An open economy New-Keynesian model of government spending across U.S. regions

Osti, Davide

Université Libre de Bruxelles, Brussels, Belgium

12 February 2015

An Open Economy New-Keynesian Model of Government Spending Across U.S. Regions*

Davide Osti
daviosti@ulb.ac.be
ECARES - Université Libre de Bruxelles

February 12, 2015

Abstract

We attempt to replicate the New-Keynesian DSGE model presented in Nakamura and Steinssón (American Economic Review 2014) in order to study the effects of a government spending shock on output and other prominent macroeconomic variables, within a simplified two-region monetary union. Two different specifications for the utility function (separable and non-separable á la Greenwood, Hercowitz, and Huffman 1988) are adopted. Perfectly flexible capital markets detained by households are introduced at a regional level first, and then firm specific capital is assumed. After calibrating for the structural parameters, the model is linearly approximated around the steady states, and impulse response functions are derived and commented.

Key words: open economy relative multiplier, leaning against the wind, Volcker - Greenspan monetary policy, fix and nominal real interest rate, military buildups.

*Paper submitted as part of the evaluation of the course *Graduate Macroeconomics III* taught by Rafael Wouters at ECARES - Université Libre de Bruxelles. We acknowledge previous comments in end of 2012 from Francesco Giavazzi as well as some exchanges on the software with Francesco Giovanardi.

1 Introduction

In the past twenty years, a relevant literature in macroeconomics started to deal systematically with the understanding of the effects of fiscal policy shocks on aggregate economic activity. Most of this literature's effort dealt with the uncovering of the transmission mechanism of fiscal policy, namely how business cycle fluctuations may be mitigated through the deployment of fiscal policy instruments. Nevertheless, the debate has not been solved yet, with a variety of empirical studies attempting to estimate the so-called *fiscal multiplier*, specifically in periods of prolonged economic downturns, such as recessions.

The greatest part of that literature suggests that a marginal increase in government spending may contribute to an output increase of about 0.5 to a little more than unity. A prominent strand of literature, begun by Barro (1981, 1990), has derived the multiplier by analyzing the reaction of output to federal military procurements. This strategy typically yielded a multiplier in the interval 0.5 - 1, as exemplified by a recent research by Hall (2009) as well as Ramey (2009), whose estimated multiplier was slightly higher, around 1.2. However, this approach delicately depends on the interrelations between spending and output during the Second World War and the Korean War, and may be biased due to the "command economy" feature which dominated during the years of armed conflict.

An alternative method, initiated by Blanchard and Perotti (2002), proceeds by identifying the government spending shocks in the context of a structural vector - autoregression. These type of studies, followed among others by Galí, Lopez-Salido and Valles (2007), proposed a multiplier of unity or a little higher. Perotti (2007) as well as Mountford and Uhlig (2008) suggested, through a cross-country evidence, a lower multiplier. In Mertens and Ravn (2010), a liquidity trap environment is modeled within a New-Keynesian framework, and a multiplier smaller than that obtained in "normal" times is obtained.

Another interesting contribution is featured by Erceg and Lindé (2010), in which a New-Keynesian DSGE model - a variation of the Christiano, Eichenbaum and Evans (2005) and Smets and Wouters (2007) - is constructed in order to analyze the effects of a fiscal stimulus during periods of liquidity trap induced recession. The special feature of this model is that the duration of the liquidity trap depends on the size of the stimulus, and is thus *endogenously* determined. The recession is generated by a strong negative taste shock on households' preferences, and the main finding is that the spending multiplier may be substantially amplified during a liquidity trap. However, Erceg and Lindé ultimately find that the multiplier is a step function of the size of the stimulus, with higher multipliers associated with small stimuli. Also, public debt is less upward pressured in a liquidity trap than in normal times, which means that larger output responses translate into much more substantial tax revenues. These results provide a relatively strong rationale for limiting the stances of government interventions in times of recessions, when the monetary policy interest rate is bound close to zero. In fact, if the level of government spending rises above a certain threshold, the multiplier begins to drop, because the time of exiting the trap accelerates, and thus the stimulus will be marginally less and less effective.

We thus examine the fiscal multiplier from the perspective of a New-Keynesian dynamic stochastic general equilibrium model, extended to an open economy scenario. Our focus is restricted to unproductive government expenditure. The government spending multiplier appears as an unknown, non-linear function of all the model parameters. Modeling features such as price rigidities, which can turn the direction of the substitution effect created by higher government spending - from negative to positive - play a role in the substantiation of a large multiplier.

The crucial aspect of our analysis depends on a policy regime in which monetary policy is actively targeting inflation, while fiscal policy is passively adjusting surplus to stabilize government debt. Active monetary policy reacts to a persistent fiscal shock and the consequent growth in inflation by neatly raising the nominal interest rate. This increases the real interest rate, which decreases the demand for consumption and investment, to alleviate the stimulative effect of fiscal expansion.

After 2007-'09, various central banks' objective shifted from targeting inflation stability to stimulating demand by means of low and constant policy interest rates - often near zero. When interest rate is made unresponsive to inflation by monetary policy, a "passive" stance, it amplifies fiscal policy's impact. By maneuvering the interest rate, monetary policy enables higher current and expected inflation to transmit into lower interest rates. Instead of containing the demand stimulus of a fiscal expansion, monetary policy amplifies it. As a consequence, lower real rates prompt higher consumption and investment demand. From government spending, a positive substitution effect is induced by lower real rates, with the substantial effect of raising output, consumption, and investment multipliers. In deep contrast with active monetary/passive fiscal policy, it is very difficult to achieve small spending multipliers from the mix of passive monetary/active fiscal regime.

The model, and its various specifications, which we are going to consider though the paper, share these common features with other DSGE models present in the literature: *i.* forward looking, optimizing agents; *ii.* households deriving utility from consumption and leisure; *iii.* production sectors using labour - and later also capital - inputs; *iv.* monopolistic competition in the goods' sector; *v.* relevant nominal and real frictions; *vi.* fiscal and monetary authorities setting their instruments using simple feedback rules; *vii.* limit cashless economy; *viii.* complete financial markets across regions.

2 The open economy

There are two countries, home (H) and foreign (F), which might be seen as two groups of states within the United States[1]. Total population is normalized to unity for simplicity, n inhabitants living at home, and $1-n$ living abroad. The representative household's utility maximization problem is the following:

$$\max_{C_t, L_t} \mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t u\big[C_t, L_t(x)\big], \tag{1}$$

where β is the subjective discount factor, and x is the type of good produced by firms belonging to a specific industry. Hence, subjects obtain utility from consumption and leisure[2]. Aggregate private consumption is:

$$C_t = \Big[\phi_H^{\frac{1}{\eta}} C_{Ht}^{\frac{\eta-1}{\eta}} + \phi_F^{\frac{1}{\eta}} C_{Ft}^{\frac{\eta-1}{\eta}}\Big]^{\frac{\eta}{\eta-1}}, \tag{2}$$

where $\eta > 0$ denotes the elasticity of substitution between home and foreign goods; ϕ_H indicates the proportion of goods produced by home industries demanded by home citizens (degree of home bias), while ϕ_F represents the share of foreign produced goods demanded at home. ϕ_H and ϕ_F are such that $\phi_H + \phi_F = 1$.
Demand curves for home and foreign goods are, respectively:

$$C_{Ht} = \Big[\int_0^1 c_{ht}(z)^{\frac{\theta-1}{\theta}} dz\Big]^{\frac{\theta}{\theta-1}}; C_{Ft} = \Big[\int_0^1 c_{ft}(z)^{\frac{\theta-1}{\theta}} dz\Big]^{\frac{\theta}{\theta-1}}, \tag{3}$$

θ being the elasticity of substitution between different varieties of goods.
$p_{ht}(z)$ features the relative price level for home produced goods, whereas

[1] One region made of ten states can be imagined as H(ome), and the the rest of the states within the federation as F(oreign), as it will be later seen in the section about calibration.

[2] With a little abuse of notation, we denote labour supplied by households at time t by L_t, instead of the more spread notation N_t. Since households are endowed by a total amount of time equal to one, leisure is simply going to be defined as $1 - L_t$.

$p_{ft}(z)$ is the analog for foreign goods. The common currency across the two regions is referred as *dollar*. In this context capital letters denote aggregate variables whereas small letters refer to relative quantities, i.e. weighted by the region. There is a continuum of firms indexed by $z \in [0, 1]$. Households' resource constraint is so defined:

$$P_t C_t + \mathbb{E}_t \Big[M_{t,t+1} B_{t+1}(x) \Big] \leq B_t(x) + (1-\tau_t) W_t L_t(x) + \int_0^1 \Xi_{ht}(z) dz - T_t \quad (4)$$

where P_t denotes price index, C_t represents aggregate households' consumption, $\mathbb{E}_t B_{t+1}(x)$is a random variable describing the value of bonds held by the households at the beginning of period $t+1$, which can take value S_t (pay) or $1 - S_t$ (not pay) according to an unknown probability distribution. $\mathbb{E}_t M_{t,t+1}$ is the stochastic discount factor related to the financial securities' holding[3]. τ_t is the labour income tax rate[4], while T_t is a lump-sum tax, and W_t is the hourly nominal wage rate. A *no-Ponzi-game* condition is set to rule out the possibility of arbitrage among the financial market actors[5]. $\int_0^1 \Xi_{ht}(z) dz$ finally denotes the overall profit of firms based in H.

2.1 Solving the households' problem

We may simply set up a Lagrangian equation:

$$\Lambda = \mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t u(C_t, L_t(x)) + \mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t \lambda_t \{ B_t(x) + (1 - \tau_t) W_t L_t(x) +$$

$$+ \int_0^1 \Xi_{ht}(z) dz - \mathbb{E}_t [M_{t|t+1} B_{t+1}(x)] - P_t C_t - T_t \},$$

obtaining the following first order conditions:

$$\frac{\partial \Lambda}{\partial C_t} = u_c \big[C_t, L_t(x) \big] = \lambda_t P_t;$$

$$\frac{\partial \Lambda}{\partial B_{t+1}} = \beta \mathbb{E}_t \lambda_{t+1} = \lambda_t \mathbb{E}_t M_{t,t+1}.$$

[3]In asset pricing jargon, the stochastic discount factor is often expressed as $M_t = \beta u'(c_{t+1})/u'(c_t)$ in the asset pricing equations *á la Lucas*.

[4]Same in both the home and foreign regions, being part of a monetary and fiscal union.

[5]"Monetary authorities allowed bubbles to grow, partly because the Standard Model said there could not be bubbles", according to Joseph Stiglitz (2011) and should make us reflect on the need of avoiding absolutely unreal and absurd assumptions in macroeconomics such as the no-Ponzi-game condition. Another sensible issue would be the inclusion of financial intermediaries and the focus on *credit cycles* instead of only on business cycles.

Combining the two first order conditions to eliminate the Lagrangian multiplier yields the following Euler equation for consumption:

$$\mathbb{E}_t \frac{u_c\big(C_{t+1}, L_{t+1}(x)\big)}{u_c\big(C_t, Lt(x)\big)} = \mathbb{E}_t \frac{M_{t,t+1}}{\beta} \frac{P_{t+1}}{P_t}, \tag{5}$$

plus a transversality condition, $\lim_{T\to\infty} K_{t+T} C_{t+T} = 0$, as a consequence of Kuhn-Tucker theorem. Let us remind that u_c is the first partial derivative of utility with respect to consumption and u_l with respect to labour. Two functional forms of utility will be adopted later, in order to compare the effects on the multiplier of an additively separable utility function in leisure and consumption, and one in which leisure and consumption are complement[6]. The first order condition for labour is:

$$\frac{\partial \Lambda}{\partial L_t} = \beta^t u_l\big(C_t, L_t(x)\big) + \beta^t \lambda_t (1-\tau_t) W_t(x).$$

Hence the labour supply equation becomes:

$$\frac{u_l\big(C_t, L_t(x)\big)}{u_c\big(C_t, L_t(x)\big)} = (1-\tau_t)\frac{W_t(x)}{P_t}. \tag{6}$$

This explains the intra - temporal optimization problem between consumption and leisure, being all of the variables featured by the same temporal index.

2.2 The demand curves

Consequently, demand curves for home and foreign countries are:

$$C_{H,t} = \phi_H C_t \Big[\frac{P_{H,t}}{P_t}\Big]^{-\eta}; C_{F,t} = \phi_F C_t \Big[\frac{P_{F,t}}{P_t}\Big]^{-\eta}, \tag{7}$$

$$c_{h,t}(z) = C_{H,t} \Big[\frac{p_t(z)}{P_{H,t}}\Big]^{-\theta}; c_{f,t} = C_t \Big[\frac{p_t(z)}{P_{F,t}}\Big]^{-\theta}. \tag{8}$$

Combining the last four equations yields:

$$c_{h,t} = \phi_H C_t \Big[\frac{P_{H,t}}{P_t}\Big]^{-\eta} \Big[\frac{p_t(z)}{P_{H,t}}\Big]^{-\theta}; c_{f,t} = \phi_F C_t \Big[\frac{P_{F,t}}{P_t}\Big]^{-\eta} C_t \Big[\frac{p_t(z)}{P_{F,t}}\Big]^{-\theta},$$

where

$$P_{H,t} = \Big[\int_0^1 p_t(z)dz\Big]^{1-\theta}; P_{F,t} = \Big[\int_0^1 p_t(z)dz\Big]^{1-\theta}. \tag{9}$$

[6] I.e. *catching up with the Joneses*'s form or *GHH*.

Now, let us specify the optimality condition for the foreign household:

$$\frac{u_c[C_t^*, L_t^*(x)]}{u_c[C_t, L_t(x)]} = \frac{P_t^*}{P_t} = Q_t, \tag{10}$$

where Q_t is the real exchange rate, obtained as the ratio between the aggregate price levels in the two regions and equation (15) is the so called Backus-Smith condition[7], representing the optimal risk sharing condition between home and foreign households. Both the two regions inhabitants have an equal initial financial wealth, denoted by B_t (which can be thought as a risk free government bond).

2.3 Solving the firms' problem

Labour is the only factor of production[8]. The production function has the *Cobb-Douglas* form $y(z) = f(L_t(z))$, increasing in its argument and concave. Thus, there are decreasing marginal returns to scale, *ceteris paribus.* Another important assumption is that labour is mobile across regions. The authors claim that the consequence of this assumption is analogous to impose equal mobility on capital and labour. If indeed labour was more mobile that capital, inward migration flows would cause a decrease in the capital-labour ratio in the home region. This would have the effect of lowering the *per-capita* government spending multipliers (and vice-versa if labour was less mobile than capital - which is far more reasonably the case). Now we can exhibit the maximization problem for the representative firm z operating in industry x:

$$\max_{y_t} \mathbb{E}_0 \sum_{j=0}^{+\infty} M_{t|t+j} \big[p_{t+j}(z) y_{t+j}(z) - W_{t+j}(x) L_{t+j}(z) \big], \tag{11}$$

with an hourly wage rate $W_t(x)$ is paid to the worker by industry x. The constraint is given by the production function. The overall demand for goods

[7] According to international macroeconomic theory, with full risk sharing or complete financial markets, relative consumption should be perfectly correlated with the real exchange rate. Hence regions where the wages are low, should then receive a transfer to take advantage of cheap consumption. Nonetheless Backus and Smith (1993) observed that correlation between consumption and real exchange rates is zero. This is known as consumption-real exchange rate anomaly or Backus-Smith puzzle.

[8] It should be remarked that this is only a simplifying assumption for the basic version of the model, which later will be relaxed, in favour of two alternative specifications where capital will enter as a factor held by households first, and by firms themselves then.

of firm z at time t is nothing but the sum of the demand of home consumers, foreign consumers and the government, i.e.:

$$y_t(z) = [nC_{H,t} + (1-n)C^*_{F,t} + nG_{H,t}]\Big(\frac{p_{ht}(z)}{P_{Ht}}\Big)^{-\theta} \leq f(L_t(z)). \tag{12}$$

We may again write up a Lagrangian equation to solve the inter temporal maximization problem:

$$\Lambda = \mathbb{E}_t \sum_{j=0}^{+\infty}\{M_{t|t+j}\big[p_{t+j}(z)y_{t+j}(z) - W_{t+j}(x)L_{t+j}(z)\big] + \mu_t[f(L_t(z)) - y_t(z)]\}.$$

The first order conditions for wages, price levels, and labour demand are:

$$\frac{\partial\Lambda}{\partial W_t} = \mathbb{E}_t \sum_{j=0}^{+\infty} M_{t|t+j}L_{t|t+j}(z) = \mu_t f(L_t(z)),$$

$$\frac{\partial\Lambda}{\partial p_t} = \mathbb{E}_t \sum_{j=0}^{+\infty} M_{t|t+j}W_{t|t+j}(x) = \mu_t(-\theta)\left[\frac{p_{h,t}(z)}{P_{H,t}}\right]^{1-\theta},$$

$$\frac{\partial\Lambda}{L_t} = \mathbb{E}_t \sum_{j=0}^{+\infty} M_{t|t+j}W_{t|t+j}(x) = \mu_t f_l(L_t(z)).$$

Eliminating the multiplier yields:

$$\mathbb{E}_t \sum_{j=0}^{+\infty} L_{t+j}(z) = \mathbb{E}_t \sum_{j=0}^{+\infty} W_{t+j}(x). \tag{13}$$

Hence,

$$\frac{\mathbb{E}_t \sum_{j=0}^{+\infty} M_{t|t+j}W_{t+j}}{f_l(L_t(z))}\left[\theta\frac{p_{h,t}(z)}{P_{H,t}}\right]^{1-\theta} = \mathbb{E}_t \sum_{j=0}^{+\infty} M_{t|t+j}y_{t+j}(z).$$

After some algebra we obtain,

$$p_t(z) = \frac{\theta}{\theta-1}\mathbb{E}_t \sum_{j=0}^{+\infty} \frac{M_{t|t+j}}{M_{t|t+j}}\frac{y_{t+j}}{W_{t+j}} f_l(L_t(z)), \tag{14}$$

which describes the optimal price setting by the firm z in period $t+j$ when it can re-optimize its prices[9].

[9] A Calvo pricing system is implicitly adopted in our *New-Keynesian* framework.

2.4 The government

It fixes monetary and fiscal policy. Total government spending follows an $AR(1)$ process. nG_{Ht} quantifies the total government spending undertaken by the home region. Moreover:

$$g_{h,t}(z) = G_{H,t}\left[\frac{p_{h,t}(z)}{P_{H,t}}\right]^{-\theta}; g_{f,t}(z) = G_{F,t}\left[\frac{p_{f,t}(z)}{P_{F,t}}\right]^{-\theta}, \tag{15}$$

meaning that relative government spending in home and foreign region takes the form of a Dixit-Stiglitz CES aggregator, with these fundamental assumptions:
i. lump-sum taxes are non-distortionary;
ii. $\exists$ a perfect risk sharing across households H and F;
iii. Ricardian equivalence holds;
iv. labour income taxes are distortionary.
Assuming a simple Taylor rule for the whole open economy provides us with the following log-linear approximation:

$$\hat{r}_t^n = \rho_r \hat{r}_{t-1}^n + (1-\rho_i)(\phi_\pi \hat{\pi}_t^{ag} + \phi_y \hat{y}_t^{ag} + \phi_g \hat{g}_t^{ag}). \tag{16}$$

Here $\hat{r}_t^n$ represents the nominal interest rate, and the apex ag stands for aggregate variable:

$$\hat{\pi}_t^{ag} = n\hat{\pi}_t + (1-n)\hat{\pi}_t^*,$$

that is, in the aggregate, overall inflation equals the weighted sum of the consumer prices for the two regions, home and foreign. Thus:

$$\hat{y}_t^{ag} = n\hat{y}_t + (1-n)\hat{y}_t^*.$$

A similar condition holds for aggregate output and government spending:

$$\hat{g}_t^{ag} = n\hat{g}_t + (1-n)\hat{g}_t^*.$$

Federal government spending is driven by an exogenous autoregressive stochastic process $AR(1)$, such that:

$$g_t = \rho g_{t-1} + \varepsilon_t \tag{17}$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$. This is the only exogenous, stochastic shock hitting our economy.

3 Calibration

Two utility functional forms are considered, one with additively separable labour and consumption and the other, as adopted by Greenwood, Hercowitz, and Huffman (1988) in which they are not. ν, the Frisch labour supply elasticity, is set equal to 1 as it is common in macroeconomics, to capture variations on the extensive margin. This value is slightly higher than the one estimated in microeconomic studies as it is meant to capture variations in labour supply on the extensive margin, due, for example, to retirement and unemployment. σ is the inter-temporal elasticity of substitution and, even in absence of agreement in the literature about how to fix its value, the authors set it equal to 1 in the separable model, in order to keep the economy on a balanced growth path. The subjective discount factor is bound to 0.99, the elasticity of substitution among varieties to 7, and the elasticity of substitution between home and foreign goods to 2, a slightly higher value than that used in Backus, Kehoe, and Kydland (1994), and Chari, Kehoe, and McGrattan (2002). Higher values of η yield lower open economy relative multipliers, since they imply higher expenditure switching among regions in response to regional shocks. With regards to the Calvo structure of prices, it is assumed that the probability of price re-optimization for firms in a given period is $\alpha = 0$ in case of perfect flexibility (e.g. as in a *plain vanilla* Neoclassical setting), and $\alpha = 0.75$ in presence of price rigidities.

The size of the home region is $n = 0.1$, which is equivalent to the size of a group of states constituting a share of about one tenth of the whole federal territory. The big ratio of government spending on output is assumed to be 0.2 in steady state. The home bias parameter ϕ_H is determined by Nakamura and Steinssón throughout the use of data from the U.S. Commodity Flow Survey and the U.S. National Income and Products Accounts. To calculate the degree of home bias, the authors observe that the CFS does not take into account services' trade, which accounts for a significant share of trade among regions[10]. A $\phi_H = 0.69$ makes the home region slightly more open that Spain and slightly less open than Portugal.

Considering the government's "fundamentals", we should point out that they represent the "choice variables" of the social planner, in the sense that monetary and fiscal policy are useful instruments to influence the union's economic performance and address it towards some positive results as a reaction to negative, unpredicted recessionary shocks. We therefore devote

[10] The share of services in the total U.S. exports represents about 20% of them.

section 5 to the specification of these government's "fundamentals".

Parameter	Description	Value
ν	labour supply elasticity	1
σ	elast. of subst. btwn leisure and consumpt.	1
β	household consumpt. disc. factor	.99
θ	elast. of subst. btwn $\neq$ varieties	7
η	elast. of subst. btwn H and F	2
a	labour share in the prod. funct.	2/3
α	probability of re-optimizing prices	.75
n	population of home region	.1
ϕ_H	home bias parameter	.69
ϕ_F	complement of the home bias	.31
ϕ_π	relative weight of inflation for int. rate	.75
ϕ_y	relative weight of output growth for int. rate	.5
ϕ_g	relative weight of govt. spend. for int. rate	.5
ρ_π	Taylor coefficient for inflation	.75
ρ_y	Taylor coefficient for output	.5
ρ_g	AR coefficient for govt. spending	.933
κ	undetermined coefficient	1
L	steady state value of labour	.788
G	steady state value of govt. consumption	.2$\times Y$
C	steady state value of hh. consumption	.2634

3.1 Separable preferences

The first utility functional form is:

$$U(C_t, L_t(z)) = \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \chi \frac{L_t^{1+\frac{1}{\nu}}(z)}{1+\frac{1}{\nu}}. \tag{18}$$

Consumption and labour are complementary. The Euler equation for consumption is:

$$\mathbb{E}_t \Big(\frac{C_t}{C_{t+1}}\Big)^{1/\sigma} = \frac{\mathbb{E}_{\mathbf{t}} M_{t|t+1}}{\beta} \frac{W_t(x)}{P_t}, \tag{19}$$

and the labour supply equation:

$$\chi L_t^{1/\nu} C_t^{1/\sigma} = -(1-\tau_t) \frac{W_t(x)}{P_t}. \tag{20}$$

3.2 Non separable preferences

Utility function now becomes:

$$U(C_t, L_t(z)) = \Big(1 - \frac{1}{\sigma}\Big)^{-1} \left[\frac{C_t - \chi L_t(z)^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right]^{1-\frac{1}{\sigma}}. \tag{21}$$

Again, we update the Euler equation:

$$\mathbb{E}_t\Big(\frac{C_t - \chi L_t(z)^{1+\frac{1}{\nu}}}{C_{t+1} - \chi L_{t+1}(z)^{1+\frac{1}{\nu}}}\Big)^{-1/\sigma} = \frac{\mathbb{E}_{\mathbf{t}} M_{t|t+1}}{\beta}\frac{P_{t+1}}{P_t}, \tag{22}$$

and the new labour supply:

$$\Big(\frac{C_t - \chi L_t(z)^{-\nu}}{C_t}\Big)^{-1/\sigma} = (1-\tau_t)\frac{W_t(x)}{P_t}. \tag{23}$$

4 Stationary model

In the basic version of the model, thirteen equations in twelve variables are representing the competitive equilibrium of this economy, as a sequence of allocations $\{C_t\}_{t=0}^{+\infty}$, prices $\{P_t\}_{t=0}^{+\infty}$, government purchases $\{G_t\}_{t=0}^{+\infty}$, labour income tax rates $\{\tau_t\}_{t=0}^{+\infty}$, lump sum taxes $\{T_t\}_{t=0}^{+\infty}$, real interest rates $\{r_t\}_{t=0}^{+\infty}$, labour hours supplied $\{L_t\}_{t=0}^{+\infty}$, labour hours demanded $\{S_t\}_{t=0}^{+\infty}$, wage levels $\{W\}_{t=0}^{+\infty}$, stochastic discount factors $\{M_{t+1}\}_{t=0}^{+\infty}$, prices of bonds $\{B_t\}_{t=0}^{+\infty}$, output levels $\{Y_t\}_{t=0}^{+\infty}$, and price level's variations, i.e. inflation, $\{\pi_t\}_{t=0}^{+\infty}$, satisfying the households', firms' and government's resource constraints at each period $t = 0, 1, 2, ..., +\infty$. The thirteen equations in steady state are[11]:

$$M = \beta \tag{24}$$

$$W = -\frac{P}{1-\tau}\frac{\chi L^{\frac{1}{\nu}}}{1+\frac{1}{\nu}} \tag{25}$$

$$C = \{B(1-\beta) + (1-\tau)WL + \Xi - T\}/P \tag{26}$$

$$C = \big[\phi_H^{\frac{1}{\eta}} C_H^{\frac{\eta-1}{\eta}} + \phi_F^{\frac{1}{\eta}} C_F^{\frac{\eta-1}{\eta}}\big]^{\frac{\eta}{\eta-1}} \tag{27}$$

[11]Capital letter variables without any index denote steady state levels, whereas small letters refer to growth levels. Furthermore, each equation should be seen as double, in the sense that those presented here are the equilibrium equations expressed in the variables associated with the home region, but there should be as many variables and equations for the foreign region.

$$P = \left[-\frac{\alpha L^{2\alpha-1}}{\theta W} \right]^{\frac{1}{\theta-1}} p \tag{28}$$

$$G = \left[\frac{Y}{n} + C^* \frac{n-1}{n} - C \right] \left[\frac{p}{P} \right]^{\theta} \tag{29}$$

$$g = \frac{\varepsilon}{1-\rho_g} \tag{30}$$

$$\tau = W \tag{31}$$

$$S = \frac{W}{\alpha L^{\alpha-1}} \tag{32}$$

$$Y = L^{\alpha} \tag{33}$$

$$p = \frac{\alpha\theta}{\theta-1} \frac{Y L^{\alpha-1}}{W} \tag{34}$$

$$r^n = \phi_\pi \pi + \phi_y y + \phi_g g \tag{35}$$

$$\tau = \frac{nPG_h + (1-n)PG_f}{W\frac{L^2}{2} - \frac{W^2}{2}L} \tag{36}$$

5 Policy rules

Three rules of monetary policy and two of tax policy are considered in the paper, thus the possible combinations of the five policy together interacted with different ones each time, yields six possible combinations of public policies.

5.1 Monetary policy

Three rules of monetary policy conduct are considered throughout the paper.
i. Volcker-Greenspan: $\rho_g = 0.933$, $\phi_\pi = 1.5$, $\phi_y = 0.5$, $\phi_g = 0$, and $\alpha = 0.75$. The central bank "leans against the wind", in response to inflationary government spending shocks it aggressively raises the interest rate.
ii. Constant real rate: $\rho_g = 0.933$, $\alpha = 0.75$, the central bank maintains a fixed real interest rate in response of a $+\Delta g_t$. But, to guarantee price-level determinacy, the bank responds aggressively to all other inflationary shocks.
iii. Constant nominal rate: $\rho_g = 0.85$; $\alpha = 0.75$; kept by the central bank in any case a positive government spending shock occurs. Again, it however responds aggressively to any other inflationary shock $\rightsquigarrow$ as in the discussion of effective monetary and fiscal policy at the zero lower bound in Eggertson

(2011).
Note that *ii.* can be considered as standing in between *i.* and *iii.*.

5.2 Fiscal policy

Two possible options are available to the government.
i. Lump sum taxes only: non distortionary.
ii. Balanced budget policy: taxes are now levied both on lump sums and labour income ones, hence it is a distortionary regime.
According to the latter, the government resource constraint becomes thus:

$$nP_{Ht}G_{Ht} + (1-n)P_{Ft}G_{Ft} = \tau_t \int_0^1 W_t(x)L_t(x)dx, \tag{37}$$

meaning that an $\uparrow$ in g_t inevitably leads to an $\uparrow$ in τ_t, distortionary taxes on labour income.

6 A linear approximation

Section 6.1 deals with the approximation of the basic version of our model; section 6.2 relates to the model with GHH preferences; sections 6.3 and 6.4 finally present the derivations for the more complex extensions of the baseline model, namely the model with regional capital market and that with firm specific capital, respectively. Note that along the next section starred variables denote a quantity peculiar to the foreign region.

6.1 Separable utility

Here we systematically derive the log-linearized equilibrium equations of the four versions of our model of open economy. We start approximating Euler equation, $\mathbb{E}_t\Big[\beta\Big(\frac{C_t}{C_{t+1}}\Big)^{\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\Big] = \frac{1}{1+r_t^n}$, in the basic version, as follows:

$$\begin{aligned}
&\log\beta + \frac{1}{\sigma}\log C_t - \frac{1}{\sigma}\mathbb{E}_t C_{t+1} + \log P_t - \log P_{t+1} = -\log(1+r_t^n)\\
\Leftrightarrow &\log\beta + \frac{1}{\sigma}\log\bar{C} + \frac{1}{\sigma\bar{C}}(C_t-\bar{C}) - \frac{1}{\sigma}\log\bar{C} - \frac{1}{\sigma\bar{C}}\mathbb{E}_t(C_{t+1}-\bar{C}) + \log\bar{P}+\\
&+\frac{1}{\bar{P}}(P_t-\bar{P}) - \log\bar{P} - \frac{1}{\bar{P}}\mathbb{E}_t(P_{t+1}-\bar{P}) = -\log(1+\bar{r}^n) - \frac{1}{1+\bar{r}}\mathbb{E}_t(r_{t+1}^n-\bar{r}^n)\\
\Leftrightarrow &\frac{1}{\sigma\bar{C}}(C_t-\bar{C}) - \frac{1}{\sigma\bar{C}}\mathbb{E}_t(C_{t+1}-\bar{C}) + \frac{1}{\bar{P}}(P_t-\bar{P}) - \frac{1}{\bar{P}}\mathbb{E}_t(P_{t+1}-\bar{P}) = -\frac{1}{1+\bar{r}}\mathbb{E}_t(r_{t+1}-\bar{r})\\
\Leftrightarrow &\frac{1}{\sigma}(\hat{C}_t-\hat{C}_{t+1}) - E_t\hat{\pi}_{t+1} = -\frac{\hat{r}_t^n}{1-\bar{r}}\\
\Leftrightarrow &\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \sigma(\hat{r}_t^n + \mathbb{E}_t\hat{\pi}_{t+1}), \qquad (38)
\end{aligned}$$

where in the second line we took logarithms, in the third line we performed a first order Taylor series expansion about the steady states for each variable, in the fourth we simplified every constant element, and from the fifth onwards we defined a log-linearized variable as $\hat{x}_t = \frac{x_t-\bar{x}}{\bar{x}}$[12].

We proceed further by log-linearizing the Backus-Smith condition $\Big[\frac{C_t}{C_t^*}\Big]^{1/\sigma} = Q_t$:

$$\begin{aligned}
&\frac{1}{\sigma}\log C_t - \frac{1}{\sigma}\log C_t^* = \log Q_t\\
\Leftrightarrow &\frac{1}{\sigma}\log\bar{C} + \frac{1}{\bar{C}}(C_t-\bar{C}) - \frac{1}{\sigma}\log\bar{C} - \frac{1}{\bar{C}^*}(C_t^*-\bar{C}^*) = \log\bar{Q} + \frac{1}{\bar{Q}}(Q_t-\bar{Q})\\
\Leftrightarrow &\frac{1}{\sigma}\hat{C}_t - \frac{1}{\sigma}\hat{C}_t^* = \hat{Q}_t\\
\Leftrightarrow &\hat{C}_t - \hat{C}_t^* = \sigma\hat{Q}_t. \qquad (39)
\end{aligned}$$

Log-linearizing, now, the first order condition for labour of the household problem, namely equation $\frac{\chi}{a}L_t(x)^{(1/\nu-a+1)/a}c_t^{1/\sigma}/(1-\tau_t) = \frac{S_t(x)}{P_t}$, we obtain the following expression,

[12] Thus hatted variables denote percentage deviation from their steady state value.

$$\log\chi - \log a + \frac{\frac{1}{\nu} - a - 1}{a}\log L_t(x) + \frac{1}{\sigma}\log C_t - \log(1-\tau_t) = \log S_t(x) - \log P_t$$

$$\Leftrightarrow \log\chi - \log a + (\frac{1}{\nu} - a + 1)\log\bar{L} + \frac{\frac{1}{\nu} - a - 1}{a\bar{L}}(L_t - \bar{L}) + \frac{1}{\sigma}\log\bar{C} + \frac{1}{\bar{C}}(C_t - \bar{C}) -$$

$$- \log(1-\bar{\tau}) - \frac{\bar{\tau}}{1-\bar{\tau}}(\tau_t - \bar{\tau}) = \log\bar{S} + \frac{1}{\bar{S}}(S_t - \bar{S}) - \log\bar{P} - \frac{1}{\bar{P}}(P_t - \bar{P})$$

$$\Leftrightarrow \frac{1/\nu - a + 1}{a\bar{L}}\hat{L}_t + \frac{1}{\sigma}\hat{C}_t + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t = \hat{S}_t - \hat{P}_t. \quad (40)$$

Rewriting this equation more compactly for both home and foreign regions, we obtain, for period $t+j$:

$$\hat{s}_{ht,t+j} = \psi_\nu \hat{y}_{ht,t+j} + \frac{1}{\sigma}\hat{c}_{t+j} + \frac{\bar{\tau}}{1+\bar{\tau}}\hat{\tau}_{t+j}, \quad (41)$$

$$\hat{s}_{ft,t+j} = \hat{q}_{t+j} + \psi_\nu \hat{y}_{ft,t+j} + \frac{1}{\sigma}\hat{c}^*_{t+j} + \frac{\bar{\tau}}{1+\bar{\tau}}\hat{\tau}_{t+j}. \quad (42)$$

Recall that in steady state:

$$\bar{P} = \bar{P}_H = \bar{P}_F = p(z),$$

and also:

$$\bar{C}_H = \phi_H\bar{C},$$
$$\bar{C}_F = \phi_F\bar{C}.$$

The same holding abroad:

$$\bar{C}^*_H = \phi_H\bar{C}^*,$$
$$\bar{C}^*_F = \phi_F\bar{C}^*.$$

Recall that $\bar{C} = \bar{C}^*$ due to our assumption on equal initial financial wealth among households inhabiting the home region and those populating the foreign one.

In addition to that, $\bar{G} = \bar{G}_h = \bar{G}_F$, and $n\bar{C}_H + (1-n)\bar{C}_F = \bar{C}$, such that the demand for H goods by F consumers in steady state is affected by:

$$\phi^*_H = \frac{n}{1-n}\phi_F.$$

Recall also that:

$$\hat{c}_{ht} = \hat{c}_t - \eta \hat{p}_{ht} \Leftrightarrow \hat{c}_{ft} = \hat{c}_t - \eta \hat{p}_{ft};$$
$$\hat{c}^*_{ht} = \hat{c}^*_t - \eta \hat{p}^*_{ht} \Leftrightarrow \hat{c}^*_{ft} = \hat{c}^*_t - \eta \hat{p}^*_{ft}.$$

Per capita home and foreign output is:

$$Y_{Ht} = \frac{1}{n} \int_0^1 y_{ht}(z)dz \Leftrightarrow Y_{Ft} = \frac{1}{1-n} \int_0^1 y_{ft}(z)dz,$$

and $Y_t = nY_{Ht} + (1-n)Y_{Ft}$ is the total output.
In steady state, we get:

$$\bar{Y}_{Ht} = \frac{\bar{y}_{ht}}{n} \Leftrightarrow \bar{Y}_{Ft} = \frac{\bar{y}_{ht}}{1-n};$$

and $\bar{Y} = n\bar{Y}_{Ht} + (1-n)\bar{Y}_{Ft}$ is the total constant - zero growth - output.

Log-linearize the equation representing consumption demand for home varieties:

$$y_{Ht}(z) = [nC_{Ht} + (1-n)C_{Ft} + nG_{Ht}]\Big(\frac{p_{ht}(z)}{P_{Ht}}\Big)^{-\theta},$$
$$\bar{y}(z) = [n\bar{C}_H + (1-n)\bar{C}_F + n\bar{G}_H]\Big(\frac{\bar{p}_h}{\bar{P}_H}\Big)^{-\theta},$$

and consider that:

$$\bar{C}_H = \phi_H \bar{C}; \bar{G}_H = \bar{G},$$
$$\bar{C}^*_H = \phi^*_H \bar{C}; \phi_H + \phi_F = 1,$$
$$\bar{C} = \bar{C}^*; \phi^*_H = \frac{n}{1-n}\phi_F,$$
$$\bar{Y}_H = \bar{C} + \bar{G}; \bar{Y}_F = \bar{C} + \bar{G}.$$

Therefore, $\bar{Y}_H = \bar{Y}_F = \bar{Y}$.

Log-linearized demand at time $t+j$ is thus:

$$\log Y_{Ht}(z) = \log n + \log C_{Ht} + \log(1-n) + \log C_{Ft} + \log n + \log G_{Ht} - \theta \log p_{ht}(z) + \theta \log P_{Ht} \Leftrightarrow$$

$$\log \bar{Y}_h + \frac{1}{\bar{Y}_H}(Y_{Ht} - \bar{Y}_H) + \log n + \log \bar{C}_H + \frac{1}{\bar{C}_H}(C_{Ht} - \bar{C}_H) - \log(1-n) + \log \bar{C}_F + \frac{1}{\bar{C}_F}(C_{Ft} - \bar{C}_F) +$$

$$+ \log \bar{G}_H + \frac{1}{\bar{G}_H}(G_{Ht} - \bar{G}_H) - \theta \log \bar{p}_{Ht}(z) - \frac{\theta}{\bar{p}_h}(p_{ht} - \bar{p}_h) + \theta \log \bar{P}_H + \frac{\theta}{\bar{P}_H}(P_{Ht} - \bar{P}_H) \Leftrightarrow$$

$$\hat{y}_{ht,t+j}(x) = \phi_H\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}_{ht+j} + \frac{1-n}{n}\phi_H^*\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}^*_{ht+j} + \hat{g}_{ht+j} -$$

$$- \theta\big[\hat{p}_{ht}(x) - \hat{p}_{ht+j} - \sum_{k=1}^{j} \pi_{t+k}\big]. \tag{43}$$

Analogously, the log-linearized demand equation for the foreign region is:

$$\begin{aligned}\hat{y}_{ft,t+j}(x) =& \frac{n}{1-n}\phi_F\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}_{ft+j} + \phi_F^*\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}^*_{ft+j} + \hat{g}_{ft+j} - \\ & - \theta\big[\hat{p}_{ft}(x) - \hat{p}_{ft+j} - \sum_{k=1}^{j} \pi_{t+k}\big]. \end{aligned} \tag{44}$$

Plugging the Calvo pricing equation in the last equation, leads to:

$$\begin{aligned} Y_{Ht} &= \frac{1}{n}\sum_{j=0}^{\infty}(1-\alpha)\alpha^j y_{ht-j,t}(x) \rightsquigarrow \\ \hat{y}_{ht} &= \phi_{Ht}\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}_t + \frac{1-n}{n}\phi_H^*\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}^*_{ft+j} + \hat{g}_{ht+j} - \theta\sum_{j=0}^{\infty}(1-\alpha)\alpha^j \times \\ &\times \big[\hat{p}_{ht-j}(x) - \hat{p}_{ht} - \sum_{k=0}^{j-1}\pi_{t-k}\big]. \end{aligned} \tag{45}$$

Linearizing equation about the law of motion of prices:

$$\begin{aligned} P_{Ht}^{1-\theta} &= \sum_{j=0}^{\infty}(1-\alpha)\alpha^j p_{ht-j}(x)^{1-\theta} \\ \hat{p}_{Ht} &= \sum_{j=0}^{\infty}(1-\alpha)\alpha^j\big[\hat{p}_{ht-j}(x) - \sum_{k=0}^{j-1}\hat{\pi}_{t-k}\big], \end{aligned} \tag{46}$$

combining this with $\hat{Y}_{Ht}$, we get:

$$\hat{y}_{Ht} = \phi_H\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}_{Ht} + \frac{1-n}{n}\phi^*_H\Big(\frac{\bar{C}}{\bar{y}}\Big)\hat{c}^*_{Ht} + \hat{g}_{Ht}, \tag{47}$$

$$\hat{y}_{Ft} = \phi_F\frac{n}{1-n}\Big(\frac{\bar{C}}{\bar{Y}}\Big)c^*_{Ft} + \phi^*_F\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}^*_{Ft} + \hat{g}_{Ht}. \tag{48}$$

Use the demand equations, plug them into the last two to eliminate $\hat{c}_{Ht}$, $\hat{c}^*_{Ht}$, $\hat{c}_{Ft}$, and $\hat{c}^*_{Ft}$:

$$\begin{aligned}\hat{y}_{Ht} =&\phi_H\Big(\frac{\bar{C}}{\bar{Y}}\Big)(\hat{c}_t - \eta\hat{p}_{Ht}) + \frac{1-n}{n}\phi^*_H\Big(\frac{\bar{C}}{\bar{Y}}\Big)\big[\hat{c}^*_t - \eta(\hat{p}_{Ht} - \hat{q}_t)\big] + \hat{g}_{Ht}\times\\ &\times \phi_H\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}_t + \frac{1-n}{n}\phi^*_H\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}^*_t - \eta\frac{\bar{C}}{\bar{Y}}\big[\phi_H + \frac{1-n}{n}\phi^*_H\big]\hat{p}_{Ht}+\\ &+\eta\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{q}_t\phi^*_H\frac{1-n}{n} + \hat{g}_{Ht},\end{aligned} \tag{49}$$

and similarly for the foreign region:

$$\begin{aligned}\hat{y}_{Ft} =&\frac{n}{1-n}\phi_F\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}_t + \phi^*_F\Big(\frac{\bar{C}}{\bar{Y}}\Big)\hat{c}^*_t - \eta\Big(\frac{\bar{C}}{\bar{Y}}\Big)\times\\ &\times\Big[\frac{n}{1-n}\phi_F + \phi^*_H\Big]\hat{p}_{Ht}\eta\Big(\frac{\bar{C}}{\bar{Y}}\Big)\phi^*_F\hat{q}_t + \hat{g}_{Ft}.\end{aligned} \tag{50}$$

Then,

$$\hat{s}_{ht,t+j}(x) = \psi_\nu\hat{y}_{Ht+j} - \psi_\nu\theta\Big[\hat{p}_{ht}(x) - \hat{p}_{Ht+j} - \sum_{k=1}^{j}\hat{\pi}_{t+k}\Big] + \frac{\hat{c}_{t+j}}{\sigma} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_{t+j}$$

and the same for the foreign state:

$$\hat{s}_{ft,t+j}(x) = \psi_\nu\hat{y}_{Ft+j} - \psi_\nu\theta\Big[\hat{p}_{ft}(x) - \hat{p}_{Ft+j} - \sum_{k=1}^{j}\hat{\pi}_{t+k}\Big] + \frac{\hat{c}_{t+j}}{\sigma} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_{t+j}.$$

A linear approximation of the contemporaneous inflation equation leads us to the following expressions:

$$\hat{\pi}_{Ht} = \frac{1-\alpha}{\alpha}\big[\hat{p}_{ht}(x) - \hat{p}_{Ht}\big], \tag{51}$$

and

$$\hat{\pi}_{Ft} = \frac{1-\alpha}{\alpha}\big[\hat{p}_{ft}(x) - \hat{p}_{Ft}\big]. \tag{52}$$

In ordered to approximate the Calvo pricing equation, we proceed as follows:

$$p_{ht}(z) = \frac{\theta}{1-\theta}\mathbb{E}_0 \frac{\sum_{j=0}^{\infty} \alpha^j M_{t,t+j} y_{ht+j}(z)}{\sum_{k=0}^{\infty} \alpha^j M_{t,t+k} y_{ht+k}(z)};$$

$$\log p_{ht}(z) = \log\left[\frac{\theta}{1-\theta}\right] + j\log\alpha + \log \mathbb{E}_t M_{t,t+j} + \log \mathbb{E}_t y_{ht+j} - k\log\alpha - \log \mathbb{E}_t M_{t,t+k} - \log \mathbb{E}_t y_{ht+k};$$

$$\log \bar{p}_h(z) + \frac{1}{\bar{p_h}}(p_{ht}(z) - \bar{p}_h(z)) = \log\left[\frac{\theta}{1-\theta}\right] + j\log\alpha + \log\bar{M} + \frac{1}{\bar{M}}(\mathbb{E}_t M_{t,t+j} + \bar{M}) + \log\bar{y_h} +$$

$$+ \frac{1}{\bar{y}_h}(\mathbb{E}_t y_{ht+j} - \bar{y}) - k\log\alpha - \log\bar{M} - \frac{1}{\bar{M}}(\mathbb{E}_t M_{t,t+k} - \bar{M}) - \log\bar{y}_h - \frac{1}{\bar{y}_h}(\mathbb{E}_t y_{h,t+k} - \bar{y});$$

$$\hat{p}_{ht}(z) = (j-k)\log\alpha + \mathbb{E}_t\hat{M}_{t,t+j} + \mathbb{E}_t\hat{y}_{t+j} - \mathbb{E}_t\hat{M}_{t,t+k} - \mathbb{E}_t\hat{y}_{t+k};$$

$$\hat{p}_{ht}(x) = (1-\alpha\beta)\sum_{j=0}^{\infty}(\alpha\beta)^j\mathbb{E}_t\hat{s}_{ht,t+j}(x) + \alpha\beta\sum_{j=1}^{\infty}(\alpha\beta)^j\mathbb{E}_t\hat{\pi}_{t+j};$$

$$\hat{p}_{ft}(x) = (1-\alpha\beta)\sum_{j=0}^{\infty}(\alpha\beta)^j\mathbb{E}_t\hat{s}_{ft,t+j}(x) + \alpha\beta\sum_{j=1}^{\infty}(\alpha\beta)^j\mathbb{E}_t\hat{\pi}_{t+j}.$$

Plugging in these last two equations the meaning of $\hat{s}_{h/ft,t+j}$ yields:

$$\hat{p}_{ht}(x) = (1-\alpha\beta)\zeta\sum_{j=0}^{\infty}(\alpha\beta)^j\mathbb{E}_t\big[\psi_\nu\hat{y}_{Ht} + \theta\psi_\nu\hat{p}_{Ht+j} + \frac{\hat{c_{t+j}}}{\sigma} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_{t+j}\big] +$$

$$+ \alpha\beta\sum_{j=0}^{\infty}(\alpha\beta)^j\mathbb{E}_t\pi_{t+j}, \tag{53}$$

$$\hat{p}_{tf}(x) = (1-\alpha\beta)\zeta\sum_{j=0}^{\infty}(\alpha\beta)^j\mathbb{E}_t\big[\psi_\nu\hat{y}_{Ft} + \theta\psi_\nu\hat{p}_{Ft+j} + \frac{\hat{c_{t+j}}}{\sigma} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_{t+j}\big] +$$

$$+ \alpha\beta\sum_{j=0}^{\infty}(\alpha\beta)^j\mathbb{E}_t\pi_{t+j}, \tag{54}$$

where $\zeta = \frac{1}{1+\psi_\nu\theta}$.
Quasi-differencing the previous expressions yields:

$$\hat{p}_{ht}(x) - \alpha\beta\mathbb{E}_t\hat{p}_{ht+1}(x) = (1-\alpha\beta)\zeta\Big[\psi_\nu\hat{y}_{Ht} + \theta\psi_\nu\hat{p}_{Ht} + \frac{1}{\sigma}\hat{c}_t + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t\Big] + \alpha\beta\mathbb{E}_t\hat{\pi}_{t+1},$$

$$\hat{p}_{ft}(x) - \alpha\beta\mathbb{E}_t\hat{p}_{ft+1}(x) = (1-\alpha\beta)\zeta\Big[\hat{q}_t + \psi_\nu\hat{y}_{Ft} + \theta\psi_\nu\hat{p}_{Ft} + \frac{1}{\sigma}\hat{c}_t^* + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t\Big] + \alpha\beta\mathbb{E}_t\hat{\pi}_{t+1}.$$

Now, we seek to eliminate the terms $\hat{p}_{ht}(x)$ and $\hat{p}_{ft}(x)$ from the last two equations by plugging in equations (51) and (52):

$$\pi_{Ht} - \alpha\beta\mathbb{E}_t\pi_{Ht+1} + \frac{1-\alpha}{\alpha}(\hat{p}_{Ht} - \alpha\beta\mathbb{E}_t\hat{p}_{Ht+1}) =$$
$$= \kappa\zeta\big[\psi_\nu\hat{y}_{Ht} + \theta\psi_\nu\hat{p}_{Ht} + \frac{\hat{c}_t}{\sigma} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t\big]+$$
$$+ (1-\alpha)\beta\mathbb{E}_t\hat{\pi}_{t+1}; \tag{55}$$

and

$$\pi_{Ft} - \alpha\beta\mathbb{E}_t\pi_{Ft+1} + \frac{1-\alpha}{\alpha}(\hat{p}_{Ft} - \alpha\beta\mathbb{E}_t\hat{p}_{Ft+1}) =$$
$$= \kappa\zeta\big[\psi_\nu\hat{y}_{Ft} + \theta\psi_\nu\hat{p}_{Ft} + \frac{\hat{c}_t}{\sigma} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t\big]+$$
$$+ (1-\alpha)\beta\mathbb{E}_t\hat{\pi}_{t+1}, \tag{56}$$

where $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}$.
Note that:

$$\hat{p}_{Ht+1} - \hat{H}t = \hat{\pi}_{Ht+1} - \hat{\pi}_{t+1} \Rightarrow$$
$$\hat{p}_{Ht+1} - \alpha\beta\hat{p}_{Ht+1} = (1-\alpha\beta)\hat{p}_{Ht} -$$
$$- \alpha\beta\mathbb{E}_t\hat{\pi}_{Ht+1} + \alpha\beta\mathbb{E}_t\hat{\pi}_{t+1}.$$

After a having plugged them in equations (55) and (56) yields,

$$\hat{\pi}_{Ht} = \beta\mathbb{E}_t\hat{\pi}_{Ht+1} + \kappa\zeta\psi_\nu\hat{y}_{Ht} + \kappa\zeta\hat{p}_{Ht} + \kappa\zeta\frac{\hat{c}_t}{\sigma} + \kappa\zeta\frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t;$$
$$\hat{\pi}_{Ft} = \beta\mathbb{E}_t\hat{\pi}_{Ft+1} + \kappa\zeta\psi_\nu\hat{y}_{Ft} + \kappa\zeta\hat{p}_{Ft} + \kappa\zeta\frac{\hat{c}_t}{\sigma} + \kappa\zeta\frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t.$$

These equations affirm that the present value of inflation today in H is explained by the expected value of inflation tomorrow in the same region, by the output growth of the region F today and the level of prices in H, as well as the overall level of consumption in the union and the labour income tax rate in the current period.
Now, take the overall level of prices in steady state,

$$P_t = \big[\phi_H P_{Ht}^{1-\eta} + \phi_F P_{Ft}^{1-\eta}\big]^{\frac{1}{1-\eta}} \Leftrightarrow$$
$$P_t^{1-\eta} = \big[\phi_H P_{Ht}^{1-\eta} + \phi_F P_{Ft}^{1-\eta}\big] \Rightarrow$$
$$\bar{P} = \phi_H\bar{P}_H + \phi_F\bar{P}_F,$$

and

$$\phi_H \hat{P}_{Ht} + \phi_F \hat{P}_{Ft} = 0 \Leftrightarrow$$
$$\hat{p}_{Ft} = -\frac{\phi_H}{\phi_F}\hat{p}_{Ht}.$$

Total inflation in the H region is equal to

$$\phi_H \hat{\pi}_{Ht} + \phi_F \hat{\pi}_{Ft} = \hat{\pi}_t, \tag{57}$$

and in the F region:

$$\phi_H^* \hat{p}_{Ht} + \phi_F^* \hat{p}_{Ft} = \hat{q}_t. \tag{58}$$

Finally,

$$\phi_H^* \hat{\pi}_{Ht} + \phi_F^* \hat{\pi}_{Ft} = \hat{\pi}_t^*. \tag{59}$$

The difference between home and foreign regions, in terms of differentials on real inflation is a function of the real exchange rate $\hat{q}_t$ obtained via the "Backus - Smith" condition.

6.2 GHH utility

The Greenwood, Hercowitz, and Huffman (1988)'s utility function looks like:

$$U(C_t, L_t(x)) = \left\{ \frac{C_t - \chi L_t(x)^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right\}^{1-\frac{1}{\sigma}} \Big/ \Big(1 - \frac{1}{\sigma}\Big).$$

Recall that ν stands for the Frisch labour supply elasticity, and the optimality conditions are:

$$u_l = -\chi L_t^{\frac{1}{\nu}} \{C_t - \chi L_t^{1+\frac{1}{\nu}}/(1+\frac{1}{\nu})\}^{-\frac{1}{\sigma}}$$
$$u_c = \{C_t - \chi L_t^{1+\frac{1}{\nu}}/(1+\frac{1}{\nu})\}$$
$$\frac{u_l}{u_c} = \frac{\frac{\partial U}{\partial L_t}}{\frac{\partial U}{\partial C_t}} = \frac{W_t(1-\tau_t)}{P_t} = \chi L_t^{\frac{1}{\nu}}$$

where the third equation is obtained by dividing the previous two, namely the f.o.c.s for labour and consumption, respectively. Log-linearizing the former equation around the steady state goes as follows:

$$
\begin{aligned}
&\frac{S_t(x)}{P_t} = \frac{1}{1-\tau_t}\frac{\chi}{a}y_{ht}^{\psi_\nu}(x) \Leftrightarrow \\
&\log S_t(x) - \log P_t = \log\frac{1}{1-\tau_t} + \log\frac{\chi}{a} + \psi_\nu y_{ht}(x) \Leftrightarrow \\
&\log \bar{S} - \log \bar{P} + \frac{1}{\bar{S}}(S_t - \bar{S}) - \frac{1}{\bar{P}}(P_t - \bar{P}) = \log\frac{1}{1-\bar{\tau}} + \\
&+ \frac{1}{1-\bar{\tau}}(\tau_t - \bar{\tau}) + \log\frac{\chi}{a} + \psi_\nu \log \bar{y_h} + \frac{1}{\bar{y_h}}(y_{ht} - \bar{y}_h) \Leftrightarrow \\
&\hat{s}_t - \hat{p}_t = \frac{1}{1-\bar{\tau}}\hat{\tau}_t + \hat{y}_{ht}.
\end{aligned} \tag{60}
$$

Then, according to Nakamura and Steinsson,

$$
\hat{s}_{ht,t+j} = \psi_\nu \hat{y}_{ht,t+j} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t, \tag{61}
$$

and from this equation we can derive the Phillips curves for H and F,

$$
\hat{\pi}_{Ht} = \beta\mathbb{E}_t\hat{\pi}_{Ht+1} + \kappa\zeta\psi_\nu\hat{y}_{Ht} - \kappa\zeta\hat{p}_{Ht} + \kappa\zeta\frac{1}{1-\bar{\tau}}\hat{\tau}_t, \tag{62}
$$

$$
\hat{\pi}_{Ft} = \beta\mathbb{E}_t\hat{\pi}_{Ft+1} + \kappa\zeta\psi_\nu\hat{y}_{Ft} - \kappa\zeta\hat{p}_{Ft} + \kappa\zeta\frac{1}{1-\bar{\tau}}\hat{\tau}_t. \tag{63}
$$

To approximate the Euler Equation for consumption in the GHH model we proceed as follows:

$$
\begin{aligned}
&\mathbb{E}_t\Big[\beta\frac{u_c(C_{t+1}, L_{t+1}(x))}{u_c(C_t, L_t(x))}\frac{P_t}{P_{t+1}}\Big] = \frac{1}{1+r_t^n} \Leftrightarrow \\
&\left\{C_{t+1} - \frac{\chi L_{t+1}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\}^{\frac{1}{\sigma}}\right\}\left\{C_t - \frac{\chi L_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right\}^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}} = \frac{1}{1+r_t^n} \Leftrightarrow \\
&u_{cc} = \frac{u_{cc}\bar{C}}{u_c}\hat{c}_t + \frac{u_{cc}\bar{L}}{u_c}\hat{L}_t(x)
\end{aligned}
$$

where:

$$u_{cc} = -\frac{1}{\sigma}\left\{\bar{C} - \chi\frac{\bar{L}^{1+\nu^{-1}}}{1+\nu^{-1}}\right\}^{-(1+\frac{1}{\sigma})},$$

$$u_{cl} = \frac{1}{\sigma}\left\{\bar{C} - \chi\frac{\bar{L}^{-1+\nu^{-1}}}{1+\nu^{-1}}\right\}^{-(1+\frac{1}{\sigma})}\chi\bar{L}^{-\nu^{-1}},$$

$$\frac{u_{cl}\bar{C}}{u_c} = -\frac{1}{\sigma}\bar{C}\left\{\bar{C} - \chi\frac{\bar{L}^{1+\nu^{-1}}}{1+\nu^{-1}}\right\}^{-1} = -\frac{1}{\sigma}\left\{1 - \frac{a}{\mu}\frac{\bar{Y}}{\bar{C}}\frac{1}{1+\nu^{-1}}\right\}^{-1}$$

$$\frac{u_{cc}\bar{L}}{u_c} = \frac{1}{\sigma}\left\{\bar{C} - \chi\frac{\bar{L}^{1+\nu^{-1}}}{1+\nu^{-1}}\right\}^{-1} = -\frac{u_{cc}\bar{C}}{u_c}\frac{\bar{L}}{\bar{C}}\chi\bar{L}^{\nu^{-1}} = -\frac{u_{cc}\bar{C}}{u_c}\frac{\bar{Y}}{\bar{C}}\frac{a}{\mu}.$$

Let $\sigma_c^{-1} = u_{cc}\bar{C}/u_c$ and $\sigma_l^{-1} = u_{cl}\bar{L}/u_l$. Therefore, the Euler equation log-linearized is:

$$\hat{c}_t - \sigma_c\sigma_l^{-1}\hat{l}_t = \mathbb{E}_t\hat{c}_{t+1} - \sigma_c\sigma_l^{-1}\mathbb{E}_t\hat{l}_{t+1} - \sigma_c(\hat{r}_t^n - \mathbb{E}_t\hat{\pi}_{t+1}), \tag{64}$$

nevertheless we may rewrite it as

$$\hat{c}_t - \xi_y\hat{y}_{Ht} = \mathbb{E}_t\hat{c}_{t+1} - \xi_y\mathbb{E}_t\hat{y}_{Ht+1} - \sigma_c(\hat{r}_n^t - \mathbb{E}_t\hat{\pi}_{t+1}), \tag{65}$$

where $\xi_y = \frac{\sigma_c}{\sigma_l} = \frac{\bar{Y}}{\bar{C}}\frac{a}{\mu}$, a constant value and we can do this because the log-linearized production function is $\hat{y}_{Ht} = a\hat{l}_t$.
The Backus - Smith condition within the model with the non separable type of preferences becomes:

$$\frac{u_c^*}{u_c} = Q_t$$

which linearly approximated looks like:

$$\hat{c}_t = -\xi_y\hat{y}_t - \hat{c}_t^* + \xi_y\hat{y}_t^* = \sigma_c\hat{q}_t. \tag{66}$$

6.3 Regional capital markets

As in Chirstiano, Eichenbaum, and Evans (2005), $\bar{K}_t$ denotes the overall households' capital stock; $\bar{I}_t$ features the investment undertaken by themselves, obtained by lending their capital wealth to firms which use it for production of final goods.

6.3.1 Households

$$\bar{K}_{t+1} = (1-\delta)\bar{K}_t + \Phi(I_t, I_{t+1}) \tag{67}$$

is the law of motion of capital, where δ is the rate of capital depreciation and Φ is a function specified as:

$$\Phi = \Big[1 - \phi\Big(\frac{I_t}{I_{t-1}}\Big)\Big], \tag{68}$$

so that $K_t = u_t\bar{K}_t$, where u_t indicates the capital utilization rate chosen by the household.
Her budget constraint thus becomes:

$$P_tC_t + P_tI_t + P_tA(u_t)\bar{K}_t + \mathbb{E}_t[M_{t,t+1B_{t+1}(x)}] \leq B_t(x) + (1-\tau_t)W_t(x)L_t(x)+$$
$$+ R_t^k u_t + \int_0^1 \Xi_{ht}(x)dz - T_t, \tag{69}$$

recall that the second term on the left hand side denotes the amount of investment spent by households themselves, A is the cost of using the capital stock, $R_t^k u_t\bar{K}_t$ is the rental income obtained by household when they lend capital to firms. The equilibrium equations are the same as in the baseline model, plus the usual transversality condition and the following new optimality conditions:

$$\Lambda = \sum_{t=0}^{\infty}\Big\{U(C_t, L_t(x)) + \gamma\Big[(1-\delta)\bar{K}_t \;\; + I_t - \phi\Big(\frac{I_t}{I_{t-1}}\Big)I_t - \bar{K}_{t+1}\Big]\Big\},$$
$$\frac{\partial\Lambda}{\partial C_t} = \Leftrightarrow \gamma_t\Phi(I_t, I_{t-1}) + \beta E_t[\gamma_{t+1}\Phi_2(I_{t+1}, I_t)] = u_c(C_t, L_t(x)),$$
$$\frac{\partial\Lambda}{\partial L_t(x)} = 0 \Leftrightarrow \beta(1-\delta)\mathbb{E}_t\gamma_{t+1} + \beta\mathbb{E}_t[(A'u_{t+1})u_{t+1} - A(u_{t+1})u_c(C_{t+1}, L_{t+1})],$$

where $A'(u_t) = \frac{R_t^k}{P_t}$.

6.3.2 Firms

$$y_t(x) = f(L_t(x), K_t(x))$$

is their production function, which, now, depends also on capital stock borrowed from households. Demand for goods x produced by firms in the H region is simply the sum of the goods demanded by H households and F

households as well as the investment in both regions and government purchases:

$$\begin{aligned}
&y_t(x) = \Big\{nC_{Ht} + (1-n)C^*_{Ht} + nI_{H,t} + (1-n)I^*_{H,t} + nG_{Ht}\Big\}\Big(\frac{p_t(x)}{P_{Ht}}\Big)^{-\theta};\\
&W_t(x) = f_l(L_t(x), K_t(x))S_t(x);\\
&R^k_t = f_k(L_t(x), K_t(x))S_t(x);
\end{aligned}$$

where the latter two equations express the partial derivative of the *Cobb - Douglas* with respect to labour and capital respectively. Combining the equations for wage and rate of return on capital yields:

$$\frac{S_t(x)}{P_t} = \Big\{\frac{1}{1-\tau_t}\Big\}\Big\{\frac{v_l(L_t(x))}{u_c(C_t)f_l(f^{-1}(L_t, K_t))}\Big\}. \tag{70}$$

$\phi(1) = \phi'(1) = 0$; $\kappa_I = \phi''(1) = 2.5 > 0$; $\bar{u} = 1$; $A_1 = 0$; $\sigma_a = \frac{A''(1)}{A'(1)} = 0.01$; $1 - a = \frac{1}{3}$, $\alpha = \frac{2}{3}$, as in Christiano et al., 2005. Let's start to linearly approximate the equilibrium equations of this specification of the model:

$$\begin{aligned}
&\log S_t(x) - \log P_t = -\log(1-\tau_t) + \log v_l(L) + \log u_c(C_t) - \log f_c(f^{-1}(C_l)) \Leftrightarrow\\
&\log \bar{S}(x) + \frac{S_t - \bar{S}}{\bar{S}} - \log \bar{P} - \frac{P_t - \bar{P}}{\bar{P}} = -\log(1-\bar{\tau}) - \frac{1-\tau_t - 1 + \bar{\tau}}{1-\bar{\tau}} + \log v_l -\\
&\frac{v_l(L_t) - v_l(\bar{L})}{v_l(\bar{L})} - \log u_c(\bar{C}) - \frac{u_c(C_t) - u_c(\bar{C})}{u_c(\bar{C})} - \log f_l(f^{-1}(\bar{C}, \bar{L})) - \frac{f_l(f^{-1}) - f_l(\bar{f^{-1}})}{f_l(\bar{f^{-1}})} \Leftrightarrow\\
&\hat{s}_t - \hat{p}_t = -\frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t + v_l(\hat{L}_t) - u_c(\hat{C}_t) - f_l(f^{-1}(\hat{C}_t, \hat{L}_t)).
\end{aligned}$$

Let us split this last equation into the share of H and F, respectively:

$$\hat{s}_t(x) = \Big(\frac{v_{ll}\hat{L}}{v_l} - \frac{f_{ll}\bar{L}}{f_l}\Big)\hat{l}_t - \frac{f_{lk}\bar{K}}{f_l}\hat{k}_t - \frac{u_{cc}\bar{C}}{u_c}\hat{c}_t + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t; \tag{71}$$

$$\hat{s}^*_t(x) = \Big(\frac{v_{ll}\hat{L}^*}{v_l} - \frac{f_{ll}\bar{L}^*}{f_l}\Big)\hat{l}^*_t - \frac{f_{lk}\bar{K}^*}{f_l}\hat{k}^*_t - \frac{u_{cc}\bar{C}^*}{u_c}\hat{c}^*_t + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t. \tag{72}$$

Now, considering the production function, and deriving its log-linearized counterparts is straightforward,

$$\hat{y}_t = a\hat{l}_t + (1-a)\hat{k}_t(x); \Leftrightarrow \hat{y}^*_t = a\hat{l}^*_t - +(1-a)\hat{k}^*_t(x), \tag{73}$$

again with the usual distinction between home and foreign region. Making $\hat{k}_t$ explicit from the previous formula leads to:

$$\hat{k} = \frac{\hat{y}_t - a\hat{l}_t}{1-a}. \tag{74}$$

Furthermore,

$$\begin{aligned}
&R_t^k = L_t^a(x)(1-a)K_t^{-a}(x) - a\log K_t + \log S_t;\\
&\log R_t^k = a\log L_t + \log(1-a) - a\log K_t + \log S_t;\\
&\log \bar{R}^k + \frac{R_t^k - \bar{R}^k}{\bar{R}^k} = a\log \bar{L} + \frac{aL_t - a\bar{L}}{a\bar{L}} + \log(1-a) - a\log \bar{K} -\\
&\frac{K_t - \bar{K}}{\bar{K}} + \log \bar{S} - \frac{S_t - \bar{S}}{\bar{S}};\\
&\hat{r}_t^k = a\hat{l}_t - a\hat{k}_t + \hat{s}_t;
\end{aligned} \tag{75}$$

and, complementarily,

$$\hat{r}_t^{k*} = a\hat{l}_t^* - a\hat{k}_t^* + \hat{s}_t^*. \tag{76}$$

Afterwards,

$$\begin{aligned}
&\hat{k}_t = \hat{l}_t - \frac{\hat{t}_t^k}{a} + \frac{\hat{s}_t}{a} = \frac{\hat{y}_t - a\hat{l}_t}{1-a},\\
&(1-a)\hat{l}_t - \frac{\hat{r}_t^k(1-a)}{a} + \frac{\hat{s}_t(1-a)}{a} = \hat{y}_t,\\
&\hat{l}_t = \hat{y}_t + \hat{r}_t^k\frac{1-a}{a} - \hat{s}_t\frac{1-a}{a},
\end{aligned} \tag{77}$$

which finally denotes the labour demand of firms.
Working a little more on the labour demand equation, one obtains at last:

$$\begin{aligned}
&\hat{l}_t = \frac{\hat{y}_t}{a} - \frac{1-a}{a}\hat{k}_t;\\
&\hat{l}_t = \frac{\hat{r}_t^k}{a} - \hat{k}_t - \frac{\hat{s}_t}{a};\\
&\frac{\hat{y}_t}{a} - \frac{1-a}{a}\hat{k}_t = \frac{\hat{r}_t^k}{a} + \hat{k}_t - \frac{\hat{s}_t}{a};\\
&\hat{y}_t - (1-a)\hat{k}_t = \hat{r}_t^k + a\hat{k}_t - \hat{s}_t;\\
&\hat{y} - \hat{k}_t - a\hat{k}_t = \hat{r}_t^k + a\hat{k}_t - \hat{s}_t;\\
&\hat{k}_t = \hat{y}_t - \hat{r}_t^k + \hat{s}_t.
\end{aligned} \tag{78}$$

Now, plug equation (77) in the labour supply equation, and get:

$$\begin{aligned}
&\hat{s}_t = \left\{\frac{v_l \bar{L}}{v_l} - \frac{f_{ll}\bar{L}}{f_l}\right\}\hat{l}_t - \frac{f_{lk}\bar{K}}{f_l}\hat{k}_t(x) - \frac{u_{cc}\bar{C}}{u_c}\hat{c}_t + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t \times \\
&\times \Big(\frac{1}{\nu} + 1 - a\Big)\left\{\hat{y}_t + \hat{r}_t^k \frac{1-a}{a} - \hat{s}_t \frac{1-a}{a}\right\} - (1-a)\big\{\hat{y}_t - \hat{r}_t^k - \hat{s}_t\big\} + \\
&+ \frac{1}{\sigma}\hat{c}_t + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t \Leftrightarrow \\
&\frac{\hat{y}_t}{\nu} + \frac{1-a}{a\nu}\hat{r}_t^k - \frac{1-a}{a\nu}\hat{s}_t + \hat{y}_t + \frac{1-a}{a}\hat{r}_t^k - \hat{s}_t\frac{1-a}{a} - \hat{y}_t a - \hat{r}_t^k(1-a) + \\
&+ \hat{s}_t(1-a) - \hat{y}_t + \hat{r}_t^k + \hat{s}_t + a\hat{t}_t - a\hat{r}_t^k - a\hat{s}_t + \frac{\hat{c}_t}{\sigma} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t \Leftrightarrow \\
&\hat{r}_t^k\Big[\frac{1-a}{a\nu} + \frac{1-a}{a} - (1-a)\Big] = \Big[\frac{1-a+\nu+\nu a - a\nu + \nu a^2}{a\nu}\Big]\hat{r}_t^k \Leftrightarrow \\
&\hat{s}_t\Big[-\frac{1-a}{a\nu} - \frac{1-a}{a} + (1-a)\Big] = \Big[\frac{-1+a-\nu+\nu a - \nu a + \nu a^2}{a\nu}\Big]\hat{s}_t \Leftrightarrow \\
&\hat{s}_t\frac{\nu a^2 + a - \nu - 1}{a\nu} = \hat{s}_t\Big(1 + \frac{1}{\nu} - \frac{1}{a}\Big) - \frac{1}{a\nu} = \hat{s}_t(a - \frac{1}{\nu} - \frac{1}{a}) - \frac{1}{a\nu}.
\end{aligned} \tag{79}$$

The previous equation can be written more compactly as:

$$\psi_\nu \hat{y}_t + \psi_c \sigma^{-1}\hat{c}_t + \psi_k \hat{r}_t^k + \psi_\tau \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t, \tag{80}$$

where: $\psi_\nu = \frac{a\nu^{-1}}{[(1-a)\nu^{-1}+1]}$; $\psi_c = \frac{a\sigma^{-1}}{[(1-a)\nu^{-1}+1]}$; $\psi_k = \frac{(1+\nu^{-1})(1-a)}{[(1-a)\nu^{-1}+1]}$; $\psi_\tau = \frac{a}{[(1-a)\nu^{-1}+1]}$.

Approximating the demand equations, whose procedure we here omit, yields the following expressions for both H and F's goods market,

$$\begin{aligned}
\hat{y}_{Ht} &= \phi_H \frac{\bar{C}}{\bar{Y}}\hat{c}_{Ht} + \phi_H \frac{\bar{I}}{\bar{Y}}\hat{i}_{Ht} + \phi_H^* \frac{1-n}{n}\frac{\bar{C}}{\bar{Y}}\hat{c}_{Ht}^* + \phi_H^* \frac{1-n}{n}\frac{\bar{I}}{\bar{Y}}\hat{i}_{Ft} + \hat{g}_{Ht}, \\
\hat{y}_{Ft} &= \phi_F^* \frac{\bar{C}}{\bar{Y}}\hat{c}_{Ft}^* + \phi_H^* \frac{\bar{I}}{\bar{Y}}\hat{i}_{Ft}^* + \phi_F \frac{n}{1-n}\frac{\bar{C}}{\bar{Y}}\hat{c}_{Ft} + \phi_F \frac{n}{1-n}\frac{\bar{I}}{\bar{Y}}\hat{i}_{Ft} + \hat{g}_{Ft},
\end{aligned}$$

where:

$$\hat{y}_{t,t+j} = \hat{y}_{Ht+j} - \theta\Big\{\hat{p}_t - \hat{p}_{Ht+j} - \sum_{k=1}^{j}\pi_{t+k}\Big\}, \tag{81}$$

$$\hat{y}_{t,t+j}^* = \hat{y}_{Ft+j} - \theta\Big\{\hat{p}_t - \hat{p}_{Ft+j} - \sum_{k=1}^{j}\pi_{t+k}\Big\}. \tag{82}$$

Consider now, both home and foreign investment:

$$I_H = \phi_H I; I_F = \phi_F I; I_H^* = \phi_H^* I^*; H_F^* = \phi_F^* I^*.$$

Matching $\hat{s}_t(x)$ and $\hat{s}_t^*(x)$ with equations (81) and (82) yields:

$$\hat{s}_{t,t+j} = \psi_\nu \hat{y}_{Ht+j} - \psi_\nu \theta \Big\{ \hat{p}_t - \hat{p}_{Ht+j} - \sum_{k=1}^{j} \pi_{t+k} \Big\} + \frac{1}{\sigma} \psi_c \hat{c}_{t+j} + \frac{\bar{\tau}}{1-\bar{\tau}} \hat{\tau}_{t+j}; \tag{83}$$

$$\hat{s}_t^*(x) = \psi_\nu \hat{y}_{Ft+j} - \psi_\nu \theta \Big\{ \hat{p}_t - p_F\hat{t+j} - \sum_{k=1}^{j} \pi_{t+k} \Big\} + \frac{1}{\sigma} \psi_c \hat{c}_{t+j}^* + \frac{\bar{\tau}}{1-\bar{\tau}} \hat{\tau}_{t+j}. \tag{84}$$

We then obtain the Phillips curves:

$$\pi_{Ht} = \beta \mathbb{E}_t \pi_{Ht+1} + \kappa\zeta \Big\{ \psi_{nu} \hat{y}_{Ht} - \hat{p}_{Ht} + \frac{\psi_\nu}{\sigma} \hat{c}_t + \psi_\nu \frac{\bar{\tau}}{1-\bar{\tau}} \hat{\tau}_t + \psi_k \hat{r}_t^k \Big\}; \tag{85}$$

$$\pi_{Ft} = \beta \mathbb{E}_t \pi_{Ft+1} + \kappa\zeta \Big\{ \psi_{nu} \hat{y}_{Ft} - \hat{p}_{Ft} + \frac{\psi_\nu}{\sigma} \hat{c}_t + \psi_\nu \frac{\bar{\tau}}{1-\bar{\tau}} \hat{\tau}_t + \psi_k \hat{r}_t^k \Big\}. \tag{86}$$

Now, we approximate the previously shown equations for investment:

$$\log I_{Ht} = \log \phi_H + \log I_t;$$

$$\log \bar{I} + \frac{I_{Ht} - \bar{I}_H}{\bar{I}_H} = \log \phi_H + \log \bar{I} + \frac{I_t - \bar{I}}{\bar{I}};$$

$$\hat{i}_{Ht} = \frac{1}{\phi_H} \hat{i}_t \Leftrightarrow \hat{i}_t = \phi_H \hat{i}_{Ht} \tag{87}$$

$$\hat{i}_t = \phi_H \hat{i}_{Ht} + \phi_F \hat{I}_{Ft}; \hat{i}_t^* = \phi_H^* \hat{i}_{Ht}^* + \phi_F^* \hat{i}_{Ft}^*, \tag{88}$$

because $I_t = \phi_H I_{Ht} + \phi_F \hat{I}_{Ft}$ and $I_t^* = \phi_H I_{Ht}^* + \phi_F I_{Ft}^*$.

Then,

$$\hat{i}_{Ht} = \hat{i}_t - \eta \hat{p}_{Ht}; \hat{i}_{Ft} = \hat{i}_t - \eta \hat{p}_{Ft}, \tag{89}$$

$$\hat{i}_{Ht}^* = \hat{i}_t^* - \eta(\hat{p}_{Ht} - \hat{q}_t), \hat{i}_{Ft}^* = \hat{i}_t^* - \eta(\hat{p}_{Ft} - \hat{q}_t). \tag{90}$$

Proceeding forwards,

$$\hat{y}_{Ht} = \frac{\bar{C}}{\bar{Y}}\phi_H\hat{c}_t + \frac{1-n}{n}\frac{\bar{C}}{\bar{Y}}\phi^*_H\hat{c}^*_t + \frac{\bar{I}}{\bar{Y}}\phi_H\hat{i}_t + \frac{1-n}{n}\phi^*_H\hat{i}^*_t - \eta\frac{\bar{C}+\bar{I}}{\bar{Y}}\Big\{\phi_H + \frac{1-n}{n}\phi^*_H\Big\}\hat{p}_{Ht}$$
$$+ \eta\frac{\bar{C}+\bar{I}}{\bar{Y}}\frac{1-n}{n}\phi^*_H\hat{q}_t + \hat{g}_{Ht}, \tag{91}$$
$$\hat{y}_{Ft} = \frac{\bar{C}}{\bar{Y}}\phi^*_H\hat{c}^*_t + \frac{n}{1-n}\frac{\bar{C}}{\bar{Y}}\phi_F\hat{c}_t + \frac{\bar{I}}{\bar{Y}}\phi^*_F\hat{i}^*_t + \frac{n}{1-n}\frac{\bar{I}}{\bar{Y}}\phi_F\hat{i}_t - \eta\frac{\bar{C}+\bar{I}}{\bar{Y}}\Big\{\frac{n}{1-n}\phi_F + \phi^*_F\Big\}\hat{p}_{Ht}$$
$$+ \eta\frac{\bar{C}+\bar{I}}{\bar{Y}}\phi^*_F\hat{q}_t + \hat{g}_{Ft}. \tag{92}$$

In addition to that,

$$\begin{aligned}
K_t &= u_t\bar{K}_t \\
\log K_t &= \log u_t + \log \bar{K}_t \\
\frac{K_t - \bar{K}}{\bar{K}} &= \frac{u_t - \bar{u}}{\bar{u}} + \frac{\bar{K}_t - \bar{K}}{\bar{K}} \\
\hat{k}_t &= \hat{u}_t + \hat{\bar{k}}_t
\end{aligned} \tag{93}$$

at home. Analogously

$$\hat{k}^*_t = \hat{u}^*_t + \hat{\bar{k}}_t \tag{94}$$

holds abroad.

$$\begin{aligned}
u_t &= \frac{R^*_t}{P_t} \\
A'(\bar{u}) &= \frac{\bar{R}^k}{\bar{P}} \\
\log A'(\bar{u}_t) &= \log R^k_t - \log P_t \\
\log A'(\bar{u}) + \frac{\bar{u}A''(\bar{u})}{A'(\bar{u})}\frac{du_t}{\bar{u}} &= \log \bar{R}^k_t + \frac{dR^k_t}{\bar{R}^k} - \log\bar{P} - \frac{dP_t}{\bar{P}},
\end{aligned}$$

where $\frac{A''(\bar{u})}{A'(\bar{u})} = \sigma_a$.
Furthermore:

$$D_t\Phi_1(I_t, I_{t-1}) + \beta\mathbb{E}_t\big[D_{t+1}\Phi_2(I_{t+1}, I_t)\big] = u_c(C_t, L_t(x))$$
$$\log D_t + \log\Phi'(I_t, I_{t+1}) + \log\beta + \log\mathbb{E}_t D_{t+1} + \log I_{t+1} + \log\Big\{I_{t+1}\phi\big(\frac{I_{t+1}}{I_t}\big)\Big\} = -\log\frac{1}{c^\sigma_t}$$
$$\log D_t + \log I_t - \log\Big\{I_t\phi\big(\frac{I_t}{I_{t-1}}\big)\Big\} + \log\beta + \log\mathbb{E}_t D_{t+1} + \log\mathbb{E}_t I_{t+1} - \log\mathbb{E}_t\Big\{I_{t+1}\phi\mathbb{E}_t\big(\frac{I_{t+1}}{I_t}\big)\Big\} = \log\frac{1}{c^\sigma_t}, \tag{95}$$

where D_t is a Lagrangian multiplier associated with the maximization problem of the household who owns capital and has to decide ho much to invest - e.g. how much capital to keep for financing her own consumption and how much to lend to firms, knowing the unitary rate of return on the loan is u_t. Further approximation leads to:

$$\log\bar{D}+\frac{dD_t}{\bar{D}}+\log\bar{I}+\frac{dI_t}{\bar{I}}-\log\bar{I}-\frac{dI_t}{\bar{I}}-\log\phi'\{\frac{\bar{I}}{\bar{I}}\}-\phi''\{\frac{\bar{I}}{\bar{I}}\}\frac{dI_{t+1}}{\bar{I}}-\log\phi'\{\frac{\bar{I}}{\bar{I}}\}-$$
$$-\log\phi''\{\frac{\bar{I}}{\bar{I}}\}\frac{dI_{t-1}}{\bar{I}}+\log\beta+\log\bar{D}+\mathbb{E}_t\frac{dD_{t+1}}{\bar{D}}-\log\phi'\{\frac{\bar{I}}{\bar{I}}\}-\log\phi''\{\frac{\bar{I}}{\bar{I}}\}\frac{dI_{t+1}}{\bar{I}}-$$
$$-\log\phi'\{\frac{\bar{I}}{\bar{I}}\}-\log\phi''\{\frac{\bar{I}}{\bar{I}}\}\frac{dI_t}{\bar{I}}=-\log\frac{1}{c_t^{\sigma}}-\log\frac{\bar{c}^{-\sigma-1}}{\sigma\bar{c}^{\sigma}}dc_t^{\sigma}\Leftrightarrow$$
$$\hat{D}_t+\phi''\big[\beta(\mathbb{E}_t\hat{I}_{t+1}-\hat{I}_t)-(\hat{I}_t-\hat{I}_{t-1})\big]+\frac{\hat{c}_t}{\sigma_c}=0, \tag{96}$$

and equivalently for the foreign region,

$$\hat{D}_t^*+\phi''\big[\beta(\mathbb{E}_t\hat{I}_{t+1}-\hat{I}_t)-(\hat{I}_t-\hat{I}_{t-1})\big]+\frac{\hat{c}_t^*}{\sigma_c}=0. \tag{97}$$

Assume $A(\bar{u})=0$, $A'(\bar{u})=\frac{\bar{R}^k}{\bar{P}}$, in steady state.
Now, let us approximate the following last expression for the regional capital market model:

$$D_t=\beta(1-\delta)\mathbb{E}_tD_{t+1}+\beta\mathbb{E}_t[(A'(u_{t+1})u_{t+1}-A(u_{t+1}))u_c(C_{t+1},L_{t+1}(x))]\Leftrightarrow$$
$$-\log D_t=\log\beta(1-\delta)+\log\mathbb{E}_tD_{t+1}+\log\beta+\log\mathbb{E}_tA'(u_{t+1})-$$
$$-\log\mathbb{E}_tA(u_{t+1})+\log u_c(C_t,L_t(x))\approx$$
$$\approx\log\bar{D}+\frac{dD_t}{\bar{D}}=\log\beta(1-\delta)+\log\bar{D}+\frac{dD_{t+1}}{\bar{D}}+\log\beta+\log A'(\bar{u})+$$
$$+\mathbb{E}_t\frac{A''(\bar{u})}{A'(\bar{u})}du_{t+1}-\log A(\bar{u})-\mathbb{E}_t\frac{A'(\bar{u})}{A(\bar{u})}du_{t+1}+\log u_c(\bar{C},\bar{L})+\frac{\sigma^{-1}\bar{c}^{-\sigma-1}}{\bar{c}^{\sigma}}dc_t^{-\sigma}\Leftrightarrow$$
$$\hat{D}_t=\log\beta(1-\delta)E_t\hat{D}_{t+1}+(1-\beta(1-\delta))[\mathbb{E}_t\hat{r}_{t+1}^k-\sigma_c\mathbb{E}_t\hat{c}_{t+1}]; \tag{98}$$
$$\hat{D}_t^*=\log\beta(1-\delta)\mathbb{E}_t\hat{D}_{t+1}^*+(1-\beta(1-\delta))[\mathbb{E}_t\hat{r}_{t+1}^{k*}-\sigma_c\mathbb{E}_t\hat{c}_{t+1}^*]. \tag{99}$$

6.4 Firm specific capital

This section portrays a version of the model which is very similar in structure to that analyzed in Woodford (2005). Households' behaviour is governed by the same equations as in the baseline GHH model of section 6.2. The production function of firms in industry x has the standard *Cobb-Douglas* form:

$$y_t(x) = L_t(x)^a K_t(x)^{1-a}.$$

Demand for output of firms in industry x is:

$$y_t(x) = [nC_{Ht} + (1-n)C^*_{Ht} + nI_{Ht} + (1-n)I^*_{Ht}]\Big\{\frac{p_{ht}(x)}{P_{Ht}}\Big\}^{-\theta}.$$

Firms' optimal choice of labour is:

$$W_t(x) = f_l(L_t(x), K_t(X))S_t(x).$$

Convex adjustment costs (CAC) for investment exist[13]. A firm willing to dispose of a capital stock $K_{t+1}(x)$ at time $t{+}1$ must invest $I(K_{t+1}(x)/K_t(x))K_t(x)$ at time t. $I(1) = \delta$; $I'(1) = 1$; $I''(1) = \varepsilon_\psi$. Optimal investment is

$$I'\Big\{\frac{K_{t+1}(x)}{K_t(x)}\Big\} + \mathbb{E}_t M_{t,t+1}\frac{P_{t+1}}{P_t}\Big\{I\Big(\frac{K_{t+2}(x)}{K_{t+1}(x)}\Big) - I'\Big(\frac{K_{t+2}(x)}{K_{t+1}(x)}\Big)\frac{K_{t+2}(x)}{K_{t+1}(x)}\Big\} =$$
$$= \mathbb{E}_t M_{t,t+1}\frac{P_{t+1}}{P_t}\frac{R^k_{t+1}}{P_{H,t+1}}\frac{P_{H,t+1}}{P_{t+1}},$$

where $R^k_{t+1}(x) = f_k(L_t(x), K_t(x))S_t(x)$.

Nakamura and Steinssón set either $\delta = 0.012$ and $\varepsilon_\psi = 3$ or $\delta = 0.025$ and $\varepsilon_\psi = 2.5$. In addition to that, as in the previous specifications of the model, $a = \frac{2}{3}$, so that $1 - a = \frac{1}{3}$[14].
Furthermore, by combining labour demand and supply, one obtains

$$aL_t(x)^{a-1}K_t(x)^{1-a}(1-\tau_t)\frac{S_t(x)}{P_{Ht}}\frac{P_{Ht}}{P_t} = \chi L_t(x)^{1/\nu}.$$

[13] A literature on theory of investment adjustment costs exists in both the micro and macro settings (see, for example, Wang and Wen (2010)). As a matter of facts, there appears to be a contrasting evidence between macroeconomic necessities of modeling investment adjustment costs through convex functions, and firm-level empirical evidence showing that investment is lumpy with very little serial correlation. It has been shown in Wang and Wen (2010) that these two apparently opposed instances can be reconciled by including in the analysis financial frictions such as collateralized borrowing at the firm level.

[14] a and $1 - a$ being the *Cobb - Douglas*' production function exponents of labour and capital respectively, namely the relative weights attributed to labour and capital in the production function.

Log-linearizing the law of motion for capital:

$$
\begin{aligned}
&K_{t+1} = (1-\delta)K_t + I_t \Leftrightarrow \\
&\log K_{t+1} = \log[(1-\delta)K_t + I_t]; \Leftrightarrow \\
&\log \bar{K} + \frac{dK_{t+1}}{\bar{K}} = \log[(1-\delta)\bar{K} + \bar{I}] + \\
&+ \frac{1-\delta}{(1-\delta)\bar{K}+\bar{I}} dK_t + \frac{1}{(1-\delta)\bar{K}+\bar{I}} dI_t \Leftrightarrow \\
&\frac{dK_{t+1}}{\bar{K}} = \frac{1-\delta}{\bar{K}} dK_t + \frac{\bar{I}}{\bar{K}} \frac{dI_t}{\bar{I}} \Leftrightarrow \\
&\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \frac{\bar{I}}{\bar{K}} \hat{I}_t.
\end{aligned} \tag{100}
$$

Approximating the labour market clearing equation:

$$
\begin{aligned}
&\log a + (a-1)\log L_t + (1-a)\log K_t + \log(1-\tau_t) + \log S_t - \log P_{Ht} + \\
&+ \log \frac{P_{Ht}}{P_t} = \log \chi + \frac{1}{\nu} \log L_t \Leftrightarrow \\
&\log a + (1-a)\log \bar{L} + \frac{(a-1)\bar{L}^{-a-2}}{\bar{L}^{a-1}} dL_t + (1-a)\log \bar{K} + \\
&+ \frac{(1-a)\bar{K}^{-a}}{\bar{K}^{1-a}} dK_t + \log(1-\tau_t) + \frac{\bar{\tau}}{1-\bar{\tau}} d\tau_t + \log \bar{S} + \frac{dS_t}{\bar{S}} - \log \bar{P}_H - \\
&- \frac{dP_{Ht}}{\bar{P}_H} + \log \bar{P} + \frac{dP_t}{\bar{P}} = \log \chi + \frac{1}{\nu} \log \bar{L} + \frac{1/\nu \bar{L}^{1/\nu - 1}}{\bar{L}^{1/\nu}} dL_t \Leftrightarrow \\
&\hat{s}_t = (\frac{1}{\nu} + 1 - a)\hat{l}_t - (1-a)\hat{k}_t - \hat{p}_{Ht} + \frac{\bar{\tau}}{1-\bar{\tau}} \hat{\tau}_t.
\end{aligned} \tag{101}
$$

Recall that, in steady state $\frac{\bar{R}^k}{\bar{P}} = \delta + \frac{1}{\beta} - 1$.
Therefore:

$$
\begin{aligned}
&\log I'\Big(\frac{K_{t+1}}{K_t}\Big) + \log \mathbb{E}_t M_{t,t+1} + \log P_{t+1} - \log P_t + \log I\Big(\frac{K_{t+2}}{K_{t+1}}\Big) - \\
&- \log I'\Big(\frac{K_{t+2}}{K_{t+1}}\Big) + \log K_{t+2} - \log K_{t+1} = \log M_{t,t+1} + \log P_{t+1} - \\
&- \log P_t + \log R^k_{t+1} - \log P_{H,t+1} + \log P_{H,t+1} - \log P_{t+1} \\
&... \\
&\Leftrightarrow \\
&\hat{u}_{c,t} + \hat{\varepsilon}_\psi \Big\{\hat{k}_{t+1} - \hat{k}_t\Big\} + \mathbb{E}_t \hat{u}_{c,t+1} + \beta \varepsilon_\nu \Big\{\mathbb{E}_t \hat{k}_{t+2} - \hat{k}_{t+1}\Big\} + \\
&+ (1 - \beta(1-\delta))\Big\{\mathbb{E}_t r^k_{t+1} + \hat{p}_{Ht+1}\Big\}.
\end{aligned}
$$

Log-linearizing the production function trivially yields:

$$\hat{y}_t(x) = a\hat{l}_t(x) + (1-a)\hat{k}_t(x). \tag{102}$$

The labour demand and supply, joint in a unique expression (labour market clearing condition), look like:

$$\hat{s}_t(x) = (1/\nu + 1 - a) * \hat{l}_t(x) + (1-a)\bar{k}_t(x) + \hat{p}_{Ht} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t. \tag{103}$$

The rental rate of capital can be expressed as:

$$\hat{r}^k_t(x) = \hat{s}_t(x) + a\hat{l}_t(x) - (1-a)\hat{k}_t(x); \tag{104}$$

Combining the two previous equations yields:

$$\begin{aligned}\hat{s}_t(x) =& \frac{1/\nu + 1 - a}{a}\hat{y}_t(x) - \frac{(1-a)(1/\nu+1-a)}{a}\hat{k}_t(x) - \hat{p}_{Ht} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t;\\ \Leftrightarrow& \hat{s}_t(x) = \bar{\omega}\hat{y}_t - (\bar{\omega}-\bar{\nu})\hat{k}_{Ht} - \hat{p}_{Ht} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t \end{aligned} \tag{105}$$

where, obviously, $\bar{\omega} = (1/\nu + 1 - a)/a$ and $\bar{\nu} = 1/\nu$. From which,

$$\hat{s}_t(x) = \hat{s}_{Ht} + \bar{\omega}[\hat{y}_t(x) - \hat{y}_{Ht}] - (\bar{\omega}-\bar{\nu})[\hat{k}_t(x) - \hat{k}_{Ht}]. \tag{106}$$

Through demand curves,

$$\hat{s}_t(x) = \hat{s}_{Ht} - \bar{\omega}\theta\hat{p}_t(x) - (\bar{\omega}-\bar{\nu})\tilde{k}_t(x) \tag{107}$$

where $\tilde{k}_t = \hat{k}_t - \hat{k}_{t-1}$.

Finally, the Calvo pricing equation is:

$$\mathbb{E}_t \sum_{j=0}^{\infty}(\alpha\beta)^j \Big\{\hat{p}_t(x) - \sum_{k=1}^{\infty}\pi_{H,t+k} - \hat{s}_{t+j(x)}\Big\}, \tag{108}$$

where, following Woodford (2005), the authors use $\mathbb{E}^x_t$ to denote an expectation conditional on a state of the world at date t, but integrating only those future states in which firms in industry x have not reset their prices since period t. For aggregate variables $\mathbb{E}^x_t = \mathbb{E}_t X_t$. For firm specific variables, this is not the case. Substituting for marginal costs in the previous equations, we obtain:

$$\begin{aligned}\mathbb{E}^x_t \sum_{j=0}^{\infty}(\alpha\beta)^j \big[\hat{p}_t(x) - \sum_{k=1}^{j}\pi_{Ht+k} - \hat{s}_{Ht+j}(x) + \bar{\omega}\theta\big(\hat{p}_t(x) - \\ - \sum_{k=1}^{j}\pi_{t+k}\big) + (\bar{\omega}-\bar{\nu})\tilde{k}_{t+j}(x)\big] = 0,\end{aligned} \tag{109}$$

thus:

$$(1-\bar{\omega}\theta)\hat{p}_t(x) = (1-\alpha\beta)\mathbb{E}_t^x \sum_{j=0}^{\infty}[\hat{s}_{Ht+j} + (1+\bar{\omega}\theta)] \sum_{k=1}^{j} \pi_{Ht+k} - (\bar{\omega}-\bar{\nu})\hat{k}_{t+j}(x). \tag{110}$$

In the heterogeneous market model, $\hat{p}_t(x)$ is not independent on x. This is due to the presence of the $\tilde{k}_{t+j}(x)$ term on the right hand side. Notice also that $\mathbb{E}_t^x\tilde{k}_{t+j}(x)$ depends on $\hat{p}_t(x)$. We therefore need to be able to solve for $\hat{p}_t(x)$. Combining equations (140) and (141) we get:

$$\hat{r}_t^k(x) = \bar{\omega}\hat{y}_t(x) - (\bar{\omega}-\bar{\nu})\hat{k}_t(x) - \hat{p}_{Ht} + a\hat{l}_t(x) + a\hat{k}_t(x) + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t. \tag{111}$$

Using the production function to eliminate $\hat{l}_t(x)$ yields:

$$\hat{r}_t^k(x) = \rho_y\hat{y}_t(x) - \rho_k\hat{k}_t(x) - \hat{p}_{Ht} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t, \tag{112}$$

where $\rho_y = \bar{\omega}+1$ and $\rho_k = \rho_y - \bar{\nu}$, again as in Woodford (2005).
Thus, aggregating, we get:

$$\begin{aligned}\hat{u}_{c,t} + \varepsilon_\psi(\hat{k}_{Ht+1} - \hat{k}_t) = \mathbb{E}_t\hat{u}_{c,t+1} + \beta\varepsilon_\psi[\mathbb{E}_t\hat{k}_{t+2} - \\ - \hat{k}_{t+1}] + [1-\beta(1-\delta)][\rho_y\mathbb{E}_t\hat{y}_{Ht+1} - \rho_k\hat{k}_{Ht+1} + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t].\end{aligned} \tag{113}$$

Here, the expression differs form Woodford (2005) on the coefficient on $\mathbb{E}_t u_{c,t+1}$. This is because we are using GHH preferences. Combining this expression with its foreign counterpart yields:

$$\begin{aligned}\varepsilon_\psi[\tilde{k}_{t+1}(x) - \tilde{k}_t(x)] = \beta\varepsilon_\psi[\mathbb{E}_t\tilde{k}_{t+2} - \tilde{k}_{t+1}] + [1-\beta(1-\delta)]\times \\ \times[\rho_y\mathbb{E}_t(\hat{y}_{t+1}(x) - \hat{y}_{Ht+1}) - \rho_k\tilde{k}_{t+1}(x)].\end{aligned} \tag{114}$$

Rearranging and using the firms' demand function yields:

$$[1-\beta(1-\delta)]\rho_y\theta\varepsilon_\psi^{-1}\mathbb{E}_t\hat{p}_{t+1}(x) = \beta\mathbb{E}\tilde{k}_{t+2}(x) - [1+\beta(1-\beta(1-\delta))\rho_k\varepsilon_\psi^{-1}]\tilde{k}_{t+1}(x) - \tilde{k}_t(x),$$

which ca be rewritten as:

$$\Theta\mathbb{E}_t\hat{p}_{t+1}(x) = \mathbb{E}_t[Q(L)\tilde{k}_{t+2(x)}], \tag{115}$$

where $\Theta = [1-\beta(1-\delta)]\rho_y\theta\varepsilon_\psi^{-1}$ and $Q(L) = \beta - [1+\beta(1-\beta(1-\delta))\rho_k\varepsilon_\psi^{-1}]L + L^2$, and L is the lag operator. Notice that $Q(0) = \beta > 0$, $Q(\beta) < 0$,

$Q(1) < 0$, and $Q(n) > 0$ for $n \uparrow$. So $Q(L) = \beta(1 - \mu_1 L)(1 - \mu_2 L)$, where $\mu_1, \mu_2 \in \mathbb{R}$, and $0 < \mu_1 < 1 < 1/\beta < \mu_2$. $Q(L)$ is a lag polynomial.

The above expression cannot be solved for the expected evolution of the relative capital stock, however one may think that as long as firm z's decision problem is locally convex, so that the first-order conditions characterize a locally unique optimal plan, the optimal decision for that firm's relative price in the event that the price is reset at date t must depend only on the firm's relative capital stock at time t and on the economy's aggregate state. Thus, a log-linear approximation of i's pricing rule must take the form:

$$\hat{p}_t(x) = \hat{p}_{Ht} - \psi \tilde{k}_t(x). \tag{116}$$

The assumption that the price of home goods depends on the aggregate state of the economy, is motivated by the fact that firms reset their price at date t conditional on the realizations of a uniform distribution $U(\theta)$ drawn from the entire population of firms. This implies that the average reset price is zero: $\tilde{k}_t(x) = 0$. $\hat{p}_t$ is also the average relative price chose by firms that reset their prices at time t, and the overall rate of inflation will be given by:

$$\pi_{Ht} = \frac{1-\alpha}{\alpha}\hat{p}_{Ht}. \tag{117}$$

We may introduce the notation $\tilde{p}_t(x)$ for a a generic relative price. This contrasts with $\hat{p}_t(x)$ which is used to denote the primal price set at time t. Notice that:

$$\mathbb{E}_t\tilde{p}_{t+1}(x) = \alpha[\tilde{p}_t(x) - \mathbb{E}_t\pi_{Ht+1}] + (1-\alpha)\mathbb{E}_t\hat{p}_{t+1}(x); \tag{118}$$

using the last equation we get:

$$\begin{aligned} \mathbb{E}_t\tilde{p}_{t+1}(x) &= \alpha[\tilde{p}_t(x) - \mathbb{E}_t\pi_{Ht+1}] + (1-\alpha)\mathbb{E}_t[\hat{p}_{Ht+1}(x) - \psi\tilde{k}_{t+1}(x)] \\ &= \alpha\tilde{p}_t(x) - (1-\alpha)\psi\mathbb{E}_t\tilde{k}_{t+1}(x). \end{aligned} \tag{119}$$

Similarly, the optimal quantity of investment in any period t must depend on i's relative capital stock in that period, on its price, and on the economy's aggregate state. Taking a linear approximation yields:

$$\tilde{k}_{t+1}(x) = \lambda\tilde{k}_t(x) - \gamma\tilde{p}_t(x), \tag{120}$$

where λ and γ are coefficients to be determined.

$$\begin{aligned} \mathbb{E}_t\tilde{k}_{t+2}(x) &= \lambda\tilde{k}_{t+1}(x) - \bar{\tau}\mathbb{E}_t\tilde{p}_{t+1}(x) = \\ &= [\lambda + (1-\alpha)\gamma\psi]\tilde{k}_{t+1}(x) - \alpha\gamma\tilde{p}_t(x) \end{aligned} \tag{121}$$

is a linear relation in $\tilde{k}_t(x)$ and $\tilde{p}_t(x)$. For convenience, let $A \equiv 1 + \beta + (1 - \beta(1 - \delta))\rho_k \varepsilon_\psi^{-1}$, thus $Q(L) = \beta - AL - L^2$. Equation (115) therefore becomes:

$$\Theta\alpha\tilde{p}_t(x) - \Theta(1-\alpha)\psi\mathbb{E}_t\tilde{k}_{t+1}(x) = \frac{1}{\lambda}\mathbb{E}_t\tilde{k}_{t+1}(x) + \frac{1}{\lambda}\gamma\hat{p}_t(x) - A\tilde{k}_{t+1}(x) + \\ + \beta[\lambda + (1-\alpha)\gamma\psi]\tilde{k}_{t+1}(x) - \alpha\beta\gamma\tilde{p}_t(x). \tag{122}$$

For the conjectured solution (116) to be consistent with this equation, we need the coefficients in front of $\tilde{p}_t(x)$ to fulfill:

$$(1 - \alpha\beta\lambda)\gamma = \Theta\alpha\lambda. \tag{123}$$

Using this equation, the coefficients in front of $\tilde{k}_{t+1}(x)$ becomes:

$$\begin{aligned} &\Theta(1-\alpha)\psi\lambda + 1 - A\lambda + B\lambda^2 + (1-\alpha)\beta\lambda\psi\gamma = 0 \\ &\Theta(1-\alpha)\psi\lambda + (1-\alpha\beta\lambda)(1 - A\lambda + \beta\lambda^2) = 0 \\ &(\frac{1}{\beta} - \alpha\lambda)Q(\beta\lambda) + (1-\alpha)\Theta\psi\lambda = 0. \end{aligned} \tag{124}$$

Now, returning to the optimal price setting, consider:

$$\mathbb{E}_t^x \sum_{j=0}^{\infty} (\alpha\beta)^j \tilde{k}_{t+j}(x). \tag{125}$$

Since,

$$\tilde{k}_{t+j+1}(x) = \lambda\mathbb{E}_t^x\tilde{k}_{t+j}(x) - \gamma\big[\tilde{p}_t(x) - \mathbb{E}_t\sum_{k=1}^{j}\pi_{Ht+k}\big], \tag{126}$$

but

$$\tilde{k}_{t+1}(x) = \lambda\tilde{k}_t(x) - \gamma\tilde{p}_t(x), \tag{127}$$

for all $j \leq 0$ and using $\tilde{p}_t(x) - \mathbb{E}_t\sum_{k=1}^{j}\pi_{Ht+k}$. Notice that:

$$\begin{aligned} \tilde{k}_{t+1}(x) &= \lambda\tilde{k}_t(x) - \gamma\tilde{p}_t(x) \\ \tilde{k}_{t+2}(x) &= \lambda^2\tilde{k}_t(x) - \gamma\lambda\tilde{p}_t(x) - \gamma\tilde{p}_t(x) \\ \tilde{k}_{t+3}(x) &= \lambda^3\tilde{k}_t(x) - \gamma\lambda^2\tilde{p}_t(x) - \gamma\lambda\tilde{p}_{t+1}(x) - \lambda\tilde{p}_{t+2}(x) \end{aligned} \tag{128}$$

so

$$\mathbb{E}_t^x \sum_{j=0}^{\infty}(\alpha\beta)^j \tilde{k}_{t+j}(x) = \frac{\tilde{k}_t(x)}{1-\alpha\beta\lambda} - \frac{\gamma\alpha\beta}{1-\alpha\beta\lambda}\mathbb{E}_t^x \sum_{j=0}^{\infty}(\alpha\beta)^j \tilde{p}_{t+j}(x)$$

$$\mathbb{E}_t^x \sum_{j=0}^{\infty}(\alpha\beta)^j \tilde{p}_{t+j}(x) = \sum_{j=0}^{\infty}(\alpha\beta)^j \big[\tilde{p}_t(x) - \mathbb{E}_t \sum_{k=1}^{j} \pi_{Ht+k}\big]. \tag{129}$$

In addition, using the fact that $\mathbb{E}_t^x \sum_{j=0}^{\infty}(\alpha\beta)^j \sum_{k=1}^{j} \pi_{Ht+k} = \frac{1}{1-\alpha\beta}\mathbb{E}_t \sum_{j=1}^{\infty}(\alpha\beta)^j \pi_{Ht+j}$, we have:

$$\mathbb{E}_t^x \sum_{j=0}^{\infty}(\alpha\beta)^j k_{t+j}(x) = \frac{\tilde{k}_t(x)}{1-\alpha\beta\lambda} - \frac{\gamma\alpha\beta}{(1-\alpha\beta)(1-\alpha\beta\lambda)}\tilde{p}_t(x) +$$

$$+ \frac{\gamma\alpha\beta}{(1-\alpha\beta)(1-\alpha\beta\lambda)}\mathbb{E}_t \sum_{j=1}^{\infty}(\alpha\beta)^j \pi_{Ht+j}. \tag{130}$$

For firms re-optimizing prices at time t, $\tilde{p}_t(x) = \hat{p}_t(x)$. Therefore, combining equation (106) with the last equation yields:

$$(1+\bar{\omega}\theta)\hat{p}_t(x) = (1-\beta\alpha)\mathbb{E}_t \sum_{j=0}^{\infty}(\alpha\beta)^j \hat{s}_{Ht+j} + (1-\alpha\beta)(1+\bar{\omega}\theta)\mathbb{E}_t \sum_{j=0}^{\infty}(\alpha\beta)^j$$

$$\sum_{k=1}^{j} \pi_{Ht+k} - \frac{(1-\alpha\beta)(\bar{\omega}-\bar{\nu})}{1-\alpha\beta\lambda}\tilde{k}_t(x) + \frac{\gamma\alpha\beta(\bar{\omega}-\bar{\nu})}{1-\alpha\beta\lambda}\mathbb{E}_t \sum_{j=1}^{\infty}(\alpha\beta)^j \pi_{t+j}. \tag{131}$$

Thus,

$$\phi\hat{p}_t(x) = (1-\alpha\beta)\sum_{j=0}^{\infty}(\alpha\beta)^j \mathbb{E}_t \hat{s}_{Ht+j} + \phi \sum_{j=1}^{\infty}(\alpha\beta)^j \mathbb{E}_t \pi_{Ht+j} -$$

$$-(\bar{\omega}-\bar{\nu})\frac{1-\alpha\beta}{1-\alpha\beta\lambda}\tilde{k}_t(x), \tag{132}$$

where $\phi = 1 + \bar{\omega}\theta - (\bar{\omega}-\bar{\nu})\frac{\gamma\alpha\beta}{1-\alpha\beta\gamma}$.

For this equation to be consistent with our conjecture (116), we need:

$$\phi\hat{p}_{Ht} = (1-\alpha\beta)\sum_{j=0}^{\infty}(\alpha\beta)^j \mathbb{E}_t \hat{s}_{Ht+j} + \phi \sum_{j=1}^{\infty}(\alpha\beta)^j \mathbb{E}_t \pi_{Ht+j}, \tag{133}$$

and:

$$\phi_\psi = (\bar{\omega}-\bar{\nu})\frac{1-\alpha\beta}{1-\alpha\beta\lambda}. \tag{134}$$

This latter equation along with equations (123) and (124) comprise a system of three equations in three unknown coefficients, λ, γ, ψ. Woodford (2005, pp. 17-18) proposes an algorithm to solve these three equations.

The following explains how to reduce this system of equations to a single equations to a single equation for λ. For $\lambda \neq 0$, (124) can be solved for ψ:

$$\psi(\lambda) = -\frac{(1/\beta - \alpha\lambda)Q(\beta\lambda)}{(1-\alpha)\Theta\lambda}. \tag{135}$$

Similarly, (160) defines a function:

$$\gamma(\lambda) = \frac{\Theta\alpha\lambda}{1-\alpha\beta\lambda}. \tag{136}$$

Substituting these functions for ψ in (135), we get an equation that solves for λ:

$$\begin{aligned} V(\lambda) =& [(1+\bar{\omega}\theta)(1-\alpha\beta\lambda)^2 - \alpha^2\beta(\bar{\omega}-\bar{\nu})\Theta\lambda]Q(\beta\lambda)+ \\ &+ \beta(1-\alpha)(1-\alpha\beta)(\bar{\omega}-\bar{\nu})\Theta\lambda = 0. \end{aligned} \tag{137}$$

Quasi-differencing the expressions for $\hat{p}_{Ht}(x)$ equation (133), we obtain:

$$\hat{p}_{Ht} - \alpha\beta\mathbb{E}_t\pi_{Ht+1} = (1-\alpha\beta)\phi^{-1}\hat{s}_{Ht} + \alpha\beta\mathbb{E}_t\pi_{Ht+1}. \tag{138}$$

Using equation (117) to plug for $\hat{p}_t$ yields:

$$\frac{\alpha}{1-\alpha}\pi_{Ht} - \frac{\alpha^2\beta}{1-\alpha}\mathbb{E}_t\pi_{Ht+1} = (1-\alpha\beta)\phi^{-1}\hat{s}_{Ht} + \alpha\beta\mathbb{E}_t\pi_{Ht+1}, \tag{139}$$

and:

$$\pi_{Ht} = \kappa\phi^{-1}\hat{s}_{Ht} + \beta\mathbb{E}_t\pi_{Ht+1}. \tag{140}$$

7 Discussion

One of the main features of monetary unions such as the United States is that the monetary authority cannot respond asymmetrically to the shocks characterizing the economy. So, when spending is raised in California compared with Illinois, national government policy is maintained fixed across these states. For example, the Fed is not able to respond by raising interest rates in California compared with Illinois, and Congress does not respond by raising tax rates in California compared with Illinois.

In contrast, monetary policy across states is not constant in response to national government spending shocks. It indeed depends on the monetary policy regime, which, in turn, has changed by much during the past twenty years. During the mandates of Paul Volcker and Alan Greenspan the Fed strongly "leaned against the wind", by offsetting aggregate government spending shocks by raising interest rates. More recently, the policy became more accommodative, until nominal interest rates reached the zero lower bound. The tax policy response changed quite harshly during the past five decades - for example, during the Korean War taxes were substantially increased.

Now, a government spending shock at the zero lower bound enables prices to rise. In a Neoclassical model with a constant monetary policy rule, prices jump on impact and start to quickly decline. As a consequence the real interest rate is increased in response to the fiscal shock, thus excluding any type of stimulus to private spending. On the contrary, in a New-Keynesian model, because prices vary stickily in the short run, the interest rate is left falling on impact, thus encouraging private spending. The different response of the real interest rate to government spending shocks - due to different degree of flexibility of price adjustments - therefore explains the differences in the multipliers across these models.

7.1 Baseline

Consider now the Backus-Smith condition of inter-regional risk sharing, $\hat{c}_t - \hat{c}_t^* = \sigma \hat{q}_t$. An increase in home government spending will raise the relative price of home goods relative to foreign ones and therefore will decrease the "real exchange rate", $Q_t = P_t^*/P_t$. By the Backus-Smith condition, home consumption should fall relative to foreign consumption. In other words, government spending "crowds-out" private spending in relative terms, implying a relative multiplier smaller than unity.

Since the relative nominal interest rate is held fixed in response to a regional spending shock, it is interesting to think at the zero lower bound scenario. In this case, a positive government spending shock in the home region will have the effect of raising overall future expected inflation, thus lowering the real interest rate in the short run but not until the zero lower bound, whereas in the long run, real interest rate should increase again above its steady state level. Analogously, the price of home good should jump at impact and for a relatively short period, to finally begin decreasing after a certain time. Therefore, by the constant real exchange rate holding within the monetary union, we may affirm that the overall effect of a $+\Delta g_{Ht}$ is to slightly increase the real interest rate in the medium term.

Consumption in the home region decreases in our basic model (as shown in figures 1 and 3 in Appendix B) because households anticipate a higher future real interest rate[15]. A relative increase of the real interest rate over the long term in our open economy would corresponds to the aggregate fixed long term real interest rate in a closed economy equivalent.

7.2 GHH preferences

The main implication of the formulation of this version of our model is that labour and consumption are complements. This means that our spending multiplier with the GHH model quite higher than in the basic model, around 1.5. A similar result is obtained by Galí, Lopez-Salido and Valles (2007) by including hand-to-mouth households, and in Monacelli and Perotti (2008). The reason why such a large multiplier is obtained is that households must work more to produce an additional unit of output: therefore consumption demand is raised due to the labour-consumption complementarity. But, to be able to achieve a higher consumption level, more production is needed, further raising the effects on output.

The quarterly persistence coefficient of the government spending stochastic process - ρ_g - is fixed to 0.933 in order to match the empirical evidence. The Neoclassical model - namely our model with the probability of firms re-optimizing prices at every period being set equal to zero - is insensitive to the specification of aggregate policies. In the New-Keynesian model with GHH preferences, instead, both relative output and consumption increase as a consequence of a government spending shock in the home region. Both output and consumption increase on impact of an amount that is slightly higher than the double of the shock. Furthermore, they both decay more rapidly that the shock itself (see figures 5 and 7). The fact that consumption and output rise of almost an analogous amount implies that the home region is running a trade surplus - considering that a part of the higher output is due to increased government orders. We may conjecture that households in the home region are willing to shift consumption towards periods of higher work effort, in correspondence with higher government spending. This idea is again based on the complementarity between labour and consumption existent in the GHH model.

[15] Alternatively, due to the Ricardian equivalence, households "internalize" the government budget constraint, thus adjusting their expectations to higher future taxes needed to finance the spending shock. If this was the case, then households would spend less of the current income on consumption, and start to save more.

How is the closed economy aggregate multiplier affected by the introduction of GHH preferences? As it is evident in figure 11, in case of a fixed nominal rate rule meant to proxy the zero lower bound scenario, this model can generate some extremely high multipliers. However, if monetary policy is very responsive to the stances of fiscal policy, as in the case of Volcker-Greenspan, the New-Keynesian model with GHH preferences yields a low closed economy aggregate multiplier. In the Neoclassical model, introducing GHH preferences induces the closed economy aggregate multiplier to decrease, due to the elimination of the wealth effects on labour supply. Differently, in the New-Keynesian model, fiscal policy shocks affect the markup of prices over marginal costs and thus affect output by moving labour demand. Analogously, the open-economy relative multiplier in the Neoclassical model with GHH preferences is low but different than zero since labour supply is shifted by government spending shock as a function of real wages.

The crucial point which explains why the open economy relative multiplier grows higher under GHH preferences in the New-Keynesian model relative to the case of separable preferences is that the monetary union allows for an accommodative monetary policy in relative terms, enough not to offset the increase in relative output. Therefore, the New-Keynesian model with GHH preferences is capable of matching the authors' empirical results, consistently with a model where demand shocks are likely to have large effects on output, whenever the monetary policy is sufficiently accommodative.

7.3 Variable capital

In sections 6.3 and 6.4 we introduced capital in the model in two different ways. The specification of section 6.3 is closely related with Christiano, Eichenbaum and Evans (2005). Households hold their own capital stock and lend it to firms, gaining a rental rate for each period. Capital markets are regional and investment is allowed to influence capital accumulation through a convex adjustment function. Households are able to decide at each period how much capital will be invested and how much will be lent to firms. The extension of section 6.4 is instead a reflection of Woodford (2003, 2005) in that, more realistically, capital is firm specific, thus each firm owns its own capital stock and faces convex costs to adjust its investment level on a period by period basis.

The multiplier obtained via calibration of the model with regional capital market is slightly lower than the one obtained via the baseline specification (we actually obtain a strongly positive effect on consumption but a negative effect output, after a positive government spending shock). This fact apparently contrasts with the increase in investment occurring in the home region subsequent to observing the spending shock. The reason for this prediction might be grasped by considering the regional nature of capital market, which is associated with a reduction in the degree of strategic complementarity among firms in the process of price fixing. Indeed, firms re-optimizing prices in a given period are able to do it without incurring in additional costs, since they benefit from the higher investment, thus needing to rent less capital. Nevertheless, the hypothesis of firms renting capital in a market without frictions on a periodic basis is fairly unrealistic. Assuming the existence of firm specific capital is more plausible, and a New-Keynesian model with such feature is able to reproduce the sticky adjustment of prices arising from empirical evidence, according to Eichenbaum and Fisher (2007).

The model with firm specific capital, although not replicated by ourselves in the current article, in the original paper of Nakamura and Steinssón (2014) yields a higher multiplier than in basic model. This is due to the fact that firms expect marginal returns on capital to be higher after a government spending shock (considering its persistency) and therefore they increase their investment on impact. It is finally worthy to point out that the model with regional capital markets underlies separable preferences, whereas the firm specific capital one is based on GHH utility function.

8 Conclusions

With this work, we attempted to replicate the theoretical model presented in Nakamura and Steinssón (2014). In particular, we constructed and solved with the help of Dynare the open economy New-Keynesian model to study the effects of government spending shocks on output in a monetary and fiscal union, such as the United States. First, we derived the equilibrium optimality conditions for the different classes of agents, namely households, firms, the government - fixing its fiscal policy rules - and the monetary authority. Government spending in the home region is assumed to be stochastic, and the relative effects of that shock are studied across the open economy's main macroeconomic variables such as consumption, output, prices, inflation, interest rate, (investment and capital). We approximated the equilibrium equations for *i.* a baseline model characterized by no capital and separable preferences, *ii.* a model with non separable preferences of the so-called GHH type, *iii.* a model with regional capital held by households and lent to firms, and *iv.* a model with firm specific capital. We finally drew impulse response functions for the specifications *i.*, *ii.*, *iii.* and discussed the results.

It would be useful to consider not solely the entity of the fiscal multiplier, rather also the type of aggregate spending the government implements. For example, it is likely that higher spending in education and health may lead to long run positive externalities, beyond the immediate impact on other macroeconomic variables. This consideration may be useful for evaluating the welfare effects of fiscal policy: households' utility function may therefore depend on a third variable, say, government spending. Abstracting from investment, heterogeneous labour markets, and price dispersion, we imagine that output might be below its optimal level, therefore it may be desirable to raise spending beyond a certain point, depending on the size of the multiplier. A higher multiplier would imply that more spending is desirable, especially if the aim is exiting from a liquidity trap. Monetary policy, being a costless instrument, should thus focus on eliminating such "output gaps", while fiscal policy ought to optimally stimulate private spending. Fiscal policy should target output gaps only when monetary policy is constrained at the zero lower bound.

A A closed economy limit

Consider the closed economy equivalent of the baseline model. For the separable utility function, a log-linear approximation is:

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \sigma(\hat{r}_t^n - \mathbb{E}_t \hat{\pi}_{t+1})$$
$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \zeta \sigma^{-1} \hat{c}_t + \kappa \zeta \psi_\nu \hat{y}_t$$
$$\hat{y}_t = \frac{\bar{C}}{\bar{Y}} \hat{c}_t + \hat{g}_t,$$

where $\zeta = \frac{1}{1+\psi_\nu}$ and $\psi_\nu = \frac{1+\nu^{-1}}{a-1}$.
Eliminating y_t using the first twoequations yields:

$$\hat{c}_t = \mathbb{E}_{t+1} \hat{c}_{t+1} - \sigma(\hat{r}_t^n - \mathbb{E}_t \hat{\pi}_t),$$
$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \zeta_c \hat{c}_t + \kappa \zeta_g \hat{g}_t,$$

where $\zeta_c = \zeta(\sigma^{-1} + \frac{\bar{C}}{\bar{Y}} \psi_\nu)$; and $\zeta_g = \zeta \psi_\nu$.

Finally, recall that $\hat{g}_t \sim AR(1) \Rightarrow \hat{g}_t = \rho_g \hat{g}_t + \varepsilon_{gt}$, where $\varepsilon_{gt} \sim WN(0, 1)$.

A.1 Fixed Real Rate

In an equilibrium with such a feature, the following relations hold:

$$
\begin{aligned}
\hat{c}_t &= E_t \hat{c}_{t+1} \\
\hat{\pi}_t &= \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \zeta_c \hat{c}_t + \kappa \zeta_g \hat{g}_t,
\end{aligned}
$$

where $\hat{c}_t^* = a_c \hat{g}_t$ and $\hat{\pi}_t^* = a_\pi \hat{g}_t$ are the conjectured solutions if undetermined coefficients method if used. In particular, it can be shown that

$$
a_c = 0, a_\pi = \kappa \frac{\zeta_g}{1 - \beta \rho_g}.
$$

The corresponding policy rule for nominal interest rate should therefore be:

$$
\begin{aligned}
\hat{r}_t^n &= \mathbb{E}_t \hat{\pi}_{t+1} + \phi_\pi (\hat{\pi}_t - \hat{\pi}_t^*) \\
&= a_\pi \rho_g \hat{g}_t + \phi_\pi \hat{\pi}_t - a_\pi \phi_\pi \hat{g}_t \\
&= \phi_\pi \hat{\pi}_t - a_\pi (\phi_\pi - \rho_g) \hat{g}_t,
\end{aligned}
$$

in which $\hat{\pi}_t$ and $\hat{\pi}_t^*$ respectively denote the nominal interest rates holding in the home and foreign regions.

A.2 Fixed Nominal Rate

In an equilibrium with such a feature, the following relations hold:

$$
\begin{aligned}
\hat{c}_t &= \mathbb{E}_t \hat{c}_{t+1} + \mathbb{E}_t \sigma \hat{\pi}_{t+1} \\
\hat{\pi}_t &= \beta \mathbb{E}_t \hat{c}_{t+1} + \kappa \zeta_c \hat{c}_t + \kappa \zeta_g \hat{g}_t
\end{aligned}
$$

again, $\hat{c}_t^* = a_c \hat{g}_t$, $\hat{\pi}_t^* = a_\pi \hat{g}_t$. Using the undetermined coefficients method it can be shown that

$$
\begin{aligned}
a_c =& \frac{\rho_g \kappa \zeta_g}{A_c}, a_\pi = \kappa \frac{\zeta_c}{1 - \beta \rho_g} a_c + \kappa \frac{\zeta_g}{1 - \beta \rho_g}, \\
A_c =& (1 - \rho_g)(1 - \beta \rho_g) - \rho_g \kappa \zeta_c.
\end{aligned}
$$

Only if $A_c > 0$, the solution is valid. If then $0 < \rho_g < 1 \Rightarrow A_c$ is $\downarrow$ in $\rho_g \Rightarrow$ in the valid interval, the monetary policy rule is

$$
\begin{aligned}
\hat{r}_t^n &= \phi_\pi (\hat{\pi}_t \hat{\pi}_t^*) \\
&= \phi_\pi \hat{\pi}_t - a_\pi \phi_\pi \hat{g}_t.
\end{aligned}
$$

B Impulse response functions

We thereby present the output of Dynare 4.4.3 in running three of the four declinations of Nakamura and Steinsson (2014)'s model.

Figure 1: IRFs of the main variables of our open economy after a $+\Delta G_t = 0.01$ in both regions H and F with the **basic version of the model**. $\rho_g = 0.85$.

iii/irf1'.pdf

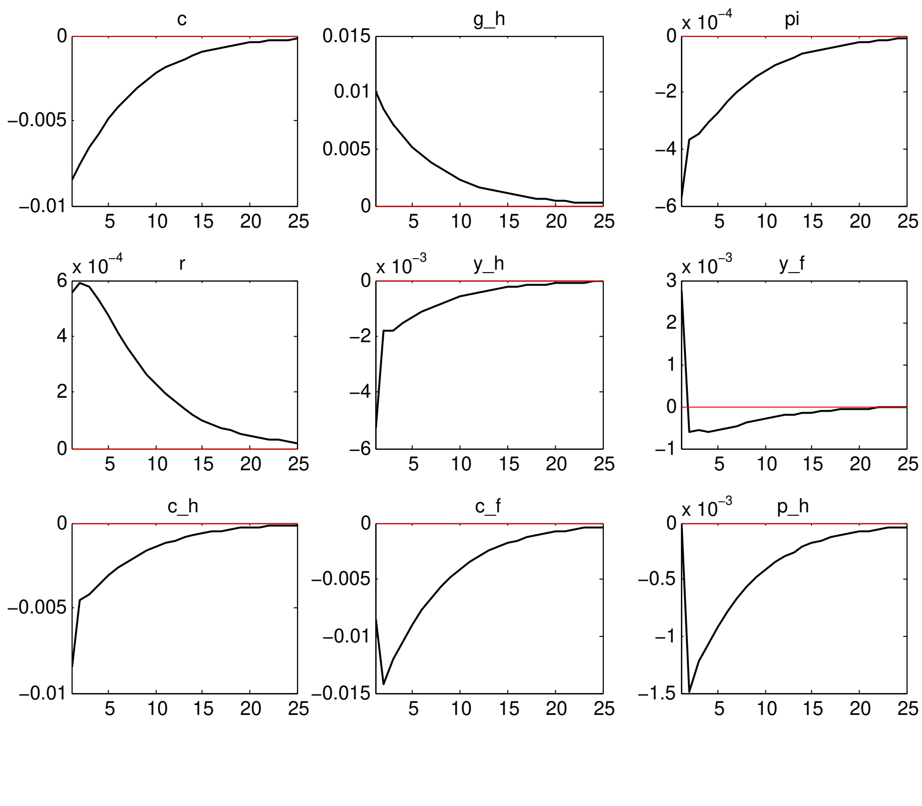


Figure 2: IRFs of the other variables of our open economy after a $+\Delta G_t = 0.01$ in both regions H and F with the **basic version of the model**. $\rho_g = 0.85$.

iii/irf2'.pdf

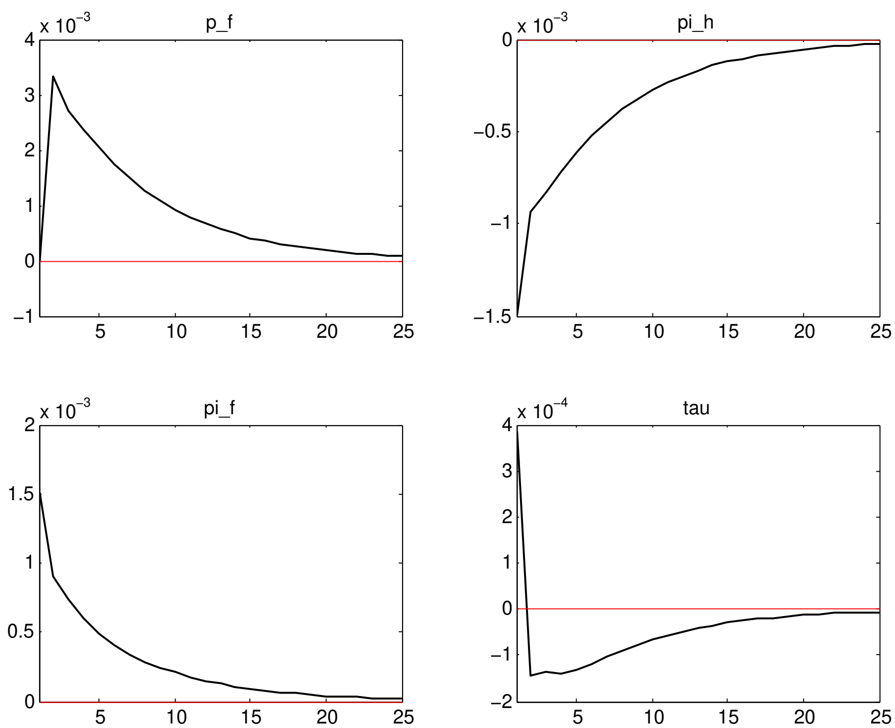


Figure 3: IRFs of the main variables of our open economy model after a $+\Delta G_t = 0.01$ in both regions H and F with the **basic version of the model**. $\rho_g = 0.933$.

iii/irf1".pdf

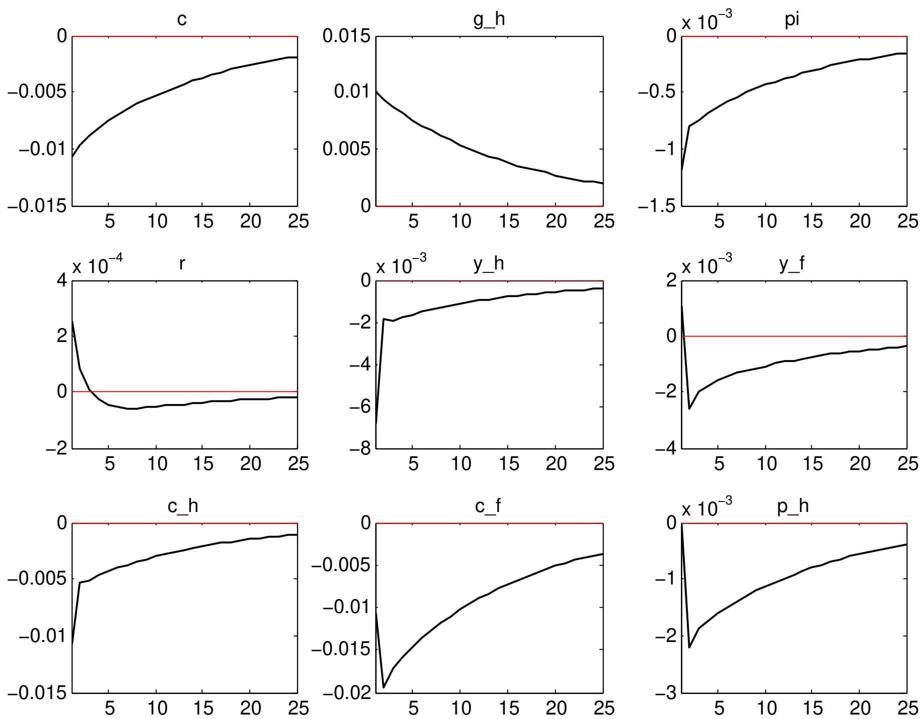


Figure 4: IRFs of the other variables of our open economy model after a $+\Delta G_t = 0.01$ in both regions H and F with the **basic version of the model**. $\rho_g = 0.933$.

iii/irf2".pdf

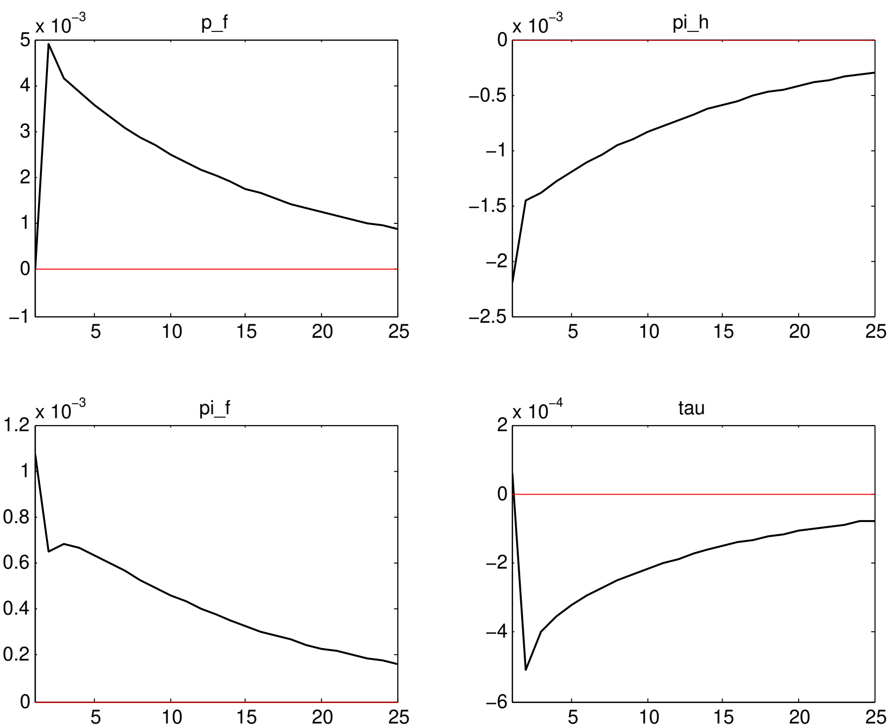


Figure 5: IRFs of the main variables of our open economy model after a $+\Delta G_t = 0.01$ in both regions H and F with the **GHH model** and a $\rho_g = 0.8$.

iii/ghhirf1'.pdf

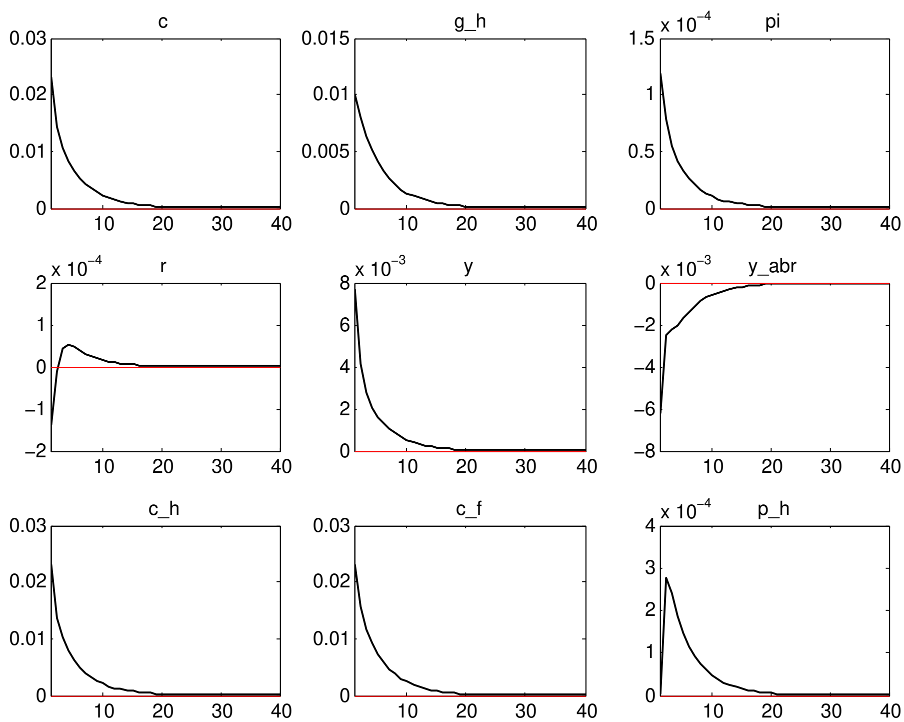


Figure 6: IRFs of the other variables of our open economy model after a $+\Delta G_t = 0.01$ in both regions H and F with the **GHH model** and a $\rho_g = 0.8$.

iii/ghhirf2'.pdf

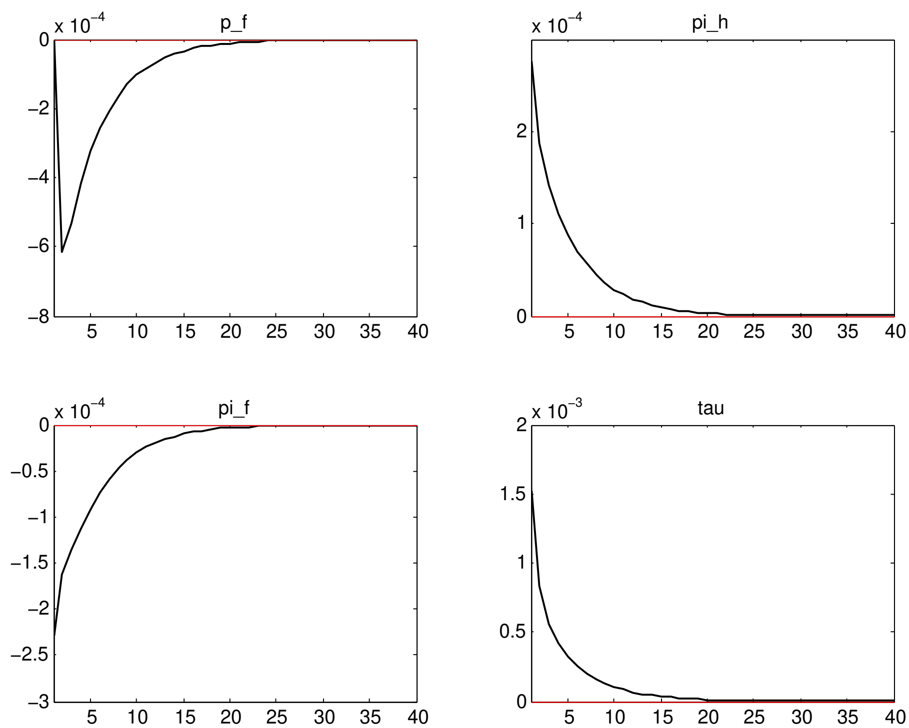


Figure 7: IRFs of the main variables of our open economy model after a $+\Delta G_t = 0.01$ in both regions H and F with the **GHH model** and a $\rho_g = 0.933$.

iii/ghhirf1"'.pdf

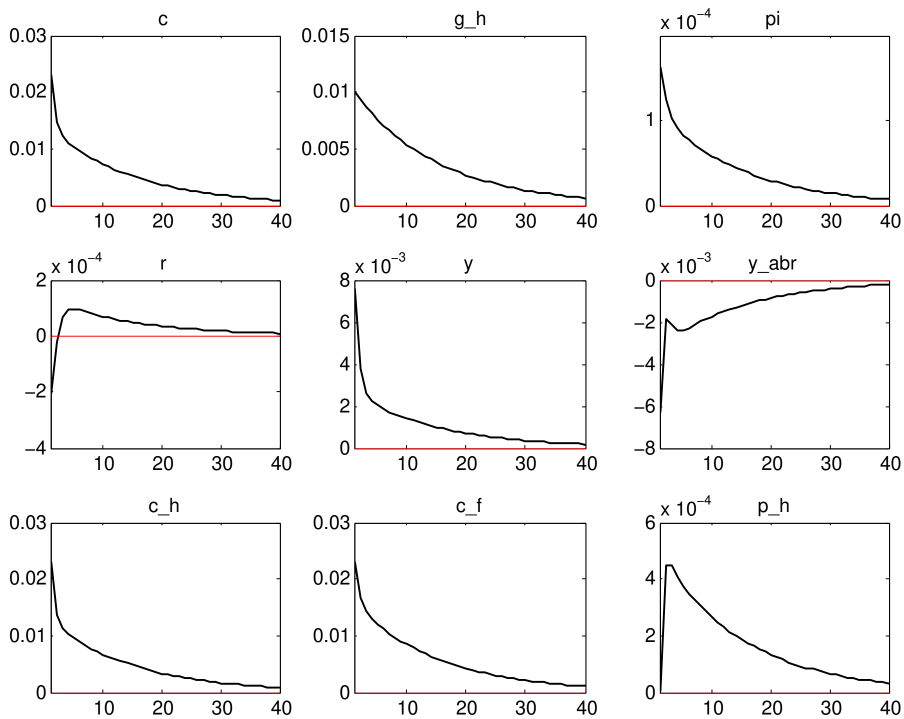


Figure 8: IRFs of the other variables of our open economy model after a $+\Delta G_t = 0.01$ in both regions H and F with the **GHH model** and a $\rho_g = 0.933$.

iii/ghhirf2"'.pdf

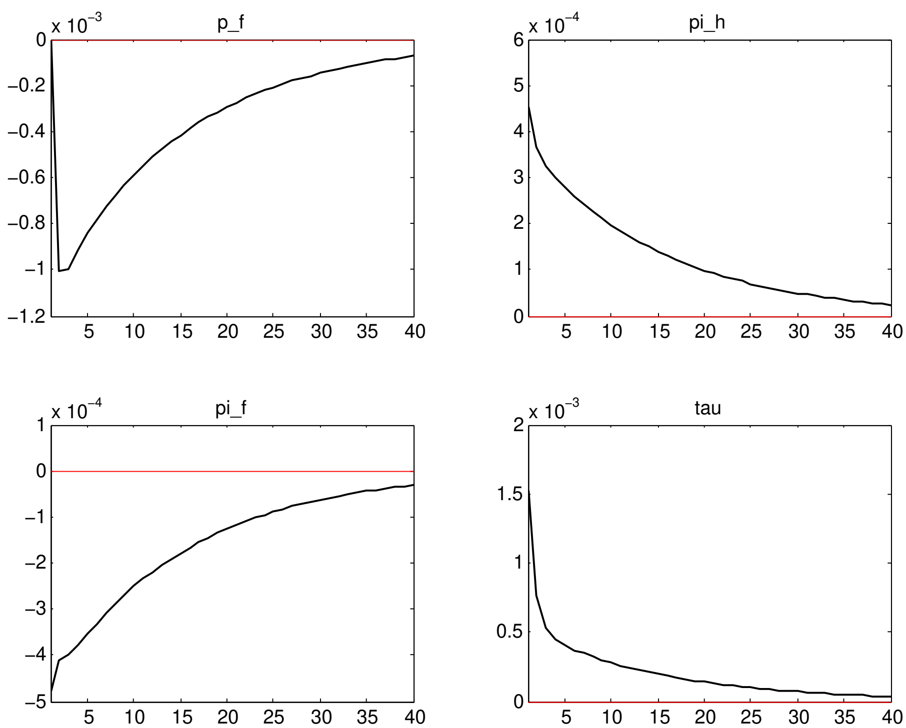


Figure 9: IRFs of the model with **regional capital market** and $\rho_g = 0.988$.
iii/regional1.pdf

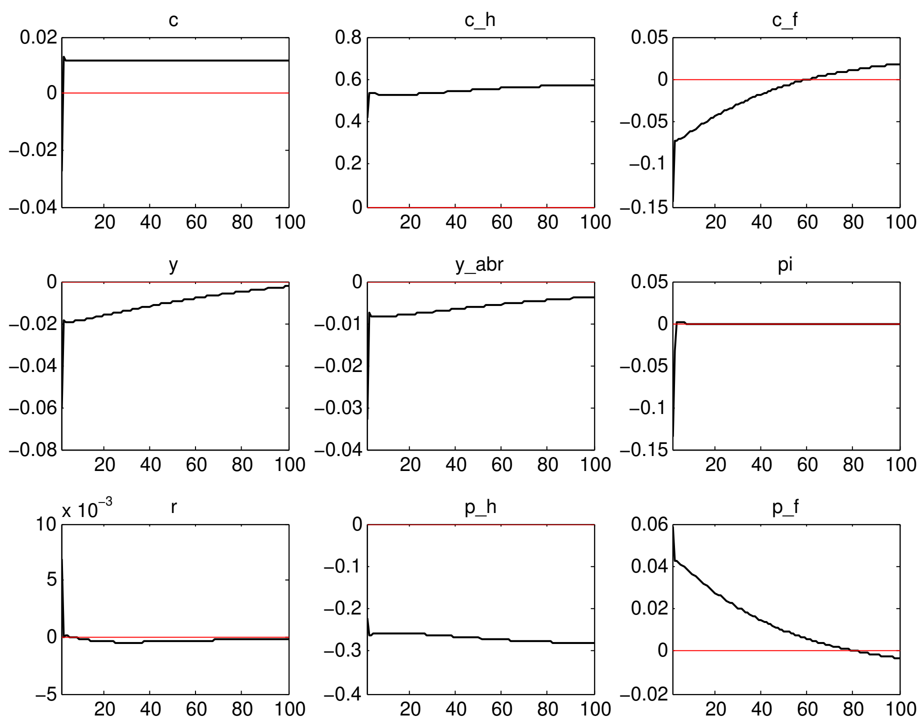


Figure 10: IRFs of the model with **regional capital market** and $\rho_g =$ 0.988.

iii/regional2.pdf

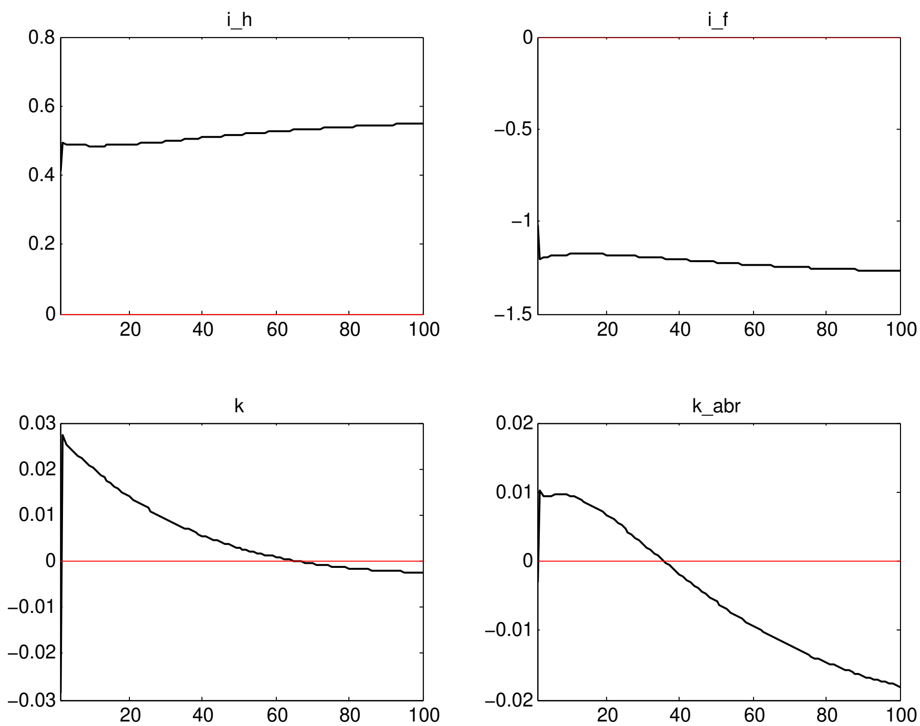
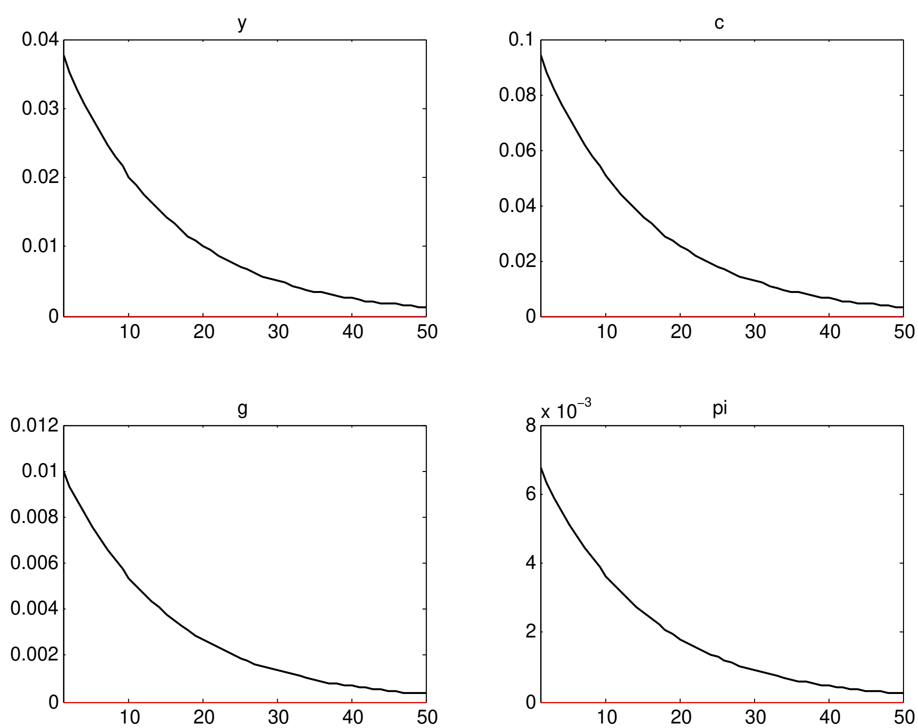


Figure 11: IRFs of the **closed economy equivalent of the GHH model**.
iii/closedeconomy.pdf



C References

1. Acconcia, Antonio, Giancarlo Corsetti, and Saverio Simonelli (2012): Mafia and Public Spending: Evidence on the Fiscal Multiplier from a Quasi-Experiment, *CSEF Working Paper* No. 281.

2. Alesina, Alberto, Carlo Ambrogio Favero, and Francesco Giavazzi (2012): The Output Effect of Fiscal Consolidations, *GRASP Working Paper Project* No. 450.

3. Barro, Robert J. and Charles J. Redlick (2011): Macroeconomic Effects from Government Purchases and Taxes,*Quarterly Journal of Economics* Vol. 126(1), pp. 51-102.

4. Baxter, Marian, and Robert G. King (1993): Fiscal Policy in General Equilibrium, *The American Economic Review* Vol. 82(3), pp. 315-334.

5. Blanchard, J. Olivier, and Daniel Leigh (2013): Growth Forecast Errors and Fiscal Multiplier, *IMF Working Paper* No. 13/1.

6. Blanchard, J. Olivier, and Roberto Perotti (2002): An Empirical Characterization of the Dynamic Effects of Changes in Government Spending and Taxes on Output, *Quarterly Journal of Economics* Volume 117, 1329-1368, Issue 4.

7. Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans (2005): Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy, *Journal of Political Economy*, Vol. 113(1), pp. 1-45.

8. Christiano, Lawrence J., Martin Eichenbaum (2011), When is the Government Spending Multiplier Large?, *Journal of Political Economy*, Vol. 119, pp. 78-121.

9. Corsetti, Giancarlo, Andre Meier, Gernot J. Muller (2012): What Determines Government Spending Multipliers, *IMF Working Paper* No. 12/250.

10. Devriers, Pete, Jaime Guajardo, Daniel Leigh, Andrea Pescatori (2012): A New Action-Based Dataset of Fiscal Consolidation, *IMF Working Paper* No. 11/128.

11. Eggertson, G.B. (2010): What Fiscal Policy is Effective at the Zero Lower Bound? *NBER Macroeconomics Annual*, ed. by D. Acemoglu, and M. Woodford, Chicago, IL, University of Chicago Press.

12. Erceg, Christopher J., Jesper Lindé (2010), Is There a Fiscal Free Lunch in a Liquidity Trap? *Board of Governors of the Federal Reserve System, International Finance Discussion Paper*, No. 1003.

13. Favero, Ambrogio Carlo, Francesco Giavazzi, Jacopo Perego (2011): Country Heterogeneity in the International Evidence of Fiscal Policy, *IMF Economic Review Palgrave Macmillan*, Vol. 59(4), pp. 652-682.

14. Guajardo, Jaime, Daniel Leigh, Andrea Pescatori (2011): Expansionary Austerity: New International Evidence, *IMF Working Paper*No. 11/158.

15. Hall, Robert E. (2011): By How Much Does GDP Rise if the Government Buys More Output?, *Brookings Papers on Economic Activity* Issue Fall, pp. 183-236.

16. Ilzetzki, Ethan, Enrique G. Mendoza, Carlos A. Végh: How Big (Small) Are Fiscal Multipliers?, *NBER Working Paper* 2010, No. 16479.

17. Kollmann, Robert (1997): The Cyclical Behaviour of Mark Ups in U.S. Manufacturing and Trade: New Empirical Evidence Based on a Model of Optimal Storage, *Economics Letters*, Vol. 57, pp. 331-337.

18. Kraay, Aart (2011): How Large is the Government Spending Multiplier? Evidence form World Bank Lending, *Working Paper, World Bank*.

19. Leeper, Eric M., Nora Traum, Todd B. Walker (2011): Clearing Up the Fiscal Multiplier Morass, *NBER Working Paper* No. 17444.

20. Mountford, Andrew, Harald Uhlig (2009): What Are the Effects of Fiscal Policy Shocks?, *Journal of Applied Econometrics*, Vol. 24(6), pp. 960-992.

21. Nakamura, Emi, Jón Steinssón (2014): Fiscal Stimulus in a Monetary Union: Evidence from US Regions, *The American Economic Review*, Vol. 104(3), pp. 753-792.

22. Perotti, Roberto: The Austerity Myth: Gain Without Pain? (2011), *IGIER-Working Paper* No. 17571.

23. Ramey, Valerie A. (2012): Government Spending and Private Activity, *NBER Working Paper* no. 17787.

24. Romer, Christina D., David H. Romer (2010): The Macroeconomic Effects of Tax Changes: Estimates Based on a New Measure of Fiscal Shocks, *The American Economic Review* Vol. 100(3), pp. 765-801.

25. Wang, Pengfei, and Yi Wen (2010): Hayashi Meets Kiyotaki and Moore: A Theory of Capital Adjustment Costs, *Research Division, Federal Reserve Bank of Saint Louis*, Working Paper Series, No. 037.

26. Smets, Frank, Raf Wouters (2003): An Estimated Stochastic General Equilibrium Model of the Euro Area, *Journal of the European Economic Association*, Vol.1(5), pp. 1123-1175.

27. Smets, Frank, Raf Wouters (2007), Shocks and Frictions in the US Business Cycle: A Bayesian DSGE Approach, *The American Economic Review*, Vol. 97(3), pp. 586-606.

28. Woodford, Michael (2003): Interest and Prices, *Princeton University Press*, Princeton, NJ.

29. Woodford, Michael (2005): Firm - Specific Capital and the New - Keynesian Phillips Curve, *International Journal of Central Banking*, Vol. 1(2), pp. 1-46.