Some estimations of the minimal magnitudes of forbidden zones in experimental data

Harin, Alexander

Modern University for the Humanities

23 July 2017

Some estimations of the minimal magnitudes of forbidden zones in experimental data

Alexander Harin
aaharin@gmail.com
Modern University for the Humanities

Suppose a random variable takes on values in an interval. The minimal distance between the expectation of the variable and the nearest boundary of the interval is considered in the present article. A question whether this distance can be neglected with respect to the standard deviation is analyzed as the main item. This minimal distance can determine the minimal magnitudes of non-zero forbidden zones and biases caused by noise for results of experiments. These non-zero forbidden zones and biases cause fundamental problems, especially in interpretations of experiments in behavioral economics and decision sciences.

Contents

1. Introduction

1.1. Bounds for functions and moments of random variables

Bounds for functions of random variables and their moments are considered in a number of works.

Bounds for the probabilities and expectations of convex functions of discrete random variables with finite support are studied in [8].

Inequalities for the expectations of functions are studied in [9]. These inequalities are based on information of the moments of discrete random variables.

A class of lower bounds on the expectation of a convex function using the first two moments of the random variable with a bounded support is considered in [1].

Bounds on the exponential moments of $\min(y, X)$ and $X\,I\{X < y\}$ using the first two moments of the random variable X are considered in [7].

1.2. Problems, their solution and the need of further research

1.2.1. Problems of applied sciences

There are some basic problems concerned with the mathematical description of the behavior of a man. They are the most actual in behavioral economics, decision sciences, social sciences and psychology. They are pointed out, e.g., in [6].

Examples of the problems are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, the four-fold pattern paradox, etc.

The essence of the problems consists in biases of preferences and decisions of a man in comparison with predictions of the probability theory.

These biases are maximal near the boundaries of the probability scale, that is, at high and low probabilities.

1.2.2. Bounds (forbidden zones) for the expectations

Bounds on the expectation of a random variable that takes on values in a finite interval are considered as well (see, e.g., [4] and [5]).

Suppose a random variable takes on values in a finite interval. An existence theorem was proven. The theorem states: if there is a non-zero lower bound on the variance of the variable, then non-zero bounds on its expectation exist near the boundaries of the interval.

The obtained non-zero bounds (or strict bounding inequalities) can be treated as non-zero forbidden zones for the expectation near the boundaries of the interval.

1.2.3. Partial solution of the problems

A non-zero noise can be associated with the non-zero minimal variance of random variables. The dispersion and noisiness of the initial data can lead to bounds (restrictions) on the expectations of experimental data. This should be taken into account when dealing with data obtained in real circumstances.

The works [2] and [3] were devoted to the well-known problems of utility and prospect theories. Such problems had been pointed out, e.g., in [6]. In [2] and [3] some examples of typical paradoxes were studied. Similar paradoxes may concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, etc. A noise and data scattering are usual circumstances of the experiments. The proposed bounds explained, at least partially, the analyzed examples of paradoxes.

1.2.4. The need of further research

However, there is a consequence of the theorem of existence of the forbidden zones: when the level of the noise and, hence, the minimal variance of variables tends to zero, then not only the width of the revealed forbidden zones, but also the ratio of this width to the standard deviation tends to zero. Therefore, in some cases these forbidden zones can be neglected at low level of the noise.

So, there is a need of a more deep consideration of the question whether, when and under what conditions this minimal distance can be neglected with respect to the standard deviation at low level of this standard deviation.

1.2.5. The aims and the practical motivation of the present article

The general aim of the present article is the consideration of the minimal distance from the nearest boundary of an interval to the expectation of a random variable that takes on values in this interval. This minimal distance is expressed here in terms of the standard deviation of the random variable.

The consideration is concentrated on the normal and similar distributions.

In this preliminary version of the article, the calculations are given as detailed as possible to be the verification for following journal articles.

The first particular aim of the article is the determination of some typical reference points for considerations of this minimal distance.

The second particular aim is to start a consideration of a question whether, when and under what conditions this minimal distance can be neglected with respect to the standard deviation of the random variable, especially when this standard deviation tends to zero.

The practical motivation of the present article is caused by the above problems of behavioral economics, decision sciences, social sciences and psychology.

The article is to provide the mathematical support for a consideration of a question whether, when and under what conditions the above influence of a noise can be neglected at low level of the noise.

1.3..General definitions and notes

For the purposes of the present article, let us define and denote some terms:

The standard deviation is referred to as **SD**.

The probability density functions are referred to as **PDFs**.

The interval boundary that is the nearest to the expectation of the variable is referred to as $b_{Boundary}$. So the minimal distance between the expectation of the variable and the nearest boundary $b_{Boundary}$ of the interval is referred to as $min(|E(X)-b_{Boundary}|)$. To avoid ambiguity, the minimal distance $min(|E(X)-b_{Boundary}|)$ between the expectation of the variable and the nearest boundary of the interval is referred to as $\boldsymbol{|E(X)-b_{Boundary}|}$. This nearest boundary is usually defined as $\boldsymbol{b_{Boundary} = 0}$.

Normal-like distributions are defined as distributions that have symmetric probability density functions f with non-increasing sides. In other words:

$$f(E(X)+a) = f(E(X)-a)$$

and if $|x_c - E(X)| \le |x_d - E(X)|$, then $f(x_c) \ge f(x_d)$.

For the conciseness, in the scope of this article, distributions with bounded or compact support are referred to as **compact distributions**. The distributions with not bounded support are referred to as **noncompact distributions**.

Usually, $\boldsymbol{h}$ denotes the value (***h***eight) of PDF, $\boldsymbol{l}$ denotes the ***l***ength. The index 1 denotes the centre of a distribution, that is $\boldsymbol{h_1} \equiv h_{Centre}$ and $\boldsymbol{l_1} \equiv l_{Centre}$. The index 2 denotes the side or tail of a distribution, that is $\boldsymbol{h_2} \equiv h_{Side} \equiv h_{Tail}$ and $\boldsymbol{l_2} \equiv l_{Side} \equiv l_{Tail}$.

The **contiguous** situation is defined as the situation when one side of distribution's support touches the boundary of a half-infinite or finite interval.

The **hypothetical reflection** situation is defined as the situation when f is modified to the hypothetical function f_{Refl} that is reflected with respect to $E(X) = 0$

$$f_{\mathrm{Re}\,fl}(x) = \theta(x) 2 f(x) .$$

The hypothetical reflection situation is, in a sense similar to the reflection of a wave of light from a mirror.

The hypothetical reflection situation can simulate and be used to analyze not normal-like distributions.

The **hypothetical adhesion** situation is modified from the hypothetical reflection situation such that the reflected part of the PDF is "adhered" to the boundary 0. In other words, a half of the reflected PDF is adhered to the point $E(X) = b_{Boundary} = 0$. In particular, in the hypothetical adhesion situation

$$E(X_{Adhes}) = \int_{-\infty}^{+\infty} f_{Adhes}(x) dx = \frac{1}{2} \int_{0}^{+\infty} f(x) dx .$$

The hypothetical situation of "adhesion" is in a sense similar to the absorption of a wave of light by a black body.

Reasons for the choice of the hypothetical situations will be considered in next articles of this series.

Note, in all hypothetic situations the standard deviation of the non-modified function is used.

2. Normal distribution

The normal distribution is one of the most important ones in the probability theory and statistics. Its PDF can be represented in a form of, e.g.,

$$f_X(x) \equiv N(0,\sigma^2) \equiv f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}.$$

Hypothetical situations

The standard deviation (SD) of the normal distribution equals σ.

One can calculate the expectation for the hypothetic situation of "reflection" from the boundary $b_{Boundary} = 0$

$$E(X) = 2\int_0^\infty xf(x)dx = 2\int_0^\infty x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2\sigma^2}} d\left(\frac{x^2}{2\sigma^2}\right) =$$
$$= \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty e^{-y} dy = -\frac{2\sigma}{\sqrt{2\pi}} e^{-y}\Big|_0^\infty = \sigma\sqrt{\frac{2}{\pi}}.$$

The ratio $|E(X)\text{-}b_{Boundary}|/SD$ is equal to

$$\frac{|E(X) - b_{Boundary}|}{SD} = \frac{E(X)}{SD} = \sqrt{\frac{2}{\pi}} \approx 0.789 \in \left(\frac{3}{4}, \frac{4}{5}\right).$$

For the hypothetic situation of "adhesion" the ratio $|E(X)\text{-}b_{Boundary}|/SD$ is equal to $1/2$ of that of the hypothetic situation of "reflection" and is equal to

$$\frac{|E(X) - b_{Boundary}|}{SD} = \frac{1}{\sqrt{2\pi}} \approx 0.399 \in \left(\frac{1}{3}, \frac{1}{2}\right).$$

So, for the hypothetic situations of both "reflection" and "adhesion," the ratio $|E(X)\text{-}b_{Boundary}|/SD$ cannot be negligibly small with respect to unity.

3. **Noncompact distributions**

3.1. Laplace distribution

One can write Laplace distribution as

$$f(x) = he^{-\frac{|x|}{\lambda}} .$$

The parameter h can be calculated from the normalizing integration

$$2\int_0^\infty f(x)dx = 2\int_0^\infty he^{-\frac{x}{\lambda}}dx = -2h\lambda\int_0^\infty e^{-\frac{x}{\lambda}}d\left(-\frac{x}{\lambda}\right) = -2h\lambda e^{-\frac{x}{\lambda}}\Big|_0^\infty = 2h\lambda = 1 .$$

So,

$$h = \frac{1}{2\lambda} \quad \text{and} \quad f(x) = \frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}} .$$

The variance can be calculated from

$$Var(X) = 2\int_0^\infty x^2 f(x)dx = 2\int_0^\infty x^2\frac{1}{2\lambda}e^{-\frac{x}{\lambda}}dx = \int_0^\infty x^2\frac{1}{\lambda}e^{-\frac{x}{\lambda}}dx =$$

$$= -x^2e^{-\frac{x}{\lambda}}\Big|_0^\infty + \int_0^\infty 2xe^{-\frac{x}{\lambda}}dx = \int_0^\infty 2xe^{-\frac{x}{\lambda}}dx = -2x\lambda e^{-\frac{x}{\lambda}}\Big|_0^\infty + \int_0^\infty 2\lambda e^{-\frac{x}{\lambda}}dx = .$$

$$= \int_0^\infty 2\lambda e^{-\frac{x}{\lambda}}dx = -2\lambda^2\int_0^\infty e^{-\frac{x}{\lambda}}d\left(-\frac{x}{\lambda}\right) = -2\lambda^2 e^{-\frac{x}{\lambda}}\Big|_0^\infty = 2\lambda^2$$

So, the standard deviation is

$$SD = \lambda\sqrt{2} .$$

Hypothetical situations

One can calculate the expectation for the hypothetic situation of "reflection"

$$E(X) = \int_0^\infty xf(x)dx = \int_0^\infty x\frac{1}{2\lambda}e^{-\frac{x}{\lambda}}dx = -x\frac{1}{2}e^{-\frac{x}{\lambda}}\Big|_0^\infty + \int_0^\infty \frac{1}{2}e^{-\frac{x}{\lambda}}dx = \frac{1}{2}\int_0^\infty e^{-\frac{x}{\lambda}}dx =$$

$$= -\frac{\lambda}{2}\int_0^\infty e^{-\frac{x}{\lambda}}d\left(-\frac{x}{\lambda}\right) = -\frac{\lambda}{2}e^{-\frac{x}{\lambda}}\Big|_0^\infty = \frac{\lambda}{2} .$$

The ratio $|E(X)\text{-}b|/SD$ is equal to

$$\frac{|E(X) - b_{Boundary}|}{SD} = \frac{\lambda}{2}\frac{1}{\lambda\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.354 \in \left(\frac{1}{3},\frac{1}{2}\right).$$

For the hypothetic situation of "adhesion" the ratio $|E(X)\text{-}b_{Boundary}|/SD$ is equal to 1/2 of that of the hypothetic situation of "reflection" and is equal to

$$\frac{|E(X) - b_{Boundary}|}{SD} = \frac{1}{4\sqrt{2}} \approx 0.177 \in \left(\frac{1}{6},\frac{1}{5}\right).$$

So, for the hypothetic situations of both "reflection" and "adhesion," the ratio $|E(X)\text{-}b_{Boundary}|/SD$ cannot be much less then unity.

3.2. A power one-step test distribution with noncompact support

Let us consider the power noncompact "one-step" test distribution. Its probability density function can be written as, e.g.,

$$f(x) = h\left(1+\alpha\,|\,x-\mu\,|\right)^{-\beta} = h\left(1+\left|\frac{x-\mu}{l}\right|\right)^{-\beta},$$

where $\mu \equiv E(X)$, $h > 0$, $\alpha = 1/l > 0$, $l > 0$ and $\beta > 3$.

The normalizing integration (under the simplifying condition $\mu = 0$) can be written as

$$2\int_0^\infty f(x)dx = 2\int_0^\infty h\left(1+\frac{x}{l}\right)^{-\beta} dx = 2hl\int_0^\infty \left(1+\frac{x}{l}\right)^{-\beta} d\frac{x}{l} =$$
$$= 2hl\frac{1}{(\beta-1)}\left(1+\frac{x}{l}\right)^{-\beta+1}\Bigg|_0^\infty = \frac{2hl}{\beta-1} = 1.$$

So

$$\frac{2hl}{\beta-1} = 1.$$

The variance can be calculated as

$$Var(X) = 2\int_0^\infty x^2 f(x)dx = 2\int_0^\infty x^2 h\left(1+\frac{x}{l}\right)^{-\beta} dx = 2h\int_0^\infty x^2\left(1+\frac{x}{l}\right)^{-\beta} dx =$$
$$= -2hx^2\frac{l}{\beta-1}\left(1+\frac{x}{l}\right)^{-\beta+1}\Bigg|_0^\infty + \frac{2hl}{\beta-1}\int_0^\infty 2x\left(1+\frac{x}{l}\right)^{-\beta+1} dx = .$$
$$= \frac{4hl}{\beta-1}\int_0^\infty x\left(1+\frac{x}{l}\right)^{-\beta+1} dx =$$
$$= -\frac{4hl}{\beta-1}x\frac{l}{\beta-2}\left(1+\frac{x}{l}\right)^{-\beta+2}\Bigg|_0^\infty + \frac{4hl^2}{(\beta-1)(\beta-2)}\int_0^\infty \left(1+\frac{x}{l}\right)^{-\beta+2} dx =$$
$$= \frac{4hl^2}{(\beta-1)(\beta-2)}\int_0^\infty \left(1+\frac{x}{l}\right)^{-\beta+2} dx =$$
$$= -\frac{4hl^2}{(\beta-1)(\beta-2)}\frac{l}{\beta-3}\left(1+\frac{x}{l}\right)^{-\beta+3}\Bigg|_0^\infty = \frac{4hl^3}{(\beta-1)(\beta-2)(\beta-3)}.$$
$$= \frac{2l^2}{(\beta-2)(\beta-3)}$$

So, the standard deviation is

$$SD = l\sqrt{\frac{2}{(\beta-2)(\beta-3)}}.$$

Hypothetical situations

For the hypothetical reflection situation, the expectation can be calculated as

$$E(X) = 2\int_0^\infty xf(x)dx = 2\int_0^\infty xh\left(1+\frac{x}{l}\right)^{-\beta} dx = 2h\int_0^\infty x^2\left(1+\frac{x}{l}\right)^{-\beta} dx =$$

$$= -2hx^2\frac{l}{\beta-1}\left(1+\frac{x}{l}\right)^{-\beta+1}\Bigg|_0^\infty + \frac{2hl}{\beta-1}\int_0^\infty\left(1+\frac{x}{l}\right)^{-\beta+1} dx =$$

$$= \frac{2hl}{\beta-1}\int_0^\infty\left(1+\frac{x}{l}\right)^{-\beta+1} dx = \qquad .$$

$$= -\frac{2hl}{\beta-1}\frac{l}{\beta-2}\left(1+\frac{x}{l}\right)^{-\beta+2}\Bigg|_0^\infty = \frac{2hl}{\beta-1}\frac{l}{\beta-2} = \frac{2hl^2}{(\beta-1)(\beta-2)} =$$

$$= \frac{l}{\beta-2}$$

The ratio $|E(X)\text{-}b_{Boundary}|/SD$ is equal to

$$\frac{|E(X)-b_{Boundary}|}{SD} = \frac{E(X)}{SD} = \frac{l}{\beta-2}\Bigg/ l\sqrt{\frac{2}{(\beta-2)(\beta-3)}} = \sqrt{\frac{1}{2}}\sqrt{\frac{\beta-3}{\beta-2}}.$$

The variance can exist only if $\beta > 3$. Let $\beta = 3+\varepsilon > 3$, where $\varepsilon \to 0$, then

$$SD = l\sqrt{\frac{2}{(\beta-2)(\beta-3)}} \xrightarrow[\varepsilon\to 0]{} l\sqrt{\frac{1}{\varepsilon}} \xrightarrow[\varepsilon\to 0]{} \infty$$

and

$$\frac{|E(X)-b_{Boundary}|}{SD} = \sqrt{\frac{1}{2}}\sqrt{\frac{3+\varepsilon-3}{3+\varepsilon-2}} = \sqrt{\frac{1}{2}}\sqrt{\frac{\varepsilon}{1+\varepsilon}} \xrightarrow[\varepsilon\to 0]{} \sqrt{\frac{\varepsilon}{2}} \xrightarrow[\varepsilon\to 0]{} 0.$$

So, if the power index tends down to three and is sufficiently close to three, then the ratio $|E(X)\text{-}b_{Boundary}|/SD$ can be much less than unity.

4. Compact distributions

4.1. General consideration of the contiguous situation

Let us consider the contiguous situation for continuous (exactly speaking, for Riemann integrable function) PDFs in general.

Suppose a normal-like continuous distribution having the PDF f, such that $E(X) = l$, $f(x) = 0$ for $x \notin [0,2l]$ and $f(l+y) = f(l-y)$ and $f(x_1) \geq f(x_2)$ if $|x_2-l| \geq |x_1-l|$. The maximal value of $f(x)$ can be denoted as $max(f(x)) = f(l) \equiv h$ and the expression for the variance can be rewritten as

$$Var(X) = \int_0^{2l} (x-l)^2 f(x)dx = 2\int_0^{l} (x-l)^2 f(x)dx =$$

$$= 2\int_0^{l} (x-l)^2 \{h-(h-f(x)]\}dx = 2h\frac{l^3}{3} - 2\int_0^{l} (x-l)^2[h-f(x)]dx = .$$

$$= l^2\frac{2hl}{3} - 2\int_0^{l} (x-l)^2[h-f(x)]dx$$

The members $(x-l)^2$ and $[h - f(x] \equiv [max(f(x)) - f(x)]$ are positive. Hence the variance is maximal when $[max(f(x)) - f(x)] = 0$, that is when $f(x) = Const = h$. This condition implies the normalization equality $2hl = 1$. Under this condition the standard deviation is equal to the well-known value

$$SD = \sqrt{l^2\frac{2hl}{3}} = l\sqrt{\frac{2hl}{3}} = \frac{l}{\sqrt{3}}.$$

Due to the symmetry of the distributions, $E(X) = l$.

The ratio $|E(X)-b_{Boundary}|/SD$ is equal to

$$\frac{|E(X)-b_{Boundary}|}{SD} = \frac{E(X)}{SD} = \frac{l}{l}\sqrt{3} = \sqrt{3}.$$

So, in the general case, for the contiguous situation, the minimal ratio $|E(X)-b_{Boundary}|/SD$ of a normal-like continuous distribution with compact support cannot tend to zero. Moreover, it is more than unity.

4.2. Power one-step test distribution with compact support

Let us consider a continuous power one-step test distribution with compact support

$$f(x) = h\left(\frac{x}{l}\right)^{\beta}[\theta(x)-\theta(x-l)] + h\left(\frac{2l-x}{l}\right)^{\beta}[\theta(x-l)-\theta(x-2l)],$$

where $\beta \geq 0$. Due to the symmetry of the distributions, $E(X) = l$.

The normalizing integration is

$$2\int_{l}^{2l} f(x)dx = 2\int_{l}^{2l} h\left(\frac{2l-x}{l}\right)^{\beta} dx = \frac{2h}{l^{\beta}}\int_{l}^{2l}(2l-x)^{\beta}dx =$$

$$= -\frac{2h}{l^{\beta}}\frac{(2l-x)^{\beta+1}}{\beta+1}\bigg|_{l}^{2l} = \frac{2h}{l^{\beta}}\frac{l^{\beta+1}}{\beta+1} = .$$

$$= \frac{2hl}{\beta+1} = 1$$

The variance equals

$$Var(X) = 2\int_{l}^{2l}(x-\mu)^2 f(x)dx = 2\int_{l}^{2l}(x-l)^2 h\left(\frac{2l-x}{l}\right)^{\beta} dx =$$

$$= \frac{2h}{l^{\beta}}\int_{l}^{2l}(x-l)^2(2l-x)^{\beta}dx =$$

$$= -\frac{2h}{l^{\beta}}(x-l)^2\frac{(2l-x)^{\beta+1}}{\beta+1}\bigg|_{l}^{2l} + \frac{2h}{l^{\beta}(\beta+1)}\int_{l}^{2l}2(x-l)(2l-x)^{\beta+1}dx = .$$

$$= \frac{4h}{l^{\beta}(\beta+1)}\int_{l}^{2l}(x-l)(2l-x)^{\beta+1}dx =$$

$$= -\frac{4h}{l^{\beta}(\beta+1)}(x-l)\frac{(2l-x)^{\beta+2}}{\beta+2}\bigg|_{l}^{2l} + \frac{4h}{l^{\beta}(\beta+1)(\beta+2)}\int_{l}^{2l}(2l-x)^{\beta+2}dx =$$

$$= \frac{4h}{l^{\beta}(\beta+1)(\beta+2)}\int_{l}^{2l}(2l-x)^{\beta+2}dx =$$

$$= -\frac{4h}{l^{\beta}(\beta+1)(\beta+2)}\frac{(2l-x)^{\beta+3}}{\beta+3}\bigg|_{l}^{2l} = \frac{4h}{l^{\beta}(\beta+1)(\beta+2)}\frac{l^{\beta+3}}{\beta+3} = .$$

$$= \frac{4hl^3}{(\beta+1)(\beta+2)(\beta+3)} = \frac{2l^2}{(\beta+2)(\beta+3)}$$

So,

$$Var(X) = \frac{2l^2}{(\beta+2)(\beta+3)}$$

and the standard deviation is

$$SD = l\sqrt{\frac{2}{(\beta+2)(\beta+3)}}\,.$$

In particular, this expression gives the well-known formulae

$$SD_{Uniform} = \frac{l}{\sqrt{3}} \quad \text{and} \quad SD_{Triangle} = \frac{l}{\sqrt{6}}\,.$$

for the uniform ($\beta = 0$) and triangle ($\beta = 1$) distributions.

The contiguous situation

The above general consideration states that minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ is more, then unity for the contiguous situation. One can see indeed that the ratio $|E(X)\text{-}b_{Boundary}|/SD$ equals

$$\frac{|E(X) - b_{Boundary}|}{SD} = \frac{E(X)}{SD} = \sqrt{\frac{(\beta+2)(\beta+3)}{2}}\,.$$

The minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ is reached at $\beta \rightarrow 0$ (the power distribution tends to the uniform one)

$$\frac{|E(X) - b_{Boundary}|}{SD} = \sqrt{\frac{(\beta+2)(\beta+3)}{2}} \xrightarrow[\beta \to 0]{} \sqrt{3}\,.$$

So, the minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ is more, then unity for the contiguous situation of the one-step power test distribution.

The hypothetic situations

One can calculate the expectation for the hypothetic situation of "reflection" (under the condition $E(x) = \mu = 0$)

$$E(X) = 2\int_0^l xf(x)dx = 2h\int_0^l xh\left(\frac{l-x}{l}\right)^\beta dx = \frac{2h}{l^\beta}\int_0^l xh\left(\frac{l-x}{l}\right)^\beta dx =$$
$$= -\frac{2h}{l^\beta} x \frac{(l-x)^{\beta+1}}{\beta+1}\Bigg|_0^l + \frac{2h}{l^\beta}\int_0^l \frac{(l-x)^{\beta+1}}{\beta+1}dx = \frac{2h}{l^\beta}\int_0^l \frac{(l-x)^{\beta+1}}{\beta+1}dx =$$
$$= -\frac{2h}{l^\beta(\beta+1)}\frac{(l-x)^{\beta+2}}{\beta+2}\Bigg|_0^l = \frac{2h}{l^\beta(\beta+1)}\frac{l^{\beta+2}}{\beta+2} = \frac{2hl}{\beta+1}\frac{l}{\beta+2} =$$
$$= \frac{l}{\beta+2}.$$

The ratio $|E(X)\text{-}b_{Boundary}|/SD \equiv R_{Ratio}(\beta)$ is equal to

$$\frac{|E(X)-b_{Boundary}|}{SD} = \frac{E(X)}{SD} = \frac{l}{\beta+2}\frac{1}{l}\sqrt{\frac{(\beta+2)(\beta+3)}{2}} =$$
$$= \frac{1}{\sqrt{2}}\sqrt{\frac{\beta+3}{\beta+2}} \equiv R_{Ratio}(\beta).$$

The derivative of $R_{Ratio}(\beta)$ with respect to β is

$$\frac{\partial R_{Ratio}(\beta)}{\partial\beta} = \frac{1}{\sqrt{2}}\left(\frac{\beta+3}{\beta+2}\right)^{-1/2}\frac{\beta+2-\beta-3}{(\beta+2)^2} =$$
$$= \frac{1}{\sqrt{2}}\sqrt{\frac{\beta+2}{\beta+3}}\frac{2-3}{(\beta+2)^2} = \frac{1}{\sqrt{2}}\sqrt{\frac{\beta+2}{\beta+3}}\frac{-1}{(\beta+2)^2} < 0.$$

The ratio $|E(X)\text{-}b_{Boundary}|/SD$ tends to the maximum at $\beta \to 0$ (the power distribution tends to the uniform one) to

$$\frac{|E(X)-b_{Boundary}|}{SD} = \frac{1}{\sqrt{2}}\sqrt{\frac{\beta+3}{\beta+2}} \xrightarrow[\beta\to 0]{} \frac{1}{\sqrt{2}}\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{2} \approx 0.87\,.$$

The minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ is reached at $\beta \to \infty$

$$\frac{|E(X)-b_{Boundary}|}{SD} \geq \frac{1}{\sqrt{2}}\sqrt{\frac{\beta+3}{\beta+2}} \xrightarrow[\beta\to\infty]{} \frac{1}{\sqrt{2}}\sqrt{\frac{\beta}{\beta}} =$$
$$= \frac{1}{\sqrt{2}} \approx 0.71 \in \left(\frac{2}{3},\frac{4}{5}\right).$$

For the hypothetic situation of "adhesion" the minimal ratio is equal to *1/2* of that of the hypothetic situation of "reflection" and is equal to

$$\frac{|E(X)-b_{Boundary}|}{SD} \geq \frac{1}{2\sqrt{2}} \approx 0.35 \in \left(\frac{1}{3},\frac{2}{5}\right).$$

So, for the hypothetic situations of both "reflection" and "adhesion," the minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ do not tend to zero when σ tend to zero.

4.3. Stepwise two-step test distribution with compact support

Let us consider the piecewise continuous two-step stepwise test distribution with compact support. Let us denote the center step by the subscript "Center" or "1," the side step by the subscript "Side" or "Tail" or, shortly, "2." So, for the contiguous situation we have

$$\begin{aligned} f(x) &= h_{Side}[\theta(x)-\theta(x-l_{Side})]+ \\ &+(h_{Side}+h_{Center})[\theta(x-l_{Side})-\theta(x-l_{Side}-2l_{Center})]+ . \\ &+h_{Side}[\theta(x-l_{Side}-2l_{Center})-\theta(x-2l_{Side}-2l_{Center})] \end{aligned}$$

or

$$\begin{aligned} f(x) &= h_{Tail}[\theta(x)-\theta(x-l_{Tail})]+ \\ &+(h_{Tail}+h_{Center})[\theta(x-l_{Tail})-\theta(x-l_{Tail}-2l_{Center})]+ . \\ &+h_{Tail}[\theta(x-l_{Tail}-2l_{Center})-\theta(x-2l_{Tail}-2l_{Center})] \end{aligned}$$

or, shortly,

$$\begin{aligned} f(x) &= h_2[\theta(x)-\theta(x-l_2)]+ \\ &+(h_2+h_1)[\theta(x-l_2)-\theta(x-l_2-2l_1)]+ . \\ &+h_2[\theta(x-l_2-2l_1)-\theta(x-2l_2-2l_1)] \end{aligned}$$

The parameters $h_{Side} \equiv h_{Tail} \equiv h_2$ and $h_{Center} \equiv h_1$, $l_{Side} \equiv l_{Tail} \equiv l_2$ and $l_{Center} \equiv l_1$ are tied by the normalizing integration

$$\begin{aligned} &2\int_0^{l_{Center}+l_{Side}} f(x)dx = 2\int_0^{l_1}(h_2+h_1)dx + 2\int_{l_1}^{l_1+l_2} h_2 dx = \\ &= 2l_1(h_2+h_1)+2l_2h_2 = 2l_2h_2+2l_1h_2+2l_1h_1 = , \\ &= 2h_2(l_2+l_1)+2l_1h_1 = 1 \end{aligned}$$

or

$$2h_{Tail}(l_{Tail}+l_{Center})+2l_{Center}h_{Center} = 1,$$

or

$$2h_{Side}(l_{Side}+l_{Center})+2l_{Center}h_{Center} = 1.$$

The variance

For the two-step stepwise test distribution the variance equals (for simplicity one can determine *E(X) = 0*)

$$Var(X)=2\int_0^{l_1+l_2}x^2f(x)dx=2\int_0^{l_1}x^2(h_2+h_1)dx+2\int_{l_1}^{l_1+l_2}x^2h_2dx=$$

$$=\frac{2(h_2+h_1)}{3}l^3{}_1+\frac{2h_2}{3}[(l_2+l_1)^3-l^3{}_1]=$$

$$=\frac{2h_1}{3}l^3{}_1+\frac{2h_2}{3}(l_2+l_1)^3=\frac{2}{3}\left[h_1l^3{}_1+h_2(l_2+l_1)^3\right]=$$

$$=\frac{2}{3}\left[l^2{}_1h_1l_1+(l_2+l_1)^2h_2(l_2+l_1)\right]$$

.

So,

$$Var(X)=\frac{2}{3}\left[l^2{}_1h_1l_1+(l_2+l_1)^2h_2(l_2+l_1)\right]$$

or

$$Var(X)=\frac{2}{3}\left[l^2{}_{Center}h_{Center}l_{Center}+(l_{Side}+l_{Center})^2h_{Side}(l_{Side}+l_{Center})\right]$$

and the standard deviation is

$$SD=\sqrt{\frac{2}{3}}\sqrt{h_1l^3{}_1+h_2(l_2+l_1)^3}\ .$$

Note, for the uniform distribution we have (for example) $h_{Side} \equiv h_{Tail} \equiv h_2 = 0$ and the variance equals

$$Var(X)=\frac{2}{3}h_1l^3{}_1=\frac{2}{3}l^2{}_1h_1l_1=\frac{2}{3}l^2{}_1\frac{1}{2}=\frac{1}{3}l^2{}_1=\frac{1}{3}l^2{}_{Center}\,.$$

Equivalently, for $h_{Center} \equiv h_1 = 0$ it equals

$$Var(X)=\frac{2}{3}h_2(l_2+l_1)^3=\frac{2}{3}(l_2+l_1)^2h_2(l_2+l_1)=\frac{2}{3}(l_2+l_1)^2\frac{1}{2}=$$

$$=\frac{1}{3}(l_2+l_1)^2=\frac{1}{3}(l_{Side}+l_{Center})^2$$

.

Due to the normalizing equality

$$2h_2(l_2+l_1)+2l_1h_1=1$$

none of these parameters can be changed independently. Using

$$h_1=\frac{1-2h_2(l_2+l_1)}{2l_1}$$

the variance can be rewritten in terms of h_2 as

$$Var(X)=\frac{2}{3}\left[h_1 l^3{}_1+h_2(l_2+l_1)^3\right]=\frac{2}{3}\left[\frac{1-2h_2(l_2+l_1)}{2l_1}l^3{}_1+h_2(l_2+l_1)^3\right]=$$

$$=\frac{1}{3}\left[[1-2h_2(l_2+l_1)]l^2{}_1+2h_2(l_2+l_1)^3\right]=$$

$$=\frac{1}{3}\left[l^2{}_1-2h_2(l_2+l_1)l^2{}_1+2h_2(l_2+l_1)^3\right]= \quad .$$

$$=\frac{1}{3}\left[l^2{}_1+2h_2(l_2+l_1)[(l_2+l_1)^2-l^2{}_1]\right]=\frac{1}{3}\left[l^2{}_1+2h_2(l_2+l_1)(l^2{}_2+2l_2l_1)\right]=$$

$$=\frac{1}{3}\left[l^2{}_1+2h_2l_2(l_2+l_1)(l_2+2l_1)\right]$$

The derivative of the variance with respect to h_2 is

$$\frac{\partial Var(X)}{\partial h_2}=\frac{2}{3}l_2(l_2+l_1)(l_2+2l_1)>0\,.$$

The variance increases when $h_{Side}\equiv h_{Tail}\equiv h_2$ increases.

Using

$$h_2 = \frac{1-2l_1h_1}{2(l_2+l_1)}$$

the variance can be rewritten in terms of h_1 as

$$Var(X) = \frac{2h_1}{3}l^3{}_1 + \frac{2h_2}{3}(l_2+l_1)^3 =$$

$$= \frac{2h_1}{3}l^3{}_1 + \frac{2}{3}\frac{1-2l_1h_1}{2(l_2+l_1)}(l_2+l_1)^3 = \frac{2h_1}{3}l^3{}_1 + \frac{1-2l_1h_1}{3}(l_2+l_1)^2 =$$

$$= \frac{2h_1}{3}l^3{}_1 + \frac{1}{3}(l_2+l_1)^2 - \frac{2l_1h_1}{3}(l_2+l_1)^2 =$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2l_1h_1\left[(l_2+l_1)^2 - l^2{}_1\right]\right\} = \qquad .$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2l_1h_1\left[l^2{}_2 + 2l_2l_1 + l_1{}^2 - l^2{}_1\right]\right\} =$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2l_1h_1\left[l^2{}_2 + 2l_2l_1\right]\right\} =$$

$$= \frac{1}{3}\left\{(l_2+l_1)^2 - 2h_1l_2l_1\left[l_2 + 2l_1\right]\right\}$$

The derivative with respect to h_1 is

$$\frac{\partial Var(X)}{\partial h_1} = \frac{2}{3}\left\{-l_2l_1\left[l_2+2l_1\right]\right\} < 0 .$$

The variance increases when $h_{Center} \equiv h_1$ decreases.

So, the derivative of the variance with respect to h_2 is positive but the derivative with respect to h_1 is negative. Remember, when h_1 increases then h_2 decreases (under the condition that other parameters are constant). So, the variance increases when $h_{Side}/h_{Center} \equiv h_{Tail}/h_{Center} \equiv h_2/h_1$ increases.

Therefore the variance is maximal at the condition $h_1 = 0$ and equals

$$Var(X) = \frac{2}{3}\left[l^2{}_1h_1l_1 + (l_2+l_1)^2h_2(l_2+l_1)\right] =$$

$$= \frac{2}{3}(l_2+l_1)^2h_2(l_2+l_1) = \frac{2}{3}(l_2+l_1)^2\frac{1}{2} =$$

$$= \frac{(l_2+l_1)^2}{3}$$

The maximum of the standard deviation is

$$\max(SD) = \frac{l_2+l_1}{\sqrt{3}} .$$

The contiguous situation

For the contiguous situation, due to the symmetry of the PDF, the expectation is

$$| E(X) - b |= E(X) = l_2 + l_1 .$$

So,

$$\frac{| E(X) - b_{Boundary} |}{SD} = \sqrt{\frac{3}{2}} \frac{l_2 + l_1}{\sqrt{h_1 l^3{}_1 + h_2 (l_2 + l_1)^3}} \leq$$

$$\leq \sqrt{\frac{3}{2}} \frac{l_2 + l_1}{\sqrt{h_2 (l_2 + l_1)^3}} = \sqrt{\frac{3}{2}} \frac{l_2 + l_1}{\sqrt{\frac{1}{2} (l_2 + l_1)^2}} = \sqrt{3} \frac{l_2 + l_1}{l_2 + l_1} = .$$

$$= \sqrt{3}$$

So, the minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ for the two-step stepwise test distribution with compact support for the contiguous situation is finite and is more than unity (and is equal to that of the uniform distribution).

The hypothetic situations

For the two-step test stepwise test distribution with compact support for the hypothetic situation of "reflection" the expectation equals

$$E(X) = 2 \int_0^{l_1 + l_2} x f(x) dx = 2 \int_0^{l_1} x (h_2 + h_1) dx + 2 \int_{l_1}^{l_1 + l_2} x h_2 dx =$$

$$= \frac{2(h_2 + h_1)}{2} l^2{}_1 + \frac{2h_2}{2} [(l_2 + l_1)^2 - l^2{}_1] = \quad ,$$

$$= h_2 l^2{}_1 + h_1 l^2{}_1 + h_2 (l_2 + l_1)^2 - h_2 l^2{}_1 = h_1 l^2{}_1 + h_2 (l_2 + l_1)^2$$

or

$$E(X) = h_1 l^2{}_1 + h_2 (l_2 + l_1)^2 =$$

$$= 2h_1 l_1 \frac{l_1}{2} + 2h_2 (l_2 + l_1) \frac{(l_2 + l_1)}{2} .$$

So,

$$E(X) = h_1 l^2{}_1 + h_2 (l_2 + l_1)^2 \equiv h_{Center} l^2{}_{Center} + h_{Side} (l_{Side} + l_{Center})^2 ,$$

Remembering that

$$SD = \sqrt{\frac{2}{3}}\sqrt{h_1 l^3{}_1 + h_2(l_2 + l_1)^3}\ ,$$

we have

$$\frac{|E(X) - b_{Boundary}|}{SD} = \sqrt{\frac{3}{2}}\frac{h_1 l^2{}_1 + h_2(l_2 + l_1)^2}{\sqrt{h_1 l^3{}_1 + h_2(l_2 + l_1)^3}}\,.$$

The ratio depends on the four parameters. The form of the ratio and preliminary calculations show that the full analysis of it is rather complicated. In addition, such an analysis is not a goal of this article.

One of simpler ways to reach this goal is a general step-by-step analysis of the ratio.

Let us analyze the three relationships

$$\frac{l_1}{l_2},\quad \frac{h_1}{h_2}\quad \text{and}\quad \frac{h_1 l_1}{h_2 l_2}.$$

The standard deviation cannot be more than $O(l_1+l_2)$. Hence the ratio can tend to zero only if $|E(X)-b_{Boundary}|$ tends to zero.

If $h_1l_1/h_2l_2 \rightarrow 0$, then, evidently, $|E(X)-b_{Boundary}| \rightarrow l_1 + l_2/2$ and the ratio is finite. This is not a step to the goal.

If $h_1l_1/h_2l_2 \rightarrow \infty$ (or, equivalently, $h_2l_2/h_1l_1 \rightarrow 0$) then, evidently, we have $|E(X)-b_{Boundary}| \rightarrow l_1/2$. Hence, if l_1/l_2 tend to zero, then $|E(X)-b_{Boundary}|$ (and the ratio $|E(X)-b_{Boundary}| / SD$) can tend to zero.

In addition, if $h_2l_2/h_1l_1 \rightarrow 0$ and $l_1/l_2 \rightarrow 0$, then

$$\frac{h_2 l_2}{h_1 l_1}\times\frac{l_1}{l_2} = \frac{h_2}{h_1} << 1\times\frac{l_1}{l_2} = \frac{l_1}{l_2} << 1$$

or

$$\frac{h_2}{h_1} << \frac{l_1}{l_2} \rightarrow 0\,,$$

So, this simple preliminary analysis proves that the ratio can tend to zero if

$$\frac{h_2}{h_1} << \frac{l_1}{l_2} \rightarrow 0 \quad \text{and} \quad \frac{h_2 l_2}{h_1 l_1} \rightarrow 0\,. \qquad (1)$$

One can refer these conditions to as the “preliminary conditions.”

We can identically rewrite the ratio as

$$\frac{|E(X)-b_{Boundary}|}{SD}=\sqrt{\frac{3}{2}}\frac{h_1 l^2{}_1+h_2(l_2+l_1)^2}{\sqrt{h_1 l^3{}_1+h_2(l_2+l_1)^3}}=$$

$$=\sqrt{\frac{3}{2}}\frac{h_1}{\sqrt{h_1}}\frac{l^2{}_1+\frac{h_2}{h_1}(l_2+l_1)^2}{\sqrt{l^3{}_1+\frac{h_2}{h_1}(l_2+l_1)^3}}=\sqrt{\frac{3}{2}}\sqrt{h_1}\frac{l^2{}_1+\frac{h_2}{h_1}(l_2+l_1)^2}{\sqrt{l^3{}_1+\frac{h_2}{h_1}(l_2+l_1)^3}}=$$

$$=\sqrt{\frac{3}{2}}\sqrt{h_1}\frac{l^2{}_2}{l^{3/2}{}_2}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}}=$$

$$=\sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}}.$$

When

$$\frac{l_1}{l_2}\to 0,$$

then the ratio tends to

$$\frac{|E(X)-b_{Boundary}|}{SD}=\sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}}\xrightarrow[\frac{l_1}{l_2}\to 0]{}.$$

$$\xrightarrow[\frac{l_1}{l_2}\to 0]{}\sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}}}$$

Further, one can consider the two simplified mutually excluding cases

$$\left(\frac{l_1}{l_2}\right)^2<<\frac{h_2}{h_1}\quad\text{or}\quad\left(\frac{l_1}{l_2}\right)^2>>\frac{h_2}{h_1}.$$

If the first case takes place and

$$\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} \to 0 \quad \text{and, due to (1),} \quad \left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} \to 0,$$

then

$$\frac{|E(X) - b_{Boundary}|}{SD} \approx \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\,\frac{\left(\frac{l_1}{l_2}\right)^2 + \frac{h_2}{h_1}}{\sqrt{\left(\frac{l_1}{l_2}\right)^3 + \frac{h_2}{h_1}}} \xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} \to 0]{}$$

$$\xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} \to 0]{} \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\,\frac{\frac{h_2}{h_1}}{\sqrt{\frac{h_2}{h_1}}} = \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\sqrt{\frac{h_2}{h_1}} = .$$

$$= \sqrt{\frac{3}{2}}\sqrt{h_2 l_2}$$

Due to the “preliminary conditions” (1),

$$h_2 l_2 << h_1 l_1.$$

Remembering

$$h_1 l_1 < 1,$$

we obtain

$$h_2 l_2 << 1.$$

Therefore it follows

$$\frac{|E(X) - b_{Boundary}|}{SD} \xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} \to 0]{} \sqrt{\frac{3}{2}}\sqrt{h_2 l_2} \xrightarrow[h_2 l_2 \to 0]{} 0.$$

A concrete example

For example, suppose the condition $a >> b$ is true, e.g., if $a \geq b*10^3$. Then the condition (1)

$$\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} << 1\,,$$

can be true if

$$\frac{l_1}{l_2} = 10^{-6} \quad \text{and} \quad \frac{h_2}{h_1} = 10^{-9}\,.$$

These values save (1) and lead to

$$h_2 l_2 = h_1 \frac{h_2}{h_1} l_1 \frac{l_2}{l_1} = h_1 l_1 \frac{h_2}{h_1}\frac{l_2}{l_1} \approx \frac{1}{2} 10^{-9} 10^{6} = \frac{10^{-3}}{2} < 10^{-3} << 1$$

and to

$$\frac{|\,E(X) - b_{Boundary}\,|}{SD} = \sqrt{\frac{3}{2}}\sqrt{h_1 l_2}\,\frac{\left(\frac{l_1}{l_2}\right)^2 + \frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^2}{\sqrt{\left(\frac{l_1}{l_2}\right)^3 + \frac{h_2}{h_1}\left(1+\frac{l_1}{l_2}\right)^3}} \approx$$

$$\approx \sqrt{\frac{3}{2}}\sqrt{\frac{10^{-3}}{2}}\,\frac{10^{-12} + 10^{-9}\left(1+10^{-6}\right)^2}{\sqrt{10^{-18} + 10^{-9}\left(1+10^{-6}\right)^3}} \approx \sqrt{\frac{3}{4}}\sqrt{10^{-3}}\,\frac{10^{-12}+10^{-9}}{\sqrt{10^{-18}+10^{-9}}} \approx .$$

$$\approx \sqrt{\frac{3}{4}}\sqrt{10^{-3}}\,\frac{10^{-9}}{\sqrt{10^{-9}}} = \sqrt{\frac{3}{4}}\,\frac{10^{-9}}{\sqrt{10^{-6}}} = \sqrt{\frac{3}{4}}10^{-6} <$$

$$< 10^{-6} << 1$$

So, the first case allows to achieve the object of the article and there is no need to investigate the more complicated and less evident second case.

In the hypothetic situation of "adhesion" the minimal ratio is, evidently, half of the above value and, hence, can be much less than unity as well.

So, it has been proven that the minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ for the piecewise continuous two-step stepwise test distribution with compact support can be much less than unity for the hypothetic situations.

4.4. Power two-step test distribution with compact support

4.4.1. General formulae

Let us consider a continuous power two-step test distribution with compact support with a PDF

$$
\begin{aligned}
f(x) &= h_2\left(\frac{x}{l_2}\right)^{\beta_2}[\theta(x)-\theta(x-l_2)]+\\
&+\left[h_2+h_1\left(\frac{x-l_2}{l_1}\right)^{\beta_1}\right][\theta(x-l_2)-\theta(x-l_2-l_1)]+\\
&+\{h_2+h_1\left[\frac{2(l_2+l_1)-x}{l_1}\right]^{\beta_2}\}[\theta[x-(l_2+2l_1)]-\theta(x-2l_2)]+\\
&+h_1\left(\frac{l_2+2l_1-x}{l_2}\right)^{\beta_2}\{\theta[x-(l_2+l_1)]-\theta[x-(l_2+2l_1)]\}
\end{aligned},
$$

where $\beta_2 \equiv \beta_{Side} \geq 0$ and $\beta_1 \equiv \beta_{Centre} \geq 0$, $h_1 \equiv h_{Centre}$ and $l_1 \equiv l_{Centre}$, $h_2 \equiv h_{Side}$ and $l_1 \equiv l_{Side}$. The above parameters are tied by the normalizing integration

$$
\begin{aligned}
&2\int_0^{l_2+l_1} f(x)dx = 2\int_0^{l_2} h_2\left(\frac{x}{l_2}\right)^{\beta_2}dx + 2\int_{l_2}^{l_2+l_1}\left[h_2+h_1\left(\frac{x-l_2}{l_1}\right)^{\beta_1}\right]dx =\\
&= 2h_2\int_0^{l_2}\left(\frac{x}{l_2}\right)^{\beta_2}dx + 2h_2\int_{l_2}^{l_2+l_1}dx + 2h_1\int_{l_2}^{l_2+l_1}\left(\frac{x-l_2}{l_1}\right)^{\beta_1}dx =\\
&= 2h_2l_2\int_0^{l_2}\left(\frac{x}{l_2}\right)^{\beta_2}d\left(\frac{x}{l_2}\right) + 2h_2\int_{l_2}^{l_2+l_1}dx + 2h_1l_1\int_{l_2}^{l_2+l_1}\left(\frac{x-l_2}{l_1}\right)^{\beta_1}d\left(\frac{x}{l_1}\right) =\\
&= \frac{2h_2l_2}{\beta_2+1}\left(\frac{x}{l_2}\right)^{\beta_2+1}\Bigg|_0^{l_2} + 2h_2l_1 + \frac{2h_1l_1}{\beta_1+1}\left(\frac{x-l_2}{l_1}\right)^{\beta_1+1}\Bigg|_{l_2}^{l_2+l_1} =\\
&= \frac{2h_2l_2}{\beta_2+1}1 + 2h_2l_1 + \frac{2h_1l_1}{\beta_1+1}\left(\frac{l_2+l_1-l_2}{l_1}\right)^{\beta_1+1} =\\
&= \frac{2h_2l_2}{\beta_2+1} + 2h_2l_1 + \frac{2h_1l_1}{\beta_1+1}\left(\frac{l_1}{l_1}\right)^{\beta_1+1} = \frac{2h_2l_2}{\beta_2+1} + 2h_2l_1 + \frac{2h_1l_1}{\beta_1+1} = 1
\end{aligned}.
$$

So,

$$
\frac{2h_2l_2}{\beta_2+1} + 2h_2l_1 + \frac{2h_1l_1}{\beta_1+1} = 1.
$$

The variance

The variance equals

$$Var(X) = 2\int_{0}^{l_1+l_2}[x-E(X)]^2 f(x)dx =$$

$$= 2\int_{0}^{l_1} x^2\left[h_2 + h_1\left(\frac{l_1-x}{l_1}\right)^{\beta_1}\right]dx + 2\int_{l_1}^{l_1+l_2} x^2 h_2\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2} dx =$$

$$= 2h_2\int_{0}^{l_1} x^2 dx + 2h_1\int_{0}^{l_1} x^2\left(\frac{l_1-x}{l_1}\right)^{\beta_1} dx + 2h_2\int_{l_1}^{l_1+l_2} x^2\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2} dx$$

and

$$2h_2\int_{0}^{l_1} x^2 dx + 2h_1\int_{0}^{l_1} x^2\left(\frac{l_1-x}{l_1}\right)^{\beta_1} dx + 2h_2\int_{l_1}^{l_1+l_2} x^2\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2} dx =$$

$$= 2h_2\left.\frac{x^3}{3}\right|_{0}^{l_1} - 2h_1\frac{l_1}{\beta_1+1}x^2\left.\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1}\right|_{0}^{l_1} + \frac{2h_1l_1}{\beta_1+1}\int_{0}^{l_1} 2x\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1} dx -$$

$$-2h_2\frac{l_2}{\beta_2+1}x^2\left.\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1}\right|_{l_1}^{l_1+l_2} + \frac{2h_2l_2}{\beta_2+1}\int_{l_1}^{l_1+l_2} 2x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1} dx =$$

$$= 2h_2\frac{l^3{}_1}{3} + \frac{4h_1l_1}{\beta_1+1}\int_{0}^{l_1} x\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1} dx -$$

$$-0 + 2h_2\frac{l_2}{\beta_2+1}l^2{}_1\left(\frac{l_1+l_2-l_1}{l_2}\right)^{\beta_2+1} + \frac{4h_2l_2}{\beta_2+1}\int_{l_1}^{l_1+l_2} x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1} dx =$$

$$= \frac{2h_2l_1}{3}l^2{}_1 + \frac{4h_1l_1}{\beta_1+1}\int_{0}^{l_1} x\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1} dx +$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1 + \frac{4h_2l_2}{\beta_2+1}\int_{l_1}^{l_1+l_2} x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1} dx$$

and

$$\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\int_0^{l_1}x\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1}dx+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\int_{l_1}^{l_1+l_2}x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1}dx=$$

$$=\frac{2h_2l_1}{3}l^2{}_1-$$

$$-\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}x\left(\frac{l_1-x}{l_1}\right)^{\beta_1+2}\Bigg|_0^{l_1}+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\int_0^{l_1}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+2}dx+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1-\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+2}\Bigg|_{l_1}^{l_1+l_2}+$$

$$+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\int_{l_1}^{l_1+l_2}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+2}dx=$$

$$=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\int_0^{l_1}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+2}dx+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1\left(\frac{l_1+l_2-l_1}{l_2}\right)^{\beta_2+2}+$$

$$+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\int_{l_1}^{l_1+l_2}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+2}dx=$$

$$=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\int_0^{l_1}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+2}dx+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\int_{l_1}^{l_1+l_2}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+2}dx$$

and

$$\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\int_0^{l_1}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+2}dx+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\int_{l_1}^{l_1+l_2}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+2}dx=$$

$$=\frac{2h_2l_1}{3}l^2{}_1-\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\frac{l_1}{\beta_1+3}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+3}\Bigg|_0^{l_1}+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1-$$

$$-\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\frac{l_2}{\beta_2+3}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+3}\Bigg|_{l_1}^{l_1+l_2}=$$

$$=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\frac{l_1}{\beta_1+3}\left(\frac{l_1}{l_1}\right)^{\beta_1+3}+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1+$$

$$+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\frac{l_2}{\beta_2+3}\left(\frac{l_1+l_2-l_1}{l_2}\right)^{\beta_2+3}=$$

$$=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\frac{l_1}{\beta_1+3}\left(\frac{l_1}{l_1}\right)^{\beta_1+3}+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1+$$

$$+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\frac{l_2}{\beta_2+3}$$

So,

$$Var(X)=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\frac{l_1}{\beta_1+3}+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\frac{l_2}{\beta_2+3}.$$

The contiguous situations

Due to the symmetry of the test distributions, $E(X) = l_2 + l_1$.
The minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ is

$$\frac{|E(X)-b_{Boundary}|}{\sqrt{Var(X)}}.$$

and this expression is rather complicated.

The hypothetic situations

One can calculate the expectation for the hypothetic situation of "reflection"

$$E(X)=2\int_0^{l_1+l_2} xf(x)dx=$$

$$=2\int_0^{l_1} x\left[h_2+h_1\left(\frac{l_1-x}{l_1}\right)^{\beta_1}\right]dx+2\int_{l_1}^{l_1+l_2} xh_2\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2}dx=$$

$$=2h_2\int_0^{l_1} xdx+2h_1\int_0^{l_1} x\left(\frac{l_1-x}{l_1}\right)^{\beta_1}dx+2h_2\int_{l_1}^{l_1+l_2} x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2}dx$$

and

$$2h_2\int_0^{l_1} xdx+2h_1\int_0^{l_1} x\left(\frac{l_1-x}{l_1}\right)^{\beta_1}dx+2h_2\int_{l_1}^{l_1+l_2} x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2}dx=$$

$$=2h_2\frac{x^2}{2}\Big|_0^{l_1}-2h_1\frac{l_1}{\beta_1+1}x\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1}\Big|_0^{l_1}+\frac{2h_1l_1}{\beta_1+1}\int_0^{l_1}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1}dx-$$

$$-2h_2\frac{l_2}{\beta_2+1}x\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1}\Big|_{l_1}^{l_1+l_2}+$$

$$+\frac{2h_2l_2}{\beta_2+1}\int_{l_1}^{l_1+l_2}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1}dx=$$

and

$$= h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1 + 1}\int_0^{l_1}\left(\frac{l_1 - x}{l_1}\right)^{\beta_1 + 1} dx +$$

$$+ \frac{2h_2 l_2}{\beta_2 + 1} l_1 \left(\frac{l_1 + l_2 - l_1}{l_2}\right)^{\beta_2 + 1} + \frac{2h_2 l_2}{\beta_2 + 1}\int_{l_1}^{l_1 + l_2}\left(\frac{l_1 + l_2 - x}{l_2}\right)^{\beta_2 + 1} dx =$$

$$= h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1 + 1}\int_0^{l_1}\left(\frac{l_1 - x}{l_1}\right)^{\beta_1 + 1} dx +$$

$$+ \frac{2h_2 l_2}{\beta_2 + 1} l_1 + \frac{2h_2 l_2}{\beta_2 + 1}\int_{l_1}^{l_1 + l_2}\left(\frac{l_1 + l_2 - x}{l_2}\right)^{\beta_2 + 1} dx$$

and

$$h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1 + 1}\int_0^{l_1}\left(\frac{l_1 - x}{l_1}\right)^{\beta_1 + 1} dx +$$

$$+ \frac{2h_2 l_2}{\beta_2 + 1} l_1 + \frac{2h_2 l_2}{\beta_2 + 1}\int_{l_1}^{l_1 + l_2}\left(\frac{l_1 + l_2 - x}{l_2}\right)^{\beta_2 + 1} dx =$$

$$= h_2 l^2{}_1 - \frac{2h_1 l_1}{\beta_1 + 1}\frac{l_1}{\beta_1 + 2}\left(\frac{l_1 - x}{l_1}\right)^{\beta_1 + 2}\Bigg|_0^{l_1} +$$

$$+ \frac{2h_2 l_2}{\beta_2 + 1} l_1 - \frac{2h_2 l_2}{\beta_2 + 1}\frac{l_2}{\beta_2 + 2}\left(\frac{l_1 + l_2 - x}{l_2}\right)^{\beta_2 + 2}\Bigg|_{l_1}^{l_1 + l_2} =$$

and

$$= h_2 l^2{}_1 - 0 + \frac{2h_1 l_1}{\beta_1 + 1}\frac{l_1}{\beta_1 + 2}\left(\frac{l_1}{l_1}\right)^{\beta_1 + 2} +$$

$$+ \frac{2h_2 l_2}{\beta_2 + 1} l_1 - 0 + \frac{2h_2 l_2}{\beta_2 + 1}\frac{l_2}{\beta_2 + 2}\left(\frac{l_1 + l_2 - l_1}{l_2}\right)^{\beta_2 + 2} =$$

$$= h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1 + 1}\frac{l_1}{\beta_1 + 2}\left(\frac{l_1}{l_1}\right)^{\beta_1 + 2} +$$

$$+ \frac{2h_2 l_2}{\beta_2 + 1} l_1 + \frac{2h_2 l_2}{\beta_2 + 1}\frac{l_2}{\beta_2 + 2}$$

and

$$h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1+1}\int_0^{l_1}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+1}dx +$$

$$+\frac{2h_2 l_2}{\beta_2+1}l_1 + \frac{2h_2 l_2}{\beta_2+1}\int_{l_1}^{l_1+l_2}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+1}dx =$$

$$= h_2 l^2{}_1 - \frac{2h_1 l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\left(\frac{l_1-x}{l_1}\right)^{\beta_1+2}\Bigg|_0^{l_1} +$$

$$+\frac{2h_2 l_2}{\beta_2+1}l_1 - \frac{2h_2 l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\left(\frac{l_1+l_2-x}{l_2}\right)^{\beta_2+2}\Bigg|_{l_1}^{l_1+l_2} =$$

$$= h_2 l^2{}_1 - 0 + \frac{2h_1 l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\left(\frac{l_1}{l_1}\right)^{\beta_1+2} +$$

$$+\frac{2h_2 l_2}{\beta_2+1}l_1 - 0 + \frac{2h_2 l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\left(\frac{l_1+l_2-l_1}{l_2}\right)^{\beta_2+2} =$$

$$= h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\left(\frac{l_1}{l_1}\right)^{\beta_1+2} +$$

$$+\frac{2h_2 l_2}{\beta_2+1}l_1 + \frac{2h_2 l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}$$

and

$$h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1+1}\frac{l_1}{\beta_1+2} + \frac{2h_2 l_2}{\beta_2+1}l_1 + \frac{2h_2 l_2}{\beta_2+1}\frac{l_2}{\beta_2+2} =$$

$$= h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1+1}\frac{l_1}{\beta_1+2} + \frac{2h_2 l_2}{\beta_2+1}\frac{1}{\beta_2+2}[l_1(\beta_2+2)+l_2].$$

So,

$$E(X) = h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1+1}\frac{l_1}{\beta_1+2} + \frac{2h_2 l_2}{\beta_2+1}\frac{1}{\beta_2+2}[l_1(\beta_2+2)+l_2].$$

General and specific formulae

The above general formulae and their analysis are rather complicated. To facilitate the achievement of the goal of the article we can consider some simple specific cases of this distribution and corresponding specific formulae. We can use also the ideas and formulae of the two-step stepwise test distribution with compact support from the preceding subsection.

4.4.2. The case of two steps. $\beta_2 = \beta_1 = 0$
Distribution and normalizing equation

The above general formula of the PDF

$$
\begin{aligned}
&f(x) = h_2\left(\frac{x}{l_2}\right)^{\beta_2}[\theta(x)-\theta(x-l_2)]+ \\
&+\left[h_2+h_1\left(\frac{x-l_2}{l_1}\right)^{\beta_1}\right][\theta(x-l_2)-\theta(x-l_2-l_1)]+ \\
&+\{h_2+h_1\left[\frac{2(l_2+l_1)-x}{l_1}\right]^{\beta_2}\}[\theta[x-(l_2+2l_1)]-\theta(x-2l_2)]+ \\
&+h_1\left(\frac{l_2+2l_1-x}{l_2}\right)^{\beta_2}\{\theta[x-(l_2+l_1)]-\theta[x-(l_2+2l_1)]\}
\end{aligned},
$$

is transformed to a specific one at $\beta_2 = \beta_1 = 0$, that is to the two-step stepwise test distribution with compact support

$$
\begin{aligned}
&f(x) = h_2[\theta(x)-\theta(x-l_2)]+ \\
&+(h_2+h_1)[\theta(x-l_2)-\theta(x-l_2-l_1)]+ \\
&+(h_2+h_1)\{\theta[x-(l_2+2l_1)]-\theta(x-2l_2)\}+ \\
&+h_1\{\theta[x-(l_2+l_1)]-\theta[x-(l_2+2l_1)]\}
\end{aligned},
$$

The above general formula of the normalizing integration

$$\frac{2h_2l_2}{\beta_2+1}+2h_2l_1+\frac{2h_1l_1}{\beta_1+1}=1$$

is transformed to a specific one at $\beta_2 = \beta_1 = 0$

$$2h_2l_2+2h_2l_1+2h_1l_1=1.$$

This expression naturally coincides with the above one of the preceding subchapter.

The variance

The above general formula

$$Var(X)=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\frac{l_1}{\beta_1+3}+$$
$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\frac{l_2}{\beta_2+3}.$$

is transformed to a specific one at $\beta_2 = \beta_1 = 0$

$$Var(X)=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{1}\frac{l_1}{2}\frac{l_1}{3}+$$
$$+\frac{2h_2l_2}{1}l^2{}_1+\frac{4h_2l_2}{1}\frac{l_2}{2}l_1+\frac{4h_2l_2}{1}\frac{l_2}{2}\frac{l_2}{3}=$$
$$=\frac{2h_2l_1}{3}l^2{}_1+\frac{2h_1l_1}{3}l^2{}_1+2h_2l_2l^2{}_1+2h_2l_2l_2l_1+\frac{2h_2l_2}{3}l^2{}_2=.$$
$$=\frac{2}{3}[h_2l_1l^2{}_1+h_1l_1l^2{}_1+3h_2l_2l^2{}_1+3h_2l_2l_2l_1+h_2l_2l^2{}_2]$$

and

$$Var(X)=\frac{2}{3}[h_2l_1l^2{}_1+h_1l_1l^2{}_1+3h_2l_2l^2{}_1+3h_2l_2l_2l_1+h_2l_2l^2{}_2]=$$
$$=\frac{2}{3}[h_1l_1l^2{}_1+h_2(l_1l^2{}_1+3l_2l^2{}_1+3l_2l_2l_1+l_2l^2{}_2)]=.$$
$$=\frac{2}{3}[h_1l_1l^2{}_1+h_2(l_1+l_2)^3]$$

This expression naturally coincides with the above one of the preceding subchapter.

The contiguous situations

Due to the symmetry of the test distribution, $E(X) = l_2 + l_1$.
So, as in the preceding subchapter, we have

$$\frac{|E(X)-b_{Boundary}|}{SD}=\sqrt{\frac{3}{2}}\frac{l_2+l_1}{\sqrt{h_1 l^3{}_1+h_2(l_2+l_1)^3}}\le\sqrt{3}\,.$$

The hypothetic situations

The above general formula

$$E(X)=h_2 l^2{}_1+\frac{2h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}+\frac{2h_2l_2}{\beta_2+1}\frac{1}{\beta_2+2}[l_1(\beta_2+2)+l_2]$$

is transformed to a specific one at $\beta_2 = \beta_1 = 0$

$$E(X)=h_2 l^2{}_1+\frac{2h_1l_1}{1}\frac{l_1}{2}+\frac{2h_2l_2}{1}\frac{1}{2}(2l_1+l_2)=$$

$$=h_2l^2{}_1+h_1l^2{}_1+2h_2l_2l_1+h_2l^2{}_2=$$

$$=h_1l^2{}_1+h_2(l_1+l_2)^2$$

So, the ratio is

$$\frac{|E(X)-b_{Boundary}|}{SD}=\sqrt{\frac{3}{2}}\frac{h_1l^2{}_1+h_2(l_1+l_2)^2}{\sqrt{h_1 l^3{}_1+h_2(l_2+l_1)^3}}\,.$$

This expression naturally coincides with the above one of the preceding subchapter. Therefore the consideration, final formula and conclusion may be the same as in the preceding subchapter. Namely,

$$\frac{|E(X)-b_{Boundary}|}{SD}\xrightarrow[\left(\frac{l_1}{l_2}\right)^2<<\frac{h_2}{h_1}\to 0]{}\sqrt{\frac{3}{2}}\sqrt{h_2l_2}\xrightarrow[\left(\frac{l_1}{l_2}\right)^2<<\frac{h_2}{h_1}<<\frac{l_1}{l_2}\to 0]{}0\,.$$

4.4.3. The case of a two-step triangle. $\beta_2 = \beta_1 = 1$
Distribution and normalizing equation

The above general formula of the PDF

$$f(x) = h_2\left(\frac{x}{l_2}\right)^{\beta_2}[\theta(x)-\theta(x-l_2)]+$$
$$+\left[h_2+h_1\left(\frac{x-l_2}{l_1}\right)^{\beta_1}\right][\theta(x-l_2)-\theta(x-l_2-l_1)]+$$
$$+\{h_2+h_1\left[\frac{2(l_2+l_1)-x}{l_1}\right]^{\beta_2}\}[\theta[x-(l_2+2l_1)]-\theta(x-2l_2)]+$$
$$+h_1\left(\frac{l_2+2l_1-x}{l_2}\right)^{\beta_2}\{\theta[x-(l_2+l_1)]-\theta[x-(l_2+2l_1)]\}$$

is transformed to a specific one at $\beta_2 = \beta_1 = 1$

$$f(x) = h_2\frac{x}{l_2}[\theta(x)-\theta(x-l_2)]+$$
$$+\left(h_2+h_1\frac{x-l_2}{l_1}\right)[\theta(x-l_2)-\theta(x-l_2-l_1)]+$$
$$+\left[h_2+h_1\left(\frac{2(l_2+l_1)-x}{l_1}\right)\right]\{\theta[x-(l_2+2l_1)]-\theta(x-2l_2)\}+ ,$$
$$+h_1\frac{l_2+2l_1-x}{l_2}\{\theta[x-(l_2+l_1)]-\theta[x-(l_2+2l_1)]\}$$

The above general formula of the normalizing integration

$$\frac{2h_2l_2}{\beta_2+1}+2h_2l_1+\frac{2h_1l_1}{\beta_1+1_1}=1.$$

is transformed to a specific one at $\beta_2 = \beta_1 = 1$

$$\frac{2h_2l_2}{1+1}+2h_2l_1+\frac{2h_1l_1}{1+1}= .$$
$$=h_2l_2+2h_2l_1+h_1l_1=1$$

The variance

The above general formula

$$Var(X)=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{\beta_1+1}\frac{l_1}{\beta_1+2}\frac{l_1}{\beta_1+3}+$$

$$+\frac{2h_2l_2}{\beta_2+1}l^2{}_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}l_1+\frac{4h_2l_2}{\beta_2+1}\frac{l_2}{\beta_2+2}\frac{l_2}{\beta_2+3}$$

is transformed to a specific one at $\beta_2 = \beta_1 = 1$

$$Var(X)=\frac{2h_2l_1}{3}l^2{}_1+\frac{4h_1l_1}{1+1}\frac{l_1}{1+2}\frac{l_1}{1+3}+$$

$$+\frac{2h_2l_2}{1+1}l^2{}_1+\frac{4h_2l_2}{1+1}\frac{l_2}{1+2}l_1+\frac{4h_2l_2}{1+1}\frac{l_2}{1+2}\frac{l_2}{1+3}=$$

$$=\frac{2h_2l_1}{3}l^2{}_1+\frac{h_1l_1}{2}\frac{l^2{}_1}{3}+h_2l_2l^2{}_1+\frac{2h_2l_2}{3}l_2l_1+\frac{h_2l_2}{2}\frac{l^2{}_2}{3}$$

and

$$Var(X)=\frac{2h_2l_1}{3}l^2{}_1+\frac{h_1l_1}{2}\frac{l^2{}_1}{3}+h_2l_2l^2{}_1+\frac{2h_2l_2}{3}l_2l_1+\frac{h_2l_2}{2}\frac{l^2{}_2}{3}=$$

$$=\frac{1}{6}[4h_2l_1l^2{}_1+h_1l_1l^2{}_1+6h_2l_2l^2{}_1+4h_2l_2l_2l_1+h_2l_2l^2{}_2]=$$

$$=\frac{1}{6}[h_1l^3{}_1+h_2(4l^3{}_1+6l_2l^2{}_1+4l^2{}_2l_1+l^3{}_2)]$$

or

$$Var(X)=\frac{1}{6}[h_1l^3{}_1+h_2(4l^3{}_1+6l_2l^2{}_1+4l^2{}_2l_1+l^3{}_2)]=$$

$$=\frac{1}{6}\{h_1l^3{}_1+h_2[4l_1(l_1+l_2)^2+l_2(l^2{}_2-2l^2{}_1)]\}$$

or

$$Var(X)=\frac{1}{6}[h_1l^3{}_1+h_2(4l^3{}_1+6l_2l^2{}_1+4l^2{}_2l_1+l^3{}_2)]=$$

$$\frac{1}{6}\{h_1l^3{}_1+h_2\{(l_1+l_2)^3+l_1[3l_1(l_1+l_2)+l^2{}_2)]\}\}.$$

So, at $\beta_2 = \beta_1 = 1$,

$$Var(X)=\frac{1}{6}[h_1l^3{}_1+h_2(4l^3{}_1+6l_2l^2{}_1+4l^2{}_2l_1+l^3{}_2)].$$

Due to the symmetry of the normal-like test distributions, $E(X) = l_2 + l_1$.

Due to the above general considerations about the contiguous situations, the ratio is minimal at

$$l_2 \to 0 \quad \text{and} \quad h_1 \to 0.$$

Under these tendencies, the normalizing integration tends to

$$h_2 l_2 + 2h_2 l_1 + h_1 l_1 \xrightarrow[l_2 \to 0;\, h_1 \to 0]{} 2h_2 l_1 = 1.$$

Under these tendencies, the variance tends to

$$Var(X) = \frac{2h_2 l_1}{3} l^2{}_1 + \frac{4h_1 l_1}{\beta_1 + 1} \frac{l_1}{\beta_1 + 2} \frac{l_1}{\beta_1 + 3} + \frac{2h_2 l_2}{\beta_2 + 1} l^2{}_1 +$$

$$+ \frac{4h_2 l_2}{\beta_2 + 1} \frac{l_2}{\beta_2 + 2} l_1 + \frac{4h_2 l_2}{\beta_2 + 1} \frac{l_2}{\beta_2 + 2} \frac{l_2}{\beta_2 + 3} \xrightarrow[l_2 \to 0;\, h_1 \to 0]{}$$

$$\xrightarrow[l_2 \to 0;\, h_1 \to 0]{} \frac{2h_2 l_1}{3} l^2{}_1 \xrightarrow[l_2 \to 0;\, h_1 \to 0]{} \frac{l^2{}_1}{3}$$

and

$$SD \xrightarrow[l_2 \to 0;\, h_1 \to 0]{} \frac{l_1}{\sqrt{3}}.$$

The ratio tends to

$$\frac{|E(X) - b_{Boundary}|}{SD} \xrightarrow[l_2 \to 0;\, h_1 \to 0]{} \frac{l_1}{\frac{l_1}{\sqrt{3}}} = \sqrt{3}.$$

This corresponds to the general limit of the ratio.

The above general formula

$$E(X) = h_2 l^2{}_1 + \frac{2h_1 l_1}{\beta_1 + 1}\frac{l_1}{\beta_1 + 2} + \frac{2h_2 l_2}{\beta_2 + 1}\frac{1}{\beta_2 + 2}[l_1(\beta_2 + 2) + l_2].$$

is transformed to a specific one at $\beta_2 = \beta_1 = 1$

$$E(X) = \frac{1}{3}[h_1 l^2{}_1 + h_2(3l^2{}_1 + 3l_2 l_1 + l^2{}_2)]$$

The ratio $|E(X)\text{-}b_{Boundary}|/SD$ is equal to

$$\frac{|E(X) - b_{Boundary}|}{SD} = \frac{E(X)}{SD} =$$

$$= \frac{\sqrt{3}\sqrt{2}}{3}\frac{h_1 l^2{}_1 + h_2(3l^2{}_1 + 3l_2 l_1 + l^2{}_2)}{\sqrt{h_1 l^3{}_1 + h_2(4l^3{}_1 + 6l_2 l^2{}_1 + 4l^2{}_2 l_1 + l^3{}_2)}} = .$$

$$= \sqrt{\frac{2}{3}}\frac{h_1 l^2{}_1 + h_2(3l^2{}_1 + 3l_2 l_1 + l^2{}_2)}{\sqrt{h_1 l^3{}_1 + h_2(4l^3{}_1 + 6l_2 l^2{}_1 + 4l^2{}_2 l_1 + l^3{}_2)}}$$

So,

$$\frac{|E(X) - b_{Boundary}|}{SD} = \sqrt{\frac{2}{3}}\frac{h_1 l^2{}_1 + h_2(3l^2{}_1 + 3l_2 l_1 + l^2{}_2)}{\sqrt{h_1 l^3{}_1 + h_2(4l^3{}_1 + 6l_2 l^2{}_1 + 4l^2{}_2 l_1 + l^3{}_2)}}.$$

One can use the consideration of the preceding subchapter. The ratio $|E(X)-b_{Boundary}|/SD$ can be identically rewritten as

$$\frac{|E(X)-b_{Boundary}|}{SD}=\sqrt{\frac{2}{3}}\frac{h_1 l^2{}_1+h_2(3l^2{}_1+3l_2 l_1+l^2{}_2)}{\sqrt{h_1 l^3{}_1+h_2(4l^3{}_1+6l_2 l^2{}_1+4l^2{}_2 l_1+l^3{}_2)}}=$$

$$=\sqrt{\frac{2}{3}}\sqrt{l_2}\frac{h_1\left(\frac{l_1}{l_2}\right)^2+h_2\left[3\left(\frac{l_1}{l_2}\right)^2+3\left(\frac{l_1}{l_2}\right)+1\right]}{\sqrt{h_1\left(\frac{l_1}{l_2}\right)^3+h_2\left[4\left(\frac{l_1}{l_2}\right)^3+6\left(\frac{l_1}{l_2}\right)^2+4\left(\frac{l_1}{l_2}\right)+1\right]}}=$$

$$=\sqrt{\frac{2}{3}}\left(\frac{1}{l_2}\right)^{1/2}\frac{h_1}{\sqrt{h_1}}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left[3\left(\frac{l_1}{l_2}\right)^2+3\left(\frac{l_1}{l_2}\right)+1\right]}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left[4\left(\frac{l_1}{l_2}\right)^3+6\left(\frac{l_1}{l_2}\right)^2+4\left(\frac{l_1}{l_2}\right)+1\right]}}=$$

$$=\sqrt{\frac{2}{3}}\sqrt{l_2}\frac{h_1}{\sqrt{h_1}}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left[3\left(\frac{l_1}{l_2}\right)^2+3\left(\frac{l_1}{l_2}\right)+1\right]}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left[4\left(\frac{l_1}{l_2}\right)^3+6\left(\frac{l_1}{l_2}\right)^2+4\left(\frac{l_1}{l_2}\right)+1\right]}}=$$

.

$$=\sqrt{\frac{2}{3}}\sqrt{h_1 l_2}\frac{\left(\frac{l_1}{l_2}\right)^2+\frac{h_2}{h_1}\left[3\left(\frac{l_1}{l_2}\right)^2+3\left(\frac{l_1}{l_2}\right)+1\right]}{\sqrt{\left(\frac{l_1}{l_2}\right)^3+\frac{h_2}{h_1}\left[4\left(\frac{l_1}{l_2}\right)^3+6\left(\frac{l_1}{l_2}\right)^2+4\left(\frac{l_1}{l_2}\right)+1\right]}}$$

The triangle and step functions are, in a sense, similar to each other. Therefore let us test (1)

$$\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} << 1 .$$

with respect to this special case.

Under the conditions (1), the ratio $|E(X)\text{-}b_{Boundary}|/SD$ tends to the limit

$$\frac{|E(X)-b_{Boundary}|}{SD} =$$

$$= \sqrt{\frac{2}{3}}\sqrt{h_1 l_2}\,\frac{\left(\frac{l_1}{l_2}\right)^2 + \frac{h_2}{h_1}\left[3\left(\frac{l_1}{l_2}\right)^2 + 3\left(\frac{l_1}{l_2}\right) + 1\right]}{\sqrt{\left(\frac{l_1}{l_2}\right)^3 + \frac{h_2}{h_1}\left[4\left(\frac{l_1}{l_2}\right)^3 + 6\left(\frac{l_1}{l_2}\right)^2 + 4\left(\frac{l_1}{l_2}\right) + 1\right]}} \xrightarrow[\left(\frac{l_1}{l_2}\right)^2 \to 0]{}$$

$$\xrightarrow[\left(\frac{l_1}{l_2}\right)^2 \to 0]{} \sqrt{\frac{2}{3}}\sqrt{h_1 l_2}\,\frac{\left(\frac{l_1}{l_2}\right)^2 + \frac{h_2}{h_1}}{\sqrt{\left(\frac{l_1}{l_2}\right)^3 + \frac{h_2}{h_1}}} \xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} \to 0]{}$$

$$\xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} \to 0]{} \sqrt{\frac{2}{3}}\sqrt{h_1 l_2}\,\frac{\frac{h_2}{h_1}}{\sqrt{\frac{h_2}{h_1}}} = \sqrt{\frac{2}{3}}\sqrt{h_1 l_2}\sqrt{\frac{h_2}{h_1}} =$$

$$= \sqrt{\frac{2}{3}}\sqrt{h_2 l_2} .$$

This limit is, indeed, similar to that of the preceding subchapter and by means of similar considerations

$$h_2 l_2 << h_1 l_1 \quad \text{and} \quad h_1 l_1 < 1 \quad \text{and} \quad h_2 l_2 << h_1 l_1 ,$$

we obtain

$$\frac{|E(X)-b_{Boundary}|}{SD} \xrightarrow[\left(\frac{l_1}{l_2}\right)^2 << \frac{h_2}{h_1} << \frac{l_1}{l_2} \to 0]{} \sqrt{\frac{2}{3}}\sqrt{h_2 l_2} \xrightarrow[h_2 l_2 \to 0]{} 0 .$$

In the hypothetic situation of “adhesion” the minimal ratio is, evidently, half of the above value and, hence, can be much less than unity as well.

So, it has been proven that the minimal ratio $|E(X)\text{-}b_{Boundary}|/SD$ for the continuous two-step power test distribution with compact support can be much less than unity for the hypothetic situations.

5. Conclusions

5.1. General

The minimal distance from the expectation of a random variable to the nearest boundary of the interval has been considered in the present article. The distance has been expressed in terms of the standard deviation (SD) of the variable.

The question whether this minimal distance can be neglected with respect to the SD has been particularly analyzed.

This minimal distance can determine the minimal magnitudes of forbidden zones caused by a noise for results of measurements near the boundaries of the intervals (see, e.g., [1] and [2]). These forbidden zones cause fundamental problems in behavioral economics and decision sciences, in utility and prospect theories.

5.2. Definitions

The interval boundary that is the nearest to the expectation of the variable is referred to as $\boldsymbol{b_{Boundary}}$. So the minimal distance between the expectation $E(X)$ of the variable and the nearest boundary $b_{Boundary}$ of the interval is referred to as $min(|E(X)-b_{Boundary}|)$. The ratio of this minimal distance to the standard deviation is referred to as $min(|E(X)-b_{Boundary}|)\,/\,SD$ or simply $|E(X)-b_{Boundary}|\,/\,SD$.

A **normal-like** distribution is defined as a distribution that has the symmetric probability density function (**PDF**) f with non-increasing sides.

Compact distributions are referred to as the distributions with bounded or compact support. **Noncompact distributions** are referred to as the distributions with not bounded support.

The **contiguous** situation is defined as the situation when one side of distribution's support touches the boundary of a half-infinite or finite interval.

The **hypothetical reflection** situation is defined as the situation when f is modified to the hypothetical function f_{Refl} that is reflected with respect to $E(X) = 0$

$$f_{\mathrm{Re}\,fl}(x) = \theta(x) 2 f(x) .$$

The **hypothetical adhesion** situation is modified from the hypothetical reflection situation such that the reflected part of the PDF is "adhered" to the boundary 0.

5.3. The scope of the considerations

The ratio of the minimal distance from the expectation of the variable to the nearest boundary of the interval to the SD $|E(X)\text{-}b_{Boundary}| / SD$ has been considered for the following situations:

The hypothetical reflection situation and the corresponding adhesion situation have been analyzed for the normal distribution.

The hypothetical reflection situation and the corresponding adhesion situation have been analyzed for the distributions having continuous probability density functions with noncompact support, namely for the Laplace and power test distributions.

The contiguous and hypothetical situations have been analyzed for the continuous and piecewise continuous "normal-like" test distributions with compact support.

5.4. The main results

The main three results of the present article are:

A priori. A priori, one can evidently state that the minimal distance between the expectation of a random variable and the nearest boundary of the interval can be equal to zero only if the support of the distribution is a sole point.

First. The normal distribution has the finite ratio $|E(X)\text{-}b_{Boundary}| / SD$ for the hypothetic situations of reflection and adhesion.

Second. For the contiguous situation, the continuous compact "normal-like" distributions have the finite ratio $|E(X)\text{-}b_{Boundary}|/SD$.

Third. For the hypothetic situations of reflection and adhesion, the existence of "normal-like" distributions with the negligibly small ratio $|E(X)\text{-}b_{Boundary}| / SD$ has been proven for noncompact continuous distributions and also for compact continuous and piecewise continuous distributions.

That is, for these distributions, there exist combinations of their parameters, such that the minimal distance between the expectation and the nearest boundary $min(|E(X)\text{-}b_{Boundary}|)$ can be neglected with respect to the standard deviation.

In addition, all the results of the present article can be treated as those supporting the need of further research to refine and generalize the conditions of finite ratios of the minimal distances between the expectations of the variables and the nearest boundaries of the intervals to the standard deviations.

References

[1] Dokov, S. P., Morton, D.P., 2005. Second-Order Lower Bounds on the Expectation of a Convex Function. *Math. Oper. Res.* **30**(3), 662–677.

[2] Harin, A. 2012. Data dispersion in economics (I) – Possibility of restrictions. *Rev. Econ. Fin.* **2** (3): 59-70.

[3] Harin, A., 2012. Data dispersion in economics (II) – Inevitability and Consequences of Restrictions, *Rev. Econ. Fin.* **2**(4), 24–36.

[4] Harin, A., 2015. General bounds in economics and engineering at data dispersion and risk, *Proceedings of the Thirteenth International Scientific School* **13**, 105–117, in Modeling and Analysis of Safety and Risk in Complex Systems (Saint-Petersburg: IPME RAS).

[5] Harin, A. Can forbidden zones for the expectation explain noise influence in behavioral economics and decision sciences? MPRA Paper No. 76240, 2017.

[6] Kahneman, D., and Thaler, R., 2006. Anomalies: Utility Maximization and Experienced Utility, *J Econ. Perspect.* **20**(1), 221–234.

[7] Pinelis, I., 2011. Exact lower bounds on the exponential moments of truncated random variables, *J Appl. Probab.* **48**(2), 547–560.

[8] Prékopa, A., 1990, The discrete moment problem and linear programming, *Discrete Appl. Math.* **27**(3), 235–254.

[9] Prékopa, A., 1992. Inequalities on Expectations Based on the Knowledge of Multivariate Moments. Shaked M, Tong YL, eds., *Stochastic Inequalities*, 309–331, number 22 in Lecture Notes-Monograph Series (Institute of Mathematical Statistics).