ARFIMAX and ARFIMAX-TARCH Realized Volatility Modeling

Degiannakis, Stavros

Department of Statistics, Athens University of Economics and Business

2008

ARFIMAX and ARFIMAX-TARCH Realized Volatility Modeling

Stavros Degiannakis
Department of Statistics, Athens University of Economics and Business

Abstract

ARFIMAX models are applied in estimating the intra-day realized volatility of the CAC40 and DAX30 indices. Volatility clustering and asymmetry characterize the logarithmic realized volatility of both indices. ARFIMAX model with time-varying conditional heteroscedasticity is the best performing specification and, at least in the case of DAX30, provides statistically superior next trading day's realized volatility forecasts.

Keywords: ARFIMAX, Realized Volatility, TARCH, Volatility Forecasting.
JEL: C22, C32, C53, G15.

1. Introduction

Andersen and Bollerslev (1998) first stated that the volatility estimates based on intra-day returns are more accurate than those based one daily data and introduced the *realized volatility*. The concept of the realized volatility is based on the *integrated volatility*. The integrated volatility, $\sigma_t^{2(IV)}$, aggregated over the time interval $(t-1,t)$:

$$\sigma_t^{2(IV)} = \int_{t-1}^{t} \sigma^2(x)dx, \tag{1}$$

is a latent variable which is not observable. Integrated volatility's volatility named *integrated quarticity* is $\sigma_t^{2(IQ)} = \int_{t-1}^{t} 2\sigma^4(x)dx$. The integrated volatility is asymptotically normally distributed:

$$\frac{\sqrt{m}\left(\sigma_t^{2(IV)} - \int_{t-1}^{t}\sigma^2(x)dx\right)}{\sqrt{\int_{t-1}^{t} 2\sigma^4(x)dx}} \xrightarrow{d} N(0,1).^{1} \tag{2}$$

The volatility at a lower frequency is computed using data available at a higher frequency. Thus, the integrated volatility over the time interval $(t-1,t)$ can be consistently estimated by the trading day's t realized volatility, $\sigma_t^{2(RV)} = \sum_{j=1}^{m-1}\left(\log\left(P_{(j+1/m),t}\right) - \log\left(P_{(j/m),t}\right)\right)^2$, which is defined as the sum of squared log-returns observed over m intra-day time intervals, for $\left\{P_{(m),t}\right\}_{j=1}^{m}$ denoting the asset prices over m intra-day time intervals of day t. The realized volatility converges in probability to the integrated volatility as $m \to \infty$, or $p\lim_{m\to\infty}\left(\sum_{j=1}^{m-1}\left(\log\left(P_{(j+1/m),t}\right) - \log\left(P_{(j/m),t}\right)\right)^2\right) = \sigma_t^{2(IV)}$.[2]

Fractionally integrated autoregressive moving average with exogenous variables (ARFIMAX) models, introduced by Granger (1980), were proposed to model the long memory property of the realized volatility. Ebens (1999) proposed the application of ARFIMAX models in realized volatility modelling and they, subsequently, applied by Bollerslev and Wright (2001), Giot and Laurent (2004), Koopman et al. (2005) and Angelidis and Degiannakis (2009) among others. As concerns the point forecasts of volatility, the findings provide evidence in favour of

[1] For details about $\sigma_t^{2(IV)}$ and $\sigma_t^{2(IQ)}$ see Barndorff-Nielsen and Shephard (2005) and references therein.

[2] See chapter 1.5 in Karatzas and Shreve (1988) and Barndorff-Nielsen and Shephard (2005).

realized volatility ARFIMAX models rather than autoregressive conditional heteroskedasticity (ARCH) framework of modeling daily log-returns. Corsi et al. (2005) noted that the volatility of S&P500 index futures volatility also exhibits time-variation and proposed the estimation of an ARFIMAX model that accounts for conditional heteroskedasticity.

In the present study, an ARFIMAX model is extended to account for volatility clustering as well as for the asymmetric relation between realized volatility and volatility of realized volatility. The unobservable term of an ARFIMAX specification is modelled as an asymmetric ARCH process. The new model framework, named ARFIMAX-TARCH model, is applied for CAC40 and DAX30 stock indices. In-sample as well as out-of-sample analysis provide statistically significant evidence in favour of the new model specification. Thus, in risk management applications, the volatility's conditional volatility should be taken into consideration.

The manuscript is divided in six sections. In section 2, descriptive information of the CAC40 and DAX30 realized volatility measures is provided. Section 3 lays out the ARFIMAX and ARFIMAX-TARCH specifications. The in-sample and out-of-sample model evaluation is investigated in sections 4 and 5, respectively, and section 6 concludes.

2. CAC40 and DAX30 Realized Volatility Properties

Tick by tick linearly interpolated prices of the CAC40 and DAX30 indices were obtained from Olsen and Associates for the period of July 1995 to December 2003. The sampling frequency should be as high as the market microstructure features do not induce bias to volatility estimator. In order to avoid market microstructure frictions without lessening the accuracy of the continuous record asymptotics, in most of the studies, such as Andersen and Bollerslev (1998), Andersen et al. (1999, 2001b) and Kayahan et al. (2002), a sampling frequency of five minutes is used.

The realized intraday volatility at day t is computed as in Martens (2002) and Koopman et al. (2005):

$$\sigma_t^{2(RV)} = \frac{\hat{\sigma}_{oc}^2 + \hat{\sigma}_{co}^2}{\hat{\sigma}_{oc}^2} \sum_{j=1}^{m-1} \left(\log\left(P_{(j+1/m),t}\right) - \log\left(P_{(j/m),t}\right) \right)^2 , \qquad (3)$$

where $\{P_{(m),t}\}_{j=1}^{m}$ are the five-minute linearly interpolated prices at trading day t with $m = 103\ (m = 134)$ observations per day for the CAC40 (DAX30) index, $\hat{\sigma}_{oc}^2 = T^{-1} \sum_{t=1}^{T} \left(\log\left(P_{(1),t}\right) - \log\left(P_{(1/m),t}\right) \right)^2$ is the open to close sample variance and

$\hat{\sigma}_{co}^2 = T^{-1}\sum_{t=1}^{T}\left(\log\left(P_{(1/m),t}\right)-\log\left(P_{(1),t-1}\right)\right)^2$ is the close to open sample variance. The factor $\sigma_{oc}^{-2}\left(\sigma_{oc}^2+\sigma_{co}^2\right)$ accounts for changes in the asset prices during the hours that the stock market is closed without inserting the noisy effect of daily returns.

[Insert Table 1 about here]

Table 1 lists descriptive information of the logarithmic realized volatility, $\log\left(\sigma_t^{2(RV)}\right)$. Lilliefors' (1967) and Anderson and Darling's (1954) statistics reject the null hypothesis that the logarithmic realized variance is normally distributed. Literature provides empirical evidence that the distribution of the daily returns, $y_t = \log\left(P_t/P_{t-1}\right)$, standardized by the square root of the realized volatility, $\left\{y_t/\sigma_t^{(RV)}\right\}_{t=1}^{T}$, is very close to the normal one, but, it is statistically distinguishable from the normal distribution[3]. The histograms of $\left\{y_t/\sigma_t^{(RV)}\right\}_{t=1}^{T}$ are plotted in Figure 1. According to Lilliefors and Aderson-Darling normality tests in Table 3, $\left\{y_t/\sigma_t^{(RV)}\right\}_{t=1}^{T}$ is normally distributed, in the case of DAX30, at 1% level of significance. In the case of CAC40, the null hypothesis, that the empirical distribution of $\left\{y_t/\sigma_t^{(RV)}\right\}_{t=1}^{T}$ is the normal one, is rejected at any reasonable level of significance.

Figure 2 presents the square root of realized volatility. On the right-hand axis, the daily index prices are plotted to present the asymmetric relationship between index log-returns and changes in realized volatility.

[Insert Figure 1 about here]

[Insert Figure 2 about here]

3. ARFIMAX and ARFIMAX-TARCH Intra-Day Volatility Models

An ARFIMAX(k,d,l) model for the realized volatility, $\sigma_t^{2(RV)}$, can be presented as:

$$\begin{gathered}\left(1-c(L)\right)(1-L)^d\left(\log\left(\sigma_t^{2(RV)}\right)-x_t'\beta\right)=\left(1+\delta(L)\right)\varepsilon_t, \\ \varepsilon_t \sim N\left(0,\sigma_\varepsilon^2\right),\end{gathered} \tag{4}$$

[3] See for example Andersen et al. (2001a).

where $c(L)=\sum_{i=1}^{k} c_i L^i$, $\delta(L)=\sum_{i=1}^{l} \delta_i L^i$, L is the backward-shift operator, $(1-L)^d=\sum_{j=0}^{\infty}\frac{\Gamma(j-d)L^j}{\Gamma(j+1)\Gamma(-d)}=\sum_{j=0}^{\infty}L^j\prod_{k=0}^{j}\frac{k-1-d}{k}$, $\Gamma(.)$ is the Gamma function, x_t is a vector of explanatory variables and β is a vector of unknown parameters.

As, we want to investigate whether the volatility of realized volatility exhibits time-variation, the ARFIMAX model is extended to account for conditional heteroskedasticity[4]. An ARFIMAX model with time varying conditional variance for the realized volatility, $\sigma_t^{2(RV)}$, can be presented as:

$$
\begin{aligned}
(1-c(L))(1-L)^d\left(\log\left(\sigma_t^{2(RV)}\right)-x_t'\beta\right)&=(1+\delta(L))\varepsilon_t \\
\varepsilon_t &= \sigma_t z_t \\
\sigma_t^2 &= g(\theta \mid I_{t-1}) \\
z_t &\sim N(0,1),
\end{aligned}
\tag{5}
$$

where I_{t-1} is the information set available in time $t-1$, σ_t^2 is a time-varying, positive and measurable function of I_{t-1}, θ is a vector of unknown parameters and $g(.)$ is a functional form of I_{t-1}. In our case, an asymmetric conditional variance specification is considered to account for asymmetric relationship between realized volatility and its volatility. Therefore an ARFIMAX(k,d,l)-TARCH(p,q) model for the realized volatility is proposed:

$$
\begin{aligned}
(1-c(L))(1-L)^d\left(\log\left(\sigma_t^{2(RV)}\right)-x_t'\beta\right)&=(1+\delta(L))\varepsilon_t, \\
\varepsilon_t &= \sigma_t z_t, \\
\sigma_t^2 &= a_0+\sum_{i=1}^{q}a_i\varepsilon_{t-i}^2+\sum_{i=1}^{q}\left(\gamma_i I(\varepsilon_{t-i}<0)\varepsilon_{t-i}^2\right)+\sum_{i=1}^{p}b_i\sigma_{t-i}^2,
\end{aligned}
\tag{6}
$$

where $z_t \sim N(0,1)$. $I(.)$ denotes the indicator function, i.e. $I(\varepsilon_{t-i}<0)=1$ if $\varepsilon_{t-i}<0$, and $I(\varepsilon_{t-i}<0)=0$, otherwise. The Threshold-ARCH(p,q), or TARCH(p,q), specification, introduced by Glosten et al. (1993), allows positive, $(\varepsilon_{t-i}>0)$, and negative, $(\varepsilon_{t-i}<0)$, innovations to have differential effects on the volatility of the realized volatility.

4. In-sample Evaluation

The ARFIMAX(k,d,l) model is considered in the form:

[4] Baillie et al. (1996) firstly proposed an ARFIMAX–ARCH model to analyze monthly consumer price index inflation. Hauser and Kunst (1998) also applied ARFIMAX–ARCH model to analyze monthly Swiss one-month Euromarket interest rates.

$$(1-c(L))(1-L)^d\left(\log\left(\sigma_t^{2(RV)}\right)-w_0-w_1y_{t-1}-w_2I(y_{t-1}>0)y_{t-1}\right)=(1+\delta(L))\varepsilon_t,\quad \varepsilon_t\sim N\left(0,\sigma_\varepsilon^2\right), \tag{7}$$

where $I(y_t>0)=1$ when $y_t>0$ and $I(y_t>0)=0$ otherwise. Parameter w_2 models the asymmetric relationship between realized volatility and previous trading day's log-return[5]. Since, $\varepsilon_t\sim N\left(0,\sigma_\varepsilon^2\right)$, the $\exp(\varepsilon_t)$ is log-normally distributed, therefore the unbiased in-sample realized volatility is estimated as:

$$\hat{\sigma}_t^{2(RV^*)}=\exp\left(\log\hat{\sigma}_t^{2(RV)}+0.5\sigma_\varepsilon^2\right). \tag{8}$$

In order to determine the optimal lag order, model (7) is estimated for $k=0,1,2$ and $l=0,1,2$. The optimal lag order is chosen due to the minimization of Schwarz's (1978) Bayesian criterion (SBC). The SBC, for model i, is computed as:

$$SBC^{(i)}=-2T^{-1}L_T\left(\{y_t\};\hat{\psi}^{(T)}\right)+\breve{\psi}T^{-1}\log(T)\ , \tag{9}$$

where $\psi=\left(c_1,...,c_k,d,w_0,w_1,w_2,\delta_1,...,\delta_l\right)'$, $L_T(.)$ is the maximized value of the log-likelihood function, $\hat{\psi}^{(T)}$ is the maximum likelihood estimator of ψ based on a sample of size T and $\breve{\psi}$ denotes the dimension of ψ. The optimal lag orders are $\text{ARFIMAX}(2,d,2)$ and $\text{ARFIMAX}(0,d,1)$ for CAC40 and DAX30, respectively. The models are estimated in Doornik and Ooms' (2006) ARFIMA 1.04 package of Ox Metrics.

The $\text{ARFIMAX}(k,d,l)$-$\text{TARCH}(p,q)$ model is also considered in the form:

$$\begin{gathered}(1-c(L))(1-L)^d\left(\log\left(\sigma_t^{2(RV)}\right)-w_0-w_1y_{t-1}-w_2I(y_{t-1}>0)y_{t-1}\right)=(1+\delta(L))\varepsilon_t,\\ \varepsilon_t=\sigma_t z_t,\\ \sigma_t^2=a_0+\sum_{i=1}^{q}a_i\varepsilon_{t-i}^2+\sum_{i=1}^{q}\left(\gamma_iI(\varepsilon_{t-i}<0)\varepsilon_{t-i}^2\right)+\sum_{i=1}^{p}b_i\sigma_{t-i}^2\ ,\\ z_t\sim N(0,1).\end{gathered} \tag{10}$$

Model (10) is estimated for $k=0,1,2$, $l=0,1,2$, $p=0,1,2$ and $q=1,2$. The SBC is computed according to equation (9) for $\psi=\left(c_1,...,c_k,d,w_0,w_1,w_2,\delta_1,...,\delta_l,a_0,a_1,...,a_q,\gamma_1,...,\gamma_q,b_1,...,b_p\right)'$. The $\text{ARFIMAX}(1,d,1)$-$\text{TARCH}(1,1)$ model achieves the minimum value of the SBC criterion for both indices. The unbiased in-sample realized volatility is also estimated according to (8). The models are estimated in Laurent and Peters' (2006) G@RCH 4.04 package for Ox Metrics.

[5] As presented in Figure 2.

The estimated parameters of the models that minimize the SBC criterion are reported in Table 2. The values of the parameters inform us that i) the asymmetric relation between past return and realized volatility is statistically significant in all cases (coefficient w_2), ii) the fractional integration parameter is statistically insignificant only for Paris stock market in the $\text{ARFIMAX}(2,d,2)$ model (coefficient d), iii) all the conditional variance parameters are statistically significant (coefficients a_1 and b_1), and iv) the asymmetric relationship between realized volatility and its volatility is also statistically significant (coefficient γ_1). The significance of the conditional variance parameters justifies the modeling of realized volatility's conditional variance[6].

[Insert Table 2 about here]

Figure 3 plots the histograms of logarithmic realized volatility, $\left\{\log\left(\sigma_t^{2(RV)}\right)\right\}_{t=1}^{T}$ and logarithmic realized volatility standardized by its standard deviation estimated by the $\text{ARFIMAX}(1,d,1)\text{-TARCH}(1,1)$ model, $\left\{\log\left(\sigma_t^{2(RV)}\right)/\hat{\sigma}_t\right\}_{t=1}^{T}$. The $\left\{\log\left(\sigma_t^{2(RV)}\right)/\hat{\sigma}_t\right\}_{t=1}^{T}$ is much closer to the normal distribution than the $\left\{\log\left(\sigma_t^{2(RV)}\right)\right\}_{t=1}^{T}$ is, but it is statistically distinguishable from it.

Figure 4 depicts the density graph of daily returns standardized by the in-sample estimated realized standard deviation, $\left\{y_t/\hat{\sigma}_t^{(RV^*)}\right\}_{t=1}^{T}$, whereas normality tests are listed in Table 3. Based on Lilliefors statistic, the hypothesis that $\left\{y_t/\hat{\sigma}_t^{(RV^*)}\right\}_{t=1}^{T}$ is normally distributed can not be rejected only in the case of DAX30 at 1% level of significance. Anderson-Darling statistic rejects the normality in any case.

[Insert Figure 3 about here]

[Insert Figure 4 about here]

[Insert Table 3 about here]

The average squared distance between realized volatility and its in-sample estimation is measured to evaluate the accuracy of the models in estimating the realized volatility:

$$MSE = T^{-1}\sum_{t=1}^{T}\left(\sigma_t^{2(RV)} - \hat{\sigma}_t^{2(RV^*)}\right)^2 . \qquad (11)$$

[6] The standardized residuals obey the assumption of autocorrelation absence.

Hansen and Lunde (2006) have stated that the MSE loss function ensures the equivalence of the ranking of volatility models that is induced by the true volatility and its proxy. The values of the MSE loss functions are reported in Table 4. For both indices, the ARFIMAX$(1,d,1)$-TARCH$(1,1)$ model has the lowest value.

Hansen's (2005) superior predictive ability (SPA) hypothesis testing is used to investigate whether the model with the lowest MSE value provides statistically superior realized volatility estimates. Let i be the model with the lowest MSE value, $MSE_t^{(i^*)}$ is the value of the MSE function at time t of a competing model i^*, for $i^* = 1,...,M$ and $X_t^{(i,i^*)} = MSE_t^{(i)} - MSE_t^{(i^*)}$. The null hypothesis that $E\left(X_t^{(i,1)}, \ . \ . \ . \ , X_t^{(i,M)}\right)' \leq 0$ is tested with the statistic $T^{SPA} = \max\limits_{i^*=1,...M} \frac{\sqrt{M}\,\overline{X}_{i^*}}{\sqrt{Var\left(\sqrt{M}\,\overline{X}_{i^*}\right)}}$, where $\overline{X}_{i^*} = T^{-1}\sum_{t=1}^{T} X_t^{(i,i^*)}$.[7]

According to Table 4, the ARFIMAX-TARCH specification is superior to the ARFIMAX one. The null hypothesis, that the ARFIMAX$(1,d,1)$-TARCH$(1,1)$ model is not outperformed by its competing model, is not rejected.

[Insert Table 4 about here]

5. *Out-of-sample Evaluation*

In the present section, the ability of the ARFIMAX(k,d,l) and ARFIMAX(k,d,l)-TARCH(p,q) models to predict next trading day's volatility is investigated. In total, for $k = 0,1,2$, $l = 0,1,2$, $p = 0,1,2$ and $q = 1,2$ lag orders, 56 model specifications are considered. Based on a rolling sample of $\breve{T} = 1000$ trading days, each model's parameter vector ψ is re-estimated every trading day and $\tilde{T}$ one-day-ahead volatility forecasts are computed[8], for $\tilde{T} = T - \breve{T}$. The one-day-ahead conditional standard deviation forecasts are computed as:

$$\sigma_{t+1|t}^{(RV^*)} = \sqrt{\exp\left(\log\left(\sigma_{t+1|t}^{2(RV)}\right) + 0.5\sigma_{\varepsilon}^{2(t)}\right)}. \qquad (12)$$

The distance between realized volatility and next day's predicted volatility is measured by the predicted mean squared error loss function:

---

[7] The estimation of $Var\left(\sqrt{M}\,\overline{X}_{i^*}\right)$ and the p-value of the T^{SPA} are obtained by using the bootstrap method of Politis and Romano (1994). Hansen (2005) provided a program for the computation of the SPA criterion, which is written in Ox Metrics package.

[8] $\tilde{T} = 1135$ for CAC40 and $\tilde{T} = 1136$ for DAX30.

$$PMSE = \tilde{T}^{-1} \sum_{t=1}^{\tilde{T}} \left(\sigma_{t+1}^{2(RV)} - \sigma_{t+1|t}^{2(RV^*)} \right)^2 , \tag{13}$$

and the SPA hypothesis test is applied. In the $\mathrm{ARFIMAX}(k,d,l)$ specification, the $PMSE$ is minimized for $k=1$ and $l=0$ in the CAC40 case, and for $k=l=2$ in the case of DAX30. In the $\mathrm{ARFIMAX}(k,d,l)$-$\mathrm{TARCH}(p,q)$ framework, the $PMSE$ is minimized for $k=1$, $l=2$, $p=2$, $q=1$ in the CAC40 case, and for $k=p=q=1$, $l=0$ in the case of DAX30. Table 5 presents the *p*-values of the SPA test for the null hypothesis that the ARFIMAX-TARCH model is not outperformed by the ARFIMAX one. In the case of the DAX30 the $\mathrm{ARFIMAX}(1,d,0)$-$\mathrm{TARCH}(1,1)$ model is superior to the $\mathrm{ARFIMAX}(2,d,2)$ one. As concerns the CAC40 index, the $\mathrm{ARFIMAX}(1,d,2)$-$\mathrm{TARCH}(2,1)$ model does not achieve the lowest value in the $PMSE$ loss function, but it is not statistically inferior to its competitor, the $\mathrm{ARFIMAX}(1,d,0)$ model.

[Insert Table 5 about here]

Figure 5 presents the returns scaled by the one-day-ahead realized standard deviation forecasts, $\left\{ y_t / \sigma_{t|t-1}^{(RV^*)} \right\}_{t=1}^{\tilde{T}}$. According to Table 3, the null hypothesis, that the daily returns standardized by the one-day-ahead realized standard deviation forecasts are normally distributed, in not rejected only for the $\mathrm{ARFIMAX}(2,d,2)$ model in the DAX30 case. Giot and Laurent (2004) noticed that for the CAC40[9] and SP500 indices, the $\left\{ y_t / \sigma_t^{(RV)} \right\}_{t=1}^{T}$ are normally distributed, but the $\left\{ y_t / \sigma_{t|t-1}^{(RV^*)} \right\}_{t=1}^{T}$ are not normally distributed. In the present dataset both series $\left\{ y_t / \sigma_t^{(RV)} \right\}_{t=1}^{T}$ and $\left\{ y_t / \sigma_{t|t-1}^{(RV^*)} \right\}_{t=1}^{T}$ are not normally distributed but they have an almost normal distribution as in Andersen et al. (2001a).

[Insert Figure 5 about here]

6. Conclusion

ARFIMAX and ARFIMAX-TARCH models were applied in estimating and forecasting the intra-day realized volatility of the CAC40 and DAX30 indices for the period of July 1995 to December 2003. ARFIMAX-TARCH model takes into consideration the dynamics of realized volatility's volatility. The integrated volatility, $\sigma_t^{2(IV)}$, is estimated by the realized volatility, $\sigma_t^{2(RV)}$, in equation (3), whereas the conditional variance of the logarithmic realized volatility,

[9] CAC40 was analyzed for the period 1995-1999 and fifteen-minute intraday prices were taken into account.

$\hat{\sigma}_t^2$, in equatrion (10), can be regarded as an estimation of integrated quarticity, $\sigma_t^{2(IQ)}$. The significance of the parameters of the TARCH specification provides evidence in favor of modeling the asymmetric relationship between realized volatility and its volatility.

The in-sample evaluation indicates that the ARFIMAX-TARCH specification clearly outperforms the ARFIMAX one. In the out-of-sample evaluation, the ARFIMAX-TARCH model is superior to the ARFIMAX one for the DAX30. In the case of the CAC40 index, the ARFIMAX-TARCH model is not statistically inferior to its competitor.

To sum up, from econometric point of view, the daily returns standardized by i) the realized standard deviation, ii) the in-sample estimated realized standard deviation, and iii) the one-day-ahead realized standard deviation forecasts are almost normally distributed but they are statistically distinguishable from the normal distribution. The logarithmic realized volatility standardized by its standard deviation is much closer to the normal distribution than the logarithmic realized volatility but also statistically distinguishable from it.

From economic point of view, in order to obtain more accurate stock index volatility estimations, it is necessary to treat ARFIMAX models for realized volatility, and ARCH modeling for volatility of realized volatility, simultaneously. Thus, when the volatility estimation is required in financial applications, such as risk management, option pricing and portfolio analysis, the time varying conditional heteroscedasticity of volatility should be taken into consideration.

References

1. **Andersen, T. and Bollerslev, T. (1998).** Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts. *International Economic Review*, 39, pp. 885-905.
2. **Andersen, T., Bollerslev, T. and Lange, S. (1999).** Forecasting Financial Market Volatility: Sample Frequency vis-à-vis Forecast Horizon. *Journal of Empirical Finance*, 6, pp. 457-477.
3. **Andersen, T., Bollerslev, T., Diebold, F.X. and Ebens, H. (2001a).** The Distribution of Realized Stock Return Volatility. *Journal of Financial Economics*, 61, pp. 43-76.
4. **Andersen, T., Bollerslev, T., Diebold, F.X. and Labys, P. (2001b).** The Distribution of Exchange Rate Volatility. *Journal of the American Statistical Association*, 96, pp. 42-55.
5. **Anderson, T.W. and Darling, D.A. (1954).** A Test of Goodness of Fit. *Journal of the American Statistical Association*, 49, pp. 765-769.

6. **Angelidis, T. and Degiannakis, S. (2009).** Volatility Forecasting: Intra-day vs. Inter-day Models. *Journal of International Financial Markets, Institutions and Money*, forthcoming.

7. **Baillie, R.T., Chung, C.F., and Tieslau, M.A. (1996).** Analysing Inflation by the Fractionally Integrated ARFIMA-GARCH Model. *Journal of Applied Econometrics*, 11, pp. 23–40.

8. **Barndorff-Nielsen, O.E. and Shephard, N. (2005).** How Accurate is the Asymptotic Approximation to the Distribution of Realised Volatility? in D. Andrews, J. Powell, P. Ruud, and J. Stock (Eds.) *Identification and Inference for Econometric Models*, Cambridge: Cambridge University Press.

9. **Bollerslev, T. and Wright, J.H. (2001).** Volatility Forecasting, High-Frequency Data and Frequency Domain Inference. *Review of Economics and Statistics*, 83, pp. 596-602.

10. **Corsi, F., Kretschmer, U., Mittnik, S. and Pigorsch, C. (2005).** The Volatility of Realised Volatility. Center for Financial Studies, Working Paper, 33.

11. **Doornik, J.A. and Ooms, M. (2006).** A Package for Estimating, Forecasting and Simulating Arfima Models: Arfima Package 1.04 for Ox. Nuffield College, Oxford, Working Paper.

12. **Ebens, H. (1999).** Realized Stock Volatility. Johns Hopkins University, Department of Economics, Working Paper, 420.

13. **Giot, P. and Laurent, S. (2004).** Modelling Daily Value-at-Risk Using Realized Volatility and ARCH Type Models. *Journal of Empirical Finance*, 11, pp. 379 – 398.

14. **Glosten, L., Jagannathan, R. and Runkle, D. (1993).** On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *Journal of Finance*, 48, pp. 1779–1801.

15. **Granger, C.W.J. (1980).** Long Memory Relationships and the Aggregation of Dynamic Models. *Journal of Econometrics*, 14, pp. 227-238.

16. **Hansen, P.R. (2005).** A Test for Superior Predictive Ability. *Journal of Business and Economic Statistics*, 23, pp. 365-380.

17. **Hansen, P.R. and Lunde, A. (2006).** Consistent Ranking of Volatility Models. *Journal of Econometrics*, 131, pp. 97-121.

18. **Hauser, M.A. and Kunst, R.M. (1998).** Fractionally Integrated Models With ARCH Errors: With an Application to the Swiss 1-Month Euromarket Interest Rate. *Review of Quantitative Finance and Accounting*, 10, pp. 95-113.

19. **Karatzas, I. and Shreve, S.E. (1988).** Brownian Motion and Stochastic Calculus. Springer Verlag.

20. **Kayahan, B., Saltoglu, T. and Stengos, T. (2002).** Intra-Day Features of Realized Volatility: Evidence from an Emerging Market. *International Journal of Business and Economics*, 1(1), pp. 17-24.

21. **Koopman, S.J., Jungbacker, B., and Hol, E. (2005).** Forecasting Daily Variability of the S&P100 Stock Index Using Historical, Realised and Implied Volatility Measurements. *Journal of Empirical Finance*, 12, pp. 445–475.

22. **Laurent S. and Peters, J.-P. (2006)**. G@RCH 4.2, Estimating and Forecasting ARCH Models, London: Timberlake Consultants Press.

23. **Lilliefors, H.W. (1967).** On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown. *Journal of the American Statistical Association*, 62, pp. 399-402.

24. **Martens, M. (2002).** Measuring and forecasting S&P500 index-futures volatility using high-frequency data. *Journal of Futures Markets*, 22, pp. 497–518.

25. **Politis, D.N. and Romano, J.P. (1994).** The Stationary Bootstrap. *Journal of the American Statistical Association*, 89, pp. 1303-1313.

26. **Schwarz, G. (1978)**. Estimating the Dimension of a Model. *Annals of Statistics*, 6, pp. 461-464.

Figures & Tables

Figure 1. Histograms of daily log-returns standardized by the realized standard deviation, or $\left\{ y_t / \sigma_t^{(RV)} \right\}_{t=1}^{T}$, where $y_t = \log\left(P_t / P_{t-1} \right)$ and $\sigma_t^{(RV)} = \sqrt{ \frac{\hat{\sigma}_{oc}^2 + \hat{\sigma}_{co}^2}{\hat{\sigma}_{oc}^2} \sum_{j=1}^{m-1} \left(\log\left(P_{(j+1/m),t} \right) - \log\left(P_{(j/m),t} \right) \right)^2 }$.

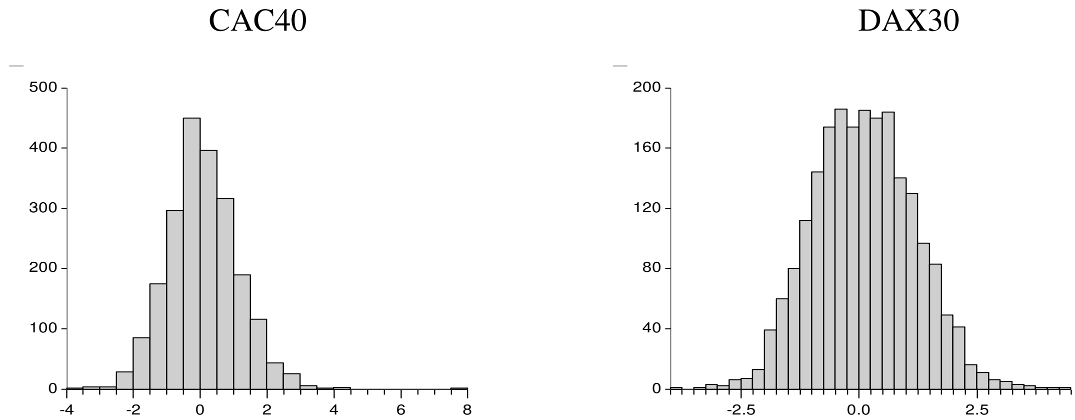


Figure 2. Realized standard deviation, $\left\{ \sigma_t^{(RV)} \right\}_{t=1}^{T}$, and index daily prices, $\left\{ P_t \right\}_{t=1}^{T}$.

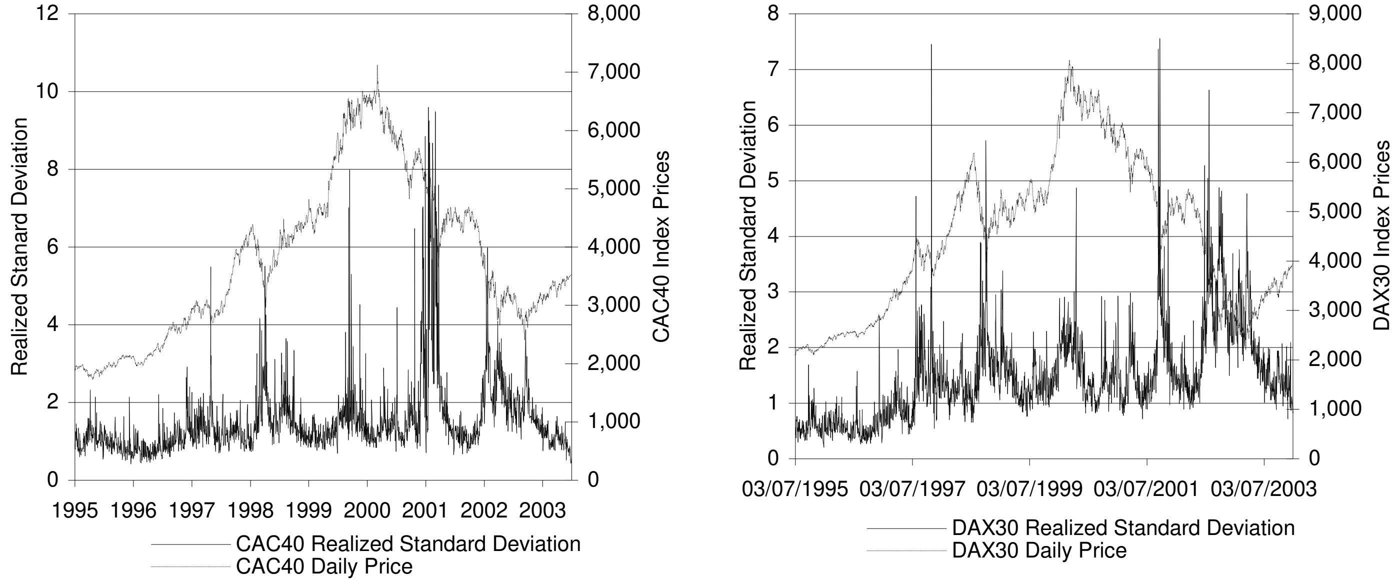
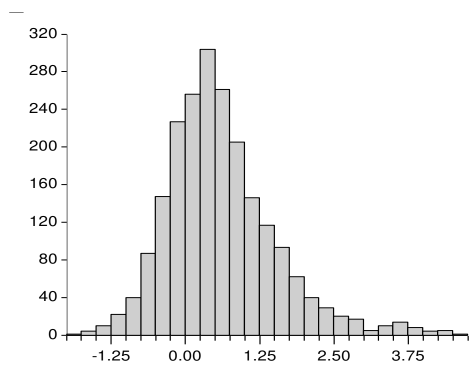
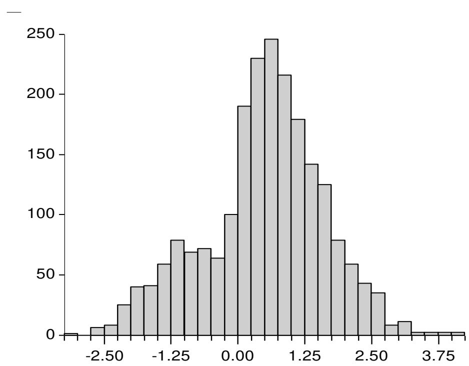


Figure 3. Histogram of logarithmic realized volatility, $\left\{\log\left(\sigma_t^{2(RV)}\right)\right\}_{t=1}^{T}$, and of logarithmic realized volatility standardized by its standard deviation, $\left\{\log\left(\sigma_t^{2(RV)}\right)/\hat{\sigma}_t\right\}_{t=1}^{T}$.

CAC40	DAX30

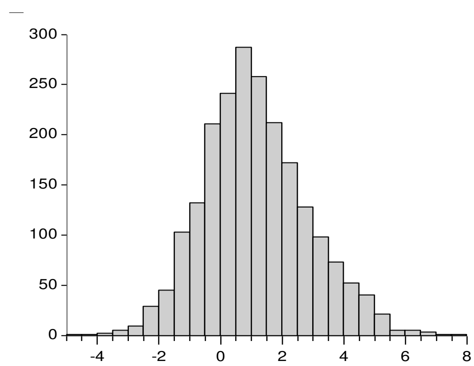
$\left\{\log\left(\sigma_t^{2(RV)}\right)\right\}_{t=1}^{T}$

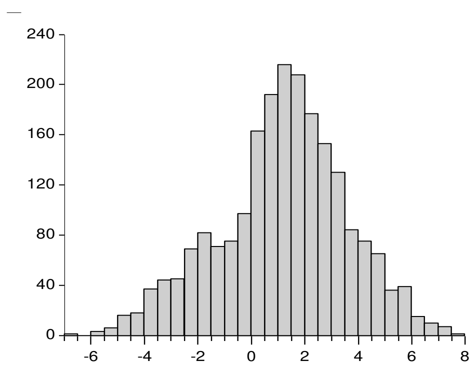




	CAC40		DAX30
Skewness	0.991222	Skewness	-0.297120
Kurtosis	4.859072	Kurtosis	3.077688
Anderson-Darling	23.3 [0.00]	Anderson-Darling	11.93 [0.00]

$\left\{\log\left(\sigma_t^{2(RV)}\right)/\hat{\sigma}_t\right\}_{t=1}^{T}$, where $\left\{\hat{\sigma}_t\right\}_{t=1}^{T}$ is estimated by the ARFIMAX$(1,d,1)$-TARCH$(1,1)$ model

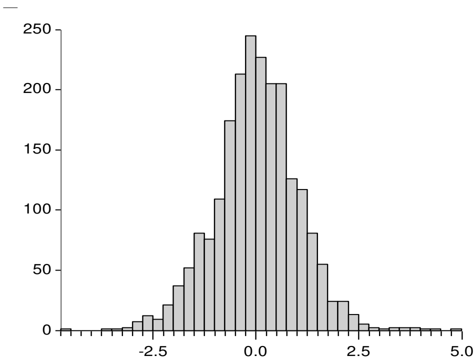


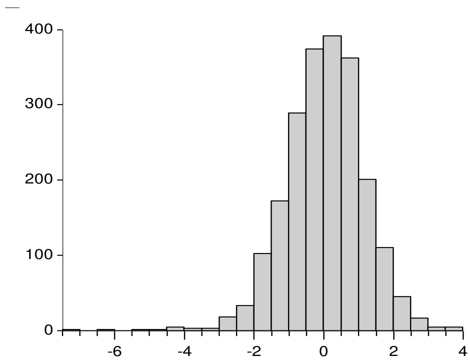


	CAC40		DAX30
Skewness	0.320103	Skewness	-0.254249
Kurtosis	3.184094	Kurtosis	2.951918
Anderson-Darling	5.04 [0.00]	Anderson-Darling	6.71 [0.00]

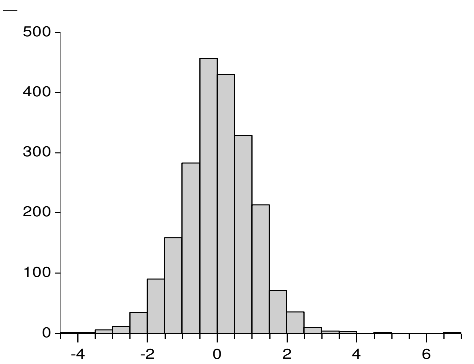
Figure 4. Histogram of daily log-returns standardized by the in-sample estimated realized standard deviation, $\left\{y_t / \hat{\sigma}_t^{(RV^*)}\right\}_{t=1}^{T}$.

CAC40	DAX30
$\text{ARFIMAX}(2,d,2)$	$\text{ARFIMAX}(0,d,1)$





CAC40	DAX30
$\text{ARFIMAX}(1,d,1)\text{-TARCH}(1,1)$	$\text{ARFIMAX}(1,d,1)\text{-TARCH}(1,1)$



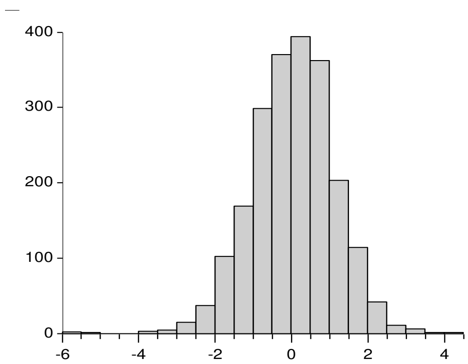
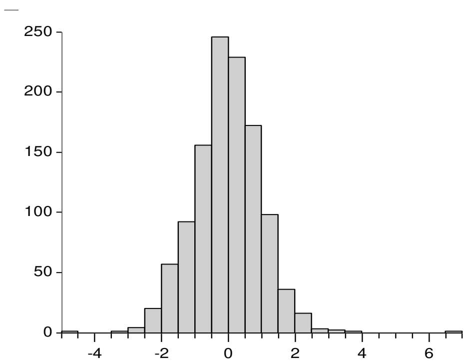
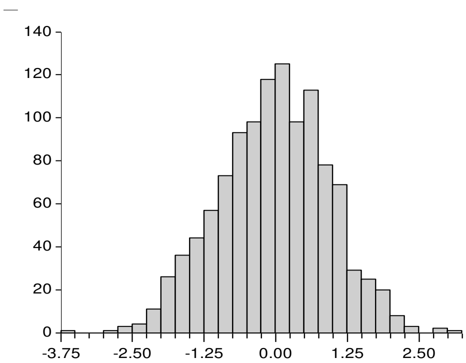


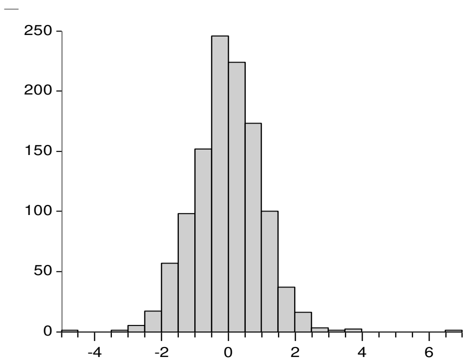
Figure 5. Histogram of daily log-returns standardized by the one-day-ahead realized standard deviation, $\left\{ y_t / \sigma_{t|t-1}^{(RV^*)} \right\}_{t=1}^{\tilde{T}}$.

CAC40	DAX30
$\text{ARFIMAX}(1,d,0)$	$\text{ARFIMAX}(2,d,2)$





$\text{ARFIMAX}(1,d,2)\text{-TARCH}(2,1)$	$\text{ARFIMAX}(1,d,0)\text{-TARCH}(1,1)$



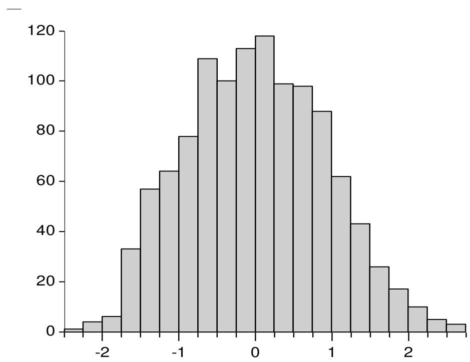


Table 1. Descriptive statistics of the logarithmic realized volatility, $\log\left(\sigma_t^{2(RV)}\right)$.

	CAC40		DAX30	
Mean	0.602		0.498	
Median	0.479		0.567	
Maximum	4.523		4.046	
Minimum	-1.751		-3.171	
Std. Dev.	0.923		1.117	
Skewness	0.9912		-0.2971	
Kurtosis	4.8590		3.0776	
Lilliefors	0.077	[0.00]	0.067	[0.00]
Anderson-Darling	23.3	[0.00]	11.93	[0.00]

P-values are displayed in squared brackets.

Table 2. Estimated parameters of the $\text{ARFIMAX}(k,d,l)$-$\text{TARCH}(p,q)$ model:

$$(1-c(L))(1-L)^d\left(\log\left(\sigma_t^{2(RV)}\right)-w_0-w_1y_{t-1}-w_2I(y_{t-1}>0)y_{t-1}\right)=(1+\delta(L))\varepsilon_t$$

$$\sigma_t^2=a_0+\sum_{i=1}^{q}a_i\varepsilon_{t-i}^2+\sum_{i=1}^{q}\left(\gamma_iI(\varepsilon_{t-i}<0)\varepsilon_{t-i}^2\right)+\sum_{i=1}^{p}b_i\sigma_{t-i}^2 .$$

	CAC40		DAX30	
	$\text{ARFIMAX}(2,d,2)$	$\text{ARFIMAX}(1,d,1)$-$\text{TARCH}(1,1)$	$\text{ARFIMAX}(0,d,1)$	$\text{ARFIMAX}(1,d,1)$-$\text{TARCH}(1,1)$
c_1	1.39680 [a] (0.3218)	0.199197 [a] (0.050252)	-	0.194364 [a] (0.045510)
c_2	-0.412986 (0.3041)	-	-	-
d	0.258789 (0.3203)	0.755740 [a] (0.068972)	0.497103 [a] (0.004017)	0.782399 [a] (0.061348)
w_0	0.416191 (0.3620)	0.040664 (0.25369)	0.198627 (2.975)	-1.331032 [a] (0.48103)
w_1	0.0229716 (0.01194)	0.021021 [b] (0.010327)	0.0350528 [a] (0.01099)	0.021232 [b] (0.0096266)
w_2	-0.118749 [a] (0.02080)	-0.104875 [a] (0.020196)	-0.128690 [a] (0.01885)	-0.098282 [a] (0.018332)
δ_1	-1.35808 [a] (0.1985)	-0.640628 [a] (0.071377)	-0.138449 [a] (0.02359)	-0.635783 [a] (0.059226)
δ_2	0.429437 [b] (0.1737)	-	-	-
a_0	-	0.005054 [b] (0.0020817)	-	0.010667 [a] (0.0039589)
a_1	-	0.058264 [a] (0.014591)	-	0.116056 [a] (0.034983)
γ_1	-	-0.054705 [a] (0.018628)	-	-0.077160 [b] (0.032481)
b_1	-	0.945498 [a] (0.017546)	-	0.872181 [a] (0.036273)
SBC	1.526597	1.411473	1.374893	1.291441

Standard errors are reported in parentheses.
[a] Indicates that the coefficient is statistically significant at 1% level of significance.
[b] Indicates that the coefficient is statistically significant at 5% level of significance.

Table 3. Lilliefors and Anderson-Darling statistics for the hypothesis that the daily returns standardized by i) the realized standard deviation, $\left\{y_t/\sigma_t^{(RV)}\right\}_{t=1}^{T}$, ii) the in-sample estimated realized standard deviation, $\left\{y_t/\hat{\sigma}_t^{(RV^*)}\right\}_{t=1}^{T}$, and ii) the one-day-ahead realized standard deviation forecast, $\left\{y_t/\sigma_{t|t-1}^{(RV^*)}\right\}_{t=1}^{\tilde{T}}$, are normally distributed.

	CAC40		DAX30	
	$\left\{y_t/\sigma_t^{(RV)}\right\}_{t=1}^{T}$			
Lilliefors	0.028 [0.00]		0.020 [0.042]	
Anderson-Darling	2.115 [0.00]		0.944 [0.014]	
	$\left\{y_t/\hat{\sigma}_t^{(RV^*)}\right\}_{t=1}^{T}$			
	ARFIMAX$(2,d,2)$	ARFIMAX$(1,d,1)$-TARCH$(1,1)$	ARFIMAX$(0,d,1)$	ARFIMAX$(1,d,1)$-TARCH$(1,1)$
Lilliefors	0.032 [0.00]	0.030 [0.00]	0.020 [0.045]	0.020 [0.041]
Anderson-Darling	3.166 [0.00]	2.695 [0.00]	2.018 [0.00]	1.381 [0.001]
	$\left\{y_t/\sigma_{t\|t-1}^{(RV^*)}\right\}_{t=1}^{\tilde{T}}$			
	ARFIMAX$(1,d,0)$	ARFIMAX$(1,d,2)$-TARCH$(2,1)$	ARFIMAX$(2,d,2)$	ARFIMAX$(1,d,0)$-TARCH$(1,1)$
Lilliefors	0.031 [0.016]	0.032 [0.011]	0.022 [>0.10]	0.029 [0.027]
Anderson-Darling	1.410 [0.001]	1.297[0.002]	0.595 [0.12]	1.413 [0.001]

P-values are displayed in squared brackets.

Table 4. *MSE* loss functions and the p-value of the SPA test for the null hypothesis that the ARFIMAX$(1,d,1)$-TARCH$(1,1)$ model provides the best in-sample realized volatility estimation.

Index	ARFIMAX$(2,d,2)$	ARFIMAX$(1,d,1)$-TARCH$(1,1)$	SPA p-value
CAC40	32.569	28.032	[0.53]
	ARFIMAX$(0,d,1)$	ARFIMAX$(1,d,1)$-TARCH$(1,1)$	SPA p-value
DAX30	9.054	7.329	[0.53]

Table 5. *PMSE* loss functions and the p-value of the SPA test for the null hypothesis that the ARFIMAX-TARCH model is not outperformed by the ARFIMAX one.

Index	ARFIMAX$(1,d,0)$	ARFIMAX$(1,d,2)$-TARCH$(2,1)$	SPA p-value
CAC40	45.438	50.838	[0.054]
	ARFIMAX$(2,d,2)$	ARFIMAX$(1,d,0)$-TARCH$(1,1)$	SPA p-value
DAX30	9.769	6.412	[0.55]