A Note on the Solow Growth Model with a CES Production Function and Declining Population

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Abstract

This study investigates the relationship between per capita output growth and population growth using the Solow growth model when population growth is negative. When the Cobb-Douglas production function is used, the per capita output growth rate is positive even if the technological progress rate is zero. In contrast, when the CES production function is used, the per capita output growth rate is zero if the technological progress rate is zero.

Keywords: Solow growth model; negative population growth; CES production function

JEL Classification: E13; E23; O41

1 Introduction

This study investigates the long-run growth rate of per capita output by using the Solow growth model when population growth is negative and the production function takes the constant-elasticity-substitution (CES) form. Almost all growth models assume that population growth is non-negative, and do not consider negative population growth. However, as long as there are countries with negative population growth such as Japan, we need to investigate a case in which population growth is negative.

Christiaans (2011) is a preceding study that introduces negative population growth into a growth model. He builds a semi-endogenous growth model with an increasing returns to scale production function and shows that the long-run growth rate of per capita output

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is positive even if population growth is negative. He also shows that when the constant returns to scale Cobb-Douglas production function is used and negative population growth is assumed, the long-run growth rate of per capita output is positive even if the exogenously given technological progress rate is zero. This result suggests that technological progress may not be so important in economies with declining population. He only investigates the case in which the Cobb-Douglas production function is used.

However, empirical studies with regard to the size of elasticity of substitution suggest that the elasticity of substitution is less than unity (Rowthorn, 1999; Antras, 2004; Klump, McAdam, and Willman, 2007; and Chirinko, 2008). Based on these empirical results, we use the CES production function. Our result shows that when the elasticity of substitution between capital and labor is less than unity, the long-run growth rate of per capita output converges to the exogenously given technological progress rate even if population growth is negative. This result implies that the engine of long-run growth is technological progress. Therefore, The result obtained from the Cobb-Douglas case is largely different from the one obtained from the CES case.

The rest of the paper is organized as follows. Section 2 investigates the long-run growth rate of per capita output when the production function takes the Cobb-Douglas form and population growth is negative. Section 3 investigates the long-run growth rate of per capita output when the production function takes the CES form according to whether population growth is positive or negative. Section 4 concludes the paper.

2 Cobb-Douglas production function with negative population growth

In this section, by using the Solow (1956) growth model, we investigate the long-run growth rate of per capita output when population growth is negative.

Suppose that the production function takes the following Cobb-Douglas form:

$$Y = K^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1}$$

where Y denotes output; K, capital stock; L, labor input; and A, an index of technology. The parameter α denotes the output elasticity with respect to capital.

Let capital stock per effective labor be $k = K/(AL)$. Suppose that a constant fraction s of total output Y is saved. We assume that the growth rate of population is constant $\dot{L}/L = n$, the growth rate of technological progress is constant $\dot{A}/A = \gamma > 0$, and the rate of capital depreciation is constant $\delta \in [0, 1]$. Then, the dynamics of k is given by

$$\dot{k} = sk^{\alpha} - (n + \gamma + \delta)k. \tag{2}$$

Following the procedure of Christiaans (2011), we investigate the case in which population growth is negative. When $n < -(\gamma + \delta) < 0$, that is, the population growth rate is negative and the absolute value of it is large, we have $\dot{k} > 0$ as long as $k > 0$. Hence, k continues to increase. Then, the growth rate of k is given by

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (n + \gamma + \delta) > 0. \tag{3}$$

With $\alpha - 1 < 0$, as k increases, we obtain

$$\lim_{k \to +\infty} \frac{\dot{k}}{k} = -(n + \gamma + \delta) > 0. \tag{4}$$

This suggests that the growth rate of k is positive even if population growth is negative.

Let g_x be the growth rate of a variable x. The growth rate of per capita output Y/L leads to

$$\begin{aligned} g^*_{Y/L} &= \gamma + g_y = \gamma + \alpha g_k \\ &= \gamma - \alpha(n + \gamma + \delta) > 0, \end{aligned} \tag{5}$$

where "$*$" denotes the long-run value.

When the technological progress rate is zero, that is, $\gamma = 0$, we obtain

$$g^*_{Y/L} = -\alpha(n + \delta) > 0. \tag{6}$$

Therefore, the long-run growth rate of per capita output is positive even if the technological progress rate is zero. In contrast, when population growth is positive, $g_{Y/L}$ is given by $g_{Y/L} = \gamma$. Accordingly, when the technological progress rate is zero, we have $g_{Y/L} = 0$. However, when population growth is negative, per capita output can continue to increase even without technological progress, which is a strong result.

The above results are summarized in Figure 1. The horizontal axis and vertical axis correspond to the growth rate of population and the long-run growth rate of per capita output, respectively.

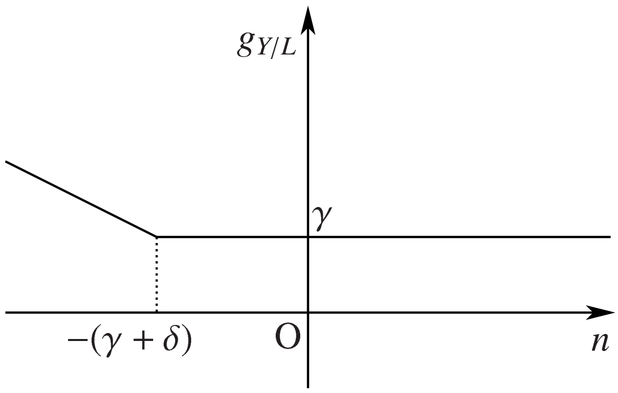

Figure 1: Relationship between n and $g_{Y/L}$: Cobb-Douglas production function

3 Solow model with a CES production function

3.1 Positive population growth

Suppose that the production function takes the following CES form.

$$Y = [\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}, \quad 0 < \alpha < 1,\ \sigma > 0, \tag{7}$$

where α denotes a constant parameter and σ, the elasticity of substitution between capital and labor.

Output per effective labor is given by.

$$y = f(k) = [\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)]^{\frac{\sigma}{\sigma-1}}. \tag{8}$$

The equation of motion of k is given by

$$\dot{k} = s[\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)]^{\frac{\sigma}{\sigma-1}} - (n+\gamma+\delta)k. \tag{9}$$

When the production function takes the CES form, Inada conditions may not hold depending on conditions (Barro and Sala-i-Martin, 2003). Then, we may not have $k > 0$ such that $\dot{k} = 0$.

The marginal productivity of capital is given by

$$f'(k) = \alpha[\alpha + (1-\alpha)k^{\frac{1-\sigma}{\sigma}}]^{\frac{1}{\sigma-1}}. \tag{10}$$

When $0 < \sigma < 1$, with regard to the marginal productivity of capital, we obtain the following relationships:

$$\lim_{k\to 0} f'(k) = \alpha^{\frac{\sigma}{\sigma-1}}, \tag{11}$$

$$\lim_{k\to \infty} f'(k) = 0. \tag{12}$$

From this, k converges to $k = 0$ if the following condition holds:

$$n + \gamma + \delta > s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{13}$$

In this case, the long-run growth rate of per capita output leads to

$$g^*_{Y/L} = s\alpha^{\frac{\sigma}{\sigma-1}} - (n + \delta), \tag{14}$$

which can be positive or negative depending on the size of n.

On the other hand, k converges to $k^* > 0$ if the following condition holds:

$$n + \gamma + \delta < s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{15}$$

Then, the long-run growth rate of per capita output leads to

$$g^*_{Y/L} = \gamma > 0. \tag{16}$$

When $\sigma > 1$, with regard to the marginal product of capital, we obtain the following relationships.

$$\lim_{k\to 0} f'(k) = \infty, \tag{17}$$

$$\lim_{k\to \infty} f'(k) = \alpha^{\frac{\sigma}{\sigma-1}}. \tag{18}$$

From this, k converges to $k^* > 0$ if the following condition holds:

$$n + \gamma + \delta > s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{19}$$

Then, the long-run growth rate of per capita output is given by

$$g^*_{Y/L} = \gamma > 0. \tag{20}$$

On the other hand, $\dot{k} > 0$ as long as $k > 0$ if the following condition holds.

$$n + \gamma + \delta < s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{21}$$

Then, we obtain

$$\lim_{k\to\infty} \frac{\dot{k}}{k} = s\alpha^{\frac{\sigma}{\sigma-1}} - (n + \gamma + \delta) > 0 \tag{22}$$

From this, the long-run growth rate of per capita output is given by

$$\begin{aligned} g^*_{Y/L} &= \gamma + \frac{\dot{k}}{k} \\ &= s\alpha^{\frac{\sigma}{\sigma-1}} - (n + \delta) > 0. \end{aligned} \tag{23}$$

3.2 Negative population growth

We consider a case in which $n < -(\gamma + \delta) < 0$.

$$\frac{\dot{k}}{k} = s[\alpha + (1-\alpha)k^{\frac{1-\sigma}{\sigma}}]^{\frac{\sigma}{\sigma-1}} - \underbrace{(n + \gamma + \delta)}_{-} > 0. \tag{24}$$

Accordingly, k continues to increase.

The growth rate of per capita output can be written as follows:

$$g_{Y/L} = \gamma + g_y \tag{25}$$

$$g_y = \frac{\dot{k}}{k} \times \pi(k) = \frac{\dot{k}}{k} \times \frac{1}{1 + \frac{1-\alpha}{\alpha} k^{\frac{1-\sigma}{\sigma}}}, \tag{26}$$

where $\pi(k)$ denotes the capital share of income.

When $0 < \sigma < 1$, we obtain

$$\lim_{k\to\infty} \frac{\dot{k}}{k} = -(n + \gamma + \delta) > 0. \tag{27}$$

Then, we obtain $\lim_{k\to\infty} g_y = 0$, and accordingly, the long-run growth rate of per capita output is given by

$$g^*_{Y/L} = \gamma > 0. \tag{28}$$

Therefore, $g^*_{Y/L}$ is equal to the technological progress rate.

When $\sigma > 1$, we obtain

$$\lim_{k \to \infty} \frac{\dot{k}}{k} = s\alpha^{\frac{\sigma}{\sigma-1}} - (n + \gamma + \delta) > 0. \tag{29}$$

Then, we have $\lim_{k\to\infty} g_y = \dot{k}/k$, and the long-run growth rate of per capita output is given by

$$g^*_{Y/L} = s\alpha^{\frac{\sigma}{\sigma-1}} - (n + \delta) > 0. \tag{30}$$

Summarizing the above results, we obtain the following two figures. In these figures, $\bar{n} = s\alpha^{\frac{\sigma}{\sigma-1}} - \gamma - \delta > 0$.

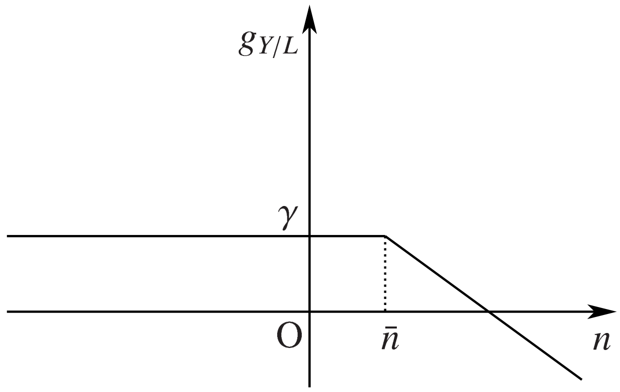

Figure 2: Relationship between n and $g_{Y/L}$: CES production function and $\sigma < 1$

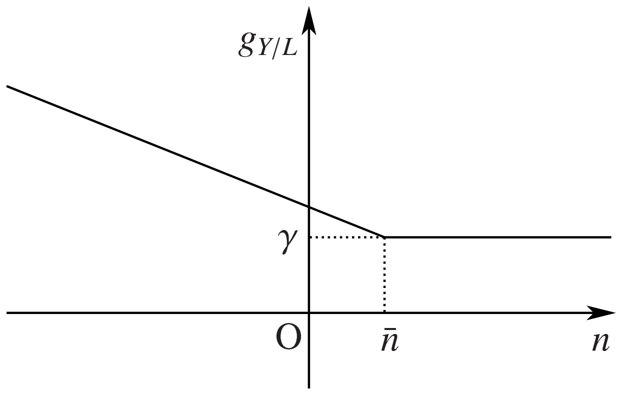

Figure 3: Relationship between n and $g_{Y/L}$: CES production function and $\sigma > 1$

4 Discussions

We explain the reason why we obtain the different results between the Cobb-Douglas case and the CES case.

The growth rate of per capita output can be rewritten as

$$g_{Y/L} = \gamma + \pi(k)\left[s \cdot \frac{Y}{K} - (n + \gamma + \delta)\right]. \tag{31}$$

Therefore, the dynamics of the capital share $\pi(k)$ and the output-capital ratio Y/K determine the value of the long-run growth rate of per capita output $g^*_{Y/L}$.

When the Cobb-Douglas production function is used and $n < 0$, the capital share remains constant, that is, $\pi(k) = \alpha$, and in addition, we obtain

$$\lim_{t\to\infty} \frac{Y}{K} = 0. \tag{32}$$

In contrast, when the production function takes the CES form and $n < 0$, we obtain the following properties.

$$\lim_{k\to\infty} \pi(k) = \begin{cases} 0 & \text{if} \quad 0 < \sigma < 1, \\ 1 & \text{if} \quad \sigma > 1. \end{cases} \tag{33}$$

$$\lim_{t\to\infty} \frac{Y}{K} = \begin{cases} 0 & \text{if} \quad 0 < \sigma < 1, \\ \alpha^{\frac{\sigma}{\sigma-1}} & \text{if} \quad \sigma > 1. \end{cases} \tag{34}$$

For instance, with $0 < \sigma < 1$, the capital share $\pi(k)$ approaches zero as time passes, and hence, $g_{Y/L}$ approaches the rate of technological progress γ. On the contrary, in the Cobb-Douglas case, the capital share is constant and α, and the output-capital ratio Y/K approaches zero as time passes. As a result, the long-run growth rate of per capita output is given by equation (5).

5 Conclusion

Our analysis shows that when the elasticity of substitution is less than unity and population growth is negative, the long-run growth rate of per capita output is given by the exogenous rate of technological progress. Therefore, as long as the technological progress rate is zero, the growth rate of per capita output is zero.

On the contrary, if the elasticity of substitution is unity, that is, the production function takes the Cobb-Douglas form, then the long-run growth rate of per capita output is positive even without technological progress when population growth is negative.

We conclude that the result that the economy can attain sustainable growth even if population growth is negative and technological progress is zero depends on the specificity of the Cobb-Douglas production function. In a more realistic case where the elasticity of substitution between capital and labor is less than unity, technological progress plays an important role in economic growth irrespective whether population growth is positive or negative.

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