Analytical Portfolio Value-at-Risk

Kaplanski, Guy

2005

Analytical Portfolio Value-at-Risk

Guy Kaplanski*

Journal of Risk, Vol. 7, Issue 2, pp. 33-54, 2005

The author acknowledges the helpful comments of the editor Philippe Jorion, the anonymous referee and Oren Bar-Ner.

* Guy Kaplanski is from Bar-Ilan University, Ramat Gan, IL 52900, Israel, e-mail: guykap@biu.ac.il.

Analytical Portfolio Value-at-Risk

Abstract

The paper develops analytical tools used to calculate the VaR of a portfolio composed of generally distributed assets. Accordingly, the VaR of a portfolio is analytically constructed from the conditional returns of the individual assets. This analytical VaR can then be used to construct optimal portfolios of generally distributed assets for the case in which the target function and/or constraints are expressed in terms of VaR. The proposed method is applicable in a wide range of practical problems such as utility maximization under a VaR constraint. The article demonstrates this method by developing a minimal VaR rule that identifies the proportions that minimize the portfolio VaR. This rule is used to compare the minimal VaR portfolio with the minimal standard deviation portfolio in the case of the lognormal distribution. This example illustrates the importance of downside risk in optimal asset allocation even under modest deviations from the normal distribution such as in the case of the lognormal distribution.

Keywords: Value-at-Risk, Risk measurement, Portfolio Optimization, Downsize Risk

Introduction

In recent years, Value-at-Risk (VaR) has become the standard tool used by financial institutions to measure and manage risk.[1] Currently, VaR is used primarily for measuring market risk. However, there has been an increasing interest in using the VaR concept as a tool for managing and regulating credit risk and as a methodology for constraining and controlling the risk exposure of a portfolio.[2] Most studies that focus on VaR in the context of a portfolio either assume a simple normal distribution or practically, use numerical calculations (see, for example, Campbell, Huisman and Koedijk (2001) and Jorion (2001)). However, the widespread adoption of VaR and other quantile measures in a portfolio framework calls for the development of analytical methods to solve the portfolio optimization problem for non-normal distributions and thereby to take full take advantage of these risk measures. These methods are also required in order to improve and further develop and explore VaR as a tool for risk measurement in a portfolio framework.

Assume, for example, the classical problem of optimal proportions between the market shares portfolio and the market bonds portfolio such as in Campbell et al. (2001). What is the portfolio VaR and how would the proportions of the two portfolios influence this overall portfolio VaR? Usually, such a common problem is solved either assuming the strong and restricting assumption of normal distribution or by using a numerical approximation method. For example, Alexander & Baptista (2002) assume either a normal distribution or a t-distribution when comparing VaR and standard deviation in the context of mean-VaR analysis. Similarly, Sentana (2001) analyzes the mean-variance frontier under a VaR constraint assuming elliptically symmetric distributions, which can be fully defined by the first two moments. However, this approach provides only an approximation in several important cases where the distribution is not symmetrical or cannot be considered elliptical. Differently, Emmer, Klüppelberg & Korn (2001), Cuoco, He & Issaenko (2001), Yiu (2004) and others analyze the impact of a VaR constraint or some other quantile constraint on asset allocation while using numerical techniques. This approach might require large calculation resources and reckons on a possibly long convergence process, especially when a range of compositions and strategies are analyzed.

This article provides an alternative analytical method to solve such problems. The article develops analytical tools for calculating the VaR of a portfolio composed of generally distributed assets. Accordingly, the VaR of a portfolio is analytically constructed from the conditional returns of the individual assets. This analytical VaR can then be used to construct optimal portfolios of generally distributed assets for the case in which the target function and/or constraints are expressed in terms of VaR. The proposed method is appropriate for a wide range of applications. For example, it can be used to analytically solve Basak & Shapiro's (2001) problem of maximizing utility under VaR constraint. Similarly, it can be used to expand the solution of Ahn, Boudoukh, Richardson & Whitelaw's (1999) problem in the realistic case of hedging a portfolio with an option that is only partially correlated with the hedged portfolio. The proposed method is demonstrated by developing a minimal VaR rule, which identifies the proportions that minimize the portfolio VaR. A numeric example with the lognormal distribution is then used to compare the minimal VaR with the minimal standard deviation portfolios. This example highlights the importance of downside risk in the context of portfolio asset allocation.

The paper is organized as follows: The next section develops the theoretical relationship between the distribution of the individual assets and the VaR of the portfolio. For simplicity, the presentation is confined to two assets (the generalization to multiple assets can be found in Appendix B). Section II applies the findings from Section I in order to develop a VaR minimization rule and to present a comprehensive numerical example. Section III shows how the previous results can be used to solve more complicated optimization problems such as Ahn, Boudoukh, Richardson and Whitelaw's (1999) problem of selecting a put option, which minimizes the portfolio VaR. Section V concludes the paper.

I. Analytical Portfolio VaR

In this section, an analytical expression of the portfolio VaR is developed. More specifically, the portfolio VaR is expressed in terms of the conditional distributions of the individual assets and their proportions in the portfolio. The information about these conditional distributions is equivalent to the information about the assets cumulative distribution functions and their mutual correlations, which is a prerequisite to solving any portfolio optimization problem. This is comparable to the information about the means and the variance-covariance matrix required for the classical Markowitz solution of the portfolio optimization problem. In order to simplify the presentation we start with a two-asset portfolio (the general solution for multiple asset portfolios together with an illustrative example are presented in Appendix B).

Denote by X and Y the risky returns on any two assets with probability density functions, $f(X)$ and $g(Y)$, and cumulative distribution functions (cdf), $F(X)$ and $G(Y)$, respectively. In the proposed method, the correlation between X and Y is realized through the use of the conditional distribution. Therefore, *without losing generality*, let us select Y as the "unconditionally-distributed" asset and X as the "conditionally-distributed" asset. This selection does not indicate anything about the assets themselves but rather implies that information about the correlation between X and Y is given by the conditional distribution of X over Y. Namely, the roles of X and Y can be inverted. Intuitively, if the specific problem is involved with the market portfolio or an index and a single asset, then selecting Y as the market portfolio corresponds to this model.

Let $X(P)$ be the P-order quantile function of $F(X)$. The quantile function is the inverse function of the cdf. Formally, $X(P)$ is the maximum value of X for which there is a probability P of being below this value in the cdf of $F(X)$ (namely, $\Pr(X \leq X(P))=P$). The quantile function is assumed to be monotonous. This monotonicity is a direct result of the cdf monotonicity, which has been proved and used by Rothschild and Stiglitz (1970) for the Second Stochastic Dominance analysis. Let $X^Y(P)$ be the conditional quantile of X on Y. Namely, $X^Y(P)$ is the inverse function of the cdf of asset X conditional on Y, $F^Y(X)$. By the same token, let $X^y(P)$ be the conditional quantile of X on a specific realization y of Y.

VaR with a $1-\hat{P}$ confidence interval, denoted as $VaR(\hat{P})$, can be defined as the loss below some reference point t over a given period of time, where there is a probability of $\hat{P}$ of incurring this loss or a larger one. In terms of the quantile function, $VaR(\hat{P})$ can simply be written as

$$VaR(\hat{P}) = t - X(\hat{P}) . \qquad (1)$$

The reference point t can be a function of the cdf of X or a constant reference point, such as the risk-free return or zero. For example, the official Basel (1996) Amendment recommends calculating the VaR as the potential loss below the current value.

Proposition 1. The VaR of a Portfolio

Without losing generality, let α be the proportion of X and 1-α the proportion of Y in a portfolio $Z_\alpha = \alpha X + (1-\alpha)Y$. *For any 0 < α,*[3] *and for some selected realization y of Y, the VaR of this portfolio is given by*

$$VaR_Z(\hat{P}) = t - \alpha\, X^y(P^*) - (1-\alpha)\, y\,, \qquad (2)$$

where t is the loss reference point, $X^y(P^*)$ is the quantile of X conditional on a given realization y, P is solved by the expression*

$$\hat{P} = \int_{-\infty}^{+\infty} F^Y\left(X^y(P^*) - \tfrac{1-\alpha}{\alpha}(Y-y)\right) g(Y)dY\,, \qquad (3)$$

and the range of probabilities for which the specific realization y of Y provides a solution of (2) is given by

$$\int_{-\infty}^{+\infty} F^Y\left(X^y(0) - \tfrac{1-\alpha}{\alpha}(Y-y)\right) g(Y)dY \le \hat{P} \le \int_{-\infty}^{+\infty} F^Y\left(X^y(1) - \tfrac{1-\alpha}{\alpha}(Y-y)\right) g(Y)dY\,. \qquad (4)$$

Proof

Denote the quantile of the portfolio $Z_\alpha = \alpha X + (1-\alpha)Y$ of order $\hat{P}$ as $Z_\alpha(\hat{P})$. Denote the cdf of the portfolio Z_α conditional on asset Y as $P^{**} \equiv H^Y(Z_\alpha)$, where $0 \le P^{**} \le 1$. For a specific selected value y of Y, denote the cdf of the portfolio return at point Z_α conditional on y as $P^* \equiv H^y(Z_\alpha)$, where $0 \le P^* \le 1$. According to Bayes' Theorem (the *"Total Probability Equation"*)

$$\hat{P} \equiv H(Z_\alpha) = \int_{-\infty}^{+\infty} g(Y) H^Y(Z_\alpha) dY\,. \qquad (5)$$

Following Levy and Kroll's (1978) quantile approach, the quantile of portfolio Z_α of order P^* *conditional* on realization y of Y can be written as

$$Z_\alpha^y(P^*) = \alpha\, X^y(P^*) + (1-\alpha)\, y\,, \qquad (6)$$

(for a proof of Levy and Kroll's (1978) quantile approach see Appendix A).
According to the previous definitions of $\hat{P}$ and P^*, we know that the $\hat{P}$-order quantile of the unconditional return on the portfolio is equal to Z_α as is the conditional quantile of order P^* over y. Hence,

$$Z_\alpha \equiv Z_\alpha(\hat{P}) = Z_\alpha^y(P^*)\,. \qquad (7)$$

From (6), (7) and the definition of P^* we can conclude that for every value of Y either there is an order $0<P^{**}<1$ such that the following holds

$$Z_\alpha \equiv \alpha X^y(P^*) + (1-\alpha)y = \alpha X^Y(P^{**}) + (1-\alpha)Y, \qquad (8)$$

or for that specific y either

$$P^{**} = 0, \qquad (8a)$$

or

$$P^{**} = 1. \qquad (8b)$$

Note that (8a) and (8b) are required as the quantile function is defined over a finite range. From (8), we get[4]

$$P^{**} = F^Y\left(X^y(P^*) + \tfrac{1-\alpha}{\alpha}(y-Y)\right). \qquad (9)$$

Combining (6) and (7) with the definition of VaR in (1) yields (2). Substituting $P^{**} \equiv H^Y(Z_\alpha)$ from (9) into (5) yields (3). Finally, from the monotonuosity of the quantile function it is sufficient to solve (5) for the two extremes $P^*=0$ and $P^*=1$ in order to find the range of probabilities in (4) for which the specific realization y of Y provides a solution to (2).

Discussion

Proposition 1 provides a method for calculating analytically the portfolio VaR based on the conditional distributions of the individual assets. First, the order P^* of the quantile of X conditional on Y is implicitly solved by (3) and then it is substituted into (2). The order P^* is required as in general the P-order quantile of a portfolio is not a linear combination of the individual quantiles.[5]

One might wonder how (2) and (3) yield the same $VaR_Z(\hat{P})$ for any selected realization y. The explanation lies in the fact that the integration in (3) is over all values of Y and the arbitrary realization y of Y serves only as a reference and starting point for this integration. Hence, the simultaneous effect of the selected y on both the order P^* in (3) and on the portfolio VaR in (2) completely offset each other such that the total impact of the selection of y on the solution of the portfolio VaR is zero. Nonetheless, the selected realization y might have an impact on the range of probabilities for which there is a solution of Proposition 1, as is defined in (4). Normally, this impact does not complicate the selection of y. This is because VaR is calculated usually over the lower left-hand side of the distribution (namely, for low order $\hat{P}$) such that (4) implies simply that the selected realization y should be sufficiently small to contain the lower range of probabilities. In other words, except for unique cases, when calculating VaR by Proposition 1 it is sufficient to choose a sufficiently small realization y and solving (4) is not required practically. This issue is further clarified in the following example.

An Illustrative Example

For illustration purposes, the following simplified example provides a graphical exposition of Proposition 1. For simplicity of presentation, let the return on Y be restricted to only two values, y_1 and y_2 with probabilities q and 1-q, respectively, where $y_1 < y_2$ (see Figure 1). In the following solution, y_1 serves as the selected realization. As has been previously mentioned, the intuition behind this selection is that VaR is calculated usually over the lower left-hand side of the distribution and therefore it is sufficient practically to simply choose a sufficiently low Y, in our case y_1, without actually solving (4). Later on, we also solve inequality (4) for a specific example.

Assuming $\alpha > 0$, the conditional cdf of the portfolio for $Y = y_1$ is given by $H^{y_1}(Z_\alpha)$. For any given order P, including the order $\hat{P}$, $H^{y_1}(Z_\alpha)$ divides the horizontal difference between $F(X)$ and y_1 according to the proportions α and (1-α) (see Figure 1 and Appendix A). Hence, according to (6), the conditional quantile is given by

$$Z_\alpha^{y_1}(P^*) = \alpha\, X^{y_1}(P^*) + (1-\alpha)y_1 . \qquad (10)$$

Similarly,

$$Z_\alpha^{y_2}(P^{**}) = \alpha\, X^{y_2}(P^{**}) + (1-\alpha)y_2 . \qquad (11)$$

Thus, from (10) and (11) we obtain

$$Z_\alpha = \alpha\, X^{y_1}(P^*) + (1-\alpha)y_1 = \alpha\, X^{y_2}(P^{**}) + (1-\alpha)y_2 , \qquad (12)$$

or $P^{**} = 0$ or $P^{**} = 1$, depending on the distribution of X and the selected realization y. According to Bayes' Theorem in its discrete form

$$\hat{P} = qP^* + (1-q)P^{**} . \qquad (13)$$

Finally, using (12) to extract P^{**} and substituting it into (13) yields (3) in its discrete form

$$\hat{P} = qP^* + (1-q)F^{y_2}\left(X^{y_1}(P^*) - \tfrac{1-\alpha}{\alpha}(y_2 - y_1)\right). \qquad (14)$$

Solving for P^*, substituting it into (6) and deducting it from the reference point t, yields (2) and produces the portfolio VaR.

For the purpose of demonstration, let us further assume that X is distributed exponentially uncorrelated with Y, namely

$$F^Y(X) = F(X) = \begin{cases} 1-\exp(-\lambda X) & 0 < X, \\ 0 & other, \end{cases} \tag{15}$$

and

$$X^Y(P) = X(P) = -\frac{1}{\lambda}\log(1-P) \quad 0 \le P \le 1. \tag{16}$$

First, as $X^y(0) = 0$ and $X^y(1) = \infty$, (4) in its discrete form yields $\sum_{Y=y_1}^{y_2} F^Y\left(0 - \frac{1-\alpha}{\alpha}(Y-y)\right)g(Y) = 0 \le \hat{P} \le \sum_{Y=y_1}^{y_2} F^Y\left(\infty - \frac{1-\alpha}{\alpha}(Y-y)\right)g(Y) = 1$, for both values of Y. Hence, in this specific case, both values of Y could be selected unrelated to the required confidence interval.

Continuing with y_1 as the arbitrarily selected realization, substituting (15) and (16) into (14) yields two possible ranges. Assuming $X^{y_1}(P^*) - \frac{1-\alpha}{\alpha}(y_2 - y_1) < 0$ or equivalently $P^* < 1 - \exp(-\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))$, (14) yields

$$P^* = \frac{\hat{P}}{q} \qquad \hat{P} < q(1-\exp(-\lambda\tfrac{1-\alpha}{\alpha}(y_2 - y_1))). \tag{17}$$

Assuming $P^* > 1 - \exp(-\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))$ yields $\hat{P} = qP^* + (1-q)(1-\exp(\log(1-P^*) + \lambda\frac{1-\alpha}{\alpha}(y_2 - y_1)))$ which yields

$$P^* = \frac{\hat{P} - (1-q)(1-\exp(\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))}{q + (1-q)\exp(\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))} \qquad \hat{P} > q(1-\exp(-\lambda\tfrac{1-\alpha}{\alpha}(y_2 - y_1))). \tag{18}$$

Substituting the order P^* from (17) and (18) into (2) yields the portfolio VaR

$$VaR_Z(\hat{P}) = \begin{cases} t + \dfrac{\alpha}{\lambda}\log(1 - \dfrac{\hat{P}}{q}) - (1-\alpha)y_1 & 0 \le \hat{P} < q(1-\exp(-\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))) \\ t + \dfrac{\alpha}{\lambda}\log(1 - \dfrac{\hat{P} - (1-q)(1-\exp(\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))}{q + (1-q)\exp(\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))}) - (1-\alpha)y_1 & q(1-\exp(-\lambda\frac{1-\alpha}{\alpha}(y_2 - y_1))) \le \hat{P} \le 1. \end{cases} \tag{19}$$

Figure 1 depicts the solution graphically for the case of assets X and Y in proportions $Z = 0.6X + 0.4Y$ (i.e. $\alpha = 0.6$). The return on asset X is exponentially distributed with parameter $\lambda = 1$ uncorrelated with the return on asset Y which is restricted to the two values, y_1= 1 and y_2= 2 with probabilities q = 0.3 and 1-q = 0.7. Figure 1 reveals that the conditional cdf of the portfolio for $Y = y_1$, $H^{y_1}(Z_\alpha)$, divides the horizontal difference between $F(X)$ and y_1 according to the proportions α = 0.6 and 1-α = 0.4. Similarly, the conditional cdf of the portfolio for $Y = y_2$, $H^{y_2}(Z_\alpha)$, divides the horizontal difference between $F(X)$ and y_2 according to the same proportions. It can also be seen in Figure 1 that $\hat{P}$ divides the vertical distance between P^* and P^{**} according to the proportions 1 - q = 0.7 and q = 0.3. Similarly, $H(Z_\alpha)$ divides the vertical distance between the conditional cdfs $H^{y_1}(Z_\alpha)$ and $H^{y_2}(Z_\alpha)$ according to the same proportions. Finally, using these characteristics, the bold curve in Figure 1 graphically depicts the solution of (19).

The lower feasible range, $(1-\alpha)y_1 \le Z_\alpha < (1-\alpha)y_2$ (i.e. $0.4 \le Z_\alpha < 0.8$), can be realized only when $Y=y_1$ (as the minimal contribution of realization y_2 to the total portfolio value is the value y_2 = 2 times its proportion in the portfolio of 0.4) . Hence, the solution divides the vertical distance between the conditional cdf $H^{y_1}(Z_\alpha)$ and $P^{**}=0$ according to the proportions 1 - q and q. Finding this vertical weighted average between $H^{y_1}(Z_\alpha)$ and $P^{**}=0$ provides the solution in this range which is given by the first range of (19). Correspondingly, the upper feasible range $Z_\alpha \ge (1-\alpha)y_2$ (i.e. $Z_\alpha \ge 0.8$) can be realized under both realizations of Y. Hence, the solution divides the vertical distance between the conditional cdfs $H^{y_1}(Z_\alpha)$ and $H^{y_2}(Z_\alpha)$ according to the same proportions, 1 - q and q. Thus, finding the vertical weighted average between $H^{y_1}(Z_\alpha)$ and $H^{y_2}(Z_\alpha)$ provides the solution in this range, which is given by the second range of (19).

To sum up, equation (2) calculates the vertical weighted average between $H^{y_1}(Z_\alpha)$ and $H^{y_2}(Z_\alpha)$ (or zero) at the order $\hat{P}$ and yields the $\hat{P}$-order VaR. Naturally, the solution of this simplified example of an asset, which is restricted to only two values, is straightforward. However, the proposed method is general and the same principles are applicable for any other, more complicated, case as is further shown in the following sections.

Generalization

Solving (3) for P^*, which may be relatively complex in the case of numerous assets (depending also on the conditional distribution), is quite simple in the case of only few assets, as it requires the solution of only two equations. Furthermore, in this case it has the advantage of yielding a unique analytical solution for any distribution and with small calculation resources. Therefore, in the case of a small number of assets, where the conditional distribution can easily be obtained, and when the distribution cannot be considered normal, Proposition 1 provides a relatively simple and straightforward solution. This case of small number of assets covers a wide range of important problems. For example, Proposition 1 is best suited for solving Campbell et al.'s (2001) problem of the optimal combination of shares and bonds portfolios. In this important problem, in which at least one of the two assets cannot be assumed to be normally distributed, it provides a unique solution, which requires solving only two equations. This is in contrast to other numerical methods, which might require a lengthy convergence process, and more seriously, might produce a solution that is path-dependent of this convergence process. Other examples, which are well suited for this method, are when optimization of VaR is required. For example, Proposition 1 can be used to analytically find the minimal VaR portfolio as is presented in the next section.

Another optimization example is analytically solving the agent's optimization problem presented in Basak & Shapiro (2001). In that problem, the agent maximizes $U(Z_\alpha)$ subject to

$$VaR_Z(\hat{P}) = t - \alpha X^y(P^*) - (1-\alpha)y \le VaR_f, \qquad (20)$$

which leads to the following constraint on the proportion of X in the portfolio

$$\alpha \ge \alpha_f = \frac{t - y - VaR_f}{X^y(P^*) - y}, \qquad (21)$$

where VaR_f is the required constraint floor, α_f is the bound on the proportion of X which is induced by the constraint and the order P^* is given in (3). Note that the order P^* in (21) is also a function of α itself such that equality (21) may define both upper and lower bounds on α. Naturally, when the optimization problem involves numerous assets the solution is more complex (see Appendix B). Another useful example of only two assets, and therefore requires the solution of only two simple equations, is when analyzing the impact of adding an asset, X, to an existing portfolio,

Y, assuming the composition of Y is unchanged. Proposition 1 enables to fully study that impact as a function of the composition of the overall portfolio. This ability is further elaborated in the next section.

II. The Minimal VaR Portfolio in Case of Continuous Distributions

In this section Proposition 1 is used to analytically find the minimal VaR portfolio in the case of two continuous and differentiable distributions. Then, an illustrative numerical example is provided. This example demonstrates the advantage of Proposition 1 over simulation techniques in calculating the VaR of a portfolio and in VaR analysis when both assets are continuously distributed. This example is further elaborated in order to compare between the minimal VaR and the minimal standard deviation in the lognormal case. This comparison illustrates also the importance of downside risk in optimal asset allocation even under modest deviations from the normal case.

Let X and Y be the returns on two risky assets as in Proposition 1. If $X^y(P)$ is differentiable for a realization y of Y over the entire range $0 \leq P \leq 1$, the proportion that leads to the minimal VaR portfolio is solved by

$$\alpha_{\min} \frac{dX^y(P^*)}{d\alpha} = y - X^y(P^*) \ , \qquad (22)$$

and the minimal VaR portfolio is given by substituting the proportion α_{min} in (2), where the order P^* is given by (3) as a function of $\hat{P}$, y and the solution α_{min}. The proof of (22) is straightforward. Differentiating the portfolio VaR, given by (2), with respect to α and equating it to zero in order to find the local minimum yields $\alpha_{\min}$, which leads to the minimal VaR portfolio. Note that the differentiability of the portfolio VaR and thus the existence of α_{min} is guaranteed as long as $X^y(P)$ is differentiable in the range $0 \leq P \leq 1$.[6]

In the following, Proposition 1 and the implied minimal VaR rule in (22) are used to analyze the VaR of a portfolio that is composed of lognormlly-distributed assets. The lognormal distribution is appealing in many economic applications. This is mainly because in contrast to the normal distribution the lognormal distribution is able to capture the empirical phenomena of positive skewness and extra kurtosis as well as the fact that risky returns are bounded from below. On the other hand, the main drawback of the lognormal distribution is that the distribution of a portfolio composed of lognormally distributed assets is not lognormally distributed and does not have an analytical expression. Thus, the VaR of the combined portfolio cannot be found straightforwardly and an approximation or a numerical technique is usually required. Proposition 1 provides a simple solution for this shortcoming as is shown below. In Appendix B this example is expanded to a more realistic case of a three-asset portfolio.

Let X and Y be multivariate lognormally distributed with expected returns and standard deviations of the logs of μ_X, μ_Y, σ_X and σ_Y, respectively, and with correlation coefficients of the logs of ρ. Namely, log(X) and log(Y) are multivariate normally distributed with the above parameters. Hence,

$$g(Y) = \frac{1}{Y\sqrt{2\pi\sigma_Y^2}}\exp(-\frac{B^2(Y)}{2\sigma_Y^2}), \qquad (23)$$

for $Y > 0$ and zero for other,

$$f^Y(X) = \frac{1}{X\sqrt{2\pi(1-\rho^2)\sigma_X^2}}\exp(-\frac{(B(X)-B(Y)\rho^{\sigma_x}/_{\sigma_y})^2}{2(1-\rho^2)\sigma_X^2}), \qquad (24)$$

and

$$F^Y(X) = F_N(\frac{B(X)-B(Y)\rho^{\sigma_X}/_{\sigma_Y}}{\sqrt{(1-\rho^2)\sigma_X^2}}), \qquad (25)$$

for $X > 0$ and zero for other, and

$$X^Y(P) = \exp(\mu_X + B(Y)\rho^{\sigma_X}/_{\sigma_Y} + N(P)\sqrt{(1-\rho^2)\sigma_X^2}), \qquad (26)$$

for $0 \le P \le 1$, where F_N and $N(P)$ are the cdf and the P-order quantile of the normal standard distribution and $B(X) \equiv \log(X) - \mu_X$. From (2), the portfolio VaR is given by

$$VaR_Z(\hat{P}) = t - \alpha\exp(\mu_X + B(y)\rho^{\sigma_X}/_{\sigma_Y} + N(P^*)\sqrt{(1-\rho^2)\sigma_X^2}) - (1-\alpha)y, \qquad (27)$$

where, from (3), the order P^* is solved implicitly by

$$\hat{P} = \int_0^{+\infty} F_N(\frac{\log(\exp(\mu_X + B(y)\rho^{\sigma_X}/_{\sigma_Y} + N(P^*)\sqrt{(1-\rho^2)\sigma_X^2}) - \frac{1-\alpha}{\alpha}(Y-y))}{\sqrt{(1-\rho^2)\sigma_X^2}}$$

$$\left.\frac{-\mu_X - B(Y)\rho \,{}^{\sigma_X}\!/\!_{\sigma_Y}}{\sqrt{(1-\rho^2)\sigma_X^2}}\right) \frac{1}{\sqrt{2\pi\sigma_Y^2 Y}} \exp\left(-\frac{B^2(Y)}{2\sigma_Y^2}\right) dY \,. \qquad (28)$$

Thus, finding the VaR of the portfolio requires solving implicitly the order P^* from (28) and substituting it in (27). This is a relatively simple task, which does not involve numerous iterations as might be in other numerical methods.

Continuing with this example, the same technique can be used to find the minimal VaR portfolio. By chain differentiation we can rewrite (22) as[7]

$$\alpha_{\min} = \frac{y - \dfrac{\int_{-\infty}^{+\infty} A(Y)YdY}{\int_{-\infty}^{+\infty} A(Y)dY}}{y - X^y(P^*)} \,, \qquad (29)$$

where $A(Y) = g(Y) f^Y\left(X^y(P^*) - \frac{1-\alpha}{\alpha}(Y-y)\right)$, $g(Y)$, $f^Y(X)$ and $X^y(P^*)$ are given by (23), (24) and (26) (only with y instead of Y), the order P^* is solved by (3) and y is any arbitrarily selected realization of Y (as the ranges of X and Y are identical).

Figure 2 presents a numerical example of the above results. Panel A plots the mean-VaR frontier of two independent lognormally-distributed assets with the following parameters: $X \sim \Lambda(2.4, 0.136)$ and $Y \sim \Lambda(2.3, 0.15)$, where for simplicity, t=0 and the confidence interval is either 99 percent or 95 percent. The VaR at each point on the curves is calculated by solving implicitly for P^* from (28) and then substituting it in (27). Thus, each point on the curves requires solving only two equations together with the trivial equation of the portfolio expected return, $E_Z = \alpha E_X + (1-\alpha)E_Y$. This is comparable with a simulation method, which might require several hundred samples for each point in order to guarantee plausible accuracy. This advantage is even more apparent when calculating the minimal VaR portfolio.

The horizontal curve in Panel A plots the minimal VaR portfolio as a function of the required confidence interval using equation (29). Once again, each point on the curve requires to solve only the two equations, (28) and (29). In contrast, a simulation method might require an iterative convergence process in which each iteration, which takes the VaR closer to the minimal VaR, might be involved also with numerous samples.

Panel B of Figure 2 plots the mean-VaR frontier for a 99 percent confidence interval assuming a correlation coefficient of ρ = -0.5, 0, 0.25 and 0.9. As in the previous example, these curves demonstrate the relative simplicity of calculating the portfolio VaR at each point using equations (27) and (28). In addition, this example shows the impact of correlation on the portfolio VaR, which, as expected, is analogous to the impact of correlation on the portfolio standard deviation. However, in spite of this similarity there are also important differences as are presented below.

Figure 3 juxtaposes the proportion of *X,* which leads to the minimal VaR portfolio, $\alpha_{\min}$, calculated by (29) with the proportion of *X,* which leads to the minimal standard deviation portfolio[8] in the lognormal case. This comparison shows that even under the modest deviation from the normal distribution of assuming a lognormal distribution there are critical differences between VaR and standard deviation and between the implied optimal asset allocation according to these risk measures. It is plausible to assume that the differences would be even larger in the case of empirical distributions. Each point on the curves in Figure 3 represents the proportion of *X* in the minimal VaR portfolio ($\alpha_{\min}$) and in the minimal standard deviation portfolio for portfolios constructed from two independent lognormally-distributed assets, *X* and *Y*. Namely, at each point the assets' returns have different parameters. Panel A plots $\alpha_{\min}$ as a function of the ratio between the expected returns of *X* and *Y* with standard deviations held constant.[9] The calculations are done for three different ratios of the standard deviations of *X* and *Y*, where the standard deviation of *X* is equal to 1.5 and the standard deviation of *Y* is 0.5, 1.5 and 2.5. Panel B plots $\alpha_{\min}$ as a function of the ratio of the standard deviations of *X* and *Y* while the expected returns of *X* and *Y* are held constant. The calculations are done for three different ratios of the expected returns of *X* and *Y*, where the expected return of *X* is equal to 10 and the expected return of *Y* is 9,10 and 11.

The results emphasize the differences between VaR and standard deviation risk measures and the impact of downside risk on assets allocation. The curves on Panel A show that for a given ratio of standard deviations, a higher expected return of *X* relative to that of *Y* does not affect $\alpha_{\min}$ which leads to the minimal standard deviation. In contrast, a higher expected return of *X* relative to that of *Y* leads to a higher $\alpha_{\min}$ which leads to the minimal VaR. For example, a difference of 20 percent between the expected returns of *X* and *Y* (namely, $E_X / E_Y \geq 1.2$) leads to a proportion of *X* that is greater than 0.95 in the minimal VaR portfolio. Furthermore, this result is almost independent of the ratio of variances of *X* and *Y*. Panel B reveals that although both values of $\alpha_{\min}$ behave similarly when the standard deviation is changed, there is a significant quantitative difference between the results, which depends on the relations between the expected returns.

In summary, Figure 3 reveals that the relationship between the minimal VaR and the minimal standard deviation in the case of the lognormal distribution is significantly different from the relationship in the case of the normal distribution. Alexander & Baptista (2002) show that in the case of a multivariate normal distribution, if the minimum VaR portfolio exists, then it lies above the minimum variance portfolio on the mean-standard deviation frontier. According to Figure 3 this does not hold in the case of the lognormal distribution.

The results of Figure 3 illustrate the basic conceptual difference between standard deviation and VaR. Theoreticians, as well as practitioners, conceptually view risk as the chance of obtaining poor results relative to a given reference point (such as expected return, the risk-free interest rate or zero). However, standard deviation measures the dispersion around the mean and reflects correctly the downward risk of two alternative prospects only when their means are equal or when distributions are symmetrical. Unlike the standard deviation, VaR measures downward risk in terms of potential loss under specifically defined probability. Thus, VaR considers the mean and dispersion as well as all higher moments. Therefore, substantial differences should be expected between the two measures and the implied assets allocation even when it is the higher moments that are being varied.

III. Analytical VaR and Ahn, Boudoukh, Richardson and Whitelaw (1999) Analysis

Ahn et al. were the first to develop an analytical VaR optimization solution. They confined themselves to the case of hedging a lognormally-distributed asset with a put option on the managed portfolios. In Ahn et al.'s analysis, X is a put option on an underlying asset Y, which is assumed to terminate in-the-money. Ahn et al. then compute the optimal strike price that minimizes VaR. Below we show how their problem can be extracted from our proposed method. Although in Ahn et al.'s specific case it is simpler to formulate the problem straightforwardly, the proposed model offers a method to formulate and solve the optimal strategic hedging problem under more complex but yet realistic conditions whereby the optimized portfolio includes various types of assets including derivative assets that are only partially correlated with the hedged portfolio.

Ahn et al. assume a fixed hedging expense of $C=hq$, where h is the number of options and q is the price of each option. They also assume that this expense is financed by a loan with a continuous interest rate r such that the amount to be repaid in the future is qe^{rt} per option. Thus, the fixed hedging expense C determines the portfolio assets allocation ($\alpha_X = h, \alpha_r = -h, \alpha_Y = 1$) and using equation (B1) we get the general problem of minimal VaR of a three asset portfolio

$$\min\,[t - h\,X^{y,r}(P^*) + hqe^{rt} - y], \qquad (30)$$

where P^* is solved from (B2). The formulation of the problem in (30) is general as it is correct for any distribution and any correlations between the three assets. In Ahn et al. specific case, the quantile of the put option conditional on the underlying asset (which is of course also conditional on the risk-free interest rate as required by (30)) is given by $X^y(P^*) = K - y$, where K is the option strike price and y is necessarily the $\hat{P}$-order quantile of Y (since in the case of a put option with full correlation $P^* = \hat{P}$ and there is only one value of X for each realization y of Y). Substituting $X^y(P^*)$ in (30) yields

$$\min[t - h(K - y) + hqe^{rt} - y] \Rightarrow \min[r - \frac{C}{q}K + Ce^{rt} - (\frac{C}{q} - 1)y] \Rightarrow$$

$$\Rightarrow \max[C\frac{K - y}{q}] \Rightarrow \max[\frac{K - y}{q}]. \qquad (31)$$

Finally, by assuming that the underlying asset Y is lognormally-distributed, such that y is equal to the Y's $\hat{P}$-order lognormal quantile, we arrive at Ahn et al.'s original minimization problem (see equation (16) in Ahn et al.).

Clearly, it is simpler in case of a full correlation to formulate the problem straightforwardly, as Ahn et al. do. However, their assumption that there is a traded put option on the underlying asset scarcely exists. In general, traded options are not written on institutional portfolios but only on specific assets and indexes. Equation (30), which is based on the proposed model, makes it possible to solve the optimal strategic hedging problem under more realistic conditions whereby the options and other derivative assets are partially correlated with the hedged portfolio. The model can also deal with more complicated problems in which the portfolio includes various types of derivatives and financing is not restricted only to debt. The following example demonstrates these advantages.

Let us assume that in a hedging problem similar to Ahn et al., the agent faces more realistic market terms. For the purpose of demonstration, suppose that the agent faces two types of put options, X_1 and X_2 written on two market indices, which are uncorrelated with each other and only partially correlated with the agent portfolio. Assume also that the agent searches for the combination of options that minimize the total VaR of her portfolio. Hence, the fixed hedging expense, C, can be used to buy two types of options. Namely, $C = h_1q_1 + h_2q_2$, where h_1 and h_2 are the number of options of the first type and the second type and q_1 and q_2 are the prices of each option, respectively. Assume, as in Ahn et al., that the hedging expense is financed by a loan with a continuous interest rate r such that the amount to be repaid in the future is $q_i e^{rt}$ per option. Thus, the fixed hedging expense C determines the portfolio assets allocation to be $\alpha_{X_1} = h_1, \alpha_{X_2} = h_2, \alpha_r = -(h_1 + h_2)$ and $\alpha_Y = 1$. Recall that the conditional quantile function of the first type of option is uncorrelated with the second type option and with r (i.e. $X_1^{x_2 y, r}(P^*) = X_1^{y}(P^*)$) and h_2=(C-h_1q_1)/ q_2, equation (B1) formulates the optimization problem to be

$$\min_{h_1} [t - h_1 X_1^{y}(P^*) + Ce^{rt} - y - \frac{C - h_1 q_1}{q_2} x_2], \tag{32}$$

where, from (B2), P^* is solved by

$$\hat{P} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_1^{Y}(X_1^{y}(P^*) - \tfrac{1}{h_1}(Y - y) - \tfrac{C - h_1 q_1}{q_2 h_1}(X_2 - x_2)) g(Y) f_2(X_2) dY dX_2 \,, \tag{33}$$

and $g(Y)$ and $f(X_2)$ are the probability density functions of Y and X_2. Finally, substituting the conditional quantile function and cdf fully formulate the optimization problem, which can then be easily solved.

IV. Concluding remarks

This paper develops analytical tools for extracting the VaR of a portfolio from the general distributions of its underlying assets. This analytical VaR can then be used to construct optimal portfolios of generally distributed assets for the case in which the target function and/or constraints are expressed in terms of VaR. The basic information required for this problem is the conditional distributions of the risky assets. This is analogous to the information about the means and the variance-covariance matrix required for the classical Markowitz optimal portfolio problem. This proposed method can be used to solve any optimization problem, which involves portfolio VaR and is applicable to any distribution, not only the problematic normal distribution.

The proposed method is used to develop a minimal VaR rule, which identifies the minimal attainable VaR. The paper presents a detailed illustrative example of a portfolio composed of two dependent lognormally-distributed assets. This example emphasizes the advantage of the proposed method since it enables overcoming the main drawback of the lognormal distribution, i.e. that the distribution of a portfolio composed of lognormally distributed assets cannot be expressed analytically. Accordingly, the proposed method makes it possible to calculate straightforwardly the portfolio VaR by solving two simple equations. In our particular example, the solution is used to compare between the minimal VaR portfolio and the minimal standard deviation portfolio in the case of the lognormal distribution. This comparison reveals that the optimal proportions that minimize the VaR depend on all moments of the distribution. This intuitive outcome highlights the importance of using the correct measure of risk and the deficiencies of the standard deviation in this regard. Thus, this example illustrates the simplicity and the efficiency of the proposed method especially in the case of a portfolio that is composed of only a small number of assets.

This case covers many practical problems in finance. For example, this method is best suited to analyze the Campbell, Huisman and Koedijk (2001) problem of the optimal combination of shares and bonds portfolios. Similarly, this method is well suited to find the minimal VaR portfolio and to analyze the impact of adding an asset to an existing portfolio on the overall portfolio VaR. Finally, the paper uses the proposed method to formulate the problem of Ahn, Boudoukh, Richardson and Whitelaw (1999) of minimizing the portfolio VaR with a put option. This additional example demonstrates the ability of this method to analytically formulate complicated, realistic VaR optimization problems such as when the hedging derivative is only partially correlated with the hedged portfolio.

Appendix A: The Quantile Function of a Portfolio Composed of Risky and Risk-Free Assets

The following proof of the quantile function of a portfolio composed of risky and risk-free assets is taken from Levy and Kroll (1978). Let X be an asset with a random return with a cdf $F(X)$ and a quantile function $X(P)$. Denote the mixture of X with the risk-free asset by Z_α. Thus, $Z_\alpha = (1-\alpha)r + \alpha X$ where $0 < \alpha$ and r is the risk-free interest rate. Recall that by definition

$$F(X(P)) = \Pr(X \leq X(P)) = P \, . \qquad \text{(A1)}$$

Thus, since α and r are constants (A1) implies that for any $0 < \alpha$ the following holds

$$\Pr((1-\alpha)r + \alpha X \leq (1-\alpha)r + \alpha X(P)) = P \, . \qquad \text{(A2)}$$

Substituting the definition of Z_α from above into (A2) yields

$$\Pr(Z_\alpha \leq (1-\alpha)r + \alpha X(P)) = P \, . \qquad \text{(A3)}$$

However, since by definition $\Pr(Z_\alpha \leq Z_\alpha(P)) = P$, where $Z_\alpha(P)$ is the quantile function of the portfolio, then by necessity

$$Z_\alpha(P) = (1-\alpha)r + \alpha X(P) \, . \qquad \text{(A4)}$$

Appendix B: Analytical VaR of Multiple Asset Portfolios

In the following the model is extended to the case of a multiple asset portfolio. The proofs in this case are identical to those of the two-assets case and will not be repeated.

Let X_i (i=1…n) be the returns on n risky assets with probability density functions $f_i(X)$ and cumulative distribution functions (cdf) $F_i(X)$. Let $Z_\alpha = \sum_{i=1}^{n} \alpha_i X_i$ be the random return on a portfolio composed of these n assets. Let $X_j^{\underline{x}_i}(P)$ be the quantile of the return on asset X_j *conditional* on the vector of realizations $\underline{x}_i$ of $\underline{X}_i$ (i=1…n, i≠j). The information about this conditional quantile is tantamount to the information about the assets' cdfs and the relationship between them and obtaining it is a prerequisite to solving any portfolio optimization problem.

Proposition B1. *For every $0\le\alpha_i$, $0<\alpha_j$ (i=1,…n, $i\neq j$) and some selected vector of realizations $\underline{x}_i$ of $\underline{X}_i$ which guarantees that $0\le P^*\le 1$, the VaR of the portfolio Z_α is given by*

$$VaR_{Z\alpha}(\hat{P}) = t - \alpha_j X_j^{\underline{x}_i}(P^*) - \sum_{\substack{i=1\\ i\neq j}}^{n} \alpha_i x_i \,, \qquad \text{(B1)}$$

where the order P^* is solved from the following equation

$$\hat{P} = \int_{-\infty}^{+\infty}\!\!\dots\!\int_{-\infty}^{+\infty} F^{\underline{X}_i}\Big(X_j^{\underline{x}_i}(P^*) - \sum_{\substack{i=1\\ i\neq j}}^{n} \tfrac{\alpha_i}{\alpha_j}(X_i - x_i)\Big) \prod_{\substack{i=1\\ i\neq j}}^{n} f_i(X_i)dX_i \,. \qquad \text{(B2)}$$

Note that the integration in (B2) is over n-1 variables. The range of probabilities for which the vector of realizations $\underline{x}_i$ of $\underline{X}_i$ provides a solution of (B2) is given by

$$\int_{-\infty}^{+\infty}\!\!\dots\!\int_{-\infty}^{+\infty} F^{\underline{X}_i}\Big(X_j^{\underline{x}_i}(0) - \sum_{\substack{i=1\\ i\neq j}}^{n} \tfrac{\alpha_i}{\alpha_j}(X_i - x_i)\Big) \prod_{\substack{i=1\\ i\neq j}}^{n} f_i(X_i)dX_i \le \hat{P} \le .$$

$$\int_{-\infty}^{+\infty}\!\!\dots\!\int_{-\infty}^{+\infty} F^{\underline{X}_i}\Big(X_j^{\underline{x}_i}(1) - \sum_{\substack{i=1\\ i\neq j}}^{n} \tfrac{\alpha_i}{\alpha_j}(X_i - x_i)\Big) \prod_{\substack{i=1\\ i\neq j}}^{n} f_i(X_i)dX_i \,. \qquad \text{(B3)}$$

Contentiously with the analogy to the two-asset case, assuming that $X_j^{\underline{x}_i}(P)$ is differentiable for any order P in the range [0,1] and for any vector of realizations $\underline{x}_i$ of $\underline{X}_i$ (i=1…n, i≠j), which is required in order to guarantee that the portfolio distribution is well behaved, the proportions of the minimal VaR portfolio, $\alpha^{min}{}_i$ (i=1,…n), are given by the equation

$$\alpha_{\mathrm{j}}^{\min}\frac{dX_{j}^{x_i}(P^*}{d\alpha_j}=x_n-X_j^{x_i}(P^*)\,, \qquad \text{(B4)}$$

together with the following *n*-2 equations

$$\alpha_{\mathrm{i}}^{\min}\frac{dX_{j}^{x_i}(P^*)}{d\alpha_i}=x_n-x_i\,, \quad i=1...n\text{-}1, \quad i\neq j\,, \qquad \text{(B5)}$$

and the trivial equation

$$\alpha_n=1-\sum_{i=1}^{n-1}\alpha_i\,, \qquad \text{(B6)}$$

where P^* is simultaneously solved from (B2) as a function of $\hat{P}$, $\alpha_i^{\min}$ and $\underline{x}_i$. Finally, the minimal VaR is obtained by substituting $\alpha_i^{\min}$ (i=1,…n) into (B1).

An Illustrative Example

Let *X* be the return on a bank commercial activity. Let *Y* and *W* be the return on the bank domestic and foreign financial investment portfolios, respectively. Let *X* and *Y* be multivariate lognormally distributed with expected returns and standard deviations of the logs of μ_X, μ_Y, σ_X and σ_Y, respectively, and with correlation coefficients of the logs of ρ. Let *W* be lognormally distributed with expected return and standard deviation of the log of μ_W and σ_W uncorrelated, as being a foreign market, with *X* and *Y*. Denote the bank total portfolio as $Z_\alpha=\alpha_1X+\alpha_2Y+\alpha_3W$, $0\leq\alpha_{2,3}$, $0<\alpha_1$.

According to preposition (B1), and using the identities of $g(Y)$ given by (23), $F^{Y,W}(X) = F^{Y}(X)$ given by (24), and $X^{y,w}(P)=X^{y}(P)$ given by (26), the bank overall VaR is given by

$$VaR_{Z\alpha}(\hat{P})=t-\alpha_1\exp(\mu_X+B(y)\rho\,{}^{\sigma_X}\!/_{\sigma_Y}+N(P^*)\sqrt{(1-\rho^2)\sigma_X^2})-\alpha_2x-\alpha_3w, \qquad \text{(B7)}$$

where the order P^* is solved from the following equation

$$\hat{P}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F^{Y,W}(X^{y,w}(P^*)-\tfrac{\alpha_2}{\alpha_1}(Y-y)-\tfrac{\alpha_3}{\alpha_1}(W-w))g(Y)h(W)dYdW= \qquad \text{(B8)}$$

$$\int_{0}^{+\infty}\int_{0}^{+\infty}F_N\Bigg(\frac{\log(\exp(\mu_X+B(y)\rho\,{}^{\sigma_X}\!/_{\sigma_Y}+N(P^*)\sqrt{(1-\rho^2)\sigma_X^2})-\frac{\alpha_2}{\alpha_1}(Y-y)-\frac{\alpha_3}{\alpha_1}(W-w))}{\sqrt{(1-\rho^2)\sigma_X^2}}$$

$$\frac{-\mu_X-B(Y)\rho\,{}^{\sigma_X}\!/_{\sigma_Y}}{\sqrt{(1-\rho^2)\sigma_X^2}}\Bigg)\frac{1}{2\pi\sigma_Y\sigma_W YW}\exp\Big(-\frac{B^2(Y)}{2\sigma_Y^2}-\frac{B^2(W)}{2\sigma_W^2}\Big)dYdW\,,$$

and where $\alpha_3 = 1 - \alpha_2 - \alpha_1$ and $B(W) \equiv \log(W) - \mu_W$.

Thus, as in the two-asset case, finding the VaR of the overall portfolio requires implicitly solving the order P^* from (B8) and substituting it in (B7). This relatively simple task can be easily used to map the impact of the proportion of the foreign portfolio on the bank overall VaR. For example, assuming the value of the bank financial investment portfolio to be 10% (i.e. $\alpha_3 + \alpha_2 = 0.1$) solving (B7) and (B8) for a vector of values in the range $\alpha_2 = 0,0.01,0.02...0.1$ will map the full scope of the impact of international diversification on the bank VaR. This is achieved by solving ten times only two equations.

REFERENCES

Ahn, D.H., Boudoukh, J., Richardson, M., and R.F., Whitelaw, "Optimal Risk Management Using Options", *J. of Finance*, 54(1), February 1999, 359-375.

Alexander, J.G., and A.M., Baptista, "Economic Implications of Using a Mean-VaR Model for Portfolio Selection: A Comparison with Mean-Variance Analysis", *Journal of Economic Dynamics and Control,* 2002, 1159-1193.

Artzner, P., Delbaen, F., Eber, J.M., and D., Heath, "Coherent Measures of Risk", *Mathematical Finance*, 9, 1999, 203-228.

Basak, S., and A., Shapiro, "Value-at-Risk Based Risk Management: Optimal Policies and Asset Prices", *Review of Financial Studies*, 14(2), 2001, 371-405.

Basel Committee on Bankig Supervision, *Amendment to the Capital Accord to Incorporate Market Risks*, January 1996.

Campbell, R., Huisman, R., and K., Koedijk, "Optimal Portfolio Selection in a Value-at-Risk Framework", *J. of Banking and Finance*, 25, 2001, 1789-1804.

Crouhy, M., Galai, D., and R., Mark, *Risk Management*, (Eds.), McGraw-Hill, 2001.

Cuoco, D., He, H., and S., Issaenko, "Optimal Dynamic Trading Strategies with Risk Limits", *University of Pennsylvania working paper*, 2001.

Duffie, D., and J., Pan, "An Overview of Value at Risk", *J. of Derivatives*, 4, Spring 1997, 7-49.

Duffie, D., and J., Pan, "Analytical Value-at-Risk with Jumps and Credit Risk"*, Finance and Stochastic*, 5, 2000, 155-180.

Emmer, S., Klüppelberg, C., and R., Korn, "Optimal Portfolios with Bounded Downside Risks", *Mathematical Finance,* 2001, 365-384.

Harvey, C.R., and A., Siddique, "Conditional Skewness in Asset Pricing Tests", *J. of Finance*, 55, June 2000, 1263-1295.

Jorion, P., *Value at Risk: The New Benchmark for Managing Financial Risk*, McGraw-Hill, December 2000.

Jorion, P., "Portfolio Optimization with Constraint on Tracking Error", *Working paper*, University of California at Irvine", 2001.

Kaplanski, G., and Y., Kroll, "VaR Risk Measures versus Traditional Risk Measures: An Analysis and a Survey ", *Journal of Risk*, 4(3), 2002, 1-27.

Levy, H., and Y., Kroll, "Ordering Uncertain Options with Borrowing and Lending", *J. of Finance*, 33(2), May 1978, 552-573.

Rothschild, M., and J.E., Stiglitz, "Increasing Risk. I: A Definition", *J. of Economic Theory*, 2, 1970, 225-243.

Sentana, E., "Mean Variance Portfolio Allocation with a Value at Risk Constraint", *C.E.P.R., Discussion Papers*, 2997, 2001.

Yiu, K.F.C., "Optimal Portfolios under a Value-at-Risk Constraint", *J. of Economic Dynamics and Control*, 28(7), April 2004, 1317-1334.

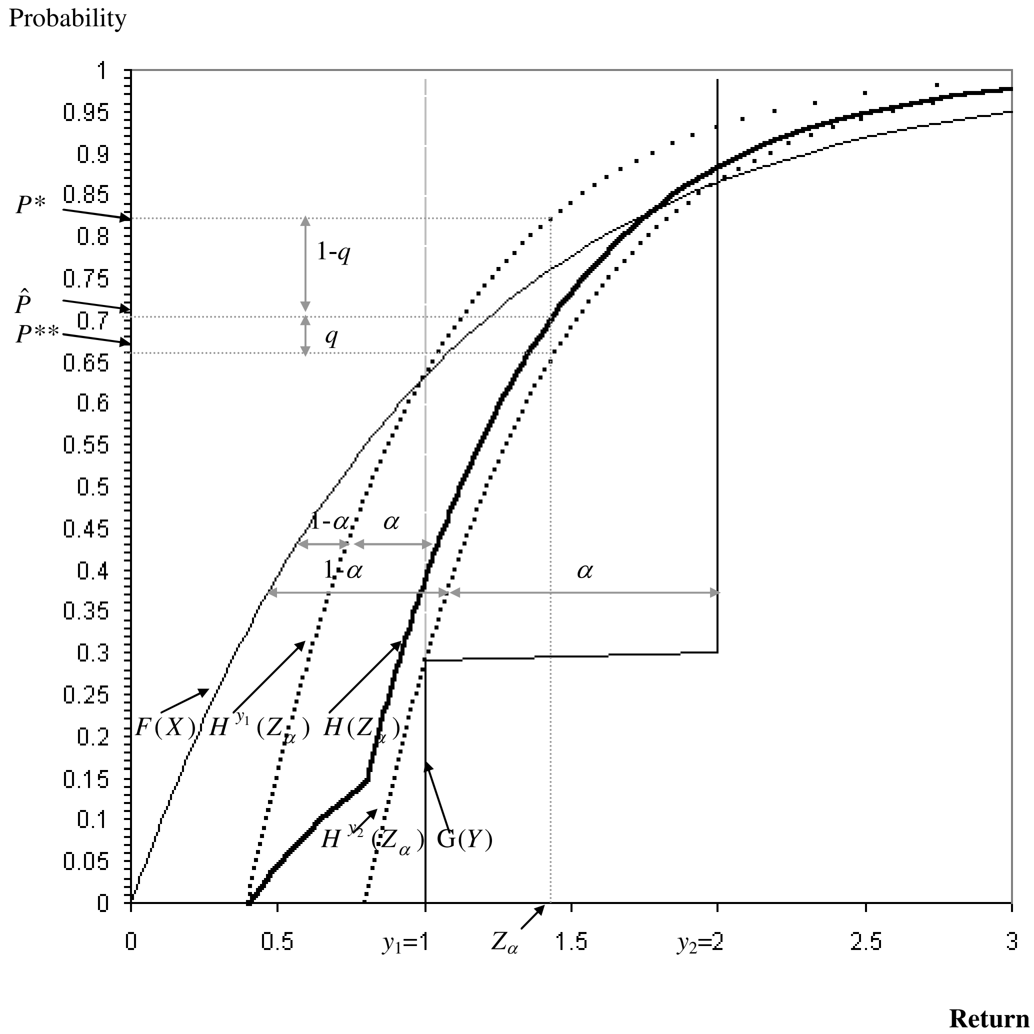

Figure 1. **A graphical exposition of Proposition 1**

The figure plots the construction of a VaR value for a portfolio composed of assets X and Y in proportions $Z = 0.6X + 0.4Y$. The return on asset X is exponentially distributed with parameter $\lambda = 1$ uncorrelated with the return on asset Y which is restricted to two values, y_1=1 and y_2=2 with probabilities q = 0.3 and 1-q = 0.7. The conditional cdf of the portfolio for $Y = y_1$, $H^{y_1}(Z_\alpha)$, divides the horizontal difference between $F(X)$ and y_1 according to the proportions α = 0.6 and 1-α = 0.4. Similarly, the conditional cdf of the portfolio for $Y = y_2$, $H^{y_2}(Z_\alpha)$, divides the horizontal difference between $F(X)$ and y_2 according to the same proportions. It can be seen in Figure 1 that $\hat{P}$ divides the vertical distance between P^* and P^{**} according to the proportions 1 - q = 0.7 and q = 0.3. Similarly, $H(Z_\alpha)$ divides the vertical distance between the conditional cdfs $H^{y_1}(Z_\alpha)$ and $H^{y_2}(Z_\alpha)$ according to the same proportions.

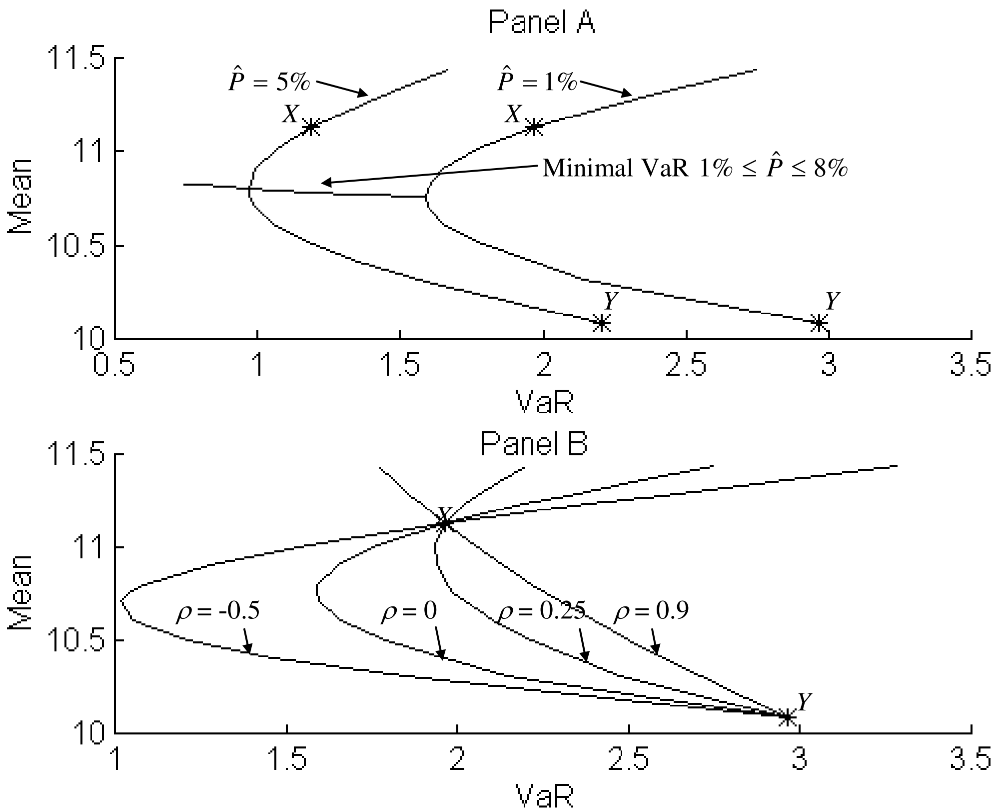

Figure 2. **Portfolio VaR of two lognormally-distributed assets**

The figure plots the mean-VaR frontier of a portfolio composed of two lognormally-distributed assets with the following parameters: $X \sim \Lambda(2.4, 0.136)$ and $Y \sim \Lambda(2.3, 0.15)$ and assuming also t=0. Panel A plots the mean-VaR frontier for 95 and 99 percent confidence intervals assuming independent distributions (namely, ρ=0). The horizontal curve plots the minimal VaR as a function of the confidence interval. Panel B plots the mean-VaR frontier of a portfolio composed of the same assets assuming a correlation coefficient of ρ=–0.5, 0, 0.25 and 0.9. Each point on the curves requires to solve only the two equations (27) and (28).

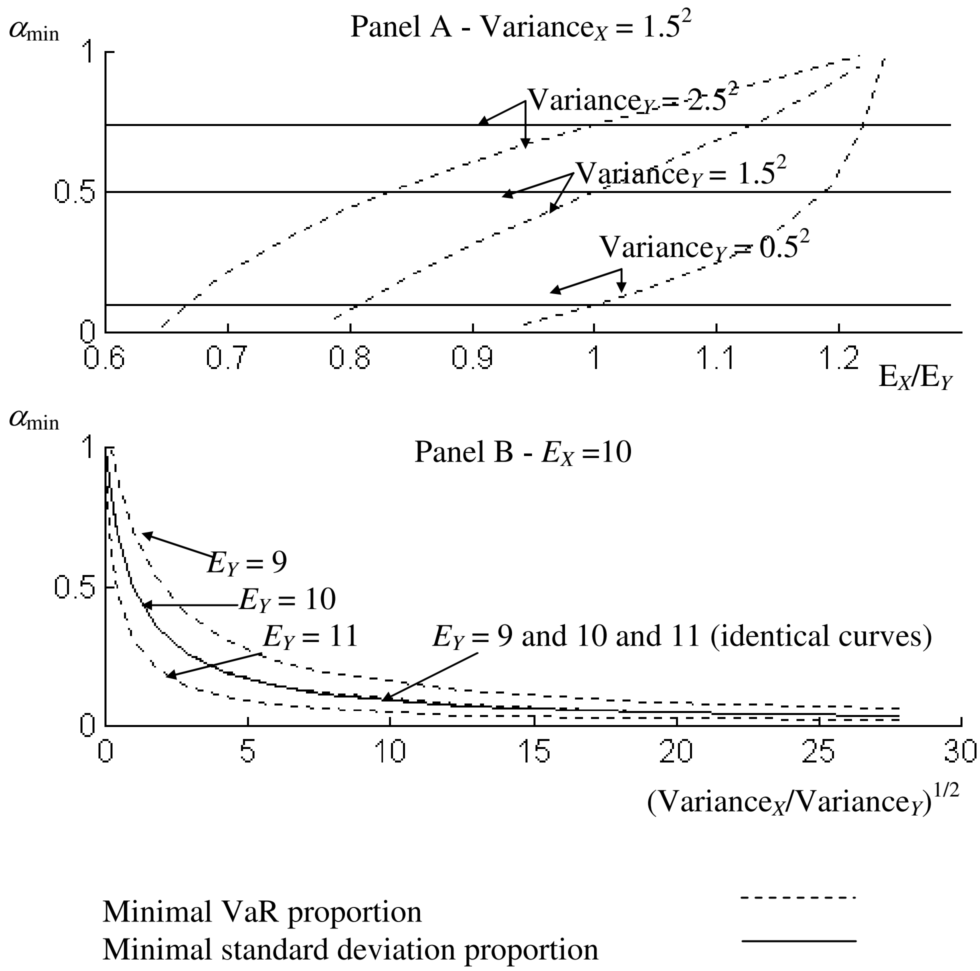

Figure 3. **The minimal VaR portfolio versus the minimal standard deviation portfolio**

Panel A plots the proportion of asset *X* in the minimal VaR portfolio versus the proportion of asset *X* in the minimal standard deviation portfolio, for different independent assets *X* and *Y,* as a function of the ratio between the expected returns of assets *X* and *Y*. Panel B plots the same as a function of the ratio between the standard deviations of *X* and *Y*. The curves illustrate the importance of downside risk in asset allocation and the critical distinction induced by VaR and standard deviation on the portfolio asset allocation even under modests deviation from the normal distribution.

[1] An introduction and overview of VaR can be found in Duffie & Pan (1997) and in the excellent books by Jorion (2000) and Crouhy, Galai & Mark (2001).

[2] For more on credit risk issues see, for example, Duffie & Pan (2000). For more on the methodology used to constrain and control risk exposure see, for example, Basak & Shapiro (2001) and Jorion (2001).

[3] Assuming $0<\alpha$ eliminates short sales of asset *X*. This assumption can easily be dropped by replacing *P** in (2) and (3) for $\alpha < 0$ by the expression 1-P*. Then, if $\alpha < 0$ the VaR of the portfolio is given by $VaR_Z(\hat{P}) = t - \alpha\, X^y(1-P^*) - (1-\alpha)y$, and *P** is solved by the expression
$\hat{P} = \int_{-\infty}^{\infty} F^Y\left(X^y(1-P^*) - \tfrac{1-\alpha}{\alpha}(Y-y)\right)g(Y)dY$.

[4] In (9) we do not need to specify separately the cases of (8a) and (8b) as, unlike the quantile function, the cdf function is defined over the entire range.

[5] The order *P* is stable in a linear combination only when the portfolio is composed of a risky asset and a risk-free asset or in the trivial case of fully correlated assets. Indeed, if *Y* is the risk-free asset *r*, then Proposition 1 converges to Levy & Kroll's (1978) solution of a portfolio of a risky asset and a risk-free asset. Substituting $Y = r$ in (3) yields
$\hat{P} = \int_{-\infty}^{+\infty} F^r\left(X^r(P^*) - \tfrac{1-\alpha}{\alpha}(r-r)\right)g(r)dr = \int_{-\infty}^{\infty} P^*\, dr = P^*$ and substituting $\hat{P} = P^*$ in (2) yields $VaR_Z(\hat{P}) = t - \alpha\, X(\hat{P}) - (1-\alpha)r$ which corresponds to the results of Kroll & Levy (see equation (3) there and in Appendix A).

[6] In (22) it is assumed that VaR is an increasing monotonic function around $\alpha_{\min}$. This assumption is correct by definition in the immediate neighborhood of $\alpha_{\min}$ as long as $\alpha_{\min}$ exists since we define $\alpha_{\min}$ as the proportion that leads to the local minimal VaR. Furthermore, $\alpha_{\min}$ always exists in our case since the assumption that the conditional quantile $X^y(P^*)$ is differentiable guarantees the monotonicity of VaR. In fact, VaR monotonicity for contiguous and differentiable quantiles derives from the monotonicity of the portfolio cdf, which, as has been previously said, is proved and used by Rothschild and Stiglitz (1970) for the case of Second Stochastic Dominance analysis. Corresponding with Artzner et al. (1999), this property does not exist for VaR in the case of discrete distributions. Nevertheless, it should be kept in mind that discrete distributions are usually empirical approximations of actual continuous distributions.

[7] The only mathematical manipulation is the use of chain differentiation in order to solve (22).
Accordingly, equation (22) is rewritten as: $\alpha_{\min} = -(y - X^y(P^*)) \Big/ \dfrac{dX^y(P^*)}{dP^*}\dfrac{d\hat{P}/d\alpha}{d\hat{P}/dP^*}$ and from (3) we obtain the following identities:

$$\frac{d\hat{P}}{dP^*} = \int_{-\infty}^{+\infty} g(Y) f^Y\left(X^y(P^*) - \tfrac{1-\alpha}{\alpha}(Y-y)\right)\frac{dX^y(P^*)}{dP^*}dY \text{ and}$$

$$\frac{d\hat{P}}{d\alpha} = \int_{-\infty}^{+\infty} g(Y) f^Y\left(X^y(P^*) - \tfrac{1-\alpha}{\alpha}(Y-y)\right)\frac{Y-y}{\alpha^2}dY .$$

[8] The proportion which leads to the minimal standard deviation portfolio is given by:

$$\alpha_{\min} = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}.$$

[9] The process of changing the assets' expected returns without changing their standard deviations or changing the assets' standard deviations without changing their expected returns is achieved by simultaneously changing both parameters of the lognormal distribution.