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VaR Risk Measures versus Traditional Risk Measures: an Analysis and Survey

Abstract

The article presents an analysis and survey regarding the validity of VaR risk measures in comparison to traditional risk measures. Individuals are assumed to either maximize their expected utility or possess a lexicographic utility function. The analysis is carried out for generally distributed functions and for the normal and log-normal distributions. The main conclusion is that although VaR is an inadequate measure within the expected utility framework, it is at least as good as other traditional risk measures. Moreover, it can be improved by modified versions such as the Accumulated-VaR (Mean-Shortfall) Assuming a lexicographic expected utility strengthens the argument for using AVaR as a legitimate risk measure especially in the case of a regulated firm.

In recent years, both the importance of risk measurement and the possibilities of efficient risk management have increased dramatically. This is a result of the globalization of the financial markets, the technological revolution in trading systems and communications and, perhaps most important, the development of derivative tools and markets. The main approach to risk measurement acknowledges the fact that portfolio management relates to the entire distribution, although specific aspects of the distribution can still be quantified by a single risk measure.

The widespread adoption of Value-at-Risk (VaR) as a risk management tool is part of this approach.[1] Formally, VaR is primarily used for measuring market risk which is defined as a decrease in the value of a position due to changes in the financial market prices. According to the Basel (1996) Amendment, financial institutions should maintain eligible capital against their market VaR in addition to the conventional capital requirements for credit risk. In addition, the Securities and Exchange Commission (SEC) allows the use of VaR in order to report market risk exposure. Recently, there has been increasing use of the VaR measure as a tool for managing and regulating credit risk and as a methodology for constraining and controlling the risk exposure of a portfolio.[2]

Most papers that have examined the validity of VaR have concentrated on its practical statistical and computational difficulties[3] and the implied perilous results (see Jorion (2000a)). Other papers have investigated VaR in the context of a portfolio.[4] Only a few pioneer papers have investigated its theoretical merits. Artzner, Delbaen, Eber & Heath (1997, 1999) analyze the fundamental requirements of risk measures. They examine whether VaR is a coherent risk measure and analyze its merits and drawbacks. Basak & Shapiro (2001) address the superiority of the Accumulate-VaR (AVaR or Mean-Shortfall), which is a variation of VaR, over VaR itself as a regulatory tool. Their important work was the first to examine the VaR constraints in terms of portfolio optimization and utility maximization.

The aim of this article is to expand the concept of VaR as a general risk measure and to examine the validity of the various VaR measures as legitimate tools for estimating specific elements of risk for decision making under uncertainty. In doing so, the article compares the various VaR measures and the traditional measures of risk and gives an overview of the relations between them. The article does not intend to analyze all aspects of VaR or to survey the huge high quality literature on the subject.

Instead, it attempts to provide a missing piece in the puzzle regarding the merits of VaR as a decision-making measure in comparison with other risk measures.

The analysis and comparison assumes the case in which individuals are risk-averse and either maximize expected utility or lexicographic expected utility. The latter is defined as a two-step process in which "safety" is considered first (by the individuals or by the regulator) and only then is expected utility maximized.

The paper shows that the mean-VaR criterion identifies alternatives which are inferior for all rational individuals. Similarly, the mean-AVaR criterion identifies alternatives which are inferior for all risk-averse individuals. The paper also investigates the type of utility functions that are consistent with VAR-derived risk measures.

Understanding these utility functions is important since they illustrate the behavior induced by VaR consideration. Surprisingly, the paper shows that the use of VaR implies irrational utility functions which do not guarantee the more-over-less preference. This drawback is substantially reduced if AVaR is used in place of VaR. These results reinforce the previous results of Artzner et al. and Basak & Shapiro (2001) in favor of AVaR over VaR both as a risk measure and as a regulatory constraint.

The paper is organized as follows: The following section provides preliminary background and presents some recently proposed VaR measures. The validity of these VaR measures is examined while assuming expected utility maximization, risk aversion and Decreasing Absolute Risk Aversion (DARA). The mean-VaR criteria for specific distributions are also developed. In section II we review the traditional measures of risk and compare them to the VaR measures. The efficiency analysis of the various risk measures in terms of mean-risk is also reviewed. Section III thoroughly analyzes the congruence of VaR measures with expected utility. In addition, this section examines the validity of VaR under simple lexicographic utility. Section IV concludes the paper.

I. The VaR Measures of Risk

Denote by X a random variable with density function f(x) and cumulative distribution function (cdf) F(X).

Define the quantile X(P) of X as the maximum value of X for which there is a probability of P to be below this value under the cdf of F(X). Formally, the definition of X(P) is: $\Pr(X \leq X(P)) = P$.[5]

Value-at-Risk at $1-\hat{P}$ confidence interval, VaR($\hat{P}$), can be defined as the loss below some reference target, η(F(X)), over a given period of time, where there exists a confidence interval of $1-\hat{P}$ of incurring this loss or a smaller one.

If η(F(X)) =E(X)≡μ_X, where μ_X is the expected mean of X, then the VaR is the loss below the expected mean, μ_X, and is denoted as VaR_e. If a constant reference point, such as the risk free-return or zero is selected, then it is denoted as VaR_t.

For example, a weekly $VaR_{t=0}$ of \$5 million at the 99 percent confidence interval means that there is a probability of 1 percent to have a loss higher than \$5 million below the current value within the next week.

In terms of the quantile function, VaR($\hat{P}$) can be written simply as:

$$VaR(\hat{P}) = \eta(F(X)) - X(\hat{P}) \qquad (1)$$

VaR calculation involves two primary steps: First, derive the forward distribution of returns.[6] Second, calculate the first $\hat{P}$ percent of this distribution. Figure 1 illustrates this process.

a.1 The VaR with Expected Mean as Reference Point (VaR_e)

Identifying a loss as being below the assumed projected mean has strong intuitive appeal. Baumol (1963, p. 174) claims that "Investment with a relatively high standard deviation will be relatively safe if its expected value is sufficiently high". Thus, he identifies the mean less k times the standard deviation as the subjective "confidence level" for the risk taken by the individual. Nevertheless, the main drawback of VaR_e (as well as any other risk measure which is based on results below the mean) is that it is unaffected by a constant shift of the whole distribution (see also Atkinson (1970, p. 253)). This drawback is particularly important for regulation since it may reduce the sensitivity of this risk measure to economic turndowns and thus reduce its efficacy. This occurs because weak economic conditions may induce a decline in the returns under all states of nature such that the decrease in the quantile function is totally offset by the decrease in the expected return of the distribution (see (1)). Thus, although the absolute loss at a certain confidence interval is higher, there is no change in the magnitude of risk as measured by VaR_e. Therefore, it is not surprising that the official Basel (1996) Amendment recommends calculating the VaR as the potential loss below the current value, i.e. VaR_t.

a.2 The VaR with a Constant Reference Point (VaR_t)

Identifying a loss as a result below some constant reference point implies the existence of some objective standard for success and failure. This was justified by Mao (1970b) for the Semi-Variance measure of risk based on his finding that executives explain risk as the chance of failing to meet their target. The same argument is also valid for the VaR measures. Correspondingly, Markowitz (1959) and Mao (1970a) show that the constant reference point's Semi-Variance is consistent with the maximization of expected utility using a utility function which guarantees the more-over-less preference and risk aversion assumptions. In section III we show that this is also true in the case of $AVaR_t$, which is presented below, although not in the case of VaR_t.

a.3 The Accumulate-VaR (AVaR)

AVaR, which is also known as Conditional-VaR or Mean-Shortfall, was introduced by Embrechts, Klueppelberg & Mikosch (1997), Artzner et al. (1997, 1999), Basak & Shapiro (2001) and Longin (2001) and was further investigated by Uryasev (2000) and others. Next section we show that a simplified version of AVaR was introduced many years ago by Domar & Musgrave (1944). This simplified version of AVaR is also a specific variation of the Fishburn (1977) α-t risk model.

The $1-\hat{P}$ confidence interval $AVaR_t$ can be defined in terms of VaR as:

$$AVaR_t(\hat{P}) = \int_0^{\hat{P}} VaR_t(P)dP \qquad (2)$$

where AVaR is usually normalized by the multiple $1/\hat{P}$. AVaR averages the VaRs with a confidence interval that ranges from $1-\hat{P}$ to 1. AVaR can be viewed as the expected loss, relative to the chosen reference point, within a constant range of probabilities 0 to $\hat{P}$.

Figure 2 presents the AVaR graphically. Analogously to VaR, the AVaR is equal to the area of probability $\hat{P}$ times the reference point ($\eta(F(X))\times\hat{P}$), minus the expected lower results which are represented by the area to the left of the cdf ($\int_0^{\hat{P}} X(P)dP$).[7]

Artzner et al. show that in contrast to VaR, AVaR fulfills the four conditions of a coherent risk measure: homogeneity, monotonicity, the risk-free condition and the sub-additive property which guarantees convexity. Other papers concentrate on its advantages as a regulatory measure. Basak & Shapiro (2001) show its superiority over VaR as a constraint on a portfolio in that it produces more reasonable results that are desirable for regulators. Longin (2001) takes it a step further by suggesting that AVaR be used as management and regulation tool for market risk during extraordinary market conditions. Uryasev (2000) introduces some practical advantages of AVaR, such as the ability to optimize it using linear programming and non-smooth optimization algorithms for empirical distributions, subject to the number of scenarios being finite.

a.4 The AAVaR

In an effort to find an optimal measure of risk, AVaR can be further modified to produce the Accumulate-AVaR (AAVaR). Define AAVaR_t with $1\text{-}\hat{P}$ confidence interval as:

$$AAVaR_t(\hat{P}) = \int_0^{\hat{P}} AVaR_t(P)dP \qquad (3)$$

AAVaR shares the same advantages of AVaR as a regulatory measure in that it is a coherent measure of risk as well as a single value which summarizes the profile of the losses beyond VaR. Moreover, its calculation does not require any additional information beyond that required for AVaR calculation. Following Longin (2001), who suggests adopting AVaR as a tool for managing risk during extraordinary market conditions, AAVaR appears to be well-suited for this purpose. Further research in the spirit of Basak & Shapiro (2001) is required to understand the intuition behind it and to investigate its merits and the repercussions of its use as a regulatory constraint. Finally, we show below that AAVaR has additional advantages as a decision making measure.

b. Mean-VaR Analysis

Throughout the paper we will use a mean-risk efficiency analysis in order to assess and compare the VaR measures as decision making criteria. Denote by $\underset{rule}{D}$ the dominance relationships between two alternatives according to some given rule. For example, $X \underset{mean-VaR}{D} Y$ states that X dominates Y according to the mean-VaR criterion which is defined as follows:

$X \underset{mean-VaR}{D} Y$ if and only if:

$$\mu_X \geq \mu_Y \qquad (4)$$

and

$$VaR(X) \leq VaR(Y) \qquad (5)$$

with at least one significant inequality.

In what follows we summarize the relations between the various mean-VaR criteria and the Stochastic Dominance rules. These rules are optimal criteria under specific assumptions about the individuals' utility functions (a detailed exposition of the Stochastic Dominance rules is given in Appendix B). The First Stochastic Dominance (FSD) rule is an optimal criterion for all rational individuals who maximize expected utility, where rationality is defined as a non-decreasing utility function. The Second Stochastic Dominance (SSD) rule is an optimal criterion for all rational risk-averse individuals who maximize expected utility, where risk aversion is defined as a non-increasing marginal utility function.

It is generally assumed by economists that the higher the wealth of an individual, the lower is his Arrow-Pratt risk aversion level. This is the Decreasing Absolute Risk Aversion (DARA). In order to fulfill the DARA property, the individual's utility function must have a positive third derivative. The Third Stochastic Dominance (TSD) rule is an optimal criterion for all rational risk-averse individuals with a positive third derivative of their utility function who maximize expected utility.

Let VaR_e, VaR_t , $AVaR_t$ and $AAVaR_t$ be defined as in (1)-(3). Then in the case of VaR_e:

$$\mathrm{X} \underset{FSD}{D} \mathrm{Y} \not\Rightarrow X \underset{mean-VaR_e}{D} Y \qquad (6)$$

In the case of VaR_t:

$$\mathrm{X} \underset{\mathrm{FSD}}{D} \mathrm{Y} \Rightarrow X \underset{mean-VaR_t}{D} Y \qquad (7)$$

$$\mathrm{X} \underset{\mathrm{SSD}}{D} \mathrm{Y} \not\Rightarrow X \underset{mean-VaR_t}{D} Y \qquad (8)$$

In the case of $AVaR_t$:

$$\mathrm{X} \underset{\mathrm{SSD}}{D} \mathrm{Y} \Rightarrow X \underset{mean-AVaR_t}{D} Y \qquad (9)$$

$$\mathrm{X} \underset{\mathrm{TSD}}{D} \mathrm{Y} \not\Rightarrow X \underset{mean-AVaR_t}{D} Y \qquad (10)$$

In the case of $AAVaR_t$:

$$\mathrm{X} \underset{\mathrm{TSD}}{D} \mathrm{Y} \Rightarrow X \underset{mean-AAVaR_t}{D} Y \qquad (11)$$

The proofs are presented in Appendix A.

The efficient set according to the mean-VaR$_t$ rule is a subset of the FSD efficient set. Thus, an inferior alternative for all rational investors is also inferior according to the mean-VaR$_t$ criterion. This does not hold for the mean-VaR$_e$ criterion. Similarly, the efficient set according to the mean-AVaR$_t$ rule is a subset of the SSD efficient set. In other words, if all risk-averse individuals prefer X over Y, then X dominates Y according to mean-AVaR$_t$. This important relation does not hold for mean-VaR criteria. Note also that VaR measures may not reflect the Rothschild & Stiglitz (1970) MPS shift of probabilities from the center to the sides of the distribution, which is the basic definition of an increase in risk. For example, if the lower probability shifts from the "center" of the distribution to any point above $\hat{P}$ then VaR is unchaged by the MPS.

Finally, if all risk-averse investors, who have a positive third derivative, prefer X over Y, then X dominates Y according to mean-AAVaR$_t$.

In order to illustrate the differences between the various mean-VaR criteria consider the folowing example: There exist two FSD efficient alternatives, X and Y, with equal means. Assume also that F(X) intersects F(Y) once from below at a probability of 4 percent, such that X(0.04)=Y(0.04). Under these assumptions, Y is inefficient according to SSD rule. However, dominance according to the mean-VaR$_t$ criterion depends on the selected confidence interval $1-\hat{P}$. For VaR($\hat{P}=0.03$) $X \underset{mean-VaR_t}{D} Y$, for VaR($\hat{P}=0.05$) $Y \underset{mean-VaR_t}{D} X$ and for VaR($\hat{P}=0.04$) there is no mean-VaR$_t$ dominance between X and Y. This example illustrates the sensitivity of VaR to the arbitrarily selected confidence interval, a drawback that was first mentioned by Artzner et al.[8] Consequently, it may rank risks incorrectly.

In contrast, AVaR$_t$ ranking remains fixed in this range. In fact, in this example X dominates Y according to SSD rule. Thus, according to (9) X dominates Y according to the mean-AVaR$_t$ criterion as well and hence one can conclude that for any $\hat{P}$ the AVaR$_t$ of X will be smaller than that of Y.

Note that the relations between the mean-VaR criteria and the Stochastic Dominance rules are analogous but not identical to Fishburn's (1977) Theorem 3 as well as Bawa's (1978) Theorem 1 regarding Lower Partial Moments. In spite of the similarity there is a fundamental difference in the basic assumptions of these measures and those of VaR measures. VaR measures assume that investors assess risk in a completly different process, in that the attitude toward risk is determined not only by the size of the loss but also by the probability of this loss to occur (see next section).

Also, note that the above relations coincide with Alexander & Baptista (2000) who show that the mean-VaR set is a subset of the mean-standard deviation set in the case of normal and t distributions. Their article analyzes the implications of using the mean-VaR criterion and the impact of the selected VaR confidence interval on the efficiency of the mean-VaR criterion.

c. Mean-VaR Efficiency Analysis for Specific Distributions

One can conclude from the previous discussion that in the case of general distributions the mean-VaR criteria are superior to the mean-standard deviation criterion since they provide necessary conditions for dominance among all expected utility maximizing individuals. We show below that in the case of normal and log-normal distributions the VaR risk measures provide a good substitute for standard deviation in the optimal efficiency criteria.

Let X and Y be normally distributed. Then:

$$X \underset{SSD}{D} Y \Leftrightarrow X \underset{mean-VaR_e}{D} Y \Leftrightarrow X \underset{\substack{mean-\text{standard}\\ \text{deviation}}}{D} Y \qquad (12)$$

If one prefers to use the VaR_t risk measure rather than VaR_e,[9] then (12) can be modified as follows:

$X \underset{SSD}{D} Y$ if and only if:

$$\mu_X \geq \mu_Y \qquad (13)$$

and

$$\mu_X + VaR_t(X) \leq \mu_Y + VaR_t(Y) \qquad (14)$$

The proofs are presented in Appendix A.

These relations support the use of the VaR in all cases where normality is assumed. For example, one can replace the beta in the CAPM by a new "VaR-beta" which is based on the VaR risk measure.

Note that in the general case, dominance according to the criterion in (13)-(14) implies dominance according to mean-VaR_t criterion. The opposite does not hold true and thus in the general case the mean-VaR_t efficient set is a subset of this criterion.

Note as well that in the normal case the AVaR measure can also be used to obtain an optimal criterion since, as Uryasev & Rockafellar (1999) show, the VaR and AVaR measures are equivalent in the case of normal distributions.

The log-normal distribution may be more appealing than the normal distribution since returns are bounded from below and it is coresponding with time-continuous trading models (see, for example, Mandelbrot (1963) and Merton (1971, 1973)). Let X and Y be log-normally distributed, $X \sim \Lambda(\hat{\mu}_X, \hat{\sigma}_X)$ and $Y \sim \Lambda(\hat{\mu}_Y, \hat{\sigma}_Y)$, such that Z_X=log (X) and Z_Y=log (Y) are normally distributed with first two moments $\hat{\mu}_X, \hat{\sigma}_X$ and $\hat{\mu}_Y, \hat{\sigma}_Y$, respectively.

Then, $X \underset{SSD}{D} Y$, if and only if:

$$\mu_X \geq \mu_Y \qquad (15)$$

and

$$\ln\left(\frac{\mu_X}{t - VaR_t(X)}\right) \leq \ln\left(\frac{\mu_Y}{t - VaR_t(Y)}\right) \qquad (16)$$

where μ_X and μ_Y are the expected values of X and Y respectively, and $\hat{\mu}_X$ and $\hat{\mu}_Y$ are the expected values of the logs of X and Y, respectively.

Inequality (16) can also be written in terms of VaR_e as follows:

$$\ln\left(\frac{\mu_X}{\mu_X - VaR_e(X)}\right) \leq \ln\left(\frac{\mu_Y}{\mu_Y - VaR_e(Y)}\right) \qquad (17)$$

The proofs are presented in Appendix A.[10]

II. VaR and Traditional Risk Measures

The various risk measures belong to one of two distinct groups, depending on the implied perception of risk. In the first group, risk is measured in terms of the probability-weighted dispersion of results around some reference point. These risk measures are affected by both negative and positive deviations from the target. Obviously, this attitude makes sense in the case of symmetrical distributions. However, in the general case positive deviations cannot be considered to be a source of risk.

In the second group, risk is measured only by results below some reference point. Below we review the most common measures in each group and compare them to VaR measures.

a. Dispersion Measures

a.1 The Standard Deviation Risk Measure: The most common risk measure in the dispersion group is given by:

$$\sigma_x = \sqrt{\int_{-\infty}^{\infty} f(x)(x-\mu_X)^2\,dx} \qquad (17)$$

Voluminous criticism of the standard deviation as a risk measure has been published, most of it relating to its inadequacy with regard to the expected utility theorem (see for example Markowitz (1959), Mao (1970a, 1970b) and many others). As previously shown, the mean-standard deviation criterion is non-optimal and inferior to the mean-VaR_t criterion since it is unable to screen out FSD inferior alternatives. On the other hand, in practice when the distribution has to be estimated from actual data, the standard deviation is much more robust than VaR measures since its calculation is based on the entire distribution.

a.2 The Coefficient of Variation Risk Measure: The Coefficient of Variation is simply the standard deviation divided by the mean. The special merit of the mean-Coefficient of Variation criterion is its optimality in the log-normal case. However, in this case it can be replaced by an optimal criterion based on the mean and the VaR (see Inequality (16)).

a.3 The Expected Absolute Deviations Risk Measure: This dispersion measure is given by:

$$AD = \int_{-\infty}^{\infty} f(x)\left|x-\mu_X\right|dx \qquad (18)$$

Atkinson (1970) discussed this dispersion measure as a measure of Inequality. More recently, Konno & Yamazaki (1991) developed a mean-Absolute Deviation optimization model which utilized this risk measure. The main advantage of their model over the mean-standard deviation model lies in the linearity of this measure and the ability to solve the optimization problem using a linear program. Note that the mean-AVaR criterion shares the same property, if we assume a finite number of scenarios.[11] Moreover, the mean-Absolute Deviation criterion is inferior to the mean-VaR criteria, since it may not screen inferior alternatives according to the relevant Stochastic Dominance rule.

a.4 The Gini Mean Difference Risk Measure: The Gini Mean Difference measures the expected value of the absolute difference between every pair of realizations of the random variable and is given by:

$$\Gamma = \frac{1}{2}\int_a^b\int_a^b |X - x| f(X) f(x) dX dx \qquad (19)$$

As with the mean-AVaR$_t$ criterion, the efficient set of the mean-Gini criterion is a subset of the SSD efficient set (see Yitzhaki (1982, 1983)). However, this criterion may screen out alternatives that can be optimal for some risk-averse individuals.[12] In addition, the mathematical complexity of this measure obscured the intuition behind it and discouraged its use.

b. The Below-a-Reference Point Risk Measures

The risk measures in this group only consider results in the lower part of the distribution and thus they are more appealing as risk measures. In Fishburn's (1977, p. 118) own words, their attractiveness in the framework of the mean-Risk analysis is their ability to "recognize the desire to come out well in the long run while avoiding potentially disastrous setbacks or embarrassing failures to perform up to standard in the short run".

b.1 Fishburn's α-t Risk Measures: Most of the traditional important measures in this group are specific cases of Fishburn's α-t model, which is defined as:

$$\int_{-\infty}^{t} (t - x)^{\alpha} f(x) dx \qquad (20)$$

where α describes different attitudes toward risk. The following risk measures are part of Fishburn's family:

b.1.1 Roy's (1952) Safety-First (SF) Risk Measure (Fishburn's α→0) : Roy's SF measure is defined as the probability of being below a reference point t.
According to Fishburn (1977) and Bawa (1978), the mean-SF criterion screens out all alternatives which are inferior according to the FSD rule. However, the mean-SF criterion may not screen out alternatives which are inferior for all risk-averse individuals. In addition, the main deficiency of SF is that it measures risk only in terms of probability, while totally ignoring the size of the loss.

b.1.2 Domar & Musgrave's (1944) Risk Measure (Fishburn's α=1)

The Domar & Musgrave (1944) risk measure (DM) is actually a simple variation of the $AVaR_t$ and is defined as:

$$DM = \int_{-\infty}^{t} (x-t) f(x) dx \qquad (21)$$

As in the case of $AVAR_t$, the mean-DM efficient set is a subset of the SSD efficient set. However, for $AVaR_t$ the integration is up to a given probability, $\hat{P}$, while the use of DM risk measure implies the comparison of two prospects over a different range of probabilities.

b.1.3 <u>Markowitz's (1959) Semi-Variance (SV) Risk Measure (Fishburn's α=2)</u>: The Markowitz (1959) constant reference point SV is defined as:

$$SV = \int_{-\infty}^{t} f(x)(x-t)^2 dx \qquad (22)$$

Mao (1970a,1970b) shows that the mean-SV criterion is consistent with managers' perception of risk. Bawa (1975) shows that the mean-SV efficient set is a subset of the TSD efficient set. Bey (1979) shows that the mean-SV criterion also identifies a substantial part of the Stochastic Dominance efficient set for both SSD and TSD rules. However, according to Fishburn (1977, p. 116) "there is no compelling a priori reason for taking α=2…". Furthermore, as Bawa (1978) noted the mean-SV criterion is not optimal for DARA utility functions.

b.1.4 <u>Worst Case Scenario (WCS) (Fishburn's α→∞)</u>: A special case in which the VaR, AVaR, AAVaR and Fishburn's risk measures overlap is the Boudoukh, Matthew & Richardson (1995) Worst-Case-Scenario measure, which can be written approximately as: WCS=t-X(0). The main deficiency of this criterion is its tendency to screen out alternatives that can be optimal for some of the investors with finite α. In addition, it is difficult in practice to estimate this measure from actual data and it may go to infinity as the sample size increases.

b.2 <u>Baumol's Risk Measure</u>: Baumol's (1963) measure is given by the expected return minus k times the standard deviation. The parameter k is an arbitrary number which is supposed to reflect the subjective level of risk aversion. The larger k is, the higher this level is and the larger the Baumol efficient set is.

The mean-Baumol efficient set is a subset of the mean-standard deviation set. Therefore, at least in the case of the normal distribution, the smaller Baumol subset may not include optimal investments for some risk averse investors.[13]

b.3 VaR Risk Measures: Like other below-a-reference risk measures, VaR measures also consider risk as being below a fixed reference point. However, VaR is differentiated from Fishburn's α-t risk measures, which weight all the results below a fixed reference point t, in that it measures risk only in terms of the loss which has a confidence interval of $1-\hat{P}$. Hence, VaR considers risk as one potential loss with a cumulative probability of occurrence of $\hat{P}$, while ignoring both larger and smaller potential losses.

AVaR is also differentiated from traditional below-a-reference risk measures. On the one hand, like those measures it weights large losses with a higher than $1-\hat{P}$ confidence interval but, on the other hand, like VaR, it ignores small losses due to results below the reference point with a smaller than $1-\hat{P}$ confidence interval. This approach may be appropriate for the regulator who wishes to insure against large losses while assuming that small losses are self-insured by the lenders.

c. Concluding Comparison of the Risk Measures

Table 1 presents the mathematical expression for each measure, discusses their main properties and summarizes the main differences between them.

In one way or another, none of the risk measures, including the VaR family, are necessarily consistent with the Von Neuman & Morgenstern (V&M) expected utility theory. However, the VaR family and in particular AVaR and AAVaR possess a few important advantages.

In the next section, we investigate the relationship between the VaR measures and expected utility theory in more detail.

III. VaR Measures, Expected Utility and the Lexicographic Expected Utility Model

In this section we analyze the congruence of VaR measures with expected utility framework. We show that while the mean-VaR_t criterion cannot fit reasonable utility functions under the expected utility analysis, the mean-$AVaR_t$ analysis can be optimal for such functions. As an alternative, we show that a lexicographic utility function is consistent with the VaR constraint analysis in the case that VaR is considered to be a "top priority goal" imposed on the agent either by shareholders or the regulator.

a. Congruence of the VaR Measures with Expected Utility

A congruence (or optimality) of a mean-Risk criterion with expected utility exists if and only if:

$$X \underset{mean-risk}{D} Y \Leftrightarrow E(U(X)) > E(U(Y)) \qquad (23)$$

where E(U(X)) and E(U(Y)) are the expected utilities of X and Y, respectively. Fishburn (1977) shows that his mean-α-t risk criterion is congruent with expected utility for the following family of utility functions:

$$U_{\alpha-t}(X) = X - \begin{cases} k(t-X)^{\alpha} & \mathrm{X} \leq \mathrm{t} \\ 0 & \text{otherwise} \end{cases} \qquad (24)$$

where α, k>0. [14] [15]

$U_{\alpha\text{-}t}$ is linear for Xs above t. The shape of $U_{\alpha\text{-}t}$ below t differs according to the value of α. As Fishburn mentions, there is mixed support in the literature for this type of utility function. Moreover, although this utility function may exhibit some risk aversion "in the small" below t and risk aversion "in the large",[16] as Bawa (1978) noted, such a function certainly does not guarantee DARA in certain ranges.

The following theorem shows that unlike Fishburn's risk measures, the mean-VaR_t criterion is congruent with expected utility theory only for utility functions that violate the basic rationality axiom of V&M expected utility theory.

<u>Theorem 1</u>: A mean-VaR_t criterion is congruent with the expected utility theory for every distribution function only for the following utility function:

$$U_{mean-VaR\,t}(X) = X - k(t-X)\delta(X - \mathrm{X}(\hat{\mathrm{P}})) \qquad (25)$$

where k>0 and δ(τ) is the Impulse Function which is defined as:[17]

$$\delta(\tau) = \begin{cases} \infty & \tau = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (26)$$

The proof is presented in Appendix A.

The left panel of Figure 3 presents $U_{mean\text{-}VaRt}$ as well as Fishburn's utility function, $U_{\alpha\text{-}t}$ for k=1 and α=1. $U_{mean\text{-}VaRt}$ is shown to be a linear function with a slope equal to 1 except for one discrete value at X= $X(\hat{P})$ at which it goes to minus infinity. Similarly to Fishburn's utility function, $U_{mean\text{-}VaRt}$ is a risk neutral function above t. In contrast to Fishburn's utility function, $U_{mean\text{-}VaRt}$ is also a risk-neutral function below t and more importantly has a discontinuity point at the value of $X(\hat{P})$ at which an increase in X decreases utility. This rather strange utility function represents irrational preferences since an increase in wealth from a certain value below $X(\hat{P})$ to $X(\hat{P})$ induces an infinite utility decrease which contradicts the rationality assumption of preferring more over less.

Basak & Shapiro (2001) have already noted that using VaR for the regulation of firms may create agency costs. We see here another aspect of this agency cost since VaR minimization is consistent with irrational expected utility maximization. In fact, this utility function simulates the preferences of Basak & Shapiro's (2001) agent as long as the constraint has not been met since the agent must decrease the portfolio VaR to a certain point no matter what the required cost is.

The main corollary from Theorem 1 is that in the case of general distributions, except from the case of the irrational utility functions such as the one on the left panel of Figure 3, the mean-VaR_t criterion cannot be justified on the grounds of the expected utility theory. In the following theorem we show that this problem is partially overcome by using $AVaR_t$ rather than VaR_t as the risk measure.

Theorem 2: A mean-$AVaR_t$ criterion is congruent with the expected utility theory for the following utility function:

$$U_{mean-AVaR_t}(X) = X - \begin{cases} k(t-X) & X \leq X(\hat{P}) \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

The proof is presented in Appendix A.

The right panel of Figure 3 depicts $U_{mean\text{-}AVaRt}$ as well as Fishburn's utility function, $U_{\alpha\text{-}t}$, for k=1 and α=1. It shows that $U_{mean\text{-}AVaRt}$ is composed of two lines with a discrete "jump" in utility at X= $X(\hat{P})$. Corresponding to Fishburn's utility function, $U_{mean\text{-}AVaRt}$ is risk-neutral in the small above the reference point and shows local risk aversion below it. Unlike Fishburn's utility function, the $U_{mean\text{-}AVaRt}$ reference point $X(\hat{P})$ has a "jump" in utility at a value (quantile) for which there is a probability of $\hat{P}$ being below this point.[18]

The "jump" at $X(\hat{P})$ may be due to additional costs that are not directly reflected in X. An example of such costs is the damage to the firm's reputation, the additional cost of liquidation which is imposed by other parties, or constraints imposed by covenant terms or regulation. A legitimate way to reflect such an increase in costs is through a jump in utility at the point $X(\hat{P})$.[19] Thus, using AVaR to regulate the firm leads shareholders (managers) to act as if they possessed an artificial utility with a jump at the $X(\hat{P})$ threshold reference point. By imposing this jump, the regulator can neutralizes the tendency of shareholders (managers) of a highly leveraged firm to take risks at the expense of depositors (lenders).

Another possible justification for the reduced level of utility below $X(\hat{P})$ is based on positive grounds. Accordingly, "optimistic" and even "pessimistic" investors may behave as if results with less than probability $\hat{P}$ of occurring are of less importance. These behavioral considerations should be empirically investigated.

In summary, according to Theorem 1 and 2 the use of VaR as a risk measure for decision making or as a constraint on the agent does not induce "rationality" and thus does not lead to optimal results for the investor. Using AVaR substantially improves "rationality" and thus induces better results for the investor. These conclusions reinforce the previous results of Artzner et al. and Basak & Shapiro (2001) in favor of AVaR over VaR both as a risk measure and as a regulatory constraint.

b. The Lexicographic Utility Approach

We have so far concluded that the mean-below-a-reference point risk measures can be inconsistent with the V&M expected utility theory. A possible intuitive behavioral explanation for this inconsistency could be that the below-a-reference point risk is a kind of "survival risk", but that it is not the only component of risk. The second component can be called the conventional "volatility risk" which deals with fluctuations that do not threaten survival or alternatively do not generate the extra costs that were previously discussed. The V&M expected utility theory could be extended to agree with these two components by assuming a different attitude towards these types of risk. Accordingly, there are hierarchical preferences between survival or avoiding disaster and obtaining an optimal mean-risk tradeoff in the traditional manner. This hierarchical preference can be quantitatively expressed in terms of the lexicographic expected utility model which was introduced by Hausner (1954) , Chipman (1960) and Fishburn (1971).[20]

On the behavioral level, the lexicographic expected utility approach is consistent with individuals' differential attitude towards various levels of loss which might be due to either differential scale of damage or any positively based behavioral pattern.[21] On the other hand, the lexicographic model suffers from the absence of substitutions between alternatives and therefore may be more relevant for modeling the behavioral pattern of a regulated firm.

In the context of a regulated firm, the imposed VaR constraints induce the shareholders (managers) to make decisions as if they possessed a "survival risk" constraint as a first priority goal in a lexicographic utility. Thus, a VaR constraint can be viewed as a survival risk constraint which lexicographically dominants the goal of maximizing V&M expected utility.

IV. Conclusions

The paper compares VaR measures and traditional measures of risk in order to place these popular risk measures in the entire map of risk analysis. The main conclusion from the analysis is that the VaR family, which is currently used for risk management purposes, is at least as good as other risk measures for decision making purposes.

In particular, it has been shown that the VaR risk measures are either a close variation or a specific case of traditional measures that consider risk in terms of results below a reference point. Some formal relationships between traditional and VaR measures have been formed and lead to conditions under which the mean-VaR analysis provides either sufficient or necessary criteria (see Appendix B).

The paper points out that VaR measures are exposed to several deficiencies. For all non-normal distributions, the mean-VaR criterion may screen out alternatives that are considered superior by some or even all risk-averse individuals. In addition, it may not identify existing dominance for all risk-averse individuals. Moreover, in the case of VaR_e (where the mean is the reference point for calculating VaR) the VaR may not identify inferior alternatives for all rational investors and at the same time screens out efficient alternatives for rational individuals.

More seriously, unless we assume normality (or log-normality), congruence with expected utility theory is obtained only for irrational utility functions. Hence, other than in the case of normal distribution (and in the case of the log-normal distribution in which a modification of the mean-VaR criterion is required.), the mean-VaR criterion cannot be justified on the grounds of expected utility theory.

Despite these conclusions, it is worth noting that most other traditional measures suffer from similar and even worse drawbacks. The mean-VaR_t criterion (where a constant t is the reference point) is superior or at least as good as the well-known mean-standard deviation criterion for the following reasons: First, both criteria are optimal in the normal case (and so are their variations in the log-normal case). Second, while the mean-VaR_t (as opposed to mean-VaR_e) criterion identifies inferior alternatives according to the FSD rule, the mean-standard deviation criterion cannot guarantee this property (except in particular cases, such as the quadratic utility function).

In light of the above deficiencies of VaR, the AVaR is superior both to the regular VaR as well as to most other traditional risk measures.

Apart from the practical and mathematical advantages of AVaR, we show that the mean-AVaR criterion identifies dominance when it exists for all risk-averse individuals, i.e. dominance by the mean-AVaR is a necessary condition for dominance by the SSD rule. This property generates regulatory advantages and implies lower agency costs. We show as well that AVaR considers all extreme loss scenarios. AVaR is also far less sensitive than VaR to the arbitrarily selected confidence interval and has a clear economic interpretation. In addition, the mean-AVaR criterion is optimal for normal distributions when assuming any rational risk aversion utility function, or for any distribution function when assuming utility functions that have a "jump" at a critical threshold point which reflects the extra cost incurred by being below this reference point.

Finally, the paper presents the idea of using the VaR risk measures in a lexicographic expected utility framework. In this model, risk is divided into two components: "survival risk" and "volatility risk" where the former takes precedence over the latter. This approach may be appropriate for behavioral models as well as the case of a regulated firm in which attaining imposed constraints dominates shareholders' preferences.

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Appendix A: Proofs

Proof of (6)

In order to prove (6) it is sufficient to provide an example. Suppose that X takes a value of either 10 or 20, each with a probability of 0.5. Similarly, Y takes a value of either 0 or 5, each with a probability of 0.5. It can easily be seen that any rational investor would prefer alternative X over Y ($\text{Min(X)} > Max(Y) \Rightarrow X \underset{FSD}{D} Y$). However, at a 50 percent or higher confidence interval ($\hat{P} < 0.5$), the VaR_es of X and Y are given by: $\text{VaR}_e(X)=5$ and $\text{VaR}_e(Y)=2.5$, respectively. Hence, both the mean and the VaR_e of X are higher than the mean and the VaR_e of Y and according to the mean-VaR_e rule there is no dominance between the two alternatives.

Proof of (7)

The mean condition: From the Stochastic Dominance necessary conditions we obtain $X \underset{FSD}{D} Y \Rightarrow \mu_X \geq \mu_Y$.

The VaR condition:[22] $X \underset{FSD}{D} Y \quad \Rightarrow \quad X(P) \geq Y(P)\,;0 \leq P \leq 1 \quad \Rightarrow$

$$\Rightarrow \quad t - X(\hat{P}) \leq t - Y(\hat{P}) \quad \Rightarrow \quad VaR_t(X) \leq VaR_t(Y).$$

Proof of (8)

In order to prove (8) it is sufficient to provide an example. Suppose that X takes a value of either 10 or 20, each with probability of 0.25, or the value of 15 with probability of 0.5. In contrast, Y takes a value of either 10 or 20, each with a probability of 0.5. It can easily be seen that any rational risk-averse investor would prefer alternative X over Y ($X \underset{SSD}{D} Y$). However, at a 75 percent or higher confidence interval ($\hat{P} < 0.25$), the VaR_t of X and Y are given by: $\text{VaR}_t(X)=t-10$ and $\text{VaR}_t(Y)=t-10$, respectively. Hence, both the mean and the VaR_t of X are equal to those of Y and according to the mean-VaR_t rule there is no dominance between the two alternatives.

Proof of (9)

The mean condition: From the Stochastic Dominance necessary conditions we obtain $X \underset{SSD}{D} Y \Rightarrow \mu_X \geq \mu_Y$.

The AVaR condition: $X \underset{SSD}{D} Y \quad \Rightarrow \quad \int_0^P X(p)dp \geq \int_0^P Y(p)dp\,;0 \leq P \leq 1 \quad \Rightarrow$

$$\Rightarrow \int_0^{\hat{P}} \left(t - X(p)\right) dp \leq \int_0^{\hat{P}} \left(t - Y(p)\right) dp \Rightarrow \text{A}VaR_t(X) \leq AVaR_t(Y).$$

Proof of (10)

In order to prove (10) it is sufficient to provide an example. Suppose that X takes a value of either 10 or 50, each with probability of 0.25, or the value of 20 with probability of 0.5. In contrast, Y takes a value of either 10 or 40, each with a probability of 0.5. It can easily be seen that any rational risk-averse investor with a positive third derivative of his utility function would prefer alternative X over Y ($X \underset{TSD}{D} Y$). However, at a 75 percent or higher confidence interval ($\hat{P} < 0.25$), the AVaR_t of X and Y are given by: $\text{AVaR}_t(X)=(t-10)\times \hat{P}$ and $\text{AVaR}_t(Y)=(t-10)\times \hat{P}$, respectively. Hence, both the mean and the AVaR_t of X are equal to those of Y and according to the mean-AVaR_t rule there is no dominance between the two alternatives.

Proof of (11)

The Mean condition: From the Stochastic Dominance necessary conditions we obtain $X \underset{TSD}{D} Y \Rightarrow \mu_X \geq \mu_Y$.

The AAVaR condition: $X \underset{TSD}{D} Y \Rightarrow \int_0^P \int_0^p X(\upsilon) d\upsilon dp \geq \int_0^P \int_0^p Y(\upsilon) d\upsilon dp \,; 0 \leq P \leq 1 \Rightarrow$

$$\Rightarrow \int_0^{\hat{P}} \int_0^p \left(t - X(\upsilon)\right) d\upsilon dp \leq \int_0^{\hat{P}} \int_0^p \left(t - Y(\upsilon)\right) d\upsilon dp \Rightarrow \text{AA}VaR_t(X) \leq AAVaR_t(Y).$$

Proof of (12)

In the normal case:

$$X \underset{SSD}{D} Y \Leftrightarrow X \underset{\substack{mean-s\tan dard \\ deviation}}{D} Y \qquad (A1)$$

and

$$X(\hat{P}) = \mu + \sigma Z(\hat{P}) \qquad (A2)$$

where $Z(\hat{P})$ is the $\hat{P}$ order value (quantile) of the normal standardized distribution, μ is the mean and σ is the standard deviation. From (A2) and the definition of VaR we find that:

$$\sigma = (X(\hat{P}) - \mu)/Z(\hat{P}) = -VaR_e / Z(\hat{P}) \qquad (A3)$$

Substituting this into the mean-standard deviation criterion produces inequality (12).

Proof of (14)

Substitute: $VaR_e = VaR_t + \mu - t$ into the mean-VaR$_e$ criterion and add the constant t to both sides in order to obtain (14).

Proof of (16)

In the log-normal case, $X \underset{SSD}{D} Y$ if and only if

$$\mu_X \geq \mu_Y \qquad (A4)$$

and

$$\hat{\sigma}_X \leq \hat{\sigma}_Y \qquad (A5)$$

where μ is the mean of the returns and $\hat{\sigma}$ is the standard deviation of the log of the returns (see Levy (1973, 1991)). Furthermore, in this case the $\hat{P}$ order value (quantile) is given by:

$$X(\hat{P}) = t - VaR_t = \exp(\hat{\mu} + \hat{\sigma} Z(\hat{P})) \qquad (A6)$$

and the mean is given by:

$$\mu = \exp(\hat{\mu} + 1/2\hat{\sigma}^2) \qquad (A7)$$

where $Z(\hat{P})$ is the $\hat{P}$ order value (quantile) of the normal standardized distribution, μ and $\hat{\sigma}$ are as defined above and $\hat{\mu}$ is the mean of the logs of the returns.

Substituting $\hat{\mu} = \ln(t - VaR_t) - Z(\hat{P})\hat{\sigma}$ from (A6) into (A7) yields:

$$\hat{\sigma}^2 - 2Z(\hat{P})\hat{\sigma} + 2\ln(\frac{t - VaR_t}{\mu}) = 0 \qquad (A8)$$

Substituting the positive solution of (A8), $\hat{\sigma} = Z(\hat{P}) + \sqrt{Z^2(\hat{P}) - 2\ln(\frac{t - VaR_t}{\mu})}$ into (A5) with a few algebraic manipulations (subject to $\hat{P} \leq 0.5$) yields (16).

Proof of Theorem 1:

Integrating $U_{\text{mean-VaRt}}$ from (25) yields:

$$\int_{-\infty}^{\infty} U(X)f(X)dX = \int_{-\infty}^{\infty} Xf(X)dX \quad -k\int_{-\infty}^{\infty}(t-X)\delta(X-\mathrm{X}(\hat{\mathrm{P}}))dX = \qquad (A9)$$

$$= \mu_X - k(t - X(\hat{P}))$$

Hence, the expected value of $U_{\text{mean-VaRt}}$ is simply the mean minus $kVaR_t$ such that:

$$X \underset{mean-VaR_t}{D} Y \Leftrightarrow \mu_X - kVaR_t(X) \geq \mu_Y - kVaR_t((Y)$$

<u>Proof of Theorem 2</u>:

Integrating $U_{\text{mean-AVaRt}}$ from (27) yields:

$$\int_{-\infty}^{\infty} U(X)f(X)dX = \int_{-\infty}^{\infty} Xf(X)dX \quad -k\int_{-\infty}^{\mathrm{X}(\hat{\mathrm{P}})}(t-X)dX = \mu_X - k\int_{-\infty}^{\mathrm{X}(\hat{\mathrm{P}})}(t-X)dX \qquad (A10)$$

Hence:

$$X \underset{mean-AVaR_t}{D} Y \Leftrightarrow \mu_X - kAVaR_t(X) \geq \mu_Y - kAVaR_t(Y)$$

Appendix B: Stochastic Dominance Rules

The Stochastic Dominance rules (see Quirk & Saposnick,(1962), Fishburn (1964), Hadar & Russell (1969), Hanoch & Levy (1969) and Whitmore (1970)) provide optimal investment criteria for several types of V&M utility functions. In the following, we present the definitions of optimal , necessary and sufficient criteria, as well as the First, the Second and the Third Stochastic Dominance rules.

Definitions:

A sufficient criterion for dominance of X over Y is one which guarantees that all the individuals under the assumed set of utility functions prefer X over Y.

A necessary criterion for dominance of X over Y is one that must be fulfilled once all individuals under the assumed set of utility functions prefer X over Y.

An optimal criterion is a necessary and sufficient criterion for dominance.

Stochastic Dominance Rules:

The First Stochastic Dominance (FSD) rule is an optimal criterion for all rational individuals who maximize expected utility where rationality is defined by a non-decreasing utility function.

According to the FSD rule, $X \underset{FSD}{D} Y$ if and only if F≠G and $F(x) \leq G(x)$ for all x, where F and G are the cdfs of X and Y.

The Second Stochastic Dominance (SSD) rule is an optimal criterion for all rational risk-averse individuals who maximize expected utility where risk aversion is defined by a non-increasing marginal utility function.

According to the SSD rule, $X \underset{SSD}{D} Y$ if and only if F≠G and $\int_0^x F(t)dt \leq \int_0^x G(t)dt$ for all x.

The Third Stochastic Dominance (TSD) rule is an optimal criterion for all rational risk-averse individuals with a positive third derivative of their utility function who maximize expected utility.

According to the TSD rule, $X \underset{TSD}{D} Y$ if and only if F≠G, $\mu_X \geq \mu_Y$ and

$\int_0^x \int_0^t F(\upsilon)d\upsilon dt \leq \int_0^x \int_0^t G(\upsilon)d\upsilon dt$ for all x.

Type of measure			Index	The arbitrary elements	Main Drawback	*SD⇒ mean-risk
Probability Dispersion			SD= $\sqrt{\int_{-\infty}^{\infty} f(x)(x-\mu_X)^2 dx}$		Accounts for positive dispersion	none
			Entropy= $-\int_{-\infty}^{\infty} f(x)\ln(f(x))dx$			none
			Abs' Dev'= $\int_{-\infty}^{\infty} f(x)\lvert x-\mu_X\rvert dx$			none
			Gini= $\frac{1}{2}\int_a^b\int_a^b \lvert R-r\rvert f(r)f(R)drdR$			SSD
			Co. Variation =σ/μ			none
Probability of Dispersion Below a Target	Fishburn= $\int_{-\infty}^{t}(t-x)^{\alpha} dF(x)$	α→0	Roy= P(X≤t) ≤ $\sigma^2/(\mu-t)^2$	t	Ignores the size of loss	FSD
		α=1	D&M= $\int_{-\infty}^{t}(x-t)f(x)dx$	t	Prospects over a changing range of probabilities	SSD
		α=2	S-Variance= $\int_{-\infty}^{t} f(x)(x-t)^2 dx$	t		TSD
		α→∞	WCS= $t - X(0)$	t	Ignores $0 < P \le \hat{P}$	TSD
	VaR		$VaR_t = t - X(\hat{P})$	$\hat{P}$, t	Ignores $P < \hat{P}$	FSD
			$VaR_e = \mu - X(\hat{P})$	$\hat{P}$, t		none
			AVaR= $\int_0^{\hat{P}} VaR(p)dp$	$\hat{P}$, t		SSD
			AAVaR= $\int_0^{\hat{P}} AVaR(p)dp$	$\hat{P}$, t		TSD
	Baumol		Baumol=μ-kσ	k	Ignores X<μ-kσ	none

Table 1: Summary of the most common risk measures, their main drawbacks and the relations in the general case between the mean-risk criterion using these risk measures and the Stochastic Dominance approach.

*The efficient set of the mean-risk criterion is a subset of the SD criterion (note that for the SD rules the following holds: FSD⇒SSD⇒TSD).

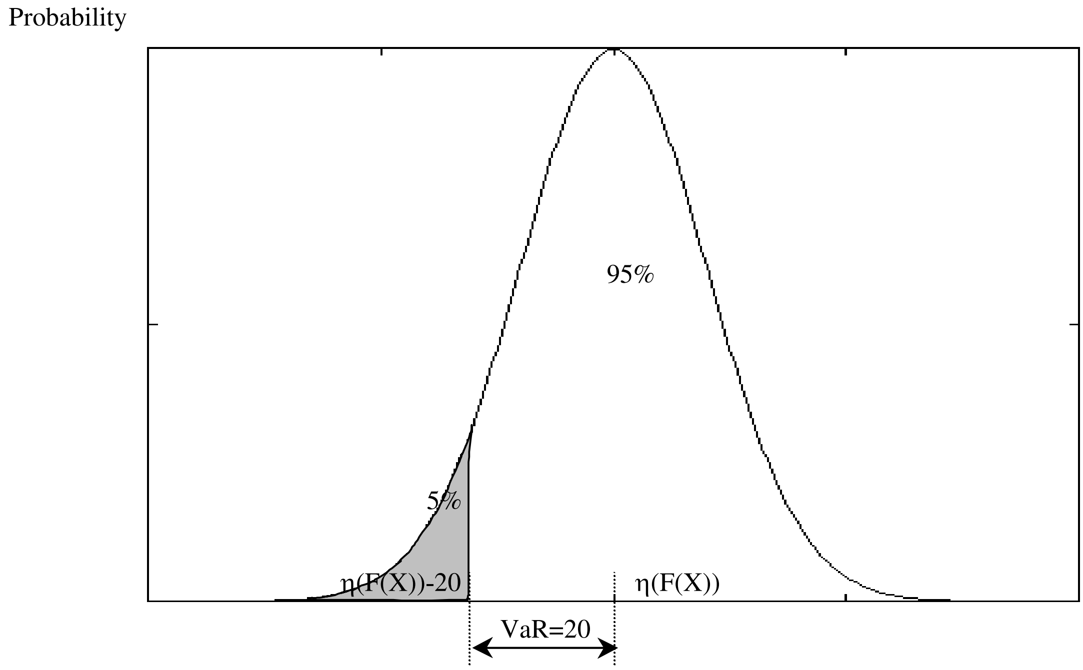

Figure 1. Value-at-Risk

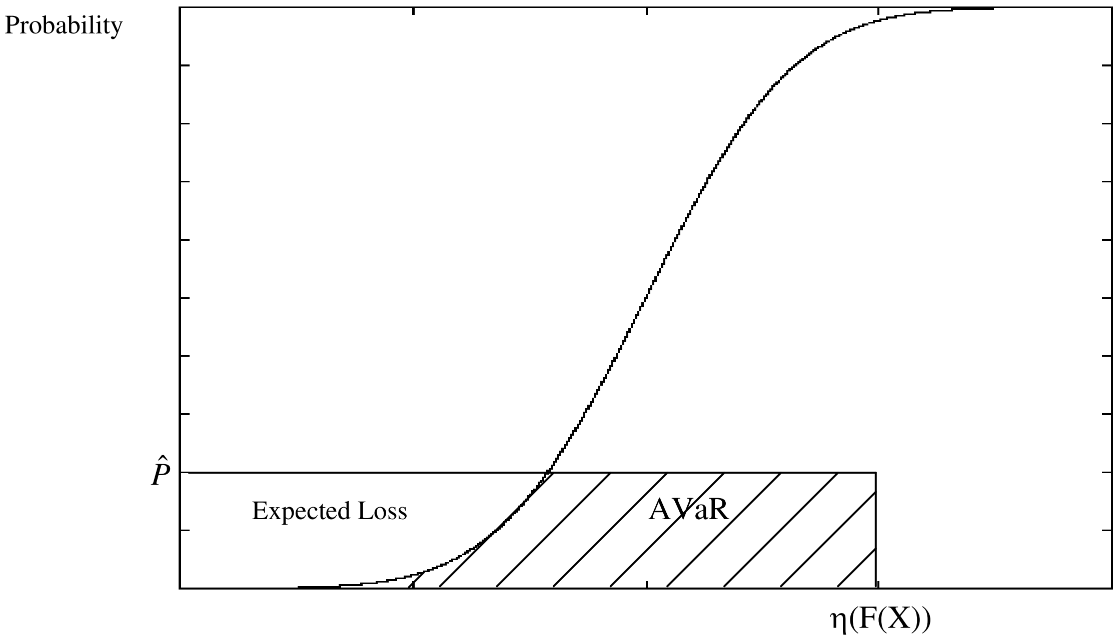

Figure 2. Accumulate-VaR

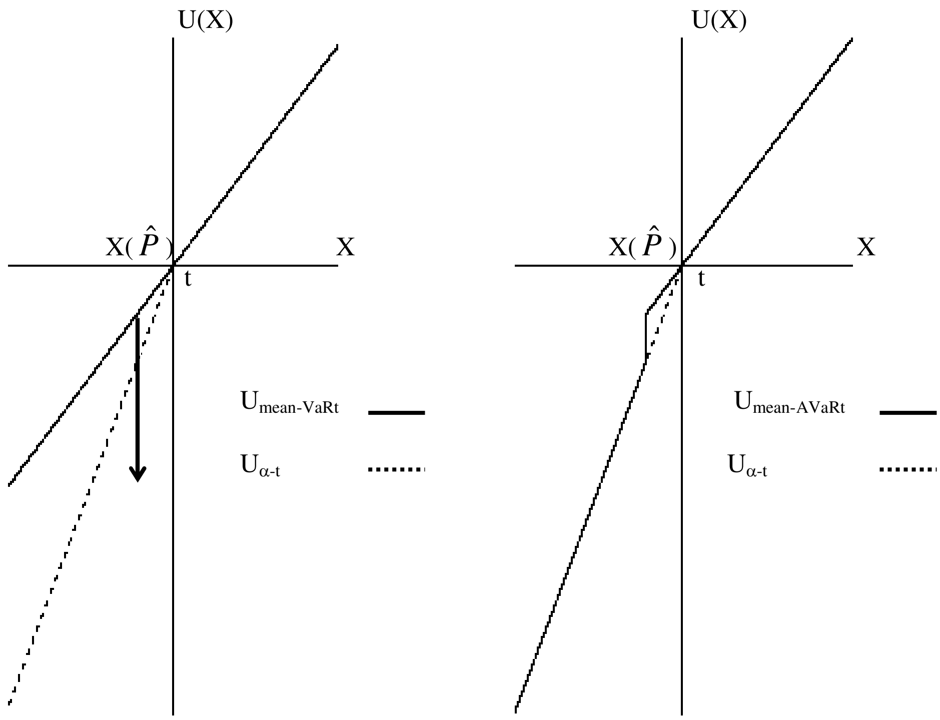

Figure 3. A comparison of Fishburn's $U_{\alpha\text{-}t}$, with $U_{Mean\text{-}VaRt}$ (left panel) and with $U_{Mean\text{-}AVaRt}$ (right panel) .

[1] An introduction and overview of VaR can be found in Linsmeier & Pearson (1996), Duffie & Pan (1997) and in the excellent books by Jorion (2000b) and Crouhy, Galai & Mark (2001).

[2] For more on credit risk issues see, for example, Duffie & Pan (2000) and Crouhy, Galai, & Mark (2000). For more on the methodology used to constrain and control risk exposure see, for example, Basak & Shapiro (2001) and Jorion (2001).

[3] See, for example, Beder (1995), Johansson, Seiler & Tjarnberg (1999) and others.

[4] See, for example, Ho, Chen & Eng (1996) and Ahn, Boudoukh, Richardson & Whitelaw (1999).

[5] Note that the requirement for the "maximum" value is relevant only in the case of discrete distributions in which there may be several values of X(P) which satisfy the following condition: $\Pr(X \leq X(P))=P$.

[6] Alternatively, VaR can be defined in terms of market values by using the relevant distribution of the forward value of the position. For the relationship between the definitions of VaR in terms of returns and in terms of market values, see Hallerbach (1999).

[7] The expected value of X in terms of the quantile function is give by: $E(X) = \int_0^1 X(P)dP$.

[8] Another aspect of the sensitivity of the selection of alternatives to the confidence interval under the mean-VaR_t criterion in the case of "fat-tails" can be found in Lucas & Klaassen (1998).

[9] The issue of the reference point of VaR, which is negligible in the traditional use of VaR for market risk during very short periods, is becoming significant with the tendency to adopt VaR for other uses such as a decision making measure.

[10] In order to compare the normal and log-normal cases, recall that $VaR_t(X) = t - X(\hat{P})$ and $VaR_t(Y) = t - Y(\hat{P})$ and rewrite (14) and (16) as $\mu_X - \mu_Y \leq X(\hat{P}) - Y(\hat{P})$ and $\ln(\mu_X) - \ln(\mu_Y) \leq \ln(X(\hat{P})) - \ln(Y(\hat{P}))$. The only difference between these inqualities is that the logs of the relevant parameters appear in the log-normal case. This should not be surprising since the logs of X and Y are normally distributed in the log-normal case.

[11] This advantage exists only for discrete empirical distributions. Kaplanski & Kroll (2001) show that in the case of general continuous distributions a simple linear program cannot be used to obtain the analytical solution for the mean-VaR optimization problem.

[12] Except for the case of at most one intersection of the alternatives' cdfs in which this criterion is also sufficient for SSD.

[13] In the context of regulation, Halpern & Kahane (1980) illustrate some basic differences between the Baumol risk measure and the "ruin constraint".

[14] A previous version of this function for the mean-Semi-Variance criterion is presented by Markowitz (1959) and further analyzed by Mao (1970b). Arzac (1974) also presents a variation for the case of mean-Safety First analysis.

[15] According to expected utility theory, this function is determined up to a positive linear transformation. Fishburn (1977) normalizes the utility function such that U(t)=t.

[16] The expressions for risk aversion "in the small" and "in the large" are adopted from Pratt (1964).

[17] The Impulse Function exhibits: $\int_{\tau-\varepsilon}^{\tau+\varepsilon} \delta(X-\tau) f(X) dX = f(\tau)$ where $f(\tau)$ is a sample of f (X) at the point τ .

[18] Throughout the analysis we assume that $X(\hat{P}) \leq t$. Though the analysis can easily be extended to cover the opposite case ($X(\hat{P}) > t$), it is unreasonable to assume a "loss" reference point, $X(\hat{P})$, in the high range where individuals tend to be risk-neutral.

[19] The alternative approach is to add the additional cost to X. However, this will lead to an "empty" span of X below the jump point.

[20] The previous use of a lexicographic safety-first rule can be found in Telser (1955), Arzac (1974) and Arzac & Bawa (1977). However, in contrast to their models, which separate the mean-risk analysis into two lexicographic components, here we suggest breaking down the risk itself into separate elements. Hence, the survival element of risk lexicographically dominates all other aspects of the distribution which can still be expressed in terms of the traditional V&M expected utility theory.

[21] Note that this approach is also consistent with Friedman & Savage's (1948) analysis which explains the selection of both lottery tickets and insurance by the same individual through different perceptions of risk at high and lowlevels of wealth. Accordingly, in the proposed lexicographic model the purchase of insurance is aimed at reducing the element of survival risk while the purchase of lottery tickets involves only the second element of wealth maximization.

[22] In the proofs of (6)-(11) we use the Stochastic Dominance rules in terms of the quantile functions X(P) and Y(P). For more details regarding this version of the Stochastic Dominance rules, see Levy & Kroll (1978).