, it is not our intention to develop a single test that outperforms all other tests in examining nonlinearity.

As nonlinear features are in general more complex and more difficult to model than a linear and independent one, it is not reasonable to restrict the form of the nonlinearity at the stage of detecting them within a sequence. Our HWBZ test as well as the BDS test satisfy this criterion and circumvent the limitation of using too many parameters like those using the Volterra expansion.

There are many criticisms on the BDS test in literature as Tsay (2010) points out that there is serious over-rejection problem in the BDS test. Our simulation studies confirm this and find that rejection frequencies are over 3 times and 2 times of test level 0.05, respectively, when sample sizes are 200 and 400 which could be very large in practice as discussed in Section 4. Our proposed test overcomes this weakness and possesses decent power. In addition, our proposed test is stable on different choice of the pre-chosen parameter and is easy to use since there is only one parameter η. Testing for nonlinear features in the sunspot data displays the applicability of our proposed HWBZ test. Both our proposed test and the BDS test draw the same conclusion which is consistent with findings in the literature that there exists nonlinearity in the Wolf's sunspots numbers.

At last, we note that our test could not only be used to detect any nonlinearity for the variable being examined. If a selected model is fitted to $\{X_t\}$ and its residuals $\{Y_t\}$ could be estimated, the HWBZ test developed in this paper could also be used to examine the appropriation of the model being used. If the null hypothesis of independence on residuals $\{Y_t\}$ is rejected, then one could conclude that the chosen model is not appropriate and need to model the residuals further.

References

[1] Brock, W.A., Dechert, W.D., Scheinkman, J.A. and LeBaron, B., 1996. A test for independence based on the correlation dimension, Econometric Reviews 15, 197-235.

[2] Box, G.E.P., Jenkins, G.M., Reinsel, G.C., 1994. Time series analysis forecasting and control. Prentice-Hall.

[3] Carlin, B.P., Polson, N.G., Stoffer, D.S., 1992. A Monte Carlo approach to nonnormal and nonlinear state space modeling. Journal of the American Statistical Association 87, 493-500.

[4] Chan, K.S., Tong, H., 1990. On likelihood ratio tests for threshold autoregression. Journal of the Royal Statistical Society B 52, 469-476.

[5] Chen, R., Tsay, R.S., 1993a. Functional-coefficient autoregressive models. Journal of the American Statistical Association 88, 298-308.

[6] Chen, R., Tsay, R.S., 1993b. Nonlinear additive ARX models. Journal of the American Statistical Association 88, 955-967.

[7] Cheng, Q., Chen, R., Li, T., 1996. Simultaneous wavelet estimation and deconvolution of reflection Seismic signals via Gibbs sampler. IEEE Transactions on Geoscience and Remote Sensing 34, 377-384.

[8] Cox, D.R., 1981. Statistical analysis of time series: some recent developments (with discussion). Scandinavian Journal of Statistics 8, 93-115.

[9] Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of U.K. inflation. Econometrica 50, 987-1008.

[10] Fan, J.Q., Yao, Q.W., 2003. Nonlinear time series: nonparametric and parametric methods. Springer-Verlag, New York.

[11] Granger, C.W.J., Andersen, A.P., 1978. An introduction to bilinear time series models. Vandenhoek and Ruprecht, Gottingen.

[12] Haggan, V., Ozaki, T., 1981. Modeling nonlinear vibrations using an amplitude-dependent autoregressive time series model. Biometrika 68, 189-196.

[13] Hamilton, J.D., 1989. A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica 57, 357-384.

[14] Hansen, B.E., 1996. Inference when a nuisance parameter is not identified under the null hypothesis. Econometrica 64, 413-430.

[15] Hsing, T., Wu, W.B., 2004. On weighted U statistics for stationary processes. Annals of Probability 32, 1600-1631.

[16] Hui, Y.C., Wong, W.K., Bai, Z.D., Zhu, Z.Z., 2017, A New Nonlinearity Test to Circumvent the Limitation of Volterra Expansion with Application, Journal of the Korean Statistical Society, forthcoming.

[17] Izenman, A.J., 1983. J.R. Wolf and H.A. Wolfer: An historical note on Zurich sunspot relative numbers. Journal of the Royal Statistical Society Series A 146, 311-318.

[18] Keenan, D.M., 1985. A Tukey non-additivity-type test for time series nonlinearity. Biometrika 72, 39-44.

[19] Lehmann, E.L., 1999. Elements of large sample theory. Springer, New York.

[20] Li, G.D., Li, W.K., 2011. Testing for linear time series models against its threshold extension. Biometrika 98(1), 243-250.

[21] Li, W.K., 1992. On the asymptotic standard errors of residual autocorrelations in nonlinear time series modeling. Biometrika 79(2), 435-7.

[22] Ljung, G.M., Box, G.E.P., 1978. On a measure of a lack of fit in time series models. Biometrika 65(2), 297-303.

[23] Moran, P.A.P., 1954. Some experiments on the prediction of sunspot numbers. Journal of the Royal Statistical Society Series B 116, 112-117.

[24] Petruccelli, J.D, Davies, N., 1986. A portmanteau test for self-exciting threshold autoregressive-type nonlinearity in time series. Biometrika 73, 687-694.

[25] Priestley, M.B., 1980. State-dependent models: a general approach to nonlinear time series analysis. Journal of Time Series Analysis 1, 47-71.

[26] Serfling, R., 1980. Approximation theorems of mathematical statistics. John Wiley & Sons, New York.

[27] Shao, X.F., Wu, W.B., 2007. Asymptotic spectral theory for nonlinear time series. Annals of Statistics 35, 1773-1801.

[28] Tiao, G.C., Tsay, R.S., 1994. Some advances in nonlinear and adaptive modeling in time series. Journal of Forecasting 13, 109-131.

[29] Tjϕstheim, D., 1994. Non-linear time series: a selective review. Scandinavian Journal of Statistics 21, 97-130.

[30] Tong, H., 1978. On a threshold model. In C. H. Chen (ed.), Pattern recognition and signal processing. Sijhoff & Noordhoff, Amsterdam.

[31] Tong, H., 1983. Threshold models in non-linear time series analysis. Springer-Verlag, New York.

[32] Tong, H., 1990. Non-linear time series: a dynamical systems approach. Oxford University Press, Oxford.

[33] Tong, H., Lim, K.S., 1980. Threshold autoregression, limit cycles and cyclical data (with discussion). Journal of the Royal Statistical Society Series B 42, 245-292.

[34] Tsay, R.S., 1986. Nonlinearity tests for time series. Biometrika 73, 461-466.

[35] Tsay, R.S., 2010. Analysis of financial time series, 3rd Edition, John Wiley & Sons.

[36] Tukey, J.W., 1949. One degree of freedom for non-additivity. Biometrics 5, 232-242.

[37] Wiener, N., 1958. Non-linear problems in random theory. Cambridge, Mass: M.I.T. Press.

[38] Wu, W.B., Min W.L., 2005. On linear processes with dependent innovations. Stochastic Processes and their Applications 115, 939-958.

[39] Yule, G.U., 1927. On a method of investigating periodicities in disturbed series with special reference to Wolfer's sunspot numbers. Philosophical Transactions of the Royal Society (London) A 226, 267-298.

[40] Zhao, Z.B., 2011. Nonparametric model validations for hidden Markov models with applications in financial econometrics. Journal of Econometrics 162(2), 225-239.

[41] Zhou, Z., 2012. Measuring nonlinear dependence in time series, a distance correlation approach. Journal of Time Series Analysis 33, 438-457.