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Comparative Statics and Dynamics of Optimal Choice Models in Hilbert Spaces

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We study properties of the solutions to a parametrized constrained optimization problem in Hilbert spaces. A special operator is studied which is of importance in economic theory; sufficient conditions are given for its existence, symmetry, and negative semidefiniteness. The techniques used are calculus and non linear functional analysis on Hilbert spaces.

INTRODUCTION

In a wide range of economic problems the equilibrium values of the variables can be regarded as solutions of a parametrized constrained maximization problem. This occurs in static as well as dynamic models; in the latter case the choice variables are often paths in certain function spaces and thus can be regarded as points in infinite dimensional spaces.

It is sometimes possible to determine qualitative properties of the solutions with respect to changes in the parameters of the model. The study of such properties is often called comparative statics; [15], [2], and [10]. Certain comparative static properties of the maxima have proven to be of particular importance for economic theory, since the works of Slutsky, Hicks, and Samuelson [15]: they have been formulated in terms of symmetry and negative semidefiniteness of a matrix, called the Slutsky–Hicks–Samuelson matrix. A discussion of this matrix and its applications is given in Section 1. The study of these properties in economic theory, however, has so far been restricted to static models where the choice variable and the parameters are elements in Euclidean spaces, and where there is only one constraint. Infinite dimensionality of the choice variables arises naturally from the underlying dynamics of the models. For example, in optimal growth models with continuous time and problems of planning with infinite horizons [4] and also from the existence of infinitely

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many characteristics of the commodities indexed, for instance, by states of nature in models with uncertainty, by location, etc. Many times these models are formalized as optimization problems with more than one constraint.

It is the purpose of this paper to extend the study of the Slutsky–Hicks–Samuelson operator ot a general class of parametrized, constrained optimization problems which appear in recent works in economic theory: the choice variables and parameters belong to infinite dimensional spaces, the objective function to be maximized depends also on parameters, and the optimization is restricted to regions given by many possibly infinite parametrized constraints, linear or not.[1] The results provide a foundation for the study of comparative statics in dynamic models such as optimal growth and other dynamic models [4].

The derivation of the Slutsky operator is more complicated in the case of many constraints, and the operator obtained is of a slightly different nature. One reason is that the "compensation" can be performed in different manners since there are many constraints, as becomes clear in the proof of Theorem I and the remark following it. Also, the existence of parameters introduces new effects that do not exist in the classical models; in general, the classical properties are not preserved. Further, since the values of the constraints may be in an infinite dimensional space of sequences (denoted C), then "generalized Lagrangian multiplier" may also be infinite dimensional, in effect, an element of the dual space of C, denoted C^*. To avoid the problem of existence of such dual elements which are not representable by sequences (e.g., purely finite additive measures [8]) and thus complicate the computations, we work on a Hilbert space of sequences C. Infinite dimensional economic models where the variables are elements of Hilbert spaces have been studied in [4] and [5].

One problem in the extension from finite to infinite dimensional choice variables and paraoptimal solutions is that closed and bounded sets in infinite dimensional spaces are not, in general, compact in certain topologies such as the L_∞ or C^* norms. To avoid this problem, one usually uses certain weak topologies in which norm bounded and closed sets are compact. However, in these topologies, the continuity of the objective functions is more difficult to obtain, and thus the usual proofs of existence of solutions by compactness-continuity arguments may restrict the class of admissible objective functions. However, using the concavity of the objective function and convexity of the set on which the optimization is performed, we prove existence of an optimal solution on norm bounded closed sets[2] or weakly compact sets without requiring

[1] Related work in infinite dimensional commodity spaces has been done for special cases of one linear constraint and no parameters in the objective function by L. Court [7] and Berger [3]. In finite dimensional models, related work for parametrized models with one constraint was done by Kalman [9], and Kalman and Intriligator [10]; Chichilnisky and Kalman studied parametrized multi-constraint problems in [6].

[2] In any reflexive Banach space or Hilbert space, norm bounded and closed sets are weakly compact [8].

the objective function to be weakly continuous, which significantly widens the choice of objective functions. Thus, the existence of a solution can be obtained in a much wider class of economic models; a useful tool here is the Banach–Saks theorem [14].

In Section 1 sufficient conditions are given for existence and uniqueness of a C^1 solution to a general optimization problem and for existence of a generalized Slutsky–Hicks–Samuelson operator which contains as a special case the operator of classical economic models. In Section 2, properties of this operator are studied: a class of objective and constrained functions is shown to preserve the classical properties of symmetry and negative semidefiniteness of the operator, which are, in general, lost in parametrized models, as seen in [10].

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We now discuss the Slutsky–Hicks–Samuelson operator and its applications. For further references, see, for instance, [15] and [10]. Consider the maximization problem:

$$\begin{aligned} &\max_x f(x, a) \\ &\text{subject to} \qquad g(x, a) = c, \end{aligned} \tag{P}$$

where f is a real valued map defined on a linear space and g is vector valued, defined on a linear space. Under cerain assumptions the optimal solution vector x denoted $h(a, c)$ is a C^1 function of the variables a and c, and, as the parameter c varies, the constraints describe a parametrized family of manifolds on which f is being maximized. In neoclassical consumer theory, for instance, f represents a utility function, x consumption of all commodities, a prices of all commodities and c income. In this theory, h is called the demand function for commodities of the consumer. In neoclassical producer theory, f represents the cost function, x inputs, a input prices, and g a production function constrained by an output requirement c; in this theory, h is called the demand function for inputs of the firm. In both these models, $c \in R^+$. Comparative static results relate to the Slutzky–Hicks–Samuelson operator, given by the derivative of the optimal solution h with respect to the parameter a restricted to the manifold given by

$$f(x, a) = r,$$

parametrized by the real number r, denoted

$$\frac{\partial}{\partial a} h(a, c)|_{f=r} .$$

This operator will also be denoted $S(a, c)$. It is a well known result that in the finite dimensional consumer model, under certain assumptions,

$$S(a, c) = \frac{\partial}{\partial a} h(a, c) + h(a, c) \frac{\partial}{\partial a} h(a, c). \tag{*}$$

Equation (*) is also called the fundamental equation of value. While in this case $S(a, c)$ is considered unobservable since it represents changes in the demand due to a price change when utility is assumed to remain constant, the right hand side represents two observable effects called the price effect and the income effect on the demand, respectively. Analogous operators are found throughout the body of economic theory. Important properties of the $S(a, c)$ operator are its symmetry and negative semidefiniteness. In addition to their empirical implications, the symmetry property (S) is related to the Frobenius property of local integrability of vector fields or preferences and the negative semidefiniteness property (N) is related to problems of stability of the equilibrium.

A natural question is whether the results of neoclassical consumer and producer theory can be obtained for the general classes of constrained optimization models described above. The results of this paper point in this general direction. However, the S and N properties of the $S(a, c)$ matrix are not, in general, preserved in parametrized models [9]; thus, one can at most hope to obtain sufficient conditions of the classes of models (objective functions and constraints) in which these properties are still satisfied. This is discussed in Section 2.

We now formally define the problem: for a given vector parameters (a, c) we study the solutions of

$$\begin{aligned} &\max_x f(x, a) \\ \text{restricted by} \qquad &g(x, a) = c. \end{aligned} \tag{1}$$

We assume that f and g are twice continuously Frechet differentiable (denoted C^2) real valued and vector valued functions, respectively. For a discussion of Frechet derivatives see, for instance, [12] or [13]. The Frechet derivative generalizes the definition of the Jacobian of a map between finite dimensional spaces. In infinite dimensional Banach spaces there are other possible definitions of derivatives, such as the Gateaux derivative which generalizes the concept of directional derivatives. For our purposes, we use the Frechet derivatives since much of the theory of ordinary derivatives extends to them, and since the implicit function theorem has a satisfactory extension in this case. In the following, all derivatives are Frechet.

We assume that the variable $x \in X$, $a \in A$, where X and A are real Hilbert spaces and that $c \in C$, an l_2 space of sequences.[3] We assume that the spaces X

[3] See, for instance, [5] for economic models defined on (weighted) $l_2[0, \infty)$ spaces, with finite measures on $[0, \infty)$, and [4] for models defined on (weighted) L_2 and Sobolev spaces.

and C have natural positive cones denoted X^+ and C^+, and we denote by X_0^+ the set of vectors in X which are strictly positive.[4] Let τ denote the weak topology on X [8], and let A_1 and C_1 be open subsets of A and C. For any (a, c) in $A_1 \times C_1$, denote by $g_{c,a}$ the set

$$\{x \in X^+ : g(x, a) \leqslant c\}.$$

The Lagrangian of (1), denoted L, is a real valued map on $X \times A_1 \times C_1 \times C^*$ (C^* the dual of C) given by

$$L(x, a, c, \lambda) = f(x, a) + \lambda(g(x, a) - c),$$

where $\lambda \in C^*$ (C is isomorphic to C^*). Let $\psi_1 \colon X_1 \times A_1 \times C_1 \to C$ be defined by $\psi_1(x, a, c) = g(x, a) - c$, and $\psi_2 \colon X_1 \times A_1 \times C_1 \times C^* \to \mathscr{L}(X, R)$ (the space of linear functionals from X to R) be defined by

$$\psi_2(x, a, c, \lambda) = \frac{\partial}{\partial x} L(x, a, c, \lambda),$$

where $(\partial/\partial x) L$ represents the partial derivative of the function L with respect to the variable x, as a function defined on $X_1 \times A_1 \times C_1 \times C^*$ with values (in view of the assumptions on f and g), on the dual space of X (denoted X^*) of continuous linear functionals on X [8]. Let $\psi \colon X_1 \times A_1 \times C_1 \times C^* \to C \times X^*$ be defined by

$$\psi(x, a, c, \lambda) = \left(g(x, a) - c, \frac{\partial}{\partial x} L(x, a, c, \lambda)\right)$$

$$= (\psi_1(x, a, c), \psi_2(x, a, c, \lambda)).$$

Let X_1 be a neighborhood of X^+.

We now briefly discuss certain special problems involved in the proof of existence of solutions and of the Slutzky–Hicks–Samuelson operator in infinite dimensional cases. In the next result we make use of necessary conditions of an optimum in order to derive the operator $S(a, c)$. These necessary conditions basically entail the existence of a separating hyperplane; in order to prove that they are satisfied in problems defined in Banach spaces one uses a Hahn-Banach type theorem which requires existence of interior points in the regions where the optimization takes place (see, for instance, the discussion in [13]). However, L_p spaces with $1 \leqslant p \leqslant \infty$ have positive cones with empty interior. In these cases,

[4] If X is, for instance, a sequence space, $x \in X$, $x = (x_t)$, $t = 1, 2, \ldots$, then x is *positive* (denoted $x > 0$), when $x_t \geqslant 0$ for all t, $(x_t) \neq (0)$, and x is *strictly positive* or $x \gg 0$, when $x_t > 0$ for all t.

When $X = L^2$, $x = (x(t))$ then $x > 0$ if $x \neq 0$ and $x(t) \geqslant 0$ a.e. $x \gg 0$ if $x(t) > 0$ a.e. Similarly, for $X = L^2(R^n)$.

however, if the function to be maximized (f) is continuous and is defined on a neighborhood X_1 of the positive cone X^+, the first order condition for a maximum can still be obtained (see footnote 10 below). An important tool for the derivation of the $S(a, c)$ operator is the implicit function theorem in Hilbert spaces [12]. This theorem requires invertibility of certain operators. In [6] the authors investigated these invertibility properties for finite dimensional models and showed that they are "generically" satisfied by using Sard's theorem. Here we assume them; one can refer, for instance, to the work of Kantorovich and Akilov [11] for sufficient conditions on the functions f and g that will yield the required invertibility of certain linear operators in infinite dimensional spaces. This is discussed further in the remarks after Theorem 1. One can also consider extensions of the results of [6] by use of the infinite dimensional version of Sards' theorem [16].

THEOREM 1. *Let $f: X_1 \times A_1 \to R$ and $g: X_1 \times A_1 \to C_1$ be (Fréchet) C^2 functions. For every $a \in A_1$, let $f(\cdot, a)$ be strictly concave and increasing on x, and g be increasing in x.*[5] *Assume*

(i) *the set $g_{c,a}$ is a nonempty convex τ-compact subset of X^+,*[6]

(ii) *g is regular as a function of x,*[7]

(iii) *for each (a, c), $(\partial/\partial(x, \lambda))\, \psi$ is a top linear isomorphism, and*

(iv) *the operator Z defined in (6') below, exists for all (x, λ) in $X_1 \times C^*$ with $\psi(x, a, c, \lambda) = 0$.*[8]

Then there exists a unique global map $h: A_1 \times C_1 \to X^+$ which is of class C^1 satisfying

$$f(h(a, c), a) = \max_{x \in g_{c,a}} f(x, a),$$

[5] f is increasing in x if $f(x_1) > f(x_2)$ when $x_1 - x_2 \in X_0^+$.

[6] $g_{c,a}$ is τ or weakly compact in X if it is closed and bounded [8]. So, basically, condition (i) can be viewed as a condition of boundedness and closedness of the "technology" represented by the feasible set $g_{c,a}$. Let $\bar{g}_{c,a} = \{x: g(x, a) = c\}$. Then when g is strictly increasing in x, given that f is strictly increasing also, the maximum of f over $g_{c,a}$ will be attained in this case at $\hat{x}$ in $\bar{g}_{c,a}$. An example in infinite dimensional spaces where the set $g_{c,a}$ is convex is provided by all the feasible consumption paths obtained from an in tial capital stock in an economy with a convex technology, in the usual optimal growth model. In these cases, the constraint g takes the form of a differential (or difference) equation with initial conditions, see [4].

[7] I.e., for all (x_0, a_0) in $X_1 \times A_1$, $(\partial/\partial x)\, g(x_0, a_0)$ is onto.

[8] This assumption is shown to be "generically" satisfied in finite dimensional versions of these problems in [6] under certain conditions. For a further discussion on the existence of the operator Z, see the remark after the theorem.

and for any choice of compensating constraint there exists a slutsky–Hicks–Samuelson operator $S: A_1 \times C_1 \to \mathscr{L}(A, X)$ *(the space of linear functionals from* A *to* X*) given by*

$$S(a, c) \equiv \frac{\partial}{\partial a} h + \frac{\partial}{\partial c} h \frac{\partial}{\partial a} g(h, a)$$

satisfying

$$S(a, c) = \frac{\partial}{\partial a} h \Big|_{\bar{f}} + \frac{\partial}{\partial c} h \left(\phi\left(\frac{\partial}{\partial a} g\right) - \mu\left(\frac{\partial}{\partial a} f\right)\right)$$
$$= -\left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right] \cdot \left(\frac{\partial^2}{\partial x \partial a} L\right),[9]$$

where the operators ϕ, μ *are defined in* (14') *below, provided these operators are well defined for all* (x, λ) *with* $\psi(x, a, c, \lambda) = 0$.

Proof. Since $g_{c,a}$ is a τ-compact subset of X_0^+ by (i), if $\{x^n\}$ is a sequence in $g_{c,a}$ with $f(x^n, a) \to \sup_{g_{c,a}} f(x, a)$, then there exists a subsequence, denoted also $\{x^n\}$, converging weakly, i.e., $\{x^n\} \to^{\tau} h$ in $g_{c,a}$ [14]. By the Banach–Saks theorem there exists a subsequence $\{x^{n_k}\}$ such that the sequence of arithmetic means $\{y^{n_k}\}$,

$$y^{n_k} = \frac{x^{n_1} + \cdots + x^{n_k}}{k}$$

converges to h in the norm. By convexity, $y^{n_k} \in g_{c,a}$, and by concavity of $f(\cdot, a)$, $\{y^{n_k}\}$ is a maximizing sequence also. Since f is continuous, h is a maximum on $g_{c,a}$. By (i), $h \in X^+$. We denote h by $h(a, c)$ also. Uniqueness follows from the assumption of strict concavity of $f(\cdot, a)$ on x. Note that, as discussed in footnote 5, h is in $\bar{g}_{c,a}$.

By [13] (Theorem 1, p. 243) and conditions (ii) and (iii), a necessary condition for $h(a, c)$ to be a maximum is that $\psi = 0$ at $(h(a, c), a, c, \lambda)$ for some $\lambda > 0$ in C^*.[10] Now by condition (iv) and by the implicit function theorem for Banach spaces (see [12]) it follows that $h(a, c)$, which is the solution of system ψ above, is of class C^1.

We now derive the $S(a, c)$ operator.[11] For each $(a, c) \in A_1 \times C_1$, the first order necessary conditions for an optimum are:

$$\begin{aligned} &\psi_1 = 0, \quad \text{i.e.,} \quad g(x, a) - c = 0 \\ \text{and} \quad &\psi_2 = 0, \quad \text{i.e.,} \quad \frac{\partial}{\partial x} L(x, a, c, \lambda) = 0 \end{aligned} \tag{2}$$

[9] We shall not distinguish between an operator and its adjoint.

[10] Note that the fact that f is continuous and defined on X_1, which is a neighborhood of X^+, replaces the condition in [3] of existence of an interior point of X^+.

[11] The approach used here generalizes the approach of Kalman and Intriligator in [10] which is done for one constraint and for finite dimensional spaces.

where, for each fixed (a, c),

$$\psi_1: X_1 \to C,$$
$$\psi_2: X_1 \times C^* \to X^*$$

so that

$$\psi: X_1 \times C^* \to C \times X^*.$$

Locally, at the maximum, the differential of (2) can be written as:

$$\left(\frac{\partial}{\partial x} g\right) dx + \left(\frac{\partial}{\partial a} g\right) da - dc = 0,$$

$$\left(\frac{\partial^2}{\partial x^2} f\right) dx + \left(\frac{\partial^2}{\partial x \partial a} f\right) da + \left(\left(\frac{\partial^2}{\partial x^2} g\right) dx\right) \lambda + \left(\left(\frac{\partial^2}{\partial x \partial a} g\right) da\right) \lambda + \left(\frac{\partial}{\partial x} g\right) d\lambda = 0, \tag{3}$$

where

$$\left(\left(\frac{\partial^2}{\partial x^2} g\right) dx\right) \lambda \quad \text{denotes} \quad \sum_j \lambda_j \left(\left(\frac{\partial^2}{\partial x^2} g^j\right) dx\right)$$

and similarly for

$$\left(\left(\frac{\partial^2}{\partial x \partial a} g\right) da\right) \lambda.$$

System (3) in turn, can be written as

$$\begin{pmatrix} 0 & \left(\frac{\partial}{\partial x} g\right) \\ \left(\frac{\partial}{\partial x} g\right) & \left(\frac{\partial^2}{\partial x^2} L\right) \end{pmatrix} \begin{pmatrix} d\lambda \\ dx \end{pmatrix} = \begin{pmatrix} -\left(\frac{\partial}{\partial a} g\right) da + dc \\ -\left(\frac{\partial^2}{\partial x \partial a} L\right) da \end{pmatrix} \tag{4}$$

where as defined above

$$\begin{aligned} &\frac{\partial}{\partial x} L: X_1 \times A_1 \times C_1 \times C^* \to X^* \\ &\frac{\partial^2}{\partial x \partial a} L: X_1 \times A_1 \times C_1 \times C^* \to \mathscr{L}(A, X^*) \end{aligned} \tag{4'}$$

and similarly

$$\frac{\partial^2}{\partial x^2} L: X_1 \times A_1 \times C_1 \times C^* \to \mathscr{L}(X, X^*)$$

so that for each a, c at the maximum $h(a, c)$ and at the corresponding λ,

$$\frac{\partial^2}{\partial x^2}(L(h, a, c, \lambda)) \in \mathscr{L}(X, X^*).$$

To simplify notation we now denote $(\partial^2/\partial x^2) L$ at $(h(a, c), a, c, \lambda)$ by $(\partial^2/\partial x^2) L$ also; by the assumption of existence of Z, $(\partial^2/\partial x^2) L$ is invertible.[12]

Thus, by (iv),

$$\begin{pmatrix} d\lambda \\ dx \end{pmatrix} = \begin{pmatrix} 0 & \left(\frac{\partial}{\partial x} g\right) \\ \left(\frac{\partial}{\partial x} g\right) & \left(\frac{\partial^2}{\partial x^2} L\right) \end{pmatrix}^{-1} \begin{pmatrix} -\left(\frac{\partial}{\partial a} g\right) da + dc \\ -\left(\frac{\partial^2}{\partial x \partial a} L\right) da \end{pmatrix}. \tag{5}$$

By results of inverting a partioned matrix we have

$$\begin{pmatrix} 0 & \left(\frac{\partial}{\partial x} g\right) \\ \left(\frac{\partial}{\partial x} g\right) & \left(\frac{\partial^2}{\partial x^2} L\right) \end{pmatrix}^{-1}$$

$$= \left(\begin{array}{c|c} Z & -Z\left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \\ \hline -\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z & \begin{array}{l} \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \right. \\ \qquad \left. \times \left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right] \end{array} \end{array}\right), \tag{6}$$

where

$$Z = -\left[\left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right)\right]^{-1}. \tag{6'}$$

From (5) and (6) we obtain

$$\begin{aligned} dx = {} & \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \left(\left(\frac{\partial}{\partial a} g\right) da - dc\right) \\ & - \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x \partial a} L\right) da. \end{aligned} \tag{7}$$

[12] Since X and A are Hilbert spaces and g is convex in the variable x, for each (a, c) the operator $(\partial^2/\partial x^2 L)$ will be negative definite at the (x, λ) which satisfy the first order conditions $\phi(x, a, c, \lambda) = 0$ when x is a maximum, and thus $(\partial^2/\partial x^2) L$ will be invertible.

From (7) we obtain

$$\frac{\partial}{\partial a}x = \left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z\left(\frac{\partial}{\partial a}g\right)$$

$$-\left[\left(\frac{\partial^2}{\partial x^2}L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z\left(\frac{\partial}{\partial x}g\right)\left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\right]\left(\frac{\partial^2}{\partial x\partial a}L\right) \tag{8}$$

and

$$\frac{\partial}{\partial c}x = -\left(\frac{\partial^2}{\partial x^2}L\right)^{-1}\left(\frac{\partial}{\partial x}g\right)Z. \tag{9}$$

We now consider the effect of a "compensated" change in the vector a, obtained by a change in the parameter c, which keeps the value of the objective function constant, i.e., when

$$df = \left(\frac{\partial}{\partial x}f\right)dx + \left(\frac{\partial}{\partial a}f\right)da = 0.$$

From (2), this implies that at the maxima,

$$-\lambda\cdot\left(\frac{\partial}{\partial x}g\right)dx + \left(\frac{\partial}{\partial a}f\right)da = 0. \tag{10}$$

Also,

$$dc = \left(\frac{\partial}{\partial x}g\right)dx + \left(\frac{\partial}{\partial a}g\right)da. \tag{11}$$

Hence, by (10) and (11), when $df = 0$

$$-\lambda\left(dc - \frac{\partial}{\partial a}g\,da\right) + \frac{\partial}{\partial a}f\,da = 0 \tag{12}$$

which implies in particular that when $df = 0$, the dc's are not all linearly independent. We now choose one of the constraints—say the ith one—to perform the "compensation," i.e., to insure that the optimal vector stays on the surface $f = r$, on which $df = 0$. Then, if c^i is the i-th component of the vector c,[13] in component form, (12) can be rewritten as[14]

$$\left(dc^i - \left(\frac{\partial}{\partial a} g^i\right) da\right) = \frac{1}{\lambda_i}\left(\frac{\partial}{da} f\right) da - \frac{1}{\lambda_i} \sum_{\nu \neq i} \lambda_\nu \left(c^\nu - \left(\frac{\partial}{\partial a} g^\nu\right) da\right). \tag{13}$$

Thus (12) and (13) imply that

$$dc - \left(\frac{\partial}{\partial a} g\right) da, \qquad \text{when} \qquad df = 0 \tag{14}$$

is

$$\mu \left(\frac{\partial}{\partial a} f\right) da + \phi \left(dc - \left(\frac{\partial}{\partial a} g\right) da\right), \tag{14'}$$

where for each (x, a, c, λ), $\mu \colon R \to C^*$ is defined by[15]

$$\mu = \Big(0, \ldots, 0, \overbrace{\frac{1}{\lambda_i}}^{i\text{th place}}, 0, \ldots\Big)$$

[13] In a basis of the Hilbert space C. Similarly, locally the dc^i are a "basis" for the cotangent bundle of C at c.

[14] If c is a real number and there is one constraint, Equation (13) becomes

$$dc - \left(\frac{\partial}{\partial a} g\right) da = \frac{1}{\lambda}\left(\frac{\partial}{\partial a} f\right) da. \tag{13'}$$

And, in the classical case, where $a = p$ (price), $g(x, a) = p \cdot x$, $c = I$ (income), x is consumption, (13) becomes

$$dI - x \cdot dp = 0. \tag{13''}$$

Note that the "compensation" has the effect of making the components of dc to be not all linearly independent on the surface $f = r$. For instance, in Equation (13), dc^i is a function of all dc^j, $j \neq i$. Note that $\phi(dc) = 0$ does not imply $dc = 0$; the analog of this situation in the classical case is the fact that dI, I = income, is not a "free" real variable any more when $f = \bar{f}$, since $dI = x \cdot dp$. In the classical consumer case the fact that Equation (14), when $df = 0$, becomes (14'), is equivalent to the classical condition that $dc - x \cdot dp$ (c denotes income) becomes zero when $df = 0$; this follows from the fact that $((\partial/\partial a) f) = 0$ in the classical consumer case (since f does not depend on a), and also that ϕ in this case is zero (see, for instance, [15]).

[15] $\mu \colon R \to C^*$ ($\cong l_2$) will be well defined if the conditions $\phi = 0$ holds for $\lambda \gg 0$ in C^* at the maximum. $\lambda \gg 0$ means $\lambda(c) > 0$ for all c in C^*. In [1] sufficient conditions are given for the existence of a strictly positive supporting hyperplane (or Lagrangian multiplier) $\lambda \gg 0$, in a different context.

and $\phi: C^* \to C^*$ ($C \cong C^*$) is defined by

$$
\begin{aligned}
\phi &= (\phi_{l,j}),\\
\phi_{l,j} &= 1 && \text{if} \quad l = j, \quad l \neq i,\\
&= 0 && \text{if} \quad l \neq j \quad \text{and} \quad l \neq i,\\
\phi_{i,j} &= -\frac{\lambda_j}{\lambda_i} && \text{if} \quad j \neq i,\\
\phi_{i,i} &= 0,
\end{aligned}
$$

and where $\phi \equiv 0$ if c is in R.[16]

Therefore, from (7), (14) and (14′) (denoting, as usual dx, when $df = 0$, by $dx\,|_{f=r}$),

$$
\begin{aligned}
dx\,|_{f=r} = {} & \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(-\mu \left(\frac{\partial}{\partial a} f\right) da - \phi \cdot dc + \phi \left(\frac{\partial}{\partial a} g\right) da\right)\\
& - \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right] \left(\frac{\partial^2}{\partial x \partial a} L\right) da
\end{aligned}
$$

and thus, when $\phi(dc) = 0$, one obtains

$$
\begin{aligned}
\left.\frac{\partial x}{\partial a}\right|_{f=r} = {} & \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(-\mu \left(\frac{\partial}{\partial a} f\right) + \phi \left(\frac{\partial}{\partial a} g\right)\right)\\
& - \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right] \left(\frac{\partial^2}{\partial x \partial a} L\right).
\end{aligned}
\tag{15}
$$

So, by (8), (9) and (15) at the maximum we obtain:

$$
\begin{aligned}
& \frac{\partial}{\partial a} h + \frac{\partial}{\partial c} h \left(\frac{\partial}{\partial a} g\right)\\
& \quad = \left.\frac{\partial h}{\partial a}\right|_{f} + \frac{\partial}{\partial c} h \left(\phi \left(\frac{\partial}{\partial a} g\right) - \mu \left(\frac{\partial}{\partial a} f\right)\right)\\
& \quad = -\left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right] \left(\frac{\partial^2}{\partial x \partial a} L\right)\\
& \quad = S(a, c),
\end{aligned}
\tag{16}
$$

which completes the proof.

[16] If C is an $l_2[0, \infty)$ space with a finite measure on $[0, \infty)$ given by the density function λ^{-t}, $t \in [0, \infty)$ (λ a constant in $(0, 1)$) as in [4] and [5], then for ϕ to be a well defined continuous operator from l_2 to l_2, a necessary and sufficient condition is that

$$\sum_{j=1}^{\infty} \lambda^{-j} \left(\frac{\lambda_i}{\lambda_j}\right)^2 < \infty$$

for all i.

Remark. Sufficient conditions for invertibility of the operators $[\partial/(\partial x, \lambda)]\,\psi$, and of

$$\left(\left(\frac{\partial}{\partial x} g\right)\left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\left(\frac{\partial}{\partial x} g\right)\right)$$

required in Theorem 1 can be obtained in certain cases for instance, by direct examination of these operators, which involve first and second order partial derivatives of the functions f and g. For instance when the spaces X, A and C are sequence spaces, these operators will be given by infinite matrices. Conditions for invertibility of infinite matrices have been studied, for instance, by Kantorovich in [11]. If X, A and C are spaces of L_2 functions on the line, one can use Fourier transform techniques as, for instance, those of [11]. However, invertibility of operators is a delicate point which requires technical considerations of its own; in this case, it requires conditions on the above operators (and thus on f and g) and on the spaces where the problem is defined, depending on the particular nature of the model. Other techniques to study general invertibility of related operators are given in [6] for finite dimensional spaces, by use of the Sard theorem. These latter results can be extended to infinite dimensional spaces, in certain cases, by use of an infinite dimensional version of the Sard theorem [16].

2

The classical property of symmetry of the Slutsky Hicks Samuelson matrix which in this framework becomes the operator $S(a, c)$ in Section 1, is, in general, not preserved [10]. For certain classes of objective functions and constraints, symmetry of $S(a, c)$ can be recovered, as seen in the next results. These classes of functions have been used in finite dimensional models of the firm, the consumer, and micromonetary models.

In what follows we assume that all spaces are Hilbert spaces of sequences.

PROPOSITION 1. *Assume the objective function $f(x, a)$ has the form*

(i) $f = \gamma[a \cdot x] + f^1(x) + f^2(a)$

and the constraints $g(x, a)$ have the form

(ii) $g^i = \delta^i[a \cdot x] + g^{i1}(x) + g^{i2}(a)$, $i = 1, 2, \ldots,$

and that the conditions of Theorem 1 of Section 1 are satisfied where $a \in A^+ \subset X^{*+}$, $c \in C^+$, $\gamma, \delta^i \in R^+$ and f, g^i have the same properties as f and g of Theorem 1. Then there exists a unique global C^1 solution for Problem (1) of Section 1, and $S(a, c)$ is symmetric.

Proof. In view of (8), (9) and (15), we obtain:

$$S(a, c) = -\left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right] \left(\frac{\partial^2}{\partial x \partial a} L\right).$$

By computing the operator $(\partial^2/\partial x \partial a) L$ for the above objective and constraint functions we obtain:

$$\frac{\partial^2}{\partial x \partial a} L = \begin{pmatrix} \gamma + \lambda\delta & & 0 & \\ & \ddots & & \\ 0 & & \gamma + \lambda\delta & \\ & & & \ddots \end{pmatrix}.$$

Note that

$$\left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right]$$

is symmetric. This completes the proof.

PROPOSITION 2. *Under the conditions of Proposition 1, $S(a, c)$ is negative semi-definite if $\gamma + \sum_{i=1}^{\infty} \lambda_i \delta^i \geq 0$.*

Proof. Negative semi-definiteness of $S(a, c)$ is obtained from the conditions for (i) and (ii) of Proposition 1 as follows:

First we prove that

$$D = \left[\left(\frac{\partial^2}{\partial x^2} L\right)^{-1} + \left(\frac{\partial^2}{\partial x^2} L\right)^{-1} \left(\frac{\partial}{\partial x} g\right) Z \left(\frac{\partial}{\partial x} g\right) \left(\frac{\partial^2}{\partial x^2} L\right)^{-1}\right]$$

is negative semi-definite.

Let z be any vector, and define a quadratic form $Q_D = z'Dz$. Let $H = ((\partial^2/\partial x^2) L)$, and $H^{-1/2}$ be the symmetric negative square root of H^{-1}. Define

$$u = H^{-1/2} v,$$

where

$$v = -\frac{\partial}{\partial x} g, \quad \text{and} \quad y = H^{-1/2} z.$$

Then,

$$Q_D = y'y - y'u(u'u)^{-1} u'y$$
$$= \| y \|^2 - \| u \|^{-2} \| u'y \|^2.$$

By the Schwarz inequality [8], $Q_D \geqslant 0$. So, $S(a, c)$ will be negative semi-definite if $(\partial^2/\partial x \partial a) L$ is positive semi-definite since under the conditions of the proposition $(\partial^2/\partial x \partial a) L$ is diagonal. But $(\partial^2/\partial x \partial a) L$ is positive semi-definite if $\gamma + \sum_{i=1}^{\infty} \lambda_i \delta^i \geqslant 0$. This completes the proof.

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