Extended Gini-type measures of risk and variability

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Abstract

The main of this paper is to introduce a family of risk measures which generalizes the Gini-type measures of risk and variability, by taking into consideration the psychological behavior. Our risk measures family is coherent and catches variability with respect to the decision-maker attitude towards risk.

JEL classification: C6, G10

Keywords: risk measure, variability measure, risk aversion, signed Choquet integral, Extended Gini Shortfall.

1 Introduction

In modern risk management, a large number of risk measures have been proposed in the literature. These measures are mappings from the set of random variables (financial losses) to real numbers. After the subprime mortgage crisis, a prominent trend associated with tail-based risk measures has emerged, especially with the most popular nowadays tail-based risk measures: the Value-at-Risk (VaR) and the Expected Shortfall (ES)

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(Embrechts et al. (2014) and the references therein).
Let $p \in (0,1)$ be a prudence level, X a risk random variable (rv), and F_X the cumulative distribution function (cdf) of X.
The VaR_p is the p-th quantile of F_X given by:

$$VaR_p(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}. \tag{1.1}$$

The ES_p is the average of VaR over large prudence levels:

$$ES_p(X) = \frac{1}{1-p}\int_p^1 VaR_q(X)dq. \tag{1.2}$$

When the cdf F_X is continuous, the ES_p risk measure and the tail Conditional Expectation (TCE) risk measure are mingled:

$$TCE_p(X) = \mathbb{E}[X/X > x_p]. \tag{1.3}$$

Where $\mathbb{E}$ is the expectation operator.

Since VaR_p and ES_p do not capture the variability of the rv X beyond the quantile x_p, Furman and Landsman (2006a) have suggested the Tail-Standard-Deviation (TSD) risk measure:

$$TSD_p^{\lambda}(X) = TCE_p(X) + \lambda\, SD_p(x), \tag{1.4}$$

where $p \in (0,1)$ is the prudence level, $\lambda \in [0,\infty)$ is the loading parameter and the Standard-Deviation measure (SD_p) is given by:

$$SD_p(X) = \sqrt{\mathbb{E}[(X - TCE_p(X))^2/X > x_p]} \tag{1.5}$$

But, this risk measure (TSD) is not monotone, not additive for co-monotonic risks and is undefined on some discrete risks violating the requirement $\mathbb{P}(X > x_p) > 0$.

Recently, in the same spirit Furman et al. (2017) introduce the Gini Shortfall (GS) risk measure which is coherent and satisfies co-monotonic additivity:

$$GS_p^{\lambda}(X) = ES_p(X) + \lambda\, TGini_p(X). \tag{1.6}$$

However, the Gini Shortfall (GS) risk measure suppose that all individuals have the same attitude towards risk. While it is obvious that people differ in the way the take personal decisions that involving risk. Such differences are described by differences in risk aversion. To incorporate psychological behavior in tail risk analysis, we introduce a new family of risk measures and this is the principal contribution of our work.

In the present paper, we set out to suggest an alternative risk measures family taking into account the notion of risk aversion. This requirement naturally leads us to Tail Extended Gini functional, that we introduce and discuss below.

The rest of the paper is organized as follows. In Section 2, we present and discuss some preliminaries such as essential properties of measures of risk and variability, and we elucidate the role of the signed Choquet integral. In section 3, we start with the Classical and Extended Gini functionals and introduce what we call Tail Extended Gini functional. In section 4, we introduce the notion of Extended Gini Shortfall and explore its various properties. We give an interpretation to our contribution in Section 5. In Section 6, we give the closed-form for elliptical distributions and derive the normal case. Finally, Section 7 concludes.

2 Preliminaries

We first introduce some basic notation. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an atomless probability space. Let L^q denote the set of all rv's on $(\Omega, \mathcal{A}, \mathbb{P})$ with finite q-th moment, $q \in [0, \infty)$ and L^∞ be the set of all essentially bounded rv's. Throughout this paper, $X \in L^0$ is a rv modeling financial losses (profits) when it has positive (negative) values. For every $X \in L^0$, F_X, denote the cdf of X, and U_X denote any uniform $[0, 1]$ rv such that the equation $F_X^{-1}(U_X) = X$ holds almost surely. The following proposition assures the existence of such rv's:

***Proposition* 2.1** (Rüschendorf (2013, Proposition 1.3))**.** *For any random variable X, there exists a $U[0, 1]$ random variable U_X such that $X = F_X^{-1}(U_X)$ almost surely.*

Throughout the present paper, we deal with several convex cones $\mathcal{X}$ of rv's, of which $\mathcal{X}$ is of particular importance and L^∞ is always contained in $\mathcal{X}$.

2.1 Measures of risk

A risk measure ρ is a function that maps a convex cone of rv's $\mathcal{X}$ to $(-\infty, \infty]$. In the context of this paper, several required properties are summarized below (we adopt the same notation as in Furman et al. (2017)):

(A) *Law-invariance:* if $X \in \mathcal{X}$ and $Y \in \mathcal{X}$ have the same distributions under $\mathbb{P}$, succinctly $X \stackrel{d}{=} Y$, then $\rho(X) = \rho(Y)$.

In the theory of coherent risk measures (Artzner et al. (1999)) the following properties are frequently used:

(B1) *Monotonicity:* $\rho(X) \leq \rho(Y)$ when $X, Y \in \mathcal{X}$ are such that $X \leq Y$ $\mathbb{P}$-almost surely.

(B2) *Translation invariance:* $\rho(X - m) = \rho(X) - m$ for all $m \in \mathbb{R}$ and $X \in \mathcal{X}$.

(A1) *Positive homogeneity:* $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda > 0$ and $X \in \mathcal{X}$.

(A2) *Subadditivity:* $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.

(A3) *Convexity:* $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$.

For interpretations of these properties, we refer the reader to Föllmer and Schied (2011, Chap 4), Delbaen (2012), and McNeil et al. (2015).

Definition 2.1 (Artzner et al. (1999))**.** *A risk measure is monetary if it satisfies properties (B1) and (B2), and it is coherent if it satisfies furthermore (A1) and (A2).*

Remark 2.1. *Any pair among three properties (A1), (A2) and (A3) implies the remaining one.*

Another important property of risk measures is *co-monotonic additivity*:

Definition 2.2 (Schmeidler(1986))**.** *Two rv's X and Y are co-monotonic when*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \text{ for } (\omega, \omega') \in \Omega \times \Omega \quad (\mathbb{P} \times \mathbb{P})\text{-almost surely.}$$

(A4) *Co-monotonic additivity:* $\rho(X + Y) = \rho(X) + \rho(Y)$ for every co-monotonic pair $X, Y \in \mathcal{X}$.

For example, both functionals VaR_p and ES_p are monetary and co-monotonically additive, whereas ES_p is even coherent (McNeil et al. (2015)).

2.2 Signed Choquet integral

The notion of signed Choquet integral plays a pivotal role thereafter. It originates from Choquet (1954), in the framework of capacities, and is further characterized and studied in decision theory by Schmeidler (1986, 1989).

Definition 2.3 (Distortion function)**.** *$h : [0, 1] \to \mathbb{R}$ is called a distortion function when it is non-decreasing and satisfies the boundary conditions $h(0) = 0$ and $h(1) = 1$.*

Let h be a distortion function, the functional defined by the equation:

$$I(X) = \int_0^{\infty} (h(1) - h(F_X(x)))dx - \int_{-\infty}^{0} h(F_X(x))dx \tag{2.1}$$

for all $X \in \mathcal{X}$ is called the (*increasing*) *Choquet integral*.

Whenever $h : [0, 1] \to \mathbb{R}$ is of finite variation and such that $h(0) = 0$, I is called the *signed Choquet integral*.
When h is right-continuous, then equation(2.1) can be rewritten as (Wang et al. (2017)):

$$I(X) = \int_0^1 F_X^{-1}(t)dh(t). \tag{2.2}$$

Furthermore, when h is absolutely continuous, with ϕ a function such that $dh(t) = \phi(t)dt$, then equation(2.2) becomes:

$$I(X) = \int_0^1 F_X^{-1}(t)\phi(t)dt. \tag{2.3}$$

In this case, ϕ is called the *weighting functional* of the signed Choquet integral I.

The signed Choquet integral, as we can readily see from representation (2.2) and the theorem below, is co-monotonically additive.

***Theorem* 2.1** (Wang et al. (2017, Theorem 2.1))**.** *A functional $I : L^{\infty} \to R$ is law-invariant, comonotonic-additive and uniformly norm-continuous if and only if I is a signed Choquet integral.*

Moreover, we know from Yaari (1987) and the theorem below, that any law-invariant risk measure is co-monotonically additive and monetary if and only if it can be represented as a Choquet integral.

***Theorem* 2.2** (Föllmer and Schied (2011, Theorem 4.88))**.** *A monetary risk measure ρ on $\mathcal{X}$ is comonotonic-additive if and only if there exists a normalized monotone set function μ on $(\Omega, \mathcal{A})$ such that*

$$\rho(X) = \int (-X)d\mu, \quad X \in \mathcal{X}$$

where, μ is given by $\mu(A) = \rho(-\mathbb{I}_A)$.

Furthermore, the functional I is sub-additive if and only if the function h is convex (cf. Yaari (1987) and Acerbi (2002)).

Remark 2.2. *The major difference between a (an increasing) Choquet integral and a signed one is that the latter, being more general, is not necessarily monotone.*

One of the practical and theoretical reasons for what we are particularly interested in signed Choquet integral is that we know that a suitable risk measure should be monotone as argued by Artzner et al. (1999), but this issue is irrelevant for a measure of variability. In other words, signed Choquet integral is relevant as long as a measure of variability is concerned.

2.3 Measures of variability

Measures of variability, denoted ν and used to quantify the magnitude of variability of rv's, are functional that map $\mathcal{X}$ to $[0, \infty]$. Desirable properties[1] for a variability measure are proposed by Furman et al. (2017):

(A) *Law-invariance.*

(C1) *Standardization:* $\nu(c) = 0$ for all $c \in \mathbb{R}$.

(C2) *Location invariance:* $\nu(X - c) = \nu(X)$ for all $c \in \mathbb{R}$ and $X \in \mathcal{X}$.

A measure of variability is coherent if it further satisfies:

(A1) *Positive homogeneity.*

(A2) *Subadditivity.*

For instance, the most classical measures of variability are the *Variance* and the *Standard Deviation*:

$$Var(X) = \mathbb{E}[(X - \mathbb{E}(X))^2], \quad X \in L^2 \tag{2.4}$$

$$SD_0(X) = \sqrt{Var(X)}, \quad X \in L^2$$

The variance functional satisfies properties (A), (C1), (C2) but not (A1) or (A2), hence it is not coherent. On the other side, the standard deviation functional, since satisfying all aforementioned properties, is coherent. Note that neither the variance nor the standard deviation is co-monotonically additive.

The following theorem is enunciated with a complete proof in Furman et al. (2017), it gives the characterization for co-monotonically additive and coherent measures of variability.

[1]Inspired from the notion of *deviation measures* of Rockafellar et al. (2006).

Theorem 2.3. *Let $\nu : L^q \to \mathbb{R}$ be any L^q-continuous functional. The following three statements are equivalent:*

(i) ν is a co-monotonically additive and coherent measure of variability.

(ii) There is a convex function $h : [0,1] \to \mathbb{R}, h(0) = h(1) = 0$, such that

$$\nu(X) = \int_0^1 F_X^{-1}(u)dh(u), \quad X \in L^q. \tag{2.5}$$

(iii) There is a non-decreasing function $g : [0,1] \to \mathbb{R}$ such that

$$\nu(X) = Cov[X, g(U_X)], \quad X \in L^q. \tag{2.6}$$

Next, we recall a few partial orders of variability that have been popular in economics, insurance, finance and probability theory:

Definition 2.4. *For $X, Y \in L^1$, X is second-order stochastically dominated (SSD) by Y, succinctly $X \prec_{SSD} Y$, if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing convex functions f. If in addition, $\mathbb{E}[X] = \mathbb{E}[Y]$, then we say that X is smaller than Y in convex order, succinctly $X \prec_{CX} Y$.*[2]

Hence, the following two properties are introduced in Furman et al. (2017):

(B3) *SSD-monotonicity:* if $X \prec_{SSD} Y$, then $\rho(X) \leq \rho(Y)$.

(C3) *CX-monotonicity:* if $X \prec_{CX} Y$, then $\nu(X) \leq \nu(Y)$.

Let $q \in [1, \infty]$, on L^q all real-valued law-invariant coherent risk measures are SSD-monotone. We refer the reader to Dana (2005), Grechuk et al. (2009), and Föllmer and Schied (2011) for proofs of the above assertions, and to Mao and Wang (2016) for a characterization of SSD-monotone risk measures.

[2] We say equally Y is a Mean Preserving Spread of X, succinctly $Y \, MPS \, X$.

3 Tail Extended Gini

3.1 Classical Gini functional

The Gini coefficient, as a measure of variability, was introduced by Corrado Gini as an alternative to the variance measure:[3]

$$Gini(X) = \mathbb{E}[\mid X^* - X^{**} \mid], \quad X \in L^1. \tag{3.1}$$

where X^* and X^{**} are two independent copies of X. The Gini coefficient has been remarkably influential in numerous research areas, applied and theoretical (e.g, Yitzhaki and Schechtman (2013) and the references therein). The Gini functional is a free-center measure of variability. Yitzhaki (1998) lists more than a dozen alternative presentations of the Gini coefficient.

For instance, the Gini functional can be written in terms of a signed Choquet integral:

$$Gini(X) = 2\int_0^1 F_X^{-1}(u)(2u-1)du, \quad X \in L^1. \tag{3.2}$$

Since all coherent measures of variability are CX-monotone and from the Theorem 2.1, it follows immediately that the Gini functional is a coherent measure of variability and it is CX-monotone.

Moreover, equation (3.2) can be written in terms of covariance (which is the most common formula of the Gini coefficient):

$$Gini(X) = 4\,Cov[F_X^{-1}(U), U], \quad X \in L^1. \tag{3.3}$$

we recall that U can be any uniformly on $[0,1]$ distributed rv.

Or,

$$Gini(X) = 4\,Cov[X, U_X], \quad X \in L^1 \tag{3.4}$$

where U_X is a uniform $[0,1]$ rv such that the equation $F_X^{-1}(U_X) = X$ holds almost.

***Remark* 3.1.** *Using the Gini functional in risk and variability measurement suppose that all individuals have the same attitude towards risk and variability.*

3.2 Extended Gini functional

Gini functional may be extended into a family of measures of variability differing from each other in the decision-maker's degree of risk aversion, which is reflected in this paper by the parameter r.

[3] e.g., Giorgi (1990, 1993) and Ceriani and Verme (2012).

The basic definition of the Extended Gini coefficient, denoted by $EGini_r$ is based on the covariance term (Yitzhaki and Schechtman (2013)):

$$EGini_r(X) = -2r\,Cov[X, (1 - F_X(X))^{r-1}], \quad r > 1 \tag{3.5}$$

we refer to Yitzhaki (1983), Shalit and Yitzhaki (1984) and Yitzhaki and Schechtman (2005) for an overview of the Extened Gini properties.

In Theorem 2.1 equation (2.6), if one sets:

$$g_r(u) = -r(1-u)^{r-1} \qquad \text{for } r > 1 \text{ and } u \in [0,1] \tag{3.6}$$

we run into:

$$\nu(X) = -r\,Cov[X, (1 - F_X(X))^{r-1}] \qquad \text{for } r > 1 \text{ and } X \in L^1. \tag{3.7}$$

Proposition 3.1. *For every $r \in (1, \infty)$, the Extended Gini functional is a signed Choquet integral given by the equation:*

$$EGini_r(X) = 2\int_0^1 F_X^{-1}(u)(1 + g_r(u))du. \tag{3.8}$$

Proof. We recall that U can be any uniformly distributed rv on $[0, 1]$ such that the equation $F_X^{-1}(U) = X$ holds almost, $\mathbb{E}[X] = m$.
We can easily verify that: $\mathbb{E}[(1 - F_X(X))^{r-1}] = 1/r$ (a more general result will be given in Lemma 3.1).

$$\begin{aligned}
EGini_r(X) &= -2r\,Cov[X, (1 - F_X(X))^{r-1}] \\
&= -2r\,\mathbb{E}[(X - \mathbb{E}(X))((1 - F_X(X))^{r-1} - \mathbb{E}[(1 - F_X(X))^{r-1}])] \\
&= -2r\,\mathbb{E}[(F_X^{-1}(u) - m)((1 - U)^{r-1} - \frac{1}{r})] \\
&= -2r\int_0^1 (F_X^{-1}(u) - m)((1 - u)^{r-1} - \frac{1}{r})du \\
&= -2r\int_0^1 F_X^{-1}(u)((1 - u)^{r-1} - \frac{1}{r})du \\
&= 2\int_0^1 F_X^{-1}(u)(1 + g_r(u))du
\end{aligned}$$

□

In this case, h_r in equation (2.7) Theorem 2.1 is given by:

$$h_r(u) = u + (1-u)^r - 1, \quad r > 1 \tag{3.9}$$

The next Corollary follows immediately from Theorem 2.1 and the fact that all coherent measures of variability are CX-monotone:

***Corollary* 3.1.** *The Extended Gini functional is a coherent measure of variability, and it is CX-monotone.*

In the spirit of Theorem 2.1 (iii), our choice of g_r gives a family of co-monotonic additive and coherent measures of variability: $EGini_r$.

3.3 Tail Extended Gini functional

In the modern financial risk management, practitioners and researchers are often interested in the tail risk. Classical risk measures like the Value-at-Risk and the Expected Shortfall are conformed to such philosophy, but they do not reflect tail variability appropriately. Therefore, in this subsection we introduce the *Tail Extended Gini functional* ($TEGini_{r,p}$).

Given any risk rv $X \in L^1$ and prudence level $p \in (0,1)$, let $F_{X,p}$ denote the cdf of rv $F_X^{-1}(U_p)$, where U_p is uniformly distributed on $[p,1]$. Then the Tail Extended Gini functional is given in terms of conditional covariance by:

$$TEGini_{r,p}(X) = \frac{-2r}{1-p} Cov[X, (1-F_X(X))^{r-1}/X > x_p], \quad r > 1 \tag{3.10}$$

***Lemma* 3.1.** *For any $r \in (1,\infty)$ and $p \in [0,1)$, let $X \in L^1$ then*

$$\mathbb{E}[(1-F_{X,p}(X))^{r-1}/X > x_p] = (1-p)^{r-1}/r. \tag{3.11}$$

Proof.

$$\begin{aligned}
\mathbb{E}[(1-F_{X,p}(X))^{r-1}/X > x_p] &= \mathbb{E}[(1-U)^{r-1}/U > p] \\
&= \frac{1}{1-p}\int_{\mathbb{R}} (1-u)^{r-1}\mathbb{1}_{[p,1]}(u)du \\
&= \frac{1}{1-p}\int_p^1 (1-u)^{r-1}du \\
&= \frac{1}{1-p}[(1-u)^r/r]_p^1 \\
&= (1-p)^{r-1}/r.
\end{aligned}$$

□

The Tail Extended Gini functional can be represented as a signed Choquet integral:

Proposition 3.2. *For every $r \in (1,\infty)$ and $p \in (0,1)$, the tail Extended Gini functional is a signed Choquet integral given by:*

$$TEGini_{r,p}(X) = \frac{2}{(1-p)^2}\int_p^1 F_X^{-1}(u)[g_r(u) + (1-p)^{r-1}]du. \tag{3.12}$$

Therefore, the Tail Extended Gini functional is co-monotonic additive.

Proof.

$$\begin{aligned}
TEGini_{r,p}(X) &= \frac{-2r}{1-p}Cov[X,(1-F_X(X))^{r-1}/X > x_p] \\
&= \frac{-2r}{1-p}\mathbb{E}[(X-m)((1-F_X(X))^{r-1} - (1-p)^{r-1}/r)/X > x_p] \\
&= \frac{-2r}{1-p}\mathbb{E}[(X-m)((1-F_X(X))^{r-1} - (1-p)^{r-1}/r)/X > x_p] \\
&= \frac{-2r}{1-p}\mathbb{E}[(F_X^{-1}(U)-m)((1-U)^{r-1} - (1-p)^{r-1}/r)/U > p] \\
&= \frac{-2r}{(1-p)^2}\int_p^1 (F_X^{-1}(u)-m)((1-u)^{r-1} - (1-p)^{r-1}/r)du \\
&= \frac{2}{(1-p)^2}\int_p^1 F_X^{-1}(u)[g_r(u) + (1-p)^{r-1}]du.
\end{aligned}$$

This completes the proof of *Proposition 3.2.* □

We can easily check that the Tail Extended Gini functional is standardized, location invariant, and positively homogeneous. However, as shown in *Proposition 3.3* below, $TEGini_{r,p}$ is not *sub-additive*. Therefore, unlike the Extended Gini functional, the $TEGini_{r,p}$ is not a coherent measure of variability.

Proposition 3.3. *For every $r \in (0,\infty)$ and $p \in (0,1)$, the $TEGini_{r,p}$ is not sub-additive.*

Proof. To establish the non-subadditivity of $TEGini_{r,p}$, we refer to a counter example (for $r = 2$) in the proof of Proposition 3.3 by Furman et al. (2017). □

Whereas $TEGini_{r,p}$ is not a coherent measure of variability, we see in the next section that a linear combination of the Expected Shortfall with the Tail Extended Gini functionals gives rise to a coherent risk measure that quantifies both the magnitude and the variability of tail risks.

4 Extended Gini Shortfall

In this section we introduce the *Extended Gini Shortfall* (*EGS*), as a linear combination of ES_p and $TEGini_{r,p}$. Namely,

$$EGS^{\lambda_{r,p}}_{r,p}(X) = ES_p(X) + \lambda_{r,p}TEGini_{r,p}(X), \quad r > 1, \lambda_{r,p} \geq 0 \tag{4.1}$$

where $p \in (0,1)$ is the prudence level and $\lambda_{r,p}$ is the *loading parameter*.

To be a reasonable risk measure, the functional *EGS* should satisfy properties of coherent risk measures listed in *Subsection* 2.1. However, as mentioned in the previous section, the functional $TEGini_{r,p}$ is not sub-additive, and as a measure of variability is not monotone. Intuitively, as suggested by Furman et al. (2017), there might be a threshold that delineates the value of $\lambda_{r,p}$ for which *EGS* is coherent. In fact, when $\lambda_{r,p}$ is zero, then *EGS* obviously inherits all the properties of the ES_p which is coherent, but when $\lambda_{r,p}$ is sufficiently large, then the *TEGini*-term starts to dominates ES_p, and thus coherence of *EGS* cannot be expected.

***Proposition* 4.1.** *Let* $p \in (0,1)$ *and* $\lambda_{r,p} \in [0.\infty)$.
The Extended Gini shortfall is a signed Choquet integral given by:

$$EGS^{\lambda_{r,p}}_{r,p}(X) = \int_0^1 F_X^{-1}(u)\,\phi^{\lambda_{r,p}}_{r,p}(u)du \tag{4.2}$$

where,

$$\phi^{\lambda_{r,p}}_{r,p}(u) = \frac{1}{(1-p)^2}[1-p+2\lambda_{r,p}(g_r(u)+(1-p)^{r-1})]\,\mathbb{1}_{[p,1]}(u), \quad u \in [0,1]. \tag{4.3}$$

Proof.

$$\begin{aligned}
EGS^{\lambda_{r,p}}_{r,p}(X) &= ES_p(X) + \lambda_{r,p}TEGini_{r,p}(X) \\
&= \frac{1}{1-p}\int_p^1 F_X^{-1}(u)du + \frac{2r}{(1-p)^2}\int_p^1 F_X^{-1}(u)[g_r(u)+(1-p)^{r-1}]du \\
&= \int_p^1 F_X^{-1}(u)[\frac{1}{1-p} + \frac{2r}{(1-p)^2}(g_r(u)+(1-p)^{r-1})]du \\
&= \int_0^1 F_X^{-1}(u)\,\phi^{\lambda_{r,p}}_{r,p}(u)du.
\end{aligned}$$

□

***Proposition* 4.2.** *The law-invariant functional EGS is translation invariant, positively homogeneous, and co-monotonically additive.*

Proof. The translation invariance of *EGS* is easily verifiable. The co-monotonic additiv-

ity and the positive homogeneity arise immediately from *Proposition* 4.1.
This completes the proof of Proposition 4.2. $\square$

Theorem 4.1. *Let $p \in (0,1)$ and $\lambda_{r,p} \in [0,\infty]$, the following statements are equivalent:*

(i) $EGS_{r,p}^{\lambda_{r,p}}$ is monotone,

(ii) $EGS_{r,p}^{\lambda_{r,p}}$ is sub-additive,

(iii) $EGS_{r,p}^{\lambda_{r,p}}$ is SSD-monotone,

(iv) $EGS_{r,p}^{\lambda_{r,p}}$ is a coherent risk measure,

(v) $\lambda_{r,p} \in [0\,,\,1/(2(r-1)(1-p)^{r-2})]$.

The following lemma will be used in the proof of the Theorem 4.1, we refer to Furman et al. (2017) for a demonstration.

Lemma 4.1. *For $\phi \in L^{\infty}([0,1])$, let the functional $\rho_\phi : L^1 \to \mathbb{R}$ be defined by:*

$$\rho_\phi(X) = \int_0^1 F_X^{-1}(u)\phi(u)du,$$

The following statements hold:
(a) ρ_ϕ is monotone if and only if $\phi \geq 0$ on $[0,1]$,
(b) ρ_ϕ is sub-additivity if and only if ϕ is non-decreasing on $[0,1]$.

Proof. (Theorem 4.1)

$(i) \Leftrightarrow (ii) \Leftrightarrow (v)$
Note that $\phi_{r,p}^{\lambda_{r,p}}$ is an increasing function on $[0,1]$, therefore $\phi_{r,p}^{\lambda_{r,p}}(u)$ is non-negative if and only if $\phi_{r,p}^{\lambda_{r,p}}(p) \geq 0$. Thus, $\phi_{r,p}^{\lambda_{r,p}}$ is non-negative if and only if $\lambda_{r,p} \in [0\,,\,1/(2(r-1)(1-p)^{r-2})]$. Hence, Lemma 4.1 implies that statements (i), (ii) and (v) are equivalent.

$(iv) \Leftrightarrow (i) + (ii)$
Since the functional EGS is translation invariant and positively homogeneous the coherent of EGS comes directely.

$(iv) \Rightarrow (iii) \Rightarrow (i)$
Corollary 4.65 in Föllmer and Schied (2011) assures $(iv) \Rightarrow (iii)$ which in turn implies (i).

This proves that all statements $(i) - (v)$ are equivalent, and finishes the entire proof of Theorem 4.1. $\square$

In the following section we give an interpretation of the previous result, especially we are going to comment the statement $\lambda_{r,p} \in [0\,,\, 1/(2(r-1)(1-p)^{r-2})]$.

5 Interpretation

For a prudence level $p \in (0,1)$ the new family of risk measures is given by equation (4.1):

$$EGS_{r,p}^{\lambda_{r,p}}(X) = ES_p(X) + \lambda_{r,p}\, TEGini_{r,p}(X), \quad r > 1, X \in L^1$$

From the Theorem 4.1 the functional EGS is coherent if and only if the loading parameter $\lambda_{r,p} \in [0\,,\, 1/(2(r-1)(1-p)^{r-2})]$.

We set:

$$B_p(r) = \frac{1}{2(r-1)(1-p)^{r-2}}, \quad r > 1 \tag{5.1}$$

Hence, $B_p(r)$ is a *threshold* that $\lambda_{r,p}$ shall not exceed in order to keep the Extended Gini Shortfall coherent. Thereafter, we are going to describe how $B_p(r)$ behaves with respect to the risk aversion degree r.

$$B_p'(r) = \frac{-1}{2} \frac{(1-p)^{r-2}[1 + (r-1)\ln(1-p)]}{[(r-1)(1-p)^{r-2}]^2}, \quad r > 1 \tag{5.2}$$

The sign of $B_p'(r)$ depends on the sign of:

$$C_p(r) = 1 + (r-1)\ln(1-p) \tag{5.3}$$

the decreasing function C_p maps $(1, \infty)$ to $(-\infty, 1)$, then there exists a unique *critical value* r_0 such as, B_p decreases over $(1, r_0]$ and increases on (r_0, ∞).

The value of r_0 is given by $C_p(r_0) = 0$, thus :

$$r_0 = 1 - \frac{1}{\ln(1-p)}. \tag{5.4}$$

Therefore, over the interval $(1, r_0]$ more the decision-maker is risk averse smallest the threshold $B_p(r)$ is. On the other hand, over (r_0, ∞) more the decision-maker has risk aversion the greatest $B_p(r)$ is.

***Remark* 5.1.** *Since p reflects the prudence level, which is usually close to 1 in practice, then r_0 (equation(5.4)) is in general close to 1.*

In the context of the Extended Gini model, we consider the risk aversion degree $r \geq 2$. Therefore, our approach deals with an increasing functional B_p.
That takes sense, since the EGS is invoicing risk according to the decision-maker's attitude towards risk, thus more the individual is risk averse more he is willing to pay for *hedging*.

For instance, an insurer using the Extended Gini Shortfall must take into account the constraint $\lambda_{r,p} \in [0\,,\, 1/(2(r-1)(1-p)^{r-2})]$, when calculating $\lambda_{r,p}$, to keep the measure of risk reasonable.

6 Extended Gini Shortfall for usual distributions

A *location-scale family* is a family of probability distributions parameterized by a location parameter and a non-negative scale parameter.
Suppose that Z is a fixed rv taking values in $\mathbb{R}$. For $\alpha \in \mathbb{R}$ and $\beta \in (0,\infty)$, let $X = \alpha + \beta Z$. The two-parameter family of distributions associated with X is called the *location-scale family* associated with the given distribution of Z; α is called the *location parameter* and β the scale parameter.

The standard form of any distribution is the form whose location and scale parameters are 0 and 1, respectively.
In this section, we restrain our attention into standardized rv's. The general case follows immediately from:

***Proposition* 6.1.** *When $X = \alpha + \beta Z$ for $\alpha \in \mathbb{R}$ and $\beta \in (0,\infty)$, then for every $p \in (0,1)$ we have:*

$$ES_p(X) = \alpha + \beta\, ES_p(Z), \tag{6.1}$$

and

$$TEGini_{r,p}(X) = \beta\, TEGini_{r,p}(Z). \tag{6.2}$$

Proof. Equation (6.3) is trivial. From equation (3.11) we have:

$$\begin{aligned}
TEGini_{r,p}(X) &= TEGini_{r,p}(\alpha + \beta Z) \\
&= \frac{2}{(1-p)^2} \int_p^1 F_X^{-1}(u)[g_r(u) + (1-p)^{r-1}]du \\
&= \frac{2}{(1-p)^2} \int_p^1 (\alpha + \beta F_Z^{-1}(u))[g_r(u) + (1-p)^{r-1}]du \\
&= \frac{2}{(1-p)^2} \int_p^1 \alpha[g_r(u) + (1-p)^{r-1}]du + \frac{2\beta}{(1-p)^2} \int_p^1 F_Z^{-1}(u)[g_r(u) + (1-p)^{r-1}]du \\
&= 0 + \beta\, TEGini_{r,p}(Z) \\
&= \beta\, TEGini_{r,p}(Z).
\end{aligned}$$

□

In what follows, we start with the general elliptical family and then specialize the obtained result to the normal distribution case.

6.1 Elliptical risks

We recall that X is an *elliptical distribution* if $X \stackrel{d}{=} \alpha + \beta Z$, where Z is a *spherical distribution.*
Let Z be a spherical rv with *characteristic generator* $\psi : [0, \infty) \to \mathbb{R}$; succinctly $Z \sim S(\psi)$.
When Z has a probability density function (pdf), then there is a *density generator* $g : [0, \infty) \to [0, \infty)$ such that $\int_0^\infty z^{-1/2} g(z)dz < \infty$, we succinctly write $Z \sim S(g)$. We can express the pdf $f : \mathbb{R} \to [0, \infty]$ of Z by:

$$f(z) = c\, g(z^2/2), \tag{6.3}$$

where $c > 0$ is the *normalising constant.* The mean $\mathbb{E}[Z]$ is finite when:

$$\int_0^\infty g(z)dz < \infty \tag{6.4}$$

in which case we have $\mathbb{E}[Z] = 0$ because the pdf f is symmetric around 0. Under condition (6.6), we define the function $\overline{G} : [0, \infty) \to [0, \infty)$ by:

$$\overline{G}(y) = c \int_y^\infty g(x)dx. \tag{6.5}$$

called the *tail generator* of Z.

Theorem 6.1. *When $Z \sim S(g)$ and $\mathbb{E}[Z]$ is finite, then for every $p \in (0,1)$ we have:*

$$ES_p(Z) = \frac{\overline{G}(z_p^2/2)}{1-p} \tag{6.6}$$

and

$$TEGini_{r,p}(Z) = \frac{2r(r-1)}{1-p}\mathbb{E}[(1-F_Z(Z))^{r-2}\overline{G}(Z^2/2)/Z > z_p] + 2[1-r(1-p)^{r-2}]ES_p(Z). \tag{6.7}$$

Proof. Part 1:

$$\begin{aligned} ES_p(Z) &= \mathbb{E}[Z/Z > z_p] \\ &= \frac{1}{1-p}\int_{z_p}^{\infty} zf(z)dz \\ &= \frac{c}{1-p}\int_{z_p}^{\infty} zg(z^2/2)dz \\ &= \frac{c}{1-p}\int_{z_p^2/2}^{\infty} g(x)dx \\ &= \frac{\overline{G}(z_p^2/2)}{1-p}. \end{aligned}$$

Part 2:

$$TEGini_{r,p}(Z) = \frac{-2r}{1-p}[\mathbb{E}[Z(1-F_Z(Z))^{r-1}/Z > z_p] - \mathbb{E}[Z/Z > z_p]\mathbb{E}[(1-F_Z(Z))^{r-1}/Z > z_p]]$$

$$\begin{aligned} \mathbb{E}[Z(1-F_Z(Z))^{r-1}/Z > z_p] &= \frac{1}{1-p}\mathbb{E}[Z(1-F_Z(Z))^{r-1}\,\mathbb{1}_{\{Z>z_p\}}] \\ &= \frac{1}{1-p}\int_{z_p}^{\infty} z(1-F_Z(z))^{r-1}f(z)dz \end{aligned}$$

Note that $zf(z)dz = -d\overline{G}(z^2/2)$ and from *Part 1*: $\overline{G}(z^2/2) = (1-p)ES_p(Z)$;
an integration by parts leads to:

$$\begin{aligned} \mathbb{E}[Z(1-F_Z(Z))^{r-1}/Z > z_p] &= \frac{1}{1-p}(1-p)^{r-1}\overline{G}(z^2/2) - (r-1)\mathbb{E}[(1-F_Z(Z))^{r-2}/Z > z_p] \\ &= (1-p)^{r-1}ES_p(Z) - (r-1)\mathbb{E}[(1-F_Z(Z))^{r-2}/Z > z_p]. \end{aligned}$$

$$\mathbb{E}[Z/Z > z_p] = ES_p(Z).$$

Finally from Lemma 3.1, we have:

$$\mathbb{E}[(1-F_Z(Z))^{r-1}/Z > z_p] = \frac{(1-p)^{r-1}}{r}$$

This completes the proof of Theorem 6.1. $\square$

Note that $Var(Z)$ is finite whenever $\int_0^\infty z^{1/2} g(z)dz < \infty$, in which case $Var(Z)$ is equal to $\int_{-\infty}^{\infty} \overline{G}(z^2/2)dz$. Hence, $f^*(z) = \overline{G}(z_p^2/2)/Var(Z)$ is a pdf.
Equation 6.9 becomes :

$$TEGini_{r,p}(Z) = \frac{2r(r-1)}{1-p} Var(Z)\mathbb{E}[(1-F_Z(Z))^{r-2} f^*(z)/Z > z_p] + 2[1-r(1-p)^{r-2}]ES_p(Z) \tag{6.8}$$

6.2 Normal risks

In this subsection, we deal with the standard normal rv $Z \sim \mathcal{N}(0,1)$ whose cdf we denote by ϕ.

***Corollary* 6.1.** *For $Z \sim \mathcal{N}(0,1)$ and every $p \in (0,1)$, we have:*

$$ES_p(Z) = \frac{\phi'(z_p)}{1-p} \tag{6.9}$$

and

$$TEGini_{r,p}(Z) = \frac{2r(r-1)}{1-p} E[(1-\phi(Z))^{r-2}\phi'(z)/Z > z_p] + 2[1-r(1-p)^{r-2}]ES_p(Z) \tag{6.10}$$

Proof. The standard normal is a spherical distribution with density generator $g(z) = exp(-z)$, $c = 1/\sqrt{2\pi}$ and $\overline{G}(z^2/2) = \phi'(z)$.
Equation (6.11) follows immediately from equation (6.8) Theorem 6.1.
To establish the second part, we use equation (6.10) with $Var(Z) = 1$. $\square$

7 Conclusion

In this paper, we have defined a generalized version of the Gini-type measures of risk and variability (introduced in Furman et al. (2017)), to consider the psychological dimension of the decision act. This risk measures family, called the Extended Gini Shortfall (EGS), captures the notion of variability, satisfies the co-monotonic additivity property, and it is coherent under a necessary and sufficient condition on the loading parameter $\lambda_{r,p}$. The consideration of the decision-maker risk aversion, joined to these properties, is exactly what we seek when searching for a suitable measure of risk.

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